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A STARK CONJECTURE “OVER Z” FOR ABELIAN L-FUNCTIONS WITH MULTIPLE ZEROS

by Karl RUBIN

INTRODUCTION

In a series of papers [10], Stark developed a conjecture about the values of Artin L-functions at $s = 1$, or equivalently (by the functional equation) the first nonvanishing derivative at $s = 0$. In the final paper Stark presented a refined conjecture (“over $\mathbf{Z}$”) for abelian L-functions with simple zeros at $s = 0$, expressing the value of the derivative at $s = 0$ in terms of logarithms of global units.

In this paper we formulate an extension of this conjecture (in the abelian case) which includes the case of L-functions with higher order zeros at $s = 0$. The conjecture is stated in §2.1, and in §3 we prove several special cases of it. In §4 we give examples to show that certain other seemingly natural generalizations of Stark's conjecture, including one given in [9], are not true in general.

This work began as an attempt to understand the connection between Stark-type conjectures and Euler systems of global units, in the sense of Kolyvagin (see [8]). In §5 and §6 we develop this connection. For example, we show that the conjecture of §2.1 is closely related to a Gras-type conjecture equating the orders of the different eigenspaces of an ideal class

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group with the index of a special subgroup in an exterior power of a group of global units (Corollary 5.4).

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1. SETUP

1.1. General notation.

Fix a number field k and a finite abelian extension K of k. If w is a place of K we write K_w for the completion of K at w and $|\ \ |_w : K_w \to \mathbf{R}^+ \cup \{0\}$ for the absolute value normalized so that

$$|\alpha|_w = \begin{cases} \pm\alpha \text{ (the usual absolute value)} & \text{if } K_w = \mathbf{R}, \\ \alpha\bar{\alpha} & \text{if } K_w = \mathbf{C}, \\ \mathbf{N}w^{-\operatorname{ord}_w(\alpha)} & \text{if } K_w \text{ is nonarchimedean} \end{cases}$$

where $\mathbf{N}w$ is the cardinality of the residue field of the finite place w.

Fix a finite set S of places of k containing all infinite places and all places ramified in K/k, and a second finite set T of places of k, disjoint from S. Define

- $S_K = \{\text{places of } K \text{ lying above places in } S\}$
- $T_K = \{\text{places of } K \text{ lying above places in } T\}$
- $\mathcal{O}_S = \{\alpha \in K : |\alpha|_w \le 1 \text{ for all } w \notin S_K\}$, the S-integers of K
- $U_{S,T} = \{\alpha \in \mathcal{O}_S^\times : \alpha \equiv 1 \pmod{w} \text{ for all } w \in T_K\}$
- $A_{S,T}$ is the 'S_K-ray class group modulo T_K', the quotient of the group of fractional ideals of $\mathcal{O}_S$ prime to T_K by the subgroup of principal ideals with a generator congruent to 1 modulo all $w \in T_K$
- $Y_S = \bigoplus_{w \in S_K} \mathbf{Z}w$, the free abelian group on S_K
- $X_S = \{\sum a_w w \in Y_S : \sum a_w = 0\}$
- $\lambda_{S,T} : U_{S,T} \to X_S \otimes \mathbf{R}$ is the map defined by $\lambda(\alpha) = \sum_{w \in S_K} -\log(|\alpha|_w) w$
- $\boldsymbol{\mu}_T$ is the group of roots of unity in $U_{S,T}$
- $R_{S,T}$ is the absolute value of the determinant of $\lambda_{S,T}$ with respect to $\mathbf{Z}$-bases of $U_{S,T}/\boldsymbol{\mu}_T$ and X_S.

Note that these objects all depend on K, but except in §6, K will generally remain fixed so we will suppress it from the notation. When necessary we will refer to $\mathcal{O}_{K,S}$, $U_{K,S,T}$, etc. If S is the set of infinite places of k and T is empty, then $\mathcal{O}_S$, $U_{S,T}$, $A_{S,T}$, and $R_{S,T}$ are the usual ring of integers, unit group, ideal class group, and regulator of K, respectively.

There is a natural exact sequence

$$0 \to U_{S,T} \to \mathcal{O}_S^\times \to \bigoplus_{w \in T_K} \mathbf{F}_w^\times \to A_{S,T} \to \mathrm{Pic}(\mathcal{O}_S) \to 0 \tag{1}$$

where $\mathbf{F}_w$ is the residue field of K at w. If we define

$$\zeta_{S,T}(s) = \prod_{\mathfrak{p} \notin S_K} (1 - \mathbf{N}\mathfrak{p}^{-s})^{-1} \prod_{\mathfrak{p} \in T_K} (1 - \mathbf{N}\mathfrak{p}^{1-s}),$$

products over primes of K, then $\operatorname{ord}_{s=0} \zeta_{S,T} = \#(S_K) - 1$ and

$$\lim_{s \to 0} s^{1-\#(S_K)} \zeta_{S,T}(s) = -\frac{\#(A_{S,T}) R_{S,T}}{\#(\boldsymbol{\mu}_T)} \tag{2}$$

(see [4]).

Let $G = \mathrm{Gal}(K/k)$ and $\widehat{G} = \mathrm{Hom}(G, \mathbf{C}^\times)$. If v is a place of k and w is a place of K above v then we will write G_v or G_w for the corresponding decomposition group in G. If $\chi \in \widehat{G}$ we define the modified Artin L-function attached to χ

$$L_{S,T}(s, \chi) = \prod_{\mathfrak{p} \notin S} (1 - \chi(\mathrm{Frob}_\mathfrak{p}) \mathbf{N}\mathfrak{p}^{-s})^{-1} \prod_{\mathfrak{p} \in T} (1 - \chi(\mathrm{Frob}_\mathfrak{p}) \mathbf{N}\mathfrak{p}^{1-s})$$

where $\mathrm{Frob}_\mathfrak{p} \in G$ is the Frobenius of the (unramified) prime $\mathfrak{p}$.

For each $\chi \in \widehat{G}$ there is an idempotent

$$e_\chi = \frac{1}{\#(G)} \sum_{\gamma \in G} \chi(\gamma) \gamma^{-1}$$

and following [11] we define the Stickelberger element

$$\Theta_{S,T}(s) = \Theta_{K/k,S,T}(s) = \sum_{\chi \in \widehat{G}} e_\chi L_{S,T}(s, \bar{\chi})$$

which we view as a $\mathbf{C}[G]$-valued meromorphic function on $\mathbf{C}$. If $r \geq 0$ and $s^{-r}\Theta_{S,T}(s)$ is holomorphic at $s = 0$ we define

$$\Theta_{S,T}^{(r)}(0) \;=\; \lim_{s \to 0} s^{-r} \Theta_{S,T}(s) \;=\; \sum_{\chi \in \widehat{G}} e_\chi \lim_{s \to 0} s^{-r} L_{S,T}(s, \bar{\chi}) \quad \in \mathbf{C}[G].$$

If $k \subset K \subset K'$ and $S \subset S'$ then $\Theta_{K'/k,S',T}$ is a $\mathbf{C}[\mathrm{Gal}(K'/k)]$-valued meromorphic function and its image under the restriction map from $\mathrm{Gal}(K'/k)$ to G satisfies

$$\Theta_{K'/k,S',T}(s)|_K = \prod_{\mathfrak{p} \in S' - S} (1 - \mathrm{Frob}_\mathfrak{p}^{-1} \mathbf{N}\mathfrak{p}^{-s}) \Theta_{K/k,S,T}(s) \tag{3}$$

(see [11] Proposition IV.1.8).

1.2. $\mathbf{Z}[G]$-modules.

Suppose M is a $\mathbf{Z}[G]$-module. We will write $\mathbf{Q}M$, $\mathbf{R}M$, and $\mathbf{C}M$ for $M \otimes \mathbf{Q}$, $M \otimes \mathbf{R}$, and $M \otimes \mathbf{C}$, respectively. If r is a nonnegative integer then $\wedge^r M$ will denote the r-th exterior power of M in the category of $\mathbf{Z}[G]$-modules. In particular $\wedge^0 M = \mathbf{Z}[G]$ and $\wedge^1 M = M$.

If M' is another $\mathbf{Z}[G]$-module then $\mathrm{Hom}(M, M')$ will mean the G-*equivariant* homomorphisms from M to M'. We view $\mathrm{Hom}(M, M')$ as a $\mathbf{Z}[G]$-module by

$$(\alpha\varphi)(m) = \varphi(\alpha m) = \alpha\varphi(m).$$

We will identify $\mathrm{Hom}(M, \mathbf{Z}[G])$ with a submodule of $\mathrm{Hom}(\mathbf{Q}M, \mathbf{Q}[G])$ in the obvious way.

Every $\varphi \in \mathrm{Hom}(M, \mathbf{Z}[G])$ induces a G-equivariant homomorphism from $\wedge^r M$ to $\wedge^{r-1} M$ for all $r \geq 1$

$$m_1 \wedge \cdots \wedge m_r \mapsto \sum_{i=1}^{r} (-1)^{i+1} \varphi(m_i) m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \cdots \wedge m_r$$

which we will also denote by φ. Iterating this construction gives a map

$$\wedge^k \mathrm{Hom}(M, \mathbf{Z}[G]) \to \mathrm{Hom}(\wedge^r M, \wedge^{r-k} M) \tag{4}$$

$$\varphi_1 \wedge \cdots \wedge \varphi_k \mapsto \varphi_k \circ \cdots \circ \varphi_1$$

for every $k \leq r$; when $k = r$ this is the map

$$(\varphi_1 \wedge \cdots \wedge \varphi_r)(m_1 \wedge \cdots \wedge m_r) = \det(\varphi_i(m_j)).$$

Definition. — *A* $\mathbf{Z}[G]$-lattice *is a finitely-generated* $\mathbf{Z}[G]$*-module which is free as a* $\mathbf{Z}$*-module.*

Definition. — *If M is a finitely generated $\mathbf{Z}[G]$-module we define its dual M^* to be the $\mathbf{Z}[G]$-lattice* $\mathrm{Hom}(M, \mathbf{Z}[G]) \subset \mathrm{Hom}(\mathbf{Q}M, \mathbf{Q}[G])$. *Equivalently, M^* is the orthogonal complement of M under the natural pairing*

$$\mathbf{Q}M \times \mathrm{Hom}(\mathbf{Q}M, \mathbf{Q}[G]) \to \mathbf{Q}[G]/\mathbf{Z}[G].$$

Proposition 1.1.

(i) *If M is a $\mathbf{Z}[G]$-lattice then there is a canonical isomorphism*

$$M^{**} = M.$$

(ii) *If*

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of $\mathbf{Z}[G]$*-lattices then so is*

$$0 \to (M'')^* \to M^* \to (M')^* \to 0.$$

Proof. — If M is a $\mathbf{Z}[G]$-lattice there is a canonical isomorphism of abelian groups

$$\operatorname{Hom}(M, \mathbf{Z}[G]) \cong \operatorname{Hom}_{\mathbf{Z}}(M, \mathbf{Z}),$$

where $\operatorname{Hom}_{\mathbf{Z}}(M, \mathbf{Z})$ denotes the group of $\mathbf{Z}$-homomorphisms from M to $\mathbf{Z}$ (see for example [1] Proposition VI.3.4). Since a $\mathbf{Z}[G]$-lattice is a free $\mathbf{Z}$-module, both assertions follow easily. □

DEFINITION. — *Suppose M is a finitely generated* $\mathbf{Z}[G]$*-module and r is a nonnegative integer. Then using the natural map*

$$\iota : \wedge^r(M^*) \to (\wedge^r M)^* \tag{5}$$

coming from (4), we define

$$\wedge_0^r M = (\iota(\wedge^r(M^*)))^* \subset \mathbf{Q}\wedge^r M.$$

Equivalently,

$$\wedge_0^r M = \{m \in \mathbf{Q}\wedge^r M : (\varphi_1 \wedge \cdots \wedge \varphi_r)(m) \in \mathbf{Z}[G] \quad \textit{for every } \varphi_1, \ldots, \varphi_r \in \operatorname{Hom}(M, \mathbf{Z}[G])\}.$$

PROPOSITION 1.2. — *Suppose M is a* $\mathbf{Z}[G]$*-lattice and $r \geq 0$. Let $\overline{\wedge^r M}$ denote the image of $\wedge^r M$ in $\mathbf{Q}\wedge^r M$ and $g = \#(G)$.*

(i) $\wedge_0^r M \supset \overline{\wedge^r M}$ *and* $[\wedge_0^r M : \overline{\wedge^r M}]$ *is finite,*

(ii) $\wedge_0^r M = \wedge^r M$ *if* $r \leq 1$,

(iii) $\mathbf{Z}[1/g]\wedge_0^r M = \mathbf{Z}[1/g]\overline{\wedge^r M}$.

Proof. — The first assertion follows easily from the definition of $\wedge_0^r M$. If $r = 0$ then $\wedge_0^0 M = \mathbf{Z}[G] = \wedge^0 M$ and if $r = 1$ then $\wedge_0^1 M = M^{**} = M$ by Proposition 1.1 (i), which proves (ii).

For any $\mathbf{Z}[G]$-module M

$$\mathbf{Z}[1/g] \operatorname{Hom}(M, \mathbf{Z}[G]) = \operatorname{Hom}(\mathbf{Z}[1/g]M, \mathbf{Z}[1/g][G]),$$

so if ι is as in (5),

$$\begin{aligned}\mathbf{Z}[1/g]\wedge_0^r M &= \mathbf{Z}[1/g]\operatorname{Hom}(\iota(\wedge^r \operatorname{Hom}(M,\mathbf{Z}[G])),\mathbf{Z}[G]) \\ &= \operatorname{Hom}(\iota(\wedge^r \operatorname{Hom}(\mathbf{Z}[1/g]M,\mathbf{Z}[1/g][G])),\mathbf{Z}[1/g][G]) \\ &= \wedge^r \mathbf{Z}[1/g]M \\ &= \mathbf{Z}[1/g]\overline{\wedge^r M},\end{aligned}$$

the third equality because $\mathbf{Z}[1/g]M$ is a projective $\mathbf{Z}[1/g][G]$-module. This proves (iii). □

Examples.

(1) If M is a free $\mathbf{Z}[G]$-module then $\wedge_0^r M = \overline{\wedge^r M}$ for every r.

(2) Suppose $M = \mathbf{Z}^s$ with trivial G-action, $s > 0$. Then for every $r > 0$, $\wedge^r M = \mathbf{Z}^{\binom{s}{r}}$ and it is easy to check that $\wedge_0^r M = \#(G)^{1-r}\overline{\wedge^r M}$. Thus if $1 < r \le s$, $\wedge_0^r M$ is strictly larger than $\overline{\wedge^r M}$.

(3) Suppose G is cyclic of odd prime power order p^n, I is the augmentation ideal of $\mathbf{Z}[G]$, and $M = I \times \cdots \times I \subset \mathbf{Z}[G]^s$. If σ is a generator of G then $(\sigma-1)^{p-1}M = pM$. If k is the smallest integer greater than or equal to $(r-1)/(p-1)$, one can show that

$$\wedge_0^r M \supset p^{-k}(\sigma-1)^{k(p-1)+1-r}\overline{\wedge^r M} \supset p^{1-k}\overline{\wedge^r M}.$$

COROLLARY 1.3. — *Suppose M is a $\mathbf{Z}[G]$-lattice and $r \ge 1$. If $\Phi \in \wedge^{r-1}\operatorname{Hom}(M,\mathbf{Z}[G])$ then Φ induces a map*

$$\wedge_0^r M \dashrightarrow M.$$

Proof. — The construction (4) shows that every $\Phi \in \wedge^{r-1}\operatorname{Hom}(M, \mathbf{Z}[G])$ induces a map from $\mathbf{Q}\wedge^r M$ to $\mathbf{Q}\wedge^1 M = \mathbf{Q}M$, and it follows easily from the definition of $\wedge_0^r M$ and Proposition 1.2 (ii) that

$$\Phi(\wedge_0^r M) \subset \wedge_0^1 M = M.$$ □

2. CONJECTURES

2.1. Statement of the conjectures.

Suppose S and T are as in §1.1 and r is a positive integer. Before stating our conjectures we record some hypotheses on S, T, and r.

HYPOTHESES 2.1. — *S and T are disjoint finite sets of places of k, and r is a nonnegative integer, satisfying*

2.1.1. *S contains all the infinite places of k,*

2.1.2. *S contains all places ramifying in K/k,*

2.1.3. *S contains at least r places which split completely in K/k,*

2.1.4. $\#(S) \geq r+1$,

2.1.5. *$U_{S,T}$ is torsion-free.*

Condition (2.1.5) means that there are no roots of unity in K congruent to 1 modulo all primes in T_K. In particular this will be satisfied if T contains primes of two different residue characteristics or one prime of sufficiently large norm. Conditions (2.1.3) and (2.1.4) ensure that $s^{-r}\Theta_{S,T}(s)$ is holomorphic at $s=0$.

Write $\lambda^{(r)} : \wedge^r U_{S,T} \to \mathbf{R}\wedge^r X_S$ for the map induced by $\lambda_{S,T}$.

CONJECTURE A. — *If S, T, and r satisfy Hypotheses 2.1, then*

$$\Theta_{S,T}^{(r)}(0)\wedge^r X_S \subset \mathbf{Q}\lambda^{(r)}(\wedge^r U_{S,T}) \quad \textit{in} \quad \mathbf{R}\wedge^r X_S.$$

CONJECTURE B. — *If S, T, and r satisfy Hypotheses 2.1, then*

$$\Theta_{S,T}^{(r)}(0)\wedge^r X_S \subset \lambda^{(r)}(\wedge_0^r U_{S,T}) \quad \textit{in} \quad \mathbf{R}\wedge^r X_S.$$

Recall $Y_S^* = \mathrm{Hom}(Y_S, \mathbf{Z}[G])$. There is a determinant pairing

$$\wedge^r X_S \times \wedge^r Y_S^* \to \mathbf{Z}[G]$$

and if $\eta \in \wedge^r Y_S^*$ we define a regulator map

$$R_\eta : \wedge^r U_{S,T} \xrightarrow{\lambda^{(r)}} \mathbf{R}\wedge^r X_S \xrightarrow{\eta} \mathbf{R}[G].$$

If $w \in S_K$ define $w^* \in Y_S^*$ by

$$w^*(w') = \sum_{\gamma w = w'} \gamma \quad \text{for } w' \in S_K.$$

If $\gamma \in G_w$ then $\gamma w^* = w^*$, so $e_\chi w^* = 0$ if χ is nontrivial on G_w.

LEMMA 2.2. — *If $u_1, \ldots, u_r \in U_{S,T}$, $w_1, \ldots, w_r \in S_K$ and $\eta = w_1^* \wedge \cdots \wedge w_r^*$, then*

$$R_\eta(u_1 \wedge \cdots \wedge u_r) = \det\Big(\sum_{\gamma\in G} \log|u_i^\gamma|_{w_j}\gamma^{-1}\Big).$$

Proof. — By definition

$$R_\eta(u_1 \wedge \cdots \wedge u_r) = \eta(\lambda(u_1) \wedge \cdots \wedge \lambda(u_r)) = \det(w_j^*(\lambda(u_i))),$$

and

$$w_j^*(\lambda(u_i)) = w_j^*\Big(\sum_{w \in S_K} \log|u_i|_w w\Big) = \sum_{\gamma \in G} \log|u_i|_{\gamma w_j}\gamma.$$

This proves the assertion, since $\log|u_i|_{\gamma w_j} = \log|u_i^{\gamma^{-1}}|_{w_j}$. □

Write $\mathbf{1}$ for the trivial character of G. For every $\chi \in \widehat{G}$ define a nonnegative integer $r(\chi) = r(\chi, S)$ by

$$\text{(6)} \qquad r(\chi) = \operatorname*{ord}_{s=0} L_{S,T}(s,\chi) = \dim_{\mathbf{C}} e_\chi \mathbf{C} X_S = \dim_{\mathbf{C}} e_\chi \mathbf{C} U_{S,T} = \begin{cases} \#(\{v \in S : \chi(G_v) = 1\}) & \text{if } \chi \neq \mathbf{1} \\ \#(S) - 1 & \text{if } \chi = \mathbf{1} \end{cases}$$

(see for example [11] Proposition I.3.4). If S, T, and r satisfy Hypotheses 2.1, then $r(\chi) \geq r$ for every χ and we define a $\mathbf{Z}[G]$-lattice $\Lambda_{S,T} = \Lambda_{K,S,T,r} \subset \mathbf{Q}\wedge^r U_{S,T}$ by

$$\Lambda_{S,T} = \{\alpha \in \wedge_0^r U_{S,T} : e_\chi \alpha = 0 \text{ in } \mathbf{C}\wedge^r U_{S,T} \text{ for every } \chi \in \widehat{G} \text{ such that } r(\chi) > r\}.$$

CONJECTURE A′. — *Suppose S, T, and r satisfy Hypotheses 2.1, and $v_1, \ldots, v_r \in S$ split completely in K/k. For each i fix a place w_i of K above v_i and let $\eta = w_1^* \wedge \cdots \wedge w_r^*$. Then there is a unique $\varepsilon_{S,T} \in \mathbf{Q}\Lambda_{S,T}$ such that*

$$R_\eta(\varepsilon_{S,T}) = \Theta_{S,T}^{(r)}(0).$$

CONJECTURE B′. — *With hypotheses as in Conjecture* A′, *there is a unique $\varepsilon_{S,T} \in \Lambda_{S,T}$ such that*

$$R_\eta(\varepsilon_{S,T}) = \Theta_{S,T}^{(r)}(0).$$

Remark. — Note that the $\varepsilon_{S,T}$ of Conjectures A′ and B′ depends (in a simple way) on the choice of η, but the truth of the conjectures does not.

2.2. Relations among the various conjectures.

We first state the relations among Conjectures A, B, A′, and B′ and the conjectures of Stark and Tate in the literature. They will be proved in the next section after some additional remarks.

PROPOSITION 2.3. — *Conjecture* A *is equivalent to Stark's conjecture "over* **Q**" *(Conjecture* I.5.1 *of* [11]*) for the characters* $\chi \in \widehat{G}$ *such that* $r(\chi) = r$.

PROPOSITION 2.4. — *Conjecture* A *is equivalent to Conjecture* A′ *and Conjecture* B *is equivalent to Conjecture* B′.

PROPOSITION 2.5. — *If* $r = 1$ *then Conjectures* B *and* B′ *(for fixed* S *and all appropriate* T*) are equivalent to the conjecture* $\mathrm{St}(K/k, S)$ *of* [11] §IV.2.

Remarks.

(1) A more obvious guess for Conjectures B and B′ might be to replace $\wedge_0^r U_{S,T}$ by the smaller lattice $\wedge^r U_{S,T}$. This turns out to be false; see §4.1.

(2) When $r > 1$, Conjecture B′ does not predict the existence of particular units of K, as it does when $r = 1$. This is unfortunate but it is to be expected, since all that the L-function gives in these cases is an $r \times r$ regulator. However, one can use Conjecture B′ to produce units in the following way. If $\varepsilon_{S,T} \in \Lambda_{S,T}$ is the element predicted by Conjecture B′, then Corollary 1.3 and Hypothesis 2.1.5 give units $\Phi(\varepsilon_{S,T}) \in U_{S,T}$ for every $\Phi \in \wedge^{r-1} \mathrm{Hom}(U_{S,T}, \mathbf{Z}[G])$. See §6.

2.3. Proofs of the relations.

Suppose S, T, and r satisfy Hypotheses 2.1. Fix $v_1, \ldots, v_r \in S$ splitting completely in K/k and for each i fix a place w_i of K above v_i. Let

$$S'_K = \{w \in S_K : w \text{ does not lie above } v_1, \ldots, v_r\},$$

which is nonempty because of (2.1.4).

LEMMA 2.6. — *Let* $w_1, \ldots, w_r$ *be as above and let* $w \in S'_K$.

(i) *If* $\chi \neq \mathbf{1}$ *or* $\#(S) > r+1$, *then* $e_\chi \Theta_{S,T}^{(r)}(0)w = 0$ *in* $\mathbf{C}Y$.

(ii) *Let* $\mathbf{x} = (w_1 - w) \wedge \cdots \wedge (w_r - w) \in \wedge^r X$. *Then*

$$\Theta_{S,T}^{(r)}(0)\wedge^r X = \Theta_{S,T}^{(r)}(0)\mathbf{Z}[G]\mathbf{x}.$$

Proof. — If either $\chi \neq \mathbf{1}$ or $\#(S) > r+1$, then by (6),

$$\chi(G_w) = 1 \Rightarrow e_\chi \Theta_{S,T}^{(r)}(0) = 0.$$

On the other hand, for every χ

$$\chi(G_w) \neq 1 \Rightarrow e_\chi w = 0.$$

This proves (i).

Clearly the left hand side of (i) contains the right hand side, so we need only show the other inclusion. Any element of X can be written in the form

$$\sum_{i=1}^{r} \alpha_i(w_i - w) + \sum_{w' \in S'_K} \beta_{w'} w' \tag{7}$$

where $\alpha_i, \beta_{w'} \in \mathbf{Z}[G]$. Thus any element $\mathbf{y} \in \wedge^r X$ can be written

$$\mathbf{y} = \alpha\mathbf{x} + \sum_{\mathbf{w}} \beta_{\mathbf{w}}\mathbf{w} \tag{8}$$

where $\alpha, \beta_{\mathbf{w}} \in \mathbf{Z}[G]$ and $\mathbf{w}$ runs over monomials $w'_1 \wedge \cdots \wedge w'_r$ where at least one of the $w'_i \in S'_K$. If $\chi \neq \mathbf{1}$ or $\#(S) > r+1$ then by (i),

$$e_\chi \Theta_{S,T}^{(r)}(0)\mathbf{y} = e_\chi \Theta_{S,T}^{(r)}(0)\alpha\mathbf{x}. \tag{9}$$

Suppose now that $\chi = \mathbf{1}$ and $\#(S) = r+1$. Then the second sum in (7) is just a single term $\beta_w w$ where $e_{\mathbf{1}}\beta_w = 0$, and it follows that $e_{\mathbf{1}}\beta_{\mathbf{w}} = 0$ for each of the coefficients $\beta_{\mathbf{w}}$ in the sum in (8). Thus (9) holds in this case as well, and (ii) follows. □

Proof of Proposition 2.3 (sketch). — Let $\Xi = \{\chi \in \widehat{G} : r(\chi) = r\}$ and suppose that Conjecture I.5.1 of [11] is true for the characters $\chi \in \Xi$. Then there is a $\mathbf{Q}[G]$-isomorphism $f : \mathbf{Q}X_S \xrightarrow{\sim} \mathbf{Q}U_{S,T}$ such that the quantities

$$a(\chi, f) = \lim_{s\to 0} s^{-r} L_{S,T}(0, \bar{\chi}) / \det(\lambda_{S,T} \circ f_\chi),$$

where $\lambda_{S,T} \circ f_\chi : e_\chi \mathbf{C}X_S \to e_\chi \mathbf{C}X_S$ is the restriction of $\lambda_{S,T} \circ f$, satisfy

$$a(\chi^\alpha, f) = a(\chi, f)^\alpha \tag{10}$$

for every $\chi \in \Xi$ and every automorphism α of $\mathbf{C}$. Define

$$\rho = \sum_{\chi \in \Xi} e_\chi a(\chi, f).$$

Since Ξ is stable under $\mathrm{Aut}(\mathbf{C})$, (10) shows that $\rho \in \mathbf{Q}[G]$. From the definition of the $a(\chi, f)$ we see also that

$$\rho \sum e_\chi \det(\lambda_{S,T} \circ f_\chi) = \Theta_{S,T}^{(r)}(0).$$

Thus if $\mathbf{x} \in \wedge^r X_S$, and $f^{(r)} : \mathbf{Q}\wedge^r X_S \xrightarrow{\sim} \mathbf{Q}\wedge^r U_{S,T}$ denotes the map induced by f,

$$\begin{aligned}\Theta_{S,T}^{(r)}(0)\wedge^r\mathbf{x} &= \rho \sum e_\chi \det(\lambda_{S,T} \circ f_\chi)\mathbf{x} \\ &= \rho \sum \lambda^{(r)} \circ f^{(r)}(e_\chi \mathbf{x}) = \rho\lambda^{(r)}(f^{(r)}(\mathbf{x})) \in \mathbf{Q}\lambda^{(r)}(\wedge^r U_{S,T})\end{aligned}$$

which proves Conjecture A. The converse is similar and (as we will not use either direction) we omit it. □

LEMMA 2.7. — *Suppose* $w_1, \ldots, w_r$ *are as above and set* $\eta = w_1^* \wedge \cdots \wedge w_r^* \in \wedge^r Y_S^*$.

(i) η *is injectve on* $\Theta_{S,T}^{(r)}(0)\mathbf{C}\wedge^r X_S = \mathbf{C}\lambda^{(r)}(\Lambda_{S,T})$.

(ii) R_η *is injective on* $\mathbf{C}\Lambda_{S,T}$.

(iii) *If* $\mathbf{u} \in \mathbf{C}\wedge^r U_{S,T}$ *satisfies* $R_\eta(\mathbf{u}) = 0$ *and* $e_\chi \mathbf{u} = 0$ *for every* $\chi \in \widehat{G}$ *such that* $r(\chi) > r$, *then* $\mathbf{u} = 0$.

Proof. — Suppose $\chi \in \widehat{G}$ and $r(\chi) = r$. Then by (6), $\dim_{\mathbf{C}}(e_\chi \mathbf{C} X_S) = r$ and so $\dim_{\mathbf{C}}(e_\chi \wedge^r \mathbf{C} X_S) = 1$. If $w \in S'_K$ then $\eta(e_\chi(w_1 - w)\wedge\cdots\wedge(w_r - w)) = e_\chi$, so η is injective on $e_\chi \mathbf{C}\wedge^r X_S$. This proves (i) and, since $R_\eta = \eta \circ \lambda^{(r)}$ and $\lambda^{(r)}$ is an isomorphism from $\mathbf{C}\wedge^r U_{S,T}$ to $\mathbf{C}\wedge^r X_S$, proves (ii) as well.

If $\mathbf{u}$ is as in (iii), then $\mathbf{u} \in \mathbf{C}\Lambda_{S,T}$ and so (ii) shows that $\mathbf{u} = 0$. □

Proof of Proposition 2.4 (see Proposition IV.2.4 of [11]). — Fix S, T, and r satisfying Hypotheses 2.1, and let $\mathbf{x}$ be as in Lemma 2.6 (ii) for some $w \in S'_K$. Then

$$\begin{aligned}\Theta_{S,T}^{(r)}(0)\wedge^r X_S \subset \lambda^{(r)}(\wedge_0^r U_{S,T}) &\Leftrightarrow \Theta_{S,T}^{(r)}(0)\mathbf{x} \in \lambda^{(r)}(\wedge_0^r U_{S,T}) \\ &\Leftrightarrow \Theta_{S,T}^{(r)}(0)\mathbf{x} \in \lambda^{(r)}(\Lambda_{S,T}) \\ &\Leftrightarrow \eta(\Theta_{S,T}^{(r)}(0)\mathbf{x}) \in \eta \circ \lambda^{(r)}(\Lambda_{S,T}) \\ &\Leftrightarrow \Theta_{S,T}^{(r)}(0) \in R_\eta(\Lambda_{S,T})\end{aligned}$$

where the equivalences come from Lemma 2.6 (ii), the injectivity of $\lambda^{(r)}$ on $\mathbf{Q}\wedge^r U_{S,T}$, (i), and the relations $R_\eta = \eta \circ \lambda^{(r)}$ and $\eta(\mathbf{x}) = 1$, respectively. This shows that Conjecture B is equivalent to Conjecture B′, with the uniqueness coming from Lemma 2.7 (iii). The proof that Conjectures A and A′ are equivalent is the same, with $\wedge_0^r U_{S,T}$ replaced by $\mathbf{Q}\wedge_0^r U_{S,T}$ and $\Lambda_{S,T}$ by $\mathbf{Q}\Lambda_{S,T}$. □

Proof of Proposition 2.5. — Fix a set S satisfying Hypotheses (2.1.1) through (2.1.4) with $r = 1$. By Lemma 1.2 (ii), Conjecture B′ asserts that $\Theta_{S,T}^{(r)}(0)X_S \subset \lambda_{S,T}(U_{S,T})$ for all T satisfying (2.1.5). To get back and forth between this statement and Conjecture IV.2.1 of Tate [11], use Proposition IV.1.2 of [11] and the relations

$$\Theta_{S,T}^{(r)}(0) = \prod_{\mathfrak{q}\in T}(1-\mathrm{Frob}_\mathfrak{q}^{-1}\mathbf{N}\mathfrak{q})\Theta_{S,\emptyset}^{(r)}(0), \quad \prod_{\mathfrak{q}\in T}(1-\mathrm{Frob}_\mathfrak{q}^{-1}\mathbf{N}\mathfrak{q})U_{S,\emptyset} \subset U_{S,T}. \; \square$$

3. SPECIAL CASES

We will denote the equivalent Conjectures B and B′ for a given S, T and r by $\mathrm{St}(K/k, S, T, r)$. In this section we prove $\mathrm{St}(K/k, S, T, r)$ in some special cases.

3.1. S contains more than r places which split completely.

PROPOSITION 3.1. — *Suppose S contains more than r places which split completely in K/k. Then $\mathrm{St}(K/k, S, T, r)$ is true.*

Proof (Compare [11] Proposition IV.3.1). — In this case (6) shows that $e_\chi\Theta_{S,T}^{(r)}(0) = 0$ if $\chi \neq \mathbf{1}$, so

$$\Theta_{S,T}^{(r)}(0) = \lim_{s\to 0} s^{-r}\zeta_{k,S,T}e_\mathbf{1}.$$

Write $A_k = A_{k,S,T}$ and $R_k = R_{k,S,T}$. If $\#(S) > r+1$ then $\Theta_{S,T}^{(r)}(0) = 0$ and $\mathrm{St}(K/k, S, T, r)$ is trivially true. Thus we may assume $\#(S) = r+1$, and by (2)

$$\Theta_{S,T}^{(r)}(0) = -\#(A_k)R_k e_\mathbf{1} = -\frac{\#(A_k)}{\#(G)}R_k\mathbf{N}_G$$

where $\mathbf{N}_G = \sum_{\gamma \in G} \gamma$.

Fix a basis $\{u_1, \ldots, u_r\}$ of the free $\mathbf{Z}$-module $U_{k,S,T}$ and define

$$\varepsilon = \frac{\#(A_k)}{\#(G)^r} u_1 \wedge \cdots \wedge u_r.$$

With η as in Conjecture B′ (for any choice of $\{w_1, \ldots, w_r\}$) Lemma 2.2 shows that

$$R_\eta(\varepsilon) = \pm \frac{\#(A_k)}{\#(G)^r} R_k \mathbf{N}_G^r = \pm \Theta_{S,T}^{(r)}(0).$$

Also $\varepsilon \in (\mathbf{Q}\wedge^r U_{K,S,T})^G$, so $e_\chi \varepsilon = 0$ for $\chi \neq \mathbf{1}$.

To complete the proof we must verify that $\varepsilon \in \wedge_0^r U_{K,S,T}$. In other words, for every $\varphi_1, \ldots, \varphi_r \in \mathrm{Hom}(U_{K,S,T}, \mathbf{Z}[G])$ we must show

$$(\varphi_1 \wedge \cdots \wedge \varphi_r)(\varepsilon) = \frac{\#(A_k)}{\#(G)^r} \det(\varphi_i(u_j)) \in \mathbf{Z}[G].$$

For every i and j,

$$\varphi_i(u_j) \in \mathbf{Z}[G]^G = \mathbf{N}_G \mathbf{Z}[G]$$

so $\det(\varphi_i(u_j)) \in \mathbf{N}_G^r \mathbf{Z}[G] = \#(G)^{r-1} \mathbf{N}_G \mathbf{Z}[G]$ and

$$(\varphi_1 \wedge \cdots \wedge \varphi_r)(\varepsilon) \in \frac{\#(A_k)}{\#(G)} \mathbf{Z}[G].$$

Since $\#(S) = r+1$, all places in S split completely in K/k. But S contains all places ramifying in K/k, so K/k is everywhere unramified. Thus by class field theory $\#(G)$ divides $\#(A_k)$, which completes the proof. □

Remarks.

(1) By Lemma 2.7 (iii), $\pm\varepsilon$ is the unique element of $\mathbf{Q}\wedge^r U_{K,S,T}$ which can satisfy Conjecture B′. It is *not* always true that $\varepsilon \in \wedge^r U_{K,S,T}$ (see §4.1), which is why we state the conjectures with $\wedge_0^r U_{K,S,T}$ instead.

(2) By Proposition 3.1 we lose no generality in Conjectures B and B′ if we assume that S has exactly r places which split completely in K/k.

COROLLARY 3.2. — *$\mathrm{St}(K/k, S, T, r)$ is true when $K = k$.*

Proof. — Since we assume that $\#(S) \geq r + 1$, this is immediate from Proposition 3.1. □

3.2. $r = 0$.

THEOREM 3.3. — $\mathrm{St}(K/k, S, T, 0)$ *is true.*

Proof. — We have $\wedge^0 X_S = \wedge_0^0 U_{S,T} = \mathbf{Z}[G]$, so Conjecture B is the assertion that $\Theta_{S,T}(0) \in \mathbf{Z}[G]$.

If k is totally real, $\Theta_{S,T}(0) \in \mathbf{Z}[G]$ by the theorem of Deligne and Ribet [2]. If k is not totally real, then S has at least one (complex) place which splits completely, and we are done by Proposition 3.1. □

3.3. Quadratic extensions.

Fix for this section S, T, and r satisfying Hypotheses 2.1. We will abbreviate $U_k = U_{k,S,T}$, $U_K = U_{K,S,T}$, $A_k = A_{k,S,T}$, $A_K = A_{K,S,T}$, $R_k = R_{k,S,T}$, and $R_K = R_{K,S,T}$. Let $h_K = \#(A_K)$ and $h_k = \#(A_k)$.

LEMMA 3.4. — *Suppose G is cyclic and S contains at least one place v such that $G_v = G$. Then*

(i) $h_k \mid h_K$,

(ii) $\#(H^1(G, U_K)) \mid h_k$,

(iii) *if $\#(G)$ is a prime power and $\widehat{H}^0(G, U_K) = H^1(G, U_K) = 0$ then h_K/h_k is prime to $\#(G)$ if and only if h_k is prime to $\#(G)$.*

Proof. — Write H_k and H_K for the (S,T)-ray class fields of k and K, respectively, so that class field theory give identifications $\mathrm{Gal}(H_k/k) = A_k$ and $\mathrm{Gal}(H_K/K) = A_K$. Since all primes in S split completely in H_k, $K \cap H_k = k$. Thus the norm map $A_K \to A_k$, which is the restriction map $\mathrm{Gal}(H_K/K) \to \mathrm{Gal}(H_k/k)$, is surjective and (i) follows.

Comparing cohomology of units, ideals, principal ideals and ideal classes of K gives an exact sequence (see for example [12] Corollary 2, which must be adapted in our case to incorporate T)

$$0 \to H^1(G, U_K) \to A_k \to A_K^G \to \widehat{H}^0(G, U_K). \tag{11}$$

This proves (ii).

Suppose now that $\widehat{H}^0(G, U_K) = H^1(G, U_K) = 0$. Then (11) shows that $A_k \cong A_K^G$. If p divides both h_k and $\#(G)$, then the cokernel of the norm map $A_K^G \to A_k$ (which we can identify with multiplication by $\#(G)$

on A_k) also has order divisible by p. Since $A_K \to A_k$ is surjective, it follows that $p \mid (h_K/h_k) = \#(A_K/A_K^G)$.

On the other hand, if h_k is prime to $\#(G)$ then so is $\#(A_K^G)$. But if G is a p-group and $p \nmid \#(A_K^G)$ then $p \nmid h_K$. This completes the proof of (iii). $\square$

THEOREM 3.5. — *If K/k is a quadratic extension then* $\mathrm{St}(K/k, S, T, r)$ *is true.*

Proof (Compare [11] Theorem IV.5.4). — Let χ denote the nontrivial character of G. If S contains more than r places which split completely then the theorem is true by Proposition 3.1. Thus we can assume that S contains exactly r places which split, and so $\operatorname*{ord}_{s=0} L_{S,T}(s,\chi) = r$.

Write $S = \{v_1, \dots, v_{r'}\}$ where $r' > r$ and $v_1, \dots, v_r$ split completely in K/k, and fix a w_i of K above each v_i. Let $\bar{w}_i$ denote the conjugate of w_i for $1 \le i \le r$. Fix a $\mathbf{Z}$-basis $\{u_1, \dots, u_{r+r'-1}\}$ of U_K such that $\{u_1, \dots, u_{r'-1}\}$ is a basis of U_k. (This is possible because our hypothesis on T ensures that U_K, U_k, and U_K/U_k are all torsion-free.) If $H^1(G, U_K) \ne 0$ then we also require that $\mathbf{N}_{K/k} u_{r'} = 1$. With respect to this basis of U_K and the places $\{w_{r+2}, \dots, w_{r'}, \bar{w}_1, \dots, \bar{w}_r, w_1, \dots, w_r\} \subset S_K$, R_K is the absolute value of the determinant of the $(r + r' - 1) \times (r + r' - 1)$ matrix $(\log |u|_w)$ which (since $|u_i|_{w_j} = |u_i|_{\bar{w}_j}$ for $i \le r' - 1$, $j \le r$) has the form

$$\begin{pmatrix} B_1 & B_2 & B_2 \\ B_3 & B_4 & B_5 \end{pmatrix}.$$

Thus

$$R_K = \pm \det \begin{pmatrix} B_1 & B_2 \end{pmatrix} \det \begin{pmatrix} B_5 - B_4 \end{pmatrix}.$$

Because of the way we normalize our valuations, for $1 \le i \le r' - 1$,

$$\log |u_i|_{w_j} = \begin{cases} \log |u_i|_{v_j} & \text{if } j \le r \\ 2 \log |u_i|_{v_j} & \text{if } j > r, \end{cases}$$

so

$$\det \begin{pmatrix} B_1 & B_2 \end{pmatrix} = \pm 2^{r'-r-1} R_k.$$

Also, if $\varepsilon_- = u_{r'} \wedge \cdots \wedge u_{r'+r-1} \in \wedge^r U_K$ and $\eta = w_1^* \wedge \cdots \wedge w_r^*$ as in Conjecture B$'$, then by Lemma 2.2

$$e_\chi R_\eta(\varepsilon_-) = \det \begin{pmatrix} B_5 - B_4 \end{pmatrix} e_\chi.$$

Using the fact that $\zeta_{K,S,T}(s) = \zeta_{k,S,T}(s) L_{S,T}(s,\chi)$, using (2) for the two zeta functions, and replacing $u_{r'}$ by $u_{r'}^{-1}$ if necessary to correct the sign, we conclude that

$$e_\chi \Theta_{S,T}^{(r)}(0) = 2^{r'-r-1} (h_K/h_k) e_\chi R_\eta(\varepsilon_-).$$

Case I: $r' > r+1$.

In this case $\operatorname*{ord}_{s=0} \zeta_{k,S,T} > r$, so $\Theta_{S,T}^{(r)}(0) = e_\chi \Theta_{S,T}^{(r)}(0)$. Define

$$\varepsilon = h_K/h_k(2^{r'-r-1}e_\chi)\varepsilon_-,$$

so $R_\eta(\varepsilon) = \Theta_{S,T}^{(r)}(0)$. Also $\varepsilon \in \wedge^r U_K$ by Lemma 3.4 (i), and clearly $e_\mathbf{1}\varepsilon = 0$ so $\varepsilon \in \Lambda_{S,T}$. Thus Conjecture B$'$ is satisfied with $\varepsilon_{S,T} = \varepsilon$.

Case II: $r' = r+1$.

Choose $\bar{u}_1, \ldots, \bar{u}_r \in U_K$ so that $\{\mathbf{N}_{K/k}\bar{u}\}$ is a basis for $\mathbf{N}_{K/k}U_K \subset U_k$. If $\widehat{H}^0(G, U_K) \neq 0$ then we also require that $\bar{u}_1$ belongs to U_k. Let $\varepsilon_+ = \bar{u}_1 \wedge \cdots \wedge \bar{u}_r \in \wedge^r U_K$. Then (replacing $\bar{u}_1$ by $\bar{u}_1^{-1}$ if necessary)

$$e_\mathbf{1} R_\eta(\varepsilon_+) = -[U_k : \mathbf{N}_{K/k}U_K] R_k e_\mathbf{1}$$

so by (2)

$$e_\mathbf{1}\Theta_{S,T}^{(r)}(0) = h_k/\#(\widehat{H}^0(G, U_K)) e_\mathbf{1} R_\eta(\varepsilon_+).$$

In this case define

$$\varepsilon = h_k/\#(\widehat{H}^0(G, U_K)) e_\mathbf{1}\varepsilon_+ + h_K/h_k e_\chi \varepsilon_- \in \mathbf{Q}\wedge^r U_K.$$

Then $R_\eta(\varepsilon) = \Theta_{S,T}^{(r)}(0)$, and in this case $\Lambda_{S,T} = \wedge_0^r U_K$. Thus we will be done if we show that ε is in the image of $\wedge^r U_K$ in $\mathbf{Q}\wedge^r U_K$.

Since $r' = r+1$, U_K has a submodule of finite index which is free of rank r over $\mathbf{Z}[G]$. Thus the Herbrand quotient shows that $\#(\widehat{H}^0(G, U_K)) = \#(H^1(G, U_K))$. Suppose first that $\widehat{H}^0(G, U_K) \neq 0$. Then our choice of $u_{r'}$ and $\bar{u}_1$ ensures that $e_\mathbf{1}\varepsilon_+ = \varepsilon_+$ and $e_\chi\varepsilon_- = \varepsilon_-$ in $\mathbf{Q}\wedge^r U_K$. Thus by Lemma 3.4

$$\varepsilon = h_k/\#(\widehat{H}^0(G, U_K))\varepsilon_+ + h_K/h_k\varepsilon_- \in \wedge^r U_K.$$

Now suppose $\widehat{H}^0(G, U_K) = 0$. It follows that U_K is a free, rank-r $\mathbf{Z}[G]$-module (see [13] Theorem 4.19), so $\{u_{r'}, \ldots, u_{r'+r-1}\}$ is a $\mathbf{Z}[G]$-basis of U_K and $e_\mathbf{1}\varepsilon_- = \pm e_\mathbf{1}\varepsilon_+$. In this case we have

$$\varepsilon = (h_K/h_k e_\chi \pm h_k e_\mathbf{1})\varepsilon_-,$$

which belongs to $\wedge^r U_K$ because $h_K/h_k \pm h_k$ lies in $2\mathbf{Z}[G]$ by Lemma 3.4. □

Remark. — Note that the proof of Theorem 3.5 shows that $\varepsilon_{S,T} \in \wedge^r U_K$, not just that $\varepsilon_{S,T} \in \Lambda_{S,T}$.

3.4. Changing S.

PROPOSITION 3.6. — *Suppose (S,T,r) satisfies Hypotheses 2.1 and $S' \supset S$ is a finite set of places of k disjoint from T. Then (S',T,r) satisfies Hypotheses 2.1 and*

$$\mathrm{St}(K/k,S,T,r) \Rightarrow \mathrm{St}(K/k,S',T,r).$$

Proof (Compare [11] Proposition IV.3.4). — That (S',T,r) satisfies Hypotheses 2.1 is immediate. By (3),

$$\Theta^{(r)}_{S',T}(0) = \prod_{v\in S'-S} (1-\mathrm{Frob}_v^{-1})\Theta^{(r)}_{S,T}(0).$$

It is easy to check that $\prod_{v\in S'-S}(1-\mathrm{Frob}_v^{-1})\Lambda_{S,T} \subset \Lambda_{S',T}$, and using Lemma 2.6 (ii) the proposition follows. □

4. (COUNTER)EXAMPLES

In this section we give examples showing that certain other plausible extensions of Stark's conjecture are not true in general.

4.1. All places in S split completely.

We first construct K, k, S, T, and r satisfying Hypotheses 2.1 such that $\Theta^{(r)}_{S,T}(0)\wedge^r X_S \not\subset \lambda^{(r)}(\wedge^r U_{S,T})$. Write A_k for the ideal class group of k and for a prime p define

$$g_p = \dim_{\mathbf{F}_p} A_k/pA_k.$$

Suppose that k and p are chosen so that $g_p > 3$ and $\boldsymbol{\mu}_p \not\subset k$. Let K be the everywhere unramified extension of k such that $G = \mathrm{Gal}(K/k)$ is identified with A_k/pA_k by class field theory; since K/k is a p-extension $\boldsymbol{\mu}_p \not\subset K$ as well.

Choose primes $\mathfrak{q}_1,\ldots,\mathfrak{q}_n$ of k, $n \geq 1$, whose classes generate pA_k. Choose primes $\mathfrak{q}'_{n+1},\ldots,\mathfrak{q}'_m$ of K of degree 1 whose classes generate the ideal class group of K, and let $\mathfrak{q}_i$ be the prime of k below $\mathfrak{q}'_i$ for $n < i \leq m$. Define

$$S = \{\text{infinite places of } k\} \cup \{\mathfrak{q}_1,\ldots,\mathfrak{q}_m\}$$

and let T be a finite set of places, disjoint from S, satisfying Hypothesis 2.1.5, such that every $\mathfrak{q}' \in T_K$ satisfies $p \nmid \mathbf{N}\mathfrak{q}' - 1$ (possible since $\boldsymbol{\mu}_p \not\subset K$). Define $r = \#(S) - 1$.

LEMMA 4.1.

(i) *S, T, and r satisfy Hypotheses 2.1,*

(ii) *$[\mathcal{O}_{K,S}^\times : U_{K,S,T}]$, $[\mathcal{O}_{k,S}^\times : U_{k,S,T}]$, and $\#(A_{k,S,T})/\#(\mathrm{Pic}(\mathcal{O}_{k,S}))$ are all prime to p,*

(iii) $\operatorname{ord}_p(\#(A_{k,S,T})) = g_p$,

(iv) $\operatorname{ord}_p([U_{k,S,T} : \mathbf{N}_{K/k} U_{K,S,T}]) = g_p(g_p - 1)/2$.

Proof. — The first assertion is immediate, and the second follows from (1) by our assumption on T. All places in S split completely in K/k, so the subgroup of A_k generated by the classes of the $\mathfrak{q}_i$ is contained in, and hence equal to, pA_k. Therefore $\mathrm{Pic}(\mathcal{O}_{k,S}) = A_k/pA_k$, so (iii) follows from (ii).

By our choice of S, $\mathrm{Pic}(\mathcal{O}_{K,S}) = 0$. It follows from a theorem of Tate ([11] Theorem II.5.1) that for every integer i,

$$\widehat{H}^i(G, \mathcal{O}_{K,S}^\times) \cong \widehat{H}^{i-2}(G, X_S).$$

Since all places in S split completely in K/k, Y_S is free of rank $r+1$ over $\mathbf{Z}[G]$ so the exact sequence

$$0 \to X_S \to Y_S \to \mathbf{Z} \to 0$$

shows that $\widehat{H}^i(G, X_S) \cong \widehat{H}^{i-1}(G, \mathbf{Z})$ for every i. Thus

$$\begin{aligned} \mathcal{O}_{k,S}^\times/\mathbf{N}_{K/k}\mathcal{O}_{K,S}^\times &= \widehat{H}^0(G, \mathcal{O}_{K,S}^\times) = \widehat{H}^{-2}(G, X_S) \\ &= \widehat{H}^{-3}(G, \mathbf{Z}) \cong (\mathbf{Z}/p\mathbf{Z})^{g_p(g_p-1)/2}, \end{aligned}$$

the last equality because $\widehat{H}^{-3}(G, \mathbf{Z}) \cong \wedge^2 G$ (exterior power as a $\mathbf{Z}$-module; see [1] Theorem V.6.4) and $G \cong (\mathbf{Z}/p\mathbf{Z})^{g_p}$. Now (iv) follows from (ii). □

PROPOSITION 4.2. — *With k, K, S, T, and r as above,*

$$\Theta_{S,T}^{(r)}(0)\wedge^r X_S \not\subset \lambda^{(r)}(\wedge^r U_{K,S,T}).$$

Proof. — Write $S = \{v_0, \dots, v_r\}$ and $\eta = w_1^* \wedge \cdots \wedge w_r^* \in Y_S^*$, where for each i, w_i is a place of K above v_i. Then $\eta(\wedge^r X_S) = \mathbf{Z}[G]$, so by (2)

$$e_{\mathbf{1}}\eta(\Theta_{S,T}^{(r)}(0)\wedge^r X_S) = e_{\mathbf{1}}\Theta_{S,T}^{(r)}(0)\mathbf{Z}[G] = -\#(A_{k,S,T})R_{k,S,T}\mathbf{Z}e_{\mathbf{1}}.$$

On the other hand, if $u_1, \dots, u_r \in U_{K,S,T}$ then by Lemma 2.2

$$e_1\eta(\lambda^{(r)}(u_1 \wedge \cdots \wedge u_r)) = e_1 \det(\log |\mathbf{N}u_i|_{v_j})$$

so

$$e_1\eta(\lambda^{(r)}(\wedge^r U_{K,S,T})) \subset [U_{k,S,T} : \mathbf{N}_{K/k}U_{K,S,T}]R_{k,S,T}\mathbf{Z}e_1.$$

Thus by Lemma 4.1, since $g_p > 3$, $e_1\Theta_{S,T}^{(r)}(0)\wedge^r X_S \not\subset e_1\lambda^{(r)}(\wedge^r U_{K,S,T})$. $\square$

4.2. Sands' conjecture.

The example of §4.1 is also a counterexample to Conjecture 2.0 of Sands [9]. There are also counterexamples of a different sort to Sands' conjecture, coming from the fact that when $\#(S) > r+1$ that conjecture requires S'-units rather than S-units, where $S' \subset S$ is the subset of primes which split completely. The following example shows this is not always possible.

Let $k = \mathbf{Q}(\sqrt{2})$ and $K = k(\boldsymbol{\mu}_7)^+$, the real subfield of $k(\boldsymbol{\mu}_7)$. Then K is a degree 2 subfield of $\mathbf{Q}(\boldsymbol{\mu}_{56})^+$, $G = \mathrm{Gal}(K/k)$ is cyclic of order 3, and K/k ramifies only at the two primes $\mathfrak{p}_7, \bar{\mathfrak{p}}_7$ above 7. Let $S = \{w, \bar{w}, \mathfrak{p}_7, \bar{\mathfrak{p}}_7\}$ where $w, \bar{w}$ are the two infinite places of k, and define $\eta = w^* \wedge \bar{w}^* \in \wedge^2 Y_S^*$. For notational convenience we will take T to be the empty set. Write $\mathcal{O}_K$ for the maximal order of K, so $\mathcal{O}_K^\times$ is the group of global units, *not* the S-units.

Fix $\chi \in \widehat{G}, \chi \neq \mathbf{1}$. In this situation, Conjecture 2.0 of Sands in [9] predicts that there are units $u_1, u_2 \in \mathcal{O}_K^\times$ such that

$$L''_{S,T}(0,\bar{\chi})e_\chi \in \mathbf{Z}[1/2][G]e_\chi R_\eta(u_1 \wedge u_2). \tag{12}$$

We will show that this cannot be the case.

Define

$$\begin{aligned} \varepsilon_{56} &= \mathbf{N}_{\mathbf{Q}(\boldsymbol{\mu}_{56})/K}(1-\zeta_{56}), \\ \varepsilon_7 &= \mathbf{N}_{\mathbf{Q}(\boldsymbol{\mu}_7)/\mathbf{Q}(\boldsymbol{\mu}_7)^+}(1-\zeta_7), \\ \varepsilon_8 &= \mathbf{N}_{\mathbf{Q}(\boldsymbol{\mu}_8)/k}(1-\zeta_8) \end{aligned}$$

where ζ_n is a primitive n-th root of unity, and the group of cyclotomic units of K

$$C_K = \mathbf{Z}[G]\varepsilon_{56} + \mathbf{Z}[G]^0\varepsilon_7 + \mathbf{Z}(1-\tau)\varepsilon_8 \subset \mathcal{O}_K^\times.$$

Here $\mathbf{Z}[G]^0$ denotes the augmentation ideal (the ideal of elements of degree 0) of $\mathbf{Z}[G]$ and τ the nontrivial automorphism of $k/\mathbf{Q}$. For convenience we write the action of $\mathbf{Z}[G]$ on $\mathcal{O}_K^\times$ additively. The class number of $\mathbf{Q}(\boldsymbol{\mu}_{56})^+$ is 1 (see [6]), so the analytic class number formula (see §2 of [3]) shows that $[\mathcal{O}_K^\times : C_K]$ is a power of 2.

Classical formulas for Dirichlet L-functions (see for example [11] §III.5), together with the factorization of $L(\bar\chi, s)$ into a product of two Dirichlet L-functions shows that

$$e_\chi R_\eta(\varepsilon_7 \wedge \varepsilon_{56}) = 4z e_\chi L''_{S,T}(0,\bar\chi)$$

with $z = \pm\gamma$ for some $\gamma \in G$ (by choosing ζ_7 and ζ_{56} and ordering w_1, w_2 carefully we could have ensured that $z = 1$).

Suppose $u_1, u_2 \in \mathcal{O}_K^\times$. Then $u_1, u_2 \in \mathbf{Z}[1/2]C_K$ so for $i = 1, 2$ we can write

$$u_i = \alpha_i \varepsilon_{56} + \beta_i \varepsilon_7 + \gamma_i \varepsilon_8$$

with $\alpha_i \in \mathbf{Z}[1/2][G]$, $\beta_i \in \mathbf{Z}[1/2][G]^0$, and $\gamma_i \in \mathbf{Z}[1/2](1-\tau)$. Then

$$e_\chi R_\eta(u_1 \wedge u_2) = e_\chi \det\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} R_\eta(\varepsilon_7 \wedge \varepsilon_{56}) \in e_\chi \mathbf{Z}[1/2][G]^0 L''_{S,T}(0,\bar\chi).$$

Since $[e_\chi \mathbf{Z}[G] : e_\chi \mathbf{Z}[G]^0] = 3$, it is impossible for (12) to be satisfied.

Remark. — Notice that this problem disappears if we allow $u_1, u_2 \in \mathcal{O}_S^\times$. In fact one can show that $\mathrm{St}(K/k, S, T, 2)$ is true for appropriate sets T.

5. CONNECTIONS WITH IDEAL CLASS GROUPS

In this section K and T will be fixed but S will vary, so we will abbreviate $U_S = U_{S,T}$, $A_S = A_{S,T}$, and $\varepsilon_S = \varepsilon_{S,T}$, the element of $\mathbf{Q}\wedge^r U_S$ predicted by Conjecture A$'$.

5.1. Changing S.

Suppose S, T, and r satisfy Hypotheses 2.1, and $v_1, \ldots, v_r \in S$ split completely in K/k. Suppose further that $v_{r+1}, \ldots, v_{r'} \notin S \cup T$ also split completely in K/k and define $S' = S \cup \{v_{r+1}, \ldots, v_{r'}\}$. For $1 \le i \le r'$ fix a place w_i of K above v_i and let $\eta = w_1^* \wedge \cdots \wedge w_r^* \in \wedge^r Y_S^*$, $\eta' = w_{r+1}^* \wedge \cdots \wedge w_{r'}^* \wedge \eta \in \wedge^{r'} Y_{S'}^*$. (Hopefully without confusion we will view $Y_S^* \subset Y_{S'}^*$ in the obvious way.)

There is an exact sequence

$$0 \to U_S \to U_{S'} \to \bigoplus_{i=r+1}^{r'} \mathbf{Z}[G]w_i \to A_{S,S'} \to 0 \tag{13}$$

where the center map sends an element of $U_{S'}$ to its $\mathcal{O}_S$-ideal and $A_{S,S'}$ is the subgroup of A_S generated by the primes above $v_{r+1}, \ldots, v_{r'}$. We have the $\mathbf{Z}[G]$-lattices $\Lambda_S = \Lambda_{S,T,r} \subset \mathbf{Q}\wedge^r U_S$ and $\Lambda_{S'} = \Lambda_{S',T,r'} \subset \mathbf{Q}\wedge^{r'} U_{S'}$ defined as in §2.1.

For $w \in S'_K$ nonarchimedean define $\tilde{w} : U_{S'} \to \mathbf{Z}[G]$ by

$$\tilde{w}(u) = \sum_{\gamma \in G} \mathrm{ord}_w(\gamma^{-1}u)\gamma, \tag{14}$$

i.e. $\tilde{w} = \log(\mathbf{N}w)^{-1}w^* \circ \lambda_{S,T}$. Let $\Phi = \tilde{w}_{r+1} \wedge \cdots \wedge \tilde{w}_{r'} \in \wedge^{r'-r} \mathrm{Hom}(U_{S'}, \mathbf{Z}[G])$. Then

$$R_{\eta'} = \prod_{i=r+1}^{r'} \log(\mathbf{N}v_i) R_\eta \circ \Phi. \tag{15}$$

If M is a finitely-generated $\mathbf{Z}[G]$-module then $\mathrm{Fitt}(M)$ will denote its Fitting ideal in $\mathbf{Z}[G]$ (everything we need about Fitting ideals, including the definition, can be found in the Appendix of [7]). Write $g = \#(G)$.

LEMMA 5.1. — *If we identify $\mathbf{Q}\wedge^r U_S$ with its image in $\mathbf{Q}\wedge^r U_{S'}$ via the inclusion of U_S in $U_{S'}$, the map $\Phi : \mathbf{Q}\wedge^{r'} U_{S'} \to \mathbf{Q}\wedge^r U_{S'}$ satisfies*

(i) *Φ is injective on $\Lambda_{S'}$,*

(ii) $\mathbf{Z}[1/g]\Phi(\Lambda_{S'}) = \mathbf{Z}[1/g]\mathrm{Fitt}(A_{S,S'})\Lambda_S$,

(iii) $\mathbf{Q}\Phi(\Lambda_{S'}) = \mathbf{Q}\Lambda_S$,

(iv) $\mathrm{Fitt}(A_{S,S'})\Lambda_S \subset \Phi(\Lambda_{S'}) \subset \Lambda_S$.

Proof. — Assertion (i) follows from Lemma 2.7 (ii) and (15).

Let M denote the image of $U_{S'}$ in $\bigoplus_{i=r+1}^{r'} \mathbf{Z}[G]w_i$ under (13) and

$$\Phi' = w^*_{r+1} \wedge \cdots \wedge w^*_{r'} \in \wedge^{r'-r} \mathrm{Hom}(M, \mathbf{Z}[G]).$$

By definition of the Fitting ideal,

$$\Phi'(\wedge^{r'-r} M) = \mathrm{Fitt}(A_{S,S'}) \subset \mathbf{Z}[G].$$

Tensoring (13) with $\mathbf{Z}[1/g]$ gives a short exact sequence

$$0 \to \mathbf{Z}[1/g]U_S \to \mathbf{Z}[1/g]U_{S'} \to \mathbf{Z}[1/g]M \to 0$$

which splits because $\mathbf{Z}[1/g][G]$ is semisimple, so we can find $\bar{M} \subset U_{S'}$ such that $\mathbf{Z}[1/g]\bar{M}$ maps isomorphically to $\mathbf{Z}[1/g]M$ and $\mathbf{Z}[1/g]U_{S'} = \mathbf{Z}[1/g]U_S \oplus \mathbf{Z}[1/g]\bar{M}$. Then

$$\mathbf{Z}[1/g]\wedge^{r'} U_{S'} = \bigoplus_{i=0}^{r'} \mathbf{Z}[1/g]\wedge^i U_S \otimes \wedge^{r'-i}\bar{M}.$$

If $i > r$ then $\Phi(\wedge^i U_S \otimes \wedge^{r'-i}\bar{M}) = 0$, and if $i < r$ then $\mathbf{Z}[1/g] \wedge^{r'-i} \bar{M} = 0$, so

$$\begin{aligned}\Phi(\mathbf{Z}[1/g]\wedge^{r'} U_{S'}) &= \mathbf{Z}[1/g]\Phi(\wedge^{r'-r}\bar{M})\wedge^r U_S \\ &= \mathbf{Z}[1/g]\Phi'(\wedge^{r'-r}M)\wedge^r U_S \\ &= \mathbf{Z}[1/g]\mathrm{Fitt}(A_{S,S'})\wedge^r U_S.\end{aligned}$$

Thus by Lemma 1.2 (iii), $\Phi(\mathbf{Z}[1/g]\wedge_0^{r'} U_{S'}) = \mathbf{Z}[1/g]\mathrm{Fitt}(A_{S,S'})\wedge_0^r U_S$. By (13) and (6) we also have

$$r(\chi, S') = r(\chi, S) + r' - r$$

for every $\chi \in \widehat{G}$, so this proves (ii). Since $\#(A_{S,S'})$ is finite, $[\mathbf{Z}[G] : \mathrm{Fitt}(A_{S,S'})]$ is finite and (iii) follows as well.

Now to prove $\Phi(\Lambda_{S'}) \subset \Lambda_S$ it is enough to show that if $\alpha \in \wedge_0^{r'} U_{S'}$ and $\varphi_1, \ldots, \varphi_r \in \mathrm{Hom}(U_S, \mathbf{Z}[G])$ then $(\varphi_1 \wedge \cdots \wedge \varphi_r)(\Phi(\alpha)) \in \mathbf{Z}[G]$. By Proposition 1.1 (ii) and (13), each φ is the restriction of a $\varphi' \in \mathrm{Hom}(U_{S'}, \mathbf{Z}[G])$, and then

$$(\varphi_1 \wedge \cdots \wedge \varphi_r)(\Phi(\alpha)) = (\tilde{w}_{r+1} \wedge \cdots \wedge \tilde{w}_{r'} \wedge \phi_1' \wedge \cdots \wedge \phi_r')(\alpha) \in \mathbf{Z}[G]$$

by definition of $\wedge_0^{r'} U_{S'}$.

Suppose $\lambda \in \mathrm{Fitt}(A_{S,S'})\Lambda_S$, i.e.

$$\lambda = \sum_i \Phi'(\mathbf{m}_i)\lambda_i$$

with $\mathbf{m}_i \in \wedge^{r'-r}M$ and $\lambda_i \in \Lambda_S$. For each i lift $\mathbf{m}_i$ to an element $\mathbf{u}_i$ of $\wedge^{r'-r}U_{S'}$ under (13). Then $\Phi(\sum \mathbf{u}_i \wedge \lambda_i) = \lambda$, and it is not difficult to see that each $\mathbf{u}_i \wedge \lambda_i \in \Lambda_{S'}$, so $\lambda \in \Phi(\Lambda_{S'})$. This completes the proof of (iv). □

PROPOSITION 5.2. — *Conjecture A′ is true for (S, T, r, η) if and only if it is true for (S', T, r', η'). If these both hold then*

$$\varepsilon_S = \Phi(\varepsilon_{S'}).$$

Proof. — By (3)

$$\Theta_{S',T}^{(r')}(0) = \prod_{i=r+1}^{r'} \log(\mathbf{N}v_i)\Theta_{S,T}^{(r)}(0).$$

Thus if $\varepsilon_{S'} \in \mathbf{Q}\Lambda_{S'}$ and $R_{\eta'}(\varepsilon_{S'}) = \Theta_{S'}^{(r')}(0)$, then $\Phi(\varepsilon_{S'}) \in \mathbf{Q}\Lambda_S$ by Lemma 5.1 (iii) and $R_\eta(\Phi(\varepsilon_{S'})) = \Theta_{S,T}^{(r)}(0)$ by (15).

Conversely suppose $\varepsilon_S \in \mathbf{Q}\Lambda_S$ and $R_\eta(\varepsilon_S) = \Theta_S^{(r)}(0)$. By Lemma 5.1 (iii) there is an element $\varepsilon_{S'} \in \mathbf{Q}\Lambda_{S'}$ satisfying $\Phi(\varepsilon_{S'}) = \varepsilon_S$, and we see again that $R_{\eta'}(\varepsilon_{S'}) = \Theta_{S',T}^{(r')}(0)$. □

THEOREM 5.3. — *Suppose Conjecture* A′ *holds for S, T, r, and η (or equivalently for S', T, r', and η'), so we have $\varepsilon_S \in \mathbf{Q}\Lambda_S$ and $\varepsilon_{S'} \in \mathbf{Q}\Lambda_{S'}$. Then*

(i) $\varepsilon_{S'} \in \Lambda_{S'} \Rightarrow \varepsilon_S \in \Lambda_S$,

(ii) $\varepsilon_{S'} \in \mathbf{Z}[1/g]\Lambda_{S'} \Leftrightarrow \varepsilon_S \in \mathbf{Z}[1/g]\mathrm{Fitt}(A_{S,S'})\Lambda_S$,

(iii) $\varepsilon_S \in \mathrm{Fitt}(A_{S,S'})\Lambda_S \Rightarrow \varepsilon_{S'} \in \Lambda_{S'}$.

Proof. — These assertions are all immediate from Proposition 5.2 and Lemma 5.1. □

COROLLARY 5.4. — *Suppose Conjecture* A′ *holds for S, T, r, and η. Then the following are equivalent:*

(i) *For every $S' = S \cup \{v_{r+1}, \dots, v_{r'}\}$ where $v_{r+1}, \dots, v_{r'} \notin S \cup T$ split completely in K/k,*

$$\varepsilon_{S'} \in \mathbf{Z}[1/g]\Lambda_{S'}.$$

(ii) $\varepsilon_S \in \mathbf{Z}[1/g]\mathrm{Fitt}(A_S)\Lambda_S$.

(iii) $\mathbf{Z}[1/g][G]\varepsilon_S = \mathbf{Z}[1/g]\mathrm{Fitt}(A_S)\Lambda_S$.

Proof.

(i) $\Rightarrow$ (ii) Choose primes $v_{r+1}, \dots, v_{r'} \notin S \cup T$, splitting completely in K/k, so that the classes of the primes of K above them generate A_S (they can be chosen to split completely because every ideal class contains infinitely many primes of degree 1). Set $S' = S \cup \{v_{r+1}, \dots, v_{r'}\}$. Then in particular $A_{S,S'} = A_S$. Applying Theorem 5.3 (ii) shows that (i) implies (ii).

(ii) $\Rightarrow$ (i) Suppose S' is as in (i). Since $A_{S,S'} \subset A_S$ and $\mathbf{Z}[1/g][G]$ is a direct sum of Dedekind domains, $\mathbf{Z}[1/g]\mathrm{Fitt}(A_S) \subset \mathbf{Z}[1/g]\mathrm{Fitt}(A_{S,S'})$. Thus Theorem 5.3 (ii) shows (ii) implies (i).

(ii) $\Rightarrow$ (iii) Define

$$e_r = \sum_{r(\chi)=r} e_\chi \in \mathbf{Z}[1/g][G]$$

and $D = e_r\mathbf{Z}[1/g][G]$. Then $\mathbf{Z}[1/g]\Lambda_S = D\Lambda_S$, $D = \oplus D_i$ with Dedekind domains D_i, and $D_i\Lambda_S$ is a torsion-free rank-one D_i-module for every i. Therefore

$$[\mathbf{Z}[1/g]\Lambda_S : \mathbf{Z}[1/g]\mathrm{Fitt}(A_S)\Lambda_S] = [D : D\mathrm{Fitt}(A_S)] = \#(e_r\mathbf{Z}[1/g]A_S).$$

On the other hand, a standard combinatorial argument using formula (2) for the zeta functions of all fields between k and K (see §5 of [8]) yields an "analytic class number formula"

$$\#(e_r\mathbf{Z}[1/g]A_S) = [\mathbf{Z}[1/g]e_r\wedge^r U_S : \mathbf{Z}[1/g][G]\varepsilon_S]$$

which is equal to $[\mathbf{Z}[1/g]\Lambda_S : \mathbf{Z}[1/g][G]\varepsilon_S]$ by Proposition 1.2 (iii). Thus (ii) implies (iii). Since (iii) clearly implies (ii) this completes the proof of the corollary. □

COROLLARY 5.5. — *Suppose $k = \mathbf{Q}$. Then Conjecture* B *is true "up to primes dividing $\#(G)$", i.e. $\varepsilon_S \in \mathbf{Z}[1/g]\Lambda_S$ for every S.*

Proof. — First suppose K is real. By Proposition 3.1 and Corollary 3.2 we may assume that $K \neq \mathbf{Q}$ and that S contains exactly r places $\{v_1 = \infty, v_2, \ldots, v_r\}$ which split completely. Let $S_0 = S - \{v_2, \ldots, v_r\}$.

By §5, Chapter III of [11], $\mathrm{St}(K/\mathbf{Q}, S_0, T, 1)$ is true with a cyclotomic unit ε_{S_0}. By Theorem 1 of §1.10 of [7], $\varepsilon_{S_0} \in \mathbf{Z}[1/g]\mathrm{Fitt}(A_{S_0})\Lambda_{S_0}$ (the "Gras conjecture"). Thus $\mathrm{St}(K/k, S, T, r)$ follows from Corollary 5.4 in this case.

If K is imaginary, the proof is similar, beginning with $S_0 = S - \{v \in S : v \text{ splits completely in } K/\mathbf{Q}\}$ and using $\mathrm{St}(K/\mathbf{Q}, S_0, T, 0)$ and Theorem 2, §1.10 of [7]. □

Remark. — Corollary 5.4 says that the "prime-to-$\#(G)$ part" of Conjecture B is essentially equivalent to a Gras-type conjecture. For primes dividing $\#(G)$ the situation is more subtle. For example, it is not at all obvious that the full Conjecture B is true when $K = \mathbf{Q}$.

6. EULER SYSTEMS

6.1. Notation.

Fix for this section a totally real field k and let $r = [k : \mathbf{Q}]$. Fix also a finite set T of primes of k containing at least one prime not dividing 2.

We will compare the elements $\varepsilon_{K,S,T} \in \wedge_0^r U_{K,S,T}$ predicted by Conjecture B′ as K varies through totally real abelian extensions of k and S through suitable sets of places of k.

Let $\mathcal{K}_\infty$ denote the set of pairs (K, S) where K is a totally real, finite abelian extension of k and S is a set of places of k such that S, T, r, and K satisfy Hypotheses 2.1. We will write $(K', S') \subset (K, S)$ if both $K' \subset K$ and $S' \subset S$. We will keep T fixed and we write $U_{K,S} = U_{K,S,T}$, $\varepsilon_{K,S} = \varepsilon_{K,S,T}$.

Remark. — If K is a totally real abelian extension of k, the r infinite places of k split completely and the only roots of unity in K are ± 1, so S, T, r, and K satisfy Hypotheses 2.1 if and only if S is disjoint from T and S contains the infinite places, the places ramifying in K/k, and at least one finite place.

For every totally real, finite abelian extension K of k define

- $G_K = \mathrm{Gal}(K/k)$,
- $\mathbf{N}_{K/F} = \sum_{\gamma \in \mathrm{Gal}(K/F)} \gamma \in \mathbf{Z}[G_K]$ if $k \subset F \subset K$,
- $\mathrm{Frob}_\mathfrak{q}$ is the Frobenius of $\mathfrak{q}$ in G_K if $\mathfrak{q}$ is a prime of k unramified in K.

For each infinite place v_i of k, $1 \le i \le r$, fix an extension w_i of v_i to $\bar{k}$ and write $w_{i,K}$ for the restriction of w_i to K. Suppose for this section that the conjecture $\mathrm{St}(K/k, S, T, r)$ is true for every $(K, S) \in \mathcal{K}_\infty$, and write $\varepsilon_{K,S} \in \Lambda_{K,S}$ for the corresponding element satisfying the conjecture with the choice $\{w_{1,K}, \ldots, w_{r,K}\}$. If $\Phi \in \wedge^{r-1} \mathrm{Hom}(U_{K,S}, \mathbf{Z}[G_K])$ then we will write $\varepsilon_{K,S,\Phi} = \Phi(\varepsilon_{K,S}) \in U_{K,S}$ for the S-unit given by Corollary 1.3 (see remark (2) at the end of §2.2).

If $(K', S) \subset (K, S) \in \mathcal{K}_\infty$ then the norm element $\mathbf{N}_{K/K'} \in \mathbf{Z}[G_K]$ induces a map

$$\mathbf{N}^r_{K/K'} : \wedge^r U_{K,S} \to \wedge^r U_{K',S}.$$

6.2. Relations.

PROPOSITION 6.1. — *Suppose $(K', S') \subset (K, S) \in \mathcal{K}_\infty$. Then*

$$\mathbf{N}^r_{K/K'}\varepsilon_{K,S} = \prod_{\mathfrak{q} \in S - S'} (1 - \mathrm{Frob}_{\mathfrak{q}}^{-1})\varepsilon_{K',S'} \quad \textit{in } \mathbf{Q}\wedge^r U_{K',S}.$$

Proof. — Write $\eta = w^*_{1,K} \wedge \cdots \wedge w^*_{r,K}$ and $\eta' = w^*_{1,K'} \wedge \cdots \wedge w^*_{r,K'}$. Then the following diagram commutes:

$$\begin{array}{ccccc}
\wedge^r U_{K,S} & \xrightarrow{\lambda^{(r)}_{K,S}} & \mathbf{R}\wedge^r X_{K,S} & \xrightarrow{\eta} & \mathbf{R}[G_K] \\
\mathbf{N}^r_{K/K'} \downarrow & & \downarrow \mathrm{res} & & \downarrow \mathrm{res} \\
\wedge^r U_{K',S} & \xrightarrow{\lambda^{(r)}_{K',S}} & \mathbf{R}\wedge^r X_{K',S} & \xrightarrow{\eta'} & \mathbf{R}[G_{K'}] \\
\mathrm{incl} \uparrow & & \uparrow \mathrm{incl} & & \uparrow \\
\wedge^r U_{K',S'} & \xrightarrow{\lambda^{(r)}_{K',S'}} & \mathbf{R}\wedge^r X_{K',S'} & \xrightarrow{\eta'} & \mathbf{R}[G_{K'}]
\end{array}$$

where 'res' and 'incl' denote the maps induced by restriction and inclusion, respectively. Thus by (3)

$$\begin{aligned}
R_{\eta'}(\mathbf{N}^r_{K/K'}\varepsilon_{K,S}) = R_\eta(\varepsilon_{K,S})|_{K'} &= \Theta^{(r)}_{K/k,S}(0)|_{K'} \\
&= \prod_{\mathfrak{q} \in S - S'} (1 - \mathrm{Frob}_{\mathfrak{q}}^{-1})\Theta^{(r)}_{K'/k,S'}(0) \\
&= R_{\eta'}\Big(\prod_{\mathfrak{q} \in S - S'} (1 - \mathrm{Frob}_{\mathfrak{q}}^{-1})\varepsilon_{K',S'}\Big).
\end{aligned}$$

Further, if $\chi \in \widehat{G_{K'}} \subset \widehat{G_K}$ and $r(\chi, S') > r$, then $r(\chi, S) > r$ so $e_\chi \mathbf{N}^r_{K/K'}\varepsilon_{K,S} = 0$. The proposition now follows from Lemma 2.7 (ii). □

Suppose $(K', S') \subset (K, S) \in \mathcal{K}_\infty$. Then there is a map

$$\mathbf{N}_{K/K'} : \mathrm{Hom}(U_{K,S}, \mathbf{Z}[G_K]) \to \mathrm{Hom}(U_{K',S'}, \mathbf{Z}[G_{K'}])$$

induced by the inclusion map $U_{K',S'} \hookrightarrow U_{K,S}^{\mathrm{Gal}(K/K')}$ and the isomorphism

$$\mathbf{Z}[G_K]^{\mathrm{Gal}(K/K')} \xrightarrow{\sim} \mathbf{Z}[G_{K'}], \quad \mathbf{N}_{K/K'} \mapsto 1.$$

If $S = S'$ one checks easily that for $\varphi \in \mathrm{Hom}(U_{K,S}, \mathbf{Z}[G_K])$ the following diagram commutes:

$$\begin{array}{ccc}
U_{K,S} & \xrightarrow{\varphi} & \mathbf{Z}[G_K] \\
\mathbf{N}_{K/K'} \downarrow & & \downarrow \text{restriction} \\
U_{K',S} & \xrightarrow{\mathbf{N}_{K/K'}} & \mathbf{Z}[G_{K'}].
\end{array} \tag{16}$$

If w is a finite place of K let $\tilde{w} : U_{K,S} \to \mathbf{Z}[G_K]$ be the map defined by (14). Let $\mathbf{Z}[G]^0$ denote the augmentation ideal of $\mathbf{Z}[G]$ and S_∞ the set of infinite places of k.

PROPOSITION 6.2. — *Suppose* $(K, S) \in \mathcal{K}_\infty$ *and*

$$\Phi \in \wedge^{r-1} \operatorname{Hom}(U_{K,S}, \mathbf{Z}[G_K]).$$

(i) *If* $\#(S) > r + 1$ *then* $\varepsilon_{K,S,\Phi} \in U_{K,S_\infty}$.

(ii) *If* $\#(S) = r + 1$ *and* $\alpha \in \mathbf{Z}[G]^0$ *then* $\alpha\varepsilon_{K,S,\Phi} \in U_{K,S_\infty}$.

(iii) *Suppose further that* $(K', S') \in \mathcal{K}_\infty$, $(K', S') \subset (K, S)$, *and let* $\Phi' = \mathbf{N}^{r-1}_{K/K'}\Phi \in \wedge^{r-1} \operatorname{Hom}(U_{K',S'}, \mathbf{Z}[G_{K'}])$. *Then*

$$\mathbf{N}_{K/K'}\varepsilon_{K,S,\Phi} = \prod_{\mathfrak{q} \in S - S'} (1 - \operatorname{Frob}_{\mathfrak{q}}^{-1})\varepsilon_{K',S',\Phi'}.$$

Proof (Compare [11] IV.2.2, IV.2.4, and IV.3.5). — By Corollary 1.3, $\varepsilon_{K,S,\Phi} \in U_{K,S}$. Suppose $v \in S - S_\infty$ and w is a place of K above v. If $r(\chi, S) > r$ then $e_\chi\tilde{w}(\varepsilon_{K,S,\Phi}) = \tilde{w}(e_\chi\varepsilon_{K,S,\Phi}) = 0$ since $\varepsilon_{K,S,\Phi} \in \Lambda_{K,S}$. If $r(\chi, S) = r$ and $\chi \neq \mathbf{1}$ then by (6), χ is nontrivial on G_v and so $e_\chi\tilde{w} = 0$. Thus $e_\chi\tilde{w}(\varepsilon_{K,S,\Phi}) = 0$ unless $\chi = \mathbf{1}$ and $r(\mathbf{1}, S) = r$, which proves (i) and (ii).

By (16) and Proposition 6.1,

$$\prod_{\mathfrak{q} \in S - S'} (1 - \operatorname{Frob}_{\mathfrak{q}}^{-1})\Phi'(\varepsilon_{K',S'}) = \Phi'(\mathbf{N}^r_{K/K'}\varepsilon_{K,S}) = \Phi(\mathbf{N}_{K/K'}\varepsilon_{K,S})$$

which is (iii). □

COROLLARY 6.3. — *Suppose* $\mathcal{K} \subset \mathcal{K}_\infty$ *and*

$$\Phi \in \varprojlim_{(K,S) \in \mathcal{K}} \wedge^{r-1} \operatorname{Hom}(U_{K,S}, \mathbf{Z}[G_K]),$$

inverse limit with respect to the maps $\mathbf{N}^{r-1}_{K/K'}$. *Then for every* $(K', S') \subset (K, S) \in \mathcal{K}$,

$$\mathbf{N}_{K/K'}\varepsilon_{K,S,\Phi} = \prod_{\mathfrak{q} \in S - S'} (1 - \operatorname{Frob}_{\mathfrak{q}}^{-1})\varepsilon_{K',S',\Phi}.$$

Proof. — This is immediate from Proposition 6.2. □

Remark. — Corollary 6.3 says that for each

$$\Phi \in \varprojlim_{(K,S) \in \mathcal{K}} \wedge^{r-1} \operatorname{Hom}(U_{K,S}, \mathbf{Z}[G_K]),$$

the elements $\varepsilon_{K,S,\Phi}$ predicted by Conjecture B$'$ form an Euler system in the sense of Kolyvagin (see [8]). Of course, this says nothing unless one can find a Φ such that these units are nontrivial.

6.3. Example.

Fix k, r, and T as above, and fix also an odd rational prime p. Define $\mathcal{K} \subset \mathcal{K}_\infty$ by

$$\mathcal{K} = \{(K,S) \in \mathcal{K}_\infty : K/k \text{ is unramified above } p, \quad S \text{ contains no primes above } p\}.$$

For each finite extension K of k let V_K denote the units congruent to 1 modulo the primes above p in $K \otimes \mathbf{Q}_p$. The following result is due to Krasner [5] (note that if K is totally real then K contains no p-th roots of unity since $p > 2$).

THEOREM 6.4 (Krasner). — *If K/k is a finite extension, unramified at primes above p, and K is totally real, then V_K is a free $\mathbf{Z}_p[G_K]$-module of rank r.*

COROLLARY 6.5. — *With notation as above,*

(i) $\varprojlim_{(K,S)\in\mathcal{K}} \mathrm{Hom}(V_K, \mathbf{Z}_p[G_K])$ *is free of rank r over* $\varprojlim_{(K,S)\in\mathcal{K}} \mathbf{Z}_p[G_K]$,

(ii) *for every K' the projection map*

$$\varprojlim_{(K,S)\in\mathcal{K}} \mathrm{Hom}(V_K, \mathbf{Z}_p[G_K]) \to \mathrm{Hom}(V_{K'}, \mathbf{Z}_p[G_{K'}])$$

is surjective.

Proof. — Immediate from Theorem 6.4. $\square$

For $(K,S) \in \mathcal{K}$ define

$$\tilde{U}_{K,S} = \{u \in U_{K,S} : e_\chi u = 0 \text{ for all } \chi \in \widehat{G_K} \text{ such that } r(\chi,S) > r\}.$$

Since S contains no primes above p, $U_{K,S} \otimes \mathbf{Z}_p$ maps canonically to V_K so there is a natural map

$$\varprojlim_{(K,S)\in\mathcal{K}} \mathrm{Hom}(V_K, \mathbf{Z}_p[G_K]) \to \varprojlim_{(K,S)\in\mathcal{K}} \mathrm{Hom}(U_{K,S}, \mathbf{Z}_p[G_K]).$$

Thus for every $\Phi \in \varprojlim_{(K,S)\in\mathcal{K}} \wedge^{r-1}\operatorname{Hom}(V_K, \mathbf{Z}_p[G_K])$ and every $(K,S) \in \mathcal{K}$ we get, as in the previous section,

$$\varepsilon_{K,S,\Phi} \in \mathbf{Z}_p U_{K,S}.$$

(In fact $\varepsilon_{K,S,\Phi} \in \tilde{U}_{K,S} \otimes \mathbf{Z}_p$ since $\varepsilon_{K,S} \in \Lambda_{K,S}$.) The following proposition says that for such a Φ these $\varepsilon_{K,S,\Phi}$ form an Euler system (see [8]).

PROPOSITION 6.6.

(i) *If*

$$\Phi \in \varprojlim_{(K,S)\in\mathcal{K}} \wedge^{r-1}\operatorname{Hom}(V_K, \mathbf{Z}_p[G_K]) \quad \textit{and} \quad (K_1, S_1) \subset (K_2, S_2) \in \mathcal{K}$$

then

$$\mathbf{N}_{K_2/K_1}\varepsilon_{K_2,S_2,\Phi} = \prod_{\mathfrak{q}\in S_2 - S_1} (1 - \operatorname{Frob}_{\mathfrak{q}}^{-1})\varepsilon_{K_1,S_1,\Phi}$$

in $\mathbf{Z}_p U_{K_1,S_1}$.

(ii) *If* $(K', S') \in \mathcal{K}$ *and the map* $\tilde{U}_{K',S'} \otimes \mathbf{Z}_p \to V_{K'}$ *is injective, then*

$$\{\varepsilon_{K',S',\Phi} : \Phi \in \varprojlim_{(K,S)\in\mathcal{K}} \wedge^{r-1}\operatorname{Hom}(V_K, \mathbf{Z}_p[G_K])\}$$

is a subgroup of finite index in $\mathbf{Z}_p\tilde{U}_{K',S'}$.

Proof. — The first assertion is just Corollary 6.3 with $\mathbf{Z}[G_K]$ replaced by $\mathbf{Z}_p[G_K]$, and the proof is the same.

Fix $(K', S') \in \mathcal{K}$. It follows from Theorem 6.4 and Corollary 6.5 that

$$\{\Phi(\mathbf{v}) : \mathbf{v} \in \wedge^r V_{K'}, \Phi \in \varprojlim_{(K,S)\in\mathcal{K}} \wedge^{r-1}\operatorname{Hom}(V_K, \mathbf{Z}_p[G_K])\} = V_{K'}.$$

Define

$$e_r = \sum_{r(\chi,S')=r} e_\chi \in \mathbf{Q}[G_{K'}].$$

Then by (6), $\mathbf{Q}\tilde{U}_{K',S'}$ is free of rank r over $e_r\mathbf{Q}[G_{K'}]$. If $\mathbf{Z}_p\tilde{U}_{K',S'} \to V_{K'}$ is injective, then comparing ranks it follows that $\mathbf{Q}_p\tilde{U}_{K',S'} \cong e_r\mathbf{Q}_pV_{K'}$, so finally

$$\{\Phi(\mathbf{u}) : \mathbf{u} \in \wedge^r\mathbf{Z}_p\tilde{U}_{K',S'}, \Phi \in \varprojlim_{(K,S)\in\mathcal{K}} \wedge^{r-1}\operatorname{Hom}(V_K, \mathbf{Z}_p[G_K])\}$$

has finite index in $\mathbf{Z}_p\tilde{U}_{K',S'}$. Also $\mathbf{Q}\wedge^r\tilde{U}_{K',S'}$ is free of rank 1 over $e_r\mathbf{Q}[G_{K'}]$, and since $e_\chi\varepsilon_{K'S'} \neq 0$ for every χ with $r(\chi, S) = r$, $\mathbf{Q}\wedge^r\tilde{U}_{K',S'} = \mathbf{Q}[G_{K'}]\varepsilon_{K',S'}$. Combining these facts proves (ii). $\square$

Remark. — The argument of the proof of Proposition 6.2 shows that $\tilde{U}_{K,S} \subset U_{K,S_\infty}$ (the global units) if $\#(S) > r+1$ and otherwise $\mathbf{Z}[G]^0\tilde{U}_{K,S} \subset U_{K,S_\infty}$. Thus the injectivity hypothesis of Proposition 6.6 (ii) follows from a form of Leopoldt's conjecture.

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