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ZETA FUNCTIONS OF JORDAN
ALGEBRAS REPRESENTATIONS

by Dehbia ACHAB

0. Introduction.

Riemann zeta function has been generalized by Epstein as follows :
let \mathfrak{S} be a symmetric positive matrix of order k , Epstein zeta function is defined by

$$\zeta_1(\mathfrak{S}, s) = \sum_{g \in \mathbf{Z}^k - \{0\}} \frac{1}{(g' \mathfrak{S} g)^s}, \quad \text{Re}(s) > \frac{k}{2}$$

where g' is the adjoint of g .

In [15], Koecher has generalized Epstein zeta function as follows :

$$\zeta_m(\mathfrak{S}, s) = \sum_{\mathfrak{U} \in [\mathbf{Z}^{k \times m} / GL(m, \mathbf{Z}), \text{rank}(\mathfrak{U})=m]} \text{Det}(\mathfrak{U}' \mathfrak{S} \mathfrak{U})^{-s}, \quad (k \geq m).$$

Koecher zeta series converges absolutely and is analytic in the half-plane $\text{Re}(s) > \frac{k}{2}$, it admits an analytic continuation as a meromorphic function on \mathbf{C} and satisfies to the functional equation

$$\mathcal{R}_m(\mathfrak{S}, s) = |\mathfrak{S}|^{-\frac{m}{2}} \mathcal{R}_m \left(\mathfrak{S}^{-1}, \frac{k}{2} - s \right)$$

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where $\mathcal{R}_m(\mathfrak{S}, s)$ is a product of $\zeta_m(\mathfrak{S}, s)$ and some gamma factor, more precisely

$$\mathcal{R}_m(\mathfrak{S}, s) = \pi^{m(\frac{m-1}{4}-s)} \Gamma(s) \Gamma\left(s - \frac{1}{2}\right) \dots \Gamma\left(s - \frac{m-1}{2}\right) \zeta_m(\mathfrak{S}, s)$$

$\Gamma(s)$ being the usual Euler gamma function.

Later, in [15], A. Krieg studied the Kœcher zeta function for the hermitian matrices with quaternionic coefficients defined by

$$\zeta(s) = \sum_{A \in GL(m, \mathcal{O}) \setminus [M(m, k, \mathcal{O}) | \text{rank}(A)=m]} \text{Det}(A\bar{A}^t)^{-s}$$

where \mathcal{O} is the ring of Hurwitz integers.

This work is situated in a more general context. In fact, we define the Kœcher zeta series associated to a self-adjoint Euclidean Jordan algebra representation and we obtain the above zeta series as particular cases of it. More precisely, let V be an Euclidean simple Jordan algebra of dimension n and rank m , E an Euclidean space of dimension N , ϕ a regular self-adjoint representation of V in the space $\text{Sym}(E)$ of symmetric morphisms of E . Let Q be the quadratic form associated to ϕ , Ω the symmetric cone associated to V and $G(\Omega)$ its automorphism group

$$G(\Omega) = \{g \in GL(V) \mid g(\Omega) = \Omega\}.$$

(H_1) We assume that V and E have \mathbf{Q} -structures $V_{\mathbf{Q}}$ and $E_{\mathbf{Q}}$ respectively and that ϕ is defined over \mathbf{Q} .

Let L be a lattice in $E_{\mathbf{Q}}$.

We define the zeta series associated to ϕ and L by the following :

$$\zeta_L(s) = \sum_{l \in \Gamma_{\circ} \backslash L'} [\det(Q(l))]^{-s}, \forall s \in \mathbf{C}$$

where $L' = \{l \in L \mid \det(Q(l)) \neq 0\}$ and Γ_{\circ} is some arithmetic subgroup of $GL(E)$ which we will precise.

Recall that the primitive rank of a Jordan algebra is the cardinality of a maximal complete system of primitive orthogonal idempotents. A Jordan algebra is said to be split if its rank equals its primitive rank.

(H_2) We assume that $V_{\mathbf{Q}}$ is split.

The fundamental results in this work are :

THEOREM 1. — *Under the assumptions (H_1) and (H_2), the zeta series converges absolutely for $\operatorname{Re}(s) > \frac{N}{2m}$.*

THEOREM 2. — *If the arithmetic subgroup Γ_0 is self-adjoint, then the zeta function ζ_L admits an analytic continuation as a meromorphic function on the whole plane \mathbf{C} and satisfies to the functional equation*

$$\zeta_L\left(\frac{N}{2m} - s\right) = \operatorname{vol}(L)\pi^{\frac{N}{2}-2ms} \frac{\Gamma_{\Omega}(s)}{\Gamma_{\Omega}\left(\frac{N}{2m} - s\right)} \zeta_{L^*}(s)$$

where $\Gamma_{\Omega}(s)$ is the Kœcher-Gindikin gamma function of the symmetric cone Ω and L^* is the dual lattice of L .

This article is composed of three parts; the first consisting in the proof of Theorem 1 by using reduction theory, the second is an adaptation of the classical method to prove Theorem 2 and the last one gives some examples.

1. Construction and convergence of the zeta series.

Let V be a simple Euclidean Jordan algebra with unity e , of dimension n and rank m , E an Euclidean space of dimension N , ϕ a representation of V in the space $\operatorname{Sym}(E)$ of self-adjoint endomorphisms of E such that

$$\forall x, y \in V, \quad \phi(xy) = \frac{1}{2}(\phi(x)\phi(y) + \phi(y)\phi(x)),$$

and $Q : E \rightarrow V$ the quadratic form associated to ϕ determined by

$$(Q(\xi) \mid x)_V = (\phi(x)\xi \mid \xi)_E \quad \forall x \in V, \quad \forall \xi \in E.$$

For $x \in V$, we denote by $L(x)$ the multiplication endomorphism, $L(x) : V \rightarrow V, y \mapsto xy$ and by $P(x)$ the quadratic representation of V , i.e $P(x) = 2L(x)^2 - L(x^2)$. Let Ω be the symmetric cone associated to V and $G(\Omega)$ the automorphism group of Ω ,

$$G(\Omega) = \{g \in GL(V) \mid g(\Omega) = \Omega\}.$$

In the sequel, we assume that ϕ is regular, that is $\exists \xi \in E$ such that $\det(Q(\xi)) \neq 0$; then $Q(E) = \bar{\Omega}$ where $\bar{\Omega}$ is the closure of Ω . We assume too that $\phi(e) = \text{id}_E$.

(H_1) Assume that V, E and ϕ are defined over \mathbf{Q} , that is there exists a \mathbf{Q} -Jordan subalgebra $V_{\mathbf{Q}}$ of V , and a \mathbf{Q} -subspace $E_{\mathbf{Q}}$ of E such that

$$V = V_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}, \quad E = E_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R},$$

and for each $x \in V_{\mathbf{Q}}$ we have $\phi(x) \in \text{Sym}(E)_{\mathbf{Q}}$.

Let L be a lattice in the space $E_{\mathbf{Q}}$. In all the sequel, we denote by \det the determinant in the Jordan algebra and by Det the usual determinant of matrices.

1.1. Arithmetic subgroups associated to ϕ and L .

Let $H = \{(\tilde{h}, h) \in GL(V) \times GL(E) \mid Q(h.\xi) = \tilde{h}.Q(\xi), \forall \xi \in E\}$.

H is non empty because, for each invertible $x \in V$,

$$Q(\phi(x)\xi) = P(x)Q(\xi).$$

It is clear that H is an algebraic subgroup of $GL(V) \times GL(E)$. As ϕ is defined over \mathbf{Q} then it is the same for H . If π_1 and π_2 are the projections of H , then the groups $\pi_1(H)$ and $\pi_2(H)$ are algebraic, defined over \mathbf{Q} , we denote them by $G(\phi)$ and $F(\phi)$ respectively.

Notice that $G(\phi) \subseteq G(\Omega)$ and that π_2 is injective. Denote by F and G the identity connected components of $F(\phi)$ and $G(\Omega)$ for the ordinary topologies respectively. Consider the map

$$\begin{aligned} \rho : F(\phi) &\rightarrow G(\phi) \\ f &\mapsto \tilde{f} \end{aligned}$$

ρ is well defined because π_2 is injective and we have :

PROPOSITION 1.1.1. — ρ satisfies to the following :

- (1) $\rho(F) = G$.
- (2) ρ is a surjective \mathbf{Q} -morphism of algebraic groups.
- (3) The groups $F(\phi)$ and $G(\phi)$ are self-adjoint $\rho(h^*) = \rho(h)^*$. So they are reductive. Moreover $|\text{Det}\rho(h)| = |\text{Det}(h)|^{\frac{2n}{N}}$.

Proof. — (1) Denote by \mathfrak{f} and \mathfrak{g} the Lie algebras of F and G respectively. It suffices to show that the differential $d\rho : \mathfrak{f} \rightarrow \mathfrak{g}$ is surjective. We start by showing the following lemma :

LEMMA 1.1.2. — For each $x \in V$ we have $\phi(x) \in \mathfrak{f}$ and $d\rho(\phi(x)) = 2L(x)$.

Proof of the lemma. — For $x \in V, \xi \in E, t \in \mathbf{R}$,

$$\begin{aligned} Q(\exp(\phi(tx)).\xi) &= Q(\phi(\exp(tx)).\xi) \\ &= P(\exp(tx)).Q(\xi) \\ &= \exp(2tL(x)).Q(\xi) \end{aligned}$$

then $\rho(\exp(t\phi(x))) = \exp(2tL(x))$, and,

$$d\rho(\phi(x)) = \frac{d}{dt}\bigg|_{t=0}[\rho(\exp(t\phi(x)))] = \frac{d}{dt}\bigg|_{t=0}[\exp(2tL(x))] = 2L(x). \quad \square$$

As \mathfrak{g} is generated by the $L(x), x \in V$, then the lemma shows that $d\rho$ is surjective.

(2) As $\rho = \pi_1 \circ i_2$ where i_2 is the injection $F(\phi) \rightarrow H, f \mapsto (\tilde{f}, f)$, then ρ is clearly a morphism of algebraic groups. Moreover, as

$$Q(f\xi) = \rho(f)Q(\xi) \forall \xi \in E \Leftrightarrow f^*\phi(x)f = \phi(\rho(f)^*x) \quad \forall x \in V,$$

then, if $(e_i)_{1 \leq i \leq n}$ is a basis of $V_{\mathbf{Q}}$, and $(\epsilon_\alpha)_{1 \leq \alpha \leq N}$ a basis of E , we have

$$\sum_{\beta, \gamma=1}^N f_{\beta\alpha} \phi(e_i)_{\beta\gamma} f_{\gamma\delta} = \phi \left(\sum \rho(f)_{ij} e_i \right) = \sum_{j=1}^n \rho(f)_{ij} \phi(e_i)_{\alpha\delta}.$$

The above formula shows that the coefficients of $\rho(f)$ are polynomials of degree 2, with rational coefficients, in the coefficients of f .

(3) Let $h \in F(\phi)$. We know that

$$Q(h\xi) = \rho(h)Q(\xi) \quad \forall \xi \in E \Leftrightarrow h^*\phi(x)h = \phi(\rho(h)^*x) \quad \forall x \in V$$

then, for each invertible $x \in V$, we have

$$h^{-1}\phi(x^{-1})h^{*-1} = \phi((\rho(h)^*x)^{-1}).$$

As $(\rho(h)^*x)^{-1} = \rho(h)^{-1}x^{-1}$, we find

$$h^{-1}\phi(x^{-1})h^{*-1} = \phi(\rho(h)^{-1}x^{-1}) \quad \forall x \in \mathcal{I}(V)$$

where $\mathcal{I}(V)$ is the set of all invertible elements of V . It follows that

$$h^{-1}\phi(x)h^{\star-1} = \phi(\rho(h)^{-1}x) \quad \forall x \in V, \text{ i.e } \rho(h^{\star})^{-1} = (\rho(h)^{\star})^{-1}.$$

The last assertion is a direct consequence of the properties

$$\text{Det}(\phi(x)) = \det(x)^{\frac{N}{m}} \text{ and } \det(gx) = \text{Det}(g)^{\frac{m}{n}} \det(x)$$

for $x \in V$ and $g \in G(\Omega)$. □

Now consider the arithmetic subgroup Γ_{\circ} of $F(\phi)$ defined by

$$\Gamma_{\circ} = \{f \in F(\phi) \mid f(L) = L\}.$$

As ρ is a surjective \mathbf{Q} -morphism of algebraic groups, then $\Gamma = \rho(\Gamma_{\circ})$ is an arithmetic subgroup of $G(\phi)$. Moreover,

$$\forall \gamma \in \Gamma_{\circ}, \det(Q(\gamma.\xi)) = \det(Q(\xi)), \quad \forall \xi \in E.$$

1.2. Reduction theory.

The following hypothesis is essential to use in this context reduction theory and to obtain some Minkowski inequality.

(H_2) In all the sequel, we assume that $V_{\mathbf{Q}}$ is a split Jordan algebra, that is, its primitive rank (which is the cardinality of maximal system of primitive orthogonal idempotents), equals its rank.

As $\text{rank}(V) = m$, then the assumption (H_2) implies that there exists in $V_{\mathbf{Q}}$ a complete system of orthogonal primitive idempotents $\{c_1, \dots, c_m\}$ which we will fix along this paper.

The corresponding Peirce decomposition $V = \bigoplus_{i \leq j} V_{ij}$ is defined over \mathbf{Q} , that is

$$V_{\mathbf{Q}} = \bigoplus_{i \leq j} V_{ij\mathbf{Q}}.$$

Let P be the subgroup of $G(\phi)$ defined by

$$P = \{g \in G(\phi) \mid (gx_{ij})_{ij} = \lambda_{ij}x_{ij} \quad \forall i, j \quad (gx_{ij})_{kl} = 0 \quad \forall (k, l) < (i, j)\}$$

where the λ_{ij} are reals, and for each y in V , the y_{ij} are the Peirce components of y , with respect to the Jordan frame $\{c_1, \dots, c_m\}$. The order on the pairs (i, j) is the lexicographic one.

PROPOSITION 1.2.1. — P is a Borel subgroup of $G(\phi)$ defined over \mathbf{Q} .

Proof — As the Peirce decomposition is defined over \mathbf{Q} , then there exists some basis of $V_{\mathbf{Q}}$ whose each element lies in some $V_{ij\mathbf{Q}}$. An element of $G(\phi)$ lies in P iff its matrix in such a basis is upper triangular. \square

Now consider the subgroup A of G defined by

$$A = \left\{ P(a) \mid a = \sum_{i=1}^m a_i \cdot c_i, \quad a_i > 0 \quad \forall i, 1 \leq i \leq m \right\}.$$

PROPOSITION 1.2.2. — A is a maximal \mathbf{Q} -split algebraic torus of P (cf. [15], chapter 2, proposition 3.5).

We denote by N the unipotent radical of P ,

$$N = \{t \in P \mid \lambda_{ij} = 1 \quad \forall i, j\}.$$

We know (cf.[15]), that

$$N = \left\{ n(z) \mid z \in \bigoplus_{j < k} V_{jk} \right\}$$

where

$$n(z) = \tau(z^{(1)}) \dots \tau(z^{(m-1)})$$

$$z^{(j)} = \sum_{k=j+1}^m z_{jk}, \quad \tau(z^{(j)}) = \exp(2z^{(j)} \square c_j)$$

the operation \square being defined by

$$x \square y = L(xy) + [L(x), L(y)]$$

(cf [15], Chapter 6, Theorem 6.3.6).

Let K be the maximal compact subgroup of $G(\phi)$ defined by

$$K = \{g \in G(\phi) \mid g.e = e\},$$

where e is the unity of V . Then we have the Iwasawa decomposition of $G(\phi)$,

$$G(\phi) = N.A.K.$$

DEFINITION 1.2.3. — A Siegel set of $G(\phi)$ (with respect to K, N, A) is the cartesian product $\mathfrak{S}_{t,u} = N_u.A_t.K$ with

$$A_t = \left\{ P(a) \in A \mid a_i \leq ta_{i+1}, \forall 1 \leq i \leq m-1, \quad a = \sum_{i=1}^m a_i c_i \right\}$$

$$N_u = \{n(z) \in N \mid \|z_{kj}\| \leq u\},$$

where t, u are two positive constants.

PROPOSITION 1.2.4. — There exist positive constants t and u and some finite subset \mathcal{B} of $G(\phi)_{\mathbf{Q}}$ such that

$$G(\phi) = \Gamma.\mathcal{B}.\mathfrak{S}_{t,u},$$

moreover, as $\Omega = G(\phi)/K$, then

$$\Omega = \Gamma\mathcal{B}.N_u.A_t.e.$$

Proof. — It is a direct consequence of Theorem 13.1 of [15], page 90, applicated to the \mathbf{Q} -reductive group $G(\phi)$ under the action of the arithmetic subgroup Γ . \square

PROPOSITION 1.2.5 (Minkowski inequality). — Let $x = \sum_{i=1}^m x_i c_i + \sum_{i < j} x_{ij}$ be the Peirce decomposition of $x \in V$. For positive reals t, u , there exists a positive constant $C_{t,u}$ such that, for each $x \in \mathfrak{S}_{t,u}.e$,

$$\prod_{i=1}^m x_i \leq C_{t,u} \cdot \det(x).$$

Proof. — Let $x = n \cdot P(a)(e)$ be an element of the Siegel set $\mathfrak{S}_{t,u}.e$ of the symmetric cone Ω , i.e.

$$n = \tau(z^{(1)}) \dots \tau(z^{(m-1)})$$

with

$$z^{(j)} = \sum_{k=j+1}^m z_{jk}, \quad \|z_{jk}\| \leq u$$

and $a = \sum_{i=1}^m a_i c_i$ such that

$$a_i \leq ta_{i+1} \quad \forall i, 1 \leq i \leq m-1.$$

The Peirce components of x are as follows :

$$x_j = a_j^2 + \frac{1}{2} \sum_{k=1}^{j-1} a_k^2 \|z_{kj}\|^2$$

$$x_{jk} = a_j^2 z_{jk} + 2 \sum_{l=1}^{j-1} a_l^2 z_{lj} z_{lk}.$$

So we find the following inequality :

$$x_j \leq a_j^2 + \frac{1}{2} a_j^2 \sum_{k=1}^{j-1} t^{2(j-k)} \|z_{kj}\|^2$$

$$\leq a_j^2 \left(1 + \frac{1}{2} \sum_{k=1}^{j-1} t^{2(j-k)} \|z_{kj}\|^2 \right) \leq a_j^2 \left(1 + \frac{1}{2} u^2 \sum_{k=1}^{j-1} t^{2(j-k)} \right).$$

Otherwise, as $\det(x) = \prod_{j=1}^m a_j^2$, we find

$$\prod_{j=1}^m x_j \leq C_{t,u} \det(x)$$

where

$$C_{t,u} = \left(1 + \frac{1}{2} u^2 \sum_{k=1}^{m-1} t^{2(j-k)} \right)^m. \quad \square$$

1.3. Convergence of the zeta series ζ_L .

The zeta series associated to the representation ϕ and the lattice L is defined by

$$\zeta_L(s) = \sum_{l \in \Gamma_0 \setminus L'} \det(Q(l))^{-s}, \quad s \in \mathbb{C}$$

where L' is the set $L' = \{l \in L \mid \det(Q(l)) \neq 0\}$.

THEOREM 1.3.1. — *Under the assumptions (H_1) (section 1.1) and (H_2) (section 1.2), the zeta series $\zeta_L(s)$ converges absolutely for $\operatorname{Re}(s) > \frac{N}{2m}$.*

Proof. — For $a \in \Omega$, we set

$$\nu(a) = \#\{l \in L' \mid Q(l) = a\}$$

$$\epsilon(a) = \#\{\gamma \in \Gamma \mid \gamma(a) = a\}$$

$$\mu(a) = \frac{\nu(a)}{\epsilon(a)}.$$

Assume s real. By Proposition 1.2.4, there exist a Siegel set $\mathfrak{S}_{t,u}$ and a finite subset \mathcal{B} of $G(\phi)_{\mathbf{Q}}$ such that $\Omega = \Gamma \mathcal{B} \mathfrak{S}_{t,u} e$ and then

$$\zeta_L(s) = \sum_{a \in \Gamma \backslash Q(L')} \mu(a) \det(a)^{-s} \leq \sum_{a \in Q(L') \cap \mathcal{B} \mathfrak{S}_{t,u} e} \nu(a) \det(a)^{-s}.$$

Before getting to the proof of the theorem, we will show the following lemma :

LEMMA 1.3.2. — The series $S = \sum_{a \in Q(L')} \nu(a) \left(\prod_{i=1}^m a_i^{-s} \right)$ converges for $s > \frac{N}{2m}$.

Proof of the lemma. — Let $E_i = \phi(c_i)E$. We have $E = \bigoplus_{i=1}^m E_i$ and this decomposition is defined over \mathbf{Q} . Then we can find lattices $R_i \subseteq (E_i)_{\mathbf{Q}}$ such that

$$L \subseteq R = \bigoplus_{i=1}^m R_i.$$

For $\xi \in E$, denote by $\xi_i = \phi(c_i)\xi \in E_i$. The series S becomes

$$S = \sum_{l \in L'} \left(\prod_{i=1}^m \|l_i\|^{-2s} \right)$$

then

$$S \leq \sum_{i=1}^m \sum_{l_i \in R_i - \{0\}} \prod_{i=1}^m \|l_i\|^{-2s} = \prod_{i=1}^m \left(\sum_{l_i \in R_i - \{0\}} \|l_i\|^{-2s} \right).$$

Each one of these series is an Epstein zeta series which converges for $s > \frac{1}{2} \dim(E_i) = \frac{N}{2m}$. \square

Let's now return to the proof of the theorem. For each equivalence class of $Q(L')$ modulo Γ , we choose a representant of the form $a = b\alpha$, $b \in \mathcal{B}$, $\alpha \in \mathfrak{S}_{t,u} e$.

By the Minkowski inequality, we have

$$\det(a) = \text{Det}(b)^{\frac{m}{n}} \det(\alpha) \geq \text{Det}(b)^{\frac{m}{n}} C_{t,u}^{-1} \prod_{i=1}^m \alpha_i$$

and if $s > 0$, then

$$\det(a)^{-s} \leq M^s \prod_{i=1}^m \alpha_i^{-s}, \quad \text{with} \quad M = C_{t,u} \sup_{b \in B} \text{Det}(b)^{-\frac{m}{n}}.$$

Set $\mathcal{B} = \{b_1, \dots, b_r\}$, $b_j = \rho(f_j)$, $L_j = f_j^{-1}(L)$. If $a = b_j \alpha$, then

$$\nu(a) = \#\{l \in L'_j \mid Q(l) = \alpha\} = \nu_j(\alpha).$$

Otherwise, if $f_1, f_2 \in F(\phi)$, then

$$(*) \quad \rho(f_1) = \rho(f_2) \Leftrightarrow \forall x \in V, \quad f_1^* \cdot \phi(x) \cdot f_1 = f_2^* \cdot \phi(x) \cdot f_2$$

so, if $b \in G(\phi)$, then

$$x \in b^{-1}(Q(L')) \Leftrightarrow \exists l \in L', \exists f \in F(\phi), \quad x = b^{-1} \cdot Q(l) = Q(fl).$$

Notice that f is not unique, but if $x = \sum_{j=1}^m x_j \cdot c_j + \sum_{k < l} x_{kl}$ is the Peirce decomposition of x , then

$$x_j = Q(f \cdot l)_j = (Q(f \cdot l) \mid c_j) = (\phi(c_j) f \cdot l \mid f \cdot l) = (f^* \cdot \phi(c_j) \cdot f \cdot l \mid l)$$

and $x_j = \|\phi(c_j) f \cdot l\|^2$ does not depend on the choice of the antecedent f of b^{-1} by the map ρ .

Finally, we find

$$\zeta_L(s) \leq M^s \sum_{j=1}^r \sum_{\alpha \in Q(L'_j)} \mu_j(\alpha) \prod_{i=1}^m \alpha_i^{-s} = M^s \sum_{j=1}^r S_j,$$

and the announced result is just a consequence of the above lemma. \square

2. Analytic continuation and the functional equation.

We use the classical method which consists to see the zeta series as the Mellin transform of the theta series associated to the representation ϕ and the lattice L , and the functional equation is a consequence of the transformation formula of the theta series.

First recall some results about zeta integrals.

2.1. Zeta integrals associated to ϕ .

For each function f in the Schwartz space $\mathcal{S}(E)$, the zeta integral associated to the representation ϕ is defined by

$$Z(f, s) = \int_E [\det Q(\xi)]^s f(\xi) d\xi, \quad \forall s \in \mathbf{C}.$$

PROPOSITION 2.1.1. — *The zeta integral $Z(f, s)$ converges absolutely for $\operatorname{Re}(s) > \frac{d}{2}(m-1) - \frac{N}{2m}$ (d denotes the dimension of the subspaces V_{ij} for $i \neq j$ in the Peirce decomposition of V). It admits an analytic continuation as a meromorphic function on the whole plane \mathbf{C} , and satisfies to the functional equation*

$$Z\left(\hat{f}, s - \frac{N}{2m}\right) = \gamma(s) Z(f, -s),$$

where

$$\gamma(s) = \pi^{\frac{N}{2} - 2ms} \frac{\Gamma_{\Omega}(s)}{\Gamma_{\Omega}\left(\frac{N}{2m} - s\right)},$$

Γ_{Ω} being the Koecher-Gindikin gamma function of the cone Ω , that is

$$\Gamma_{\Omega}(s) = \int_{\Omega} e^{-tr(x)} \det(x)^{s - \frac{n}{m}} dx,$$

and

$$\hat{f}(\xi) = \int_E e^{-2\pi i(\xi|\eta)} f(\eta) d\eta$$

is the Fourier transform of f (cf [15], Chapter 16, Theorem 16.4.3).

2.2. Theta series associated to ϕ and L .

For each $f \in \mathcal{S}(E)$, the theta series associated to ϕ and L is defined by

$$\Theta(x, f, L) = \sum_{l \in L} f[\phi(x^{\frac{1}{2}})l], \quad \forall x \in \Omega.$$

It is clear that this series converges absolutely for $x \in \Omega$.

PROPOSITION 2.2.1 (Transformation formula).

$$\Theta(x^{-1}, f, L) = \text{vol}(L)^{-1} \det(x)^{\frac{N}{2m}} \Theta(x, \hat{f}, L^*),$$

where \hat{f} is the Fourier transform of f and L^* is the dual lattice of L , that is

$$L^* = \{b \in E \mid (b \mid a) \in \mathbf{Z}, \quad \forall a \in L\},$$

and $\text{vol}(L) = \text{vol}(E/L)$.

Proof. — It is a consequence of the Poisson summation formula. If $\psi \in \mathcal{S}(E)$, then

$$\sum_{l \in L} \psi(l) = \text{vol}(L)^{-1} \sum_{l \in L^*} \hat{\psi}(l).$$

If $\psi(\xi) = f[\phi(x^{-\frac{1}{2}})\xi]$, then

$$\hat{\psi}(\eta) = \text{Det}(\phi(x^{\frac{1}{2}})) \hat{f}[\phi(x^{\frac{1}{2}})\eta] = \det(x)^{\frac{N}{2m}} \hat{f}[\phi(x^{\frac{1}{2}})\eta]. \quad \square$$

2.3. Invariance property of theta series.

LEMMA 2.3.1. — If F is a K -invariant function defined on $\bar{\Omega}$, then there exists a kernel $F'(x, y)$ defined on $\bar{\Omega} \times \bar{\Omega}$ such that

$$\begin{aligned} F'(x, e) &= F(x), \\ F'(gx, y) &= F'(x, g^*y), \\ F'(x, y) &= F'(y, x), \quad \forall x, y \in \bar{\Omega}, \forall g \in G(\Omega). \end{aligned}$$

Proof. — The function F_1 defined on $\bar{\Omega} \times G(\Omega)$ by $F_1(x, g) = F(g^*x)$ is right-invariant by K as a function of g .

The function F' defined by $F'(x, g \cdot e) = F_1(x, g)$ satisfies to the announced properties; in fact, it is clear that $F'(x, e) = F(x)$ and

$$\begin{aligned} F'(g_1 \cdot x, g \cdot e) &= F_1(g_1 \cdot x, g) = F(g^*g_1 \cdot x) \\ &= F((g_1^*g)^* \cdot x) = F_1(x, g_1^*g) \\ &= F'(x, g_1^*g \cdot e). \end{aligned}$$

Moreover, as there exists $k \in K$ such that

$$P(x^{\frac{1}{2}})y = kP(y^{\frac{1}{2}})x$$

(cf.[15], Chapter 14, Lemma 14.1.2), then

$$F'(x, y) = F(P(x^{\frac{1}{2}})y) = F(P(y^{\frac{1}{2}})x) = F'(y, x). \quad \square$$

PROPOSITION 2.3.2. — *Let $F \in \mathcal{S}(\bar{\Omega})$, a K -invariant function on $\bar{\Omega}$ and let f be defined by $f(\xi) = F(Q(\xi))$. If the arithmetic subgroup Γ_o is self-adjoint, then the theta series $\Theta(x, f, L)$ is Γ -invariant i.e.*

$$\Theta(\gamma x, f, L) = \Theta(x, f, L) \quad \forall \gamma \in \Gamma.$$

Proof. — Let F' be the kernel of Lemma 2.3.1. We have

$$\begin{aligned} f(\phi(x^{\frac{1}{2}})\xi) &= F(Q(\phi(x^{\frac{1}{2}})\xi)) = F(P(x^{\frac{1}{2}})Q(\xi)) \\ &= F'(P(x^{\frac{1}{2}})Q(\xi), e) = F'(Q(\xi), x), \end{aligned}$$

and then

$$\Theta(x, f, L) = \sum_{a \in L} F'(x, Q(a)).$$

Moreover, for h in $F(\phi)$,

$$\begin{aligned} \Theta(\rho(h)x, f, L) &= \sum_{a \in L} F'(\rho(h)x, Q(a)) = \sum_{a \in L} F'(x, \rho(h)^*Q(a)) \\ &= \sum_{a \in L} F'(x, Q(h^*a)) = \Theta(x, f, h^*L), \end{aligned}$$

and, as we assumed that $\Gamma_o^* = \Gamma_o$, then $\Theta(x, f, L)$ is Γ -invariant. \square

2.4. Mellin transform of theta series.

The Mellin transform of the theta series $\Theta(x, f, L)$ is defined by

$$\Xi(s, f, L) = \int_{\Gamma \backslash \Omega} \Theta(x, f, L) \det(x)^s d^*x, \quad \forall s \in \mathbf{C}$$

where d^*x is the G -invariant measure on Ω , $d^*x = \det(x)^{-\frac{n}{m}} dx$, dx denoting the Euclidean measure on V .

(H_3) In all the sequel, we assume that $N > m(m-1)d$ and then the image of the Euclidean measure on E under the quadratic form \mathbf{Q} has a density with respect to the Euclidean measure of V .

PROPOSITION 2.4.1. — *Let $F \in \mathcal{S}(\bar{\Omega})$, K -invariant, null on $\partial\Omega$. Let f be the function defined by $f(\xi) = F(Q(\xi))$. For $s \in \mathbf{C}$, if $\operatorname{Re}(s) > \max \left\{ \frac{N}{2m}, (m-1)\frac{d}{2} \right\}$, then the integral $\Xi(s, f, L)$ converges absolutely and satisfies to*

$$\Xi(s, f, L) = \frac{\Gamma_{\Omega} \left(\frac{N}{2m} \right)}{\pi^{\frac{N}{2}}} \zeta_L(s) \cdot Z \left(f, s - \frac{N}{2m} \right).$$

Proof. — Assume f positive and s real, then

$$\Theta(x, f, L) = \sum_{a \in Q(L')} \nu(a) F'(x, a) = \sum_{a \in \Gamma \backslash Q(L')} \mu(a) \sum_{b \in \Gamma \cdot a} F'(x, b),$$

and as

$$\begin{aligned} \int_{\Gamma/\Omega} \left[\sum_{b \in \Gamma \cdot a} F'(x, b) \right] \det(x)^s d^*x &= \int_{\Omega} F'(x, a) \det(x)^s d^*x \\ &= \det(a)^{-s} \int_{\Omega} F(x) \det(x)^s d^*x < \infty, \end{aligned}$$

then

$$\begin{aligned}
 \Xi(s, f, L) &= \int_{\Gamma \backslash \Omega} \left[\sum_{a \in \Gamma \backslash Q(L')} \mu(a) \left(\sum_{b \in \Gamma \cdot a} F'(x, b) \right) \right] \det(x)^s d^*x \\
 &= \sum_{a \in \Gamma \backslash Q(L')} \mu(a) \int_{\Gamma \backslash \Omega} \left[\sum_{b \in \Gamma \cdot a} F'(x, b) \right] \det(x)^s d^*x \\
 &= \sum_{a \in \Gamma \backslash Q(L')} \mu(a) \det(a)^{-s} \int_{\Omega} F(x) \det(x)^s d^*x \\
 &= \zeta_L(s) \int_{\Omega} F(x) \det(x)^s d^*x < +\infty,
 \end{aligned}$$

and we find

$$\Xi(s, f, L) = \zeta_L(s) \cdot \int_{\Omega} F(x) \det(x)^s d^*x.$$

Recall that for $N > m(m-1)d$, the image of the measure $d\xi$ under \mathbf{Q} is

$$d\mu(x) = \frac{\pi^{\frac{N}{2}}}{\Gamma_{\Omega}\left(\frac{N}{2m}\right)} \det(x)^{\frac{N}{2m} - \frac{n}{m}} dx,$$

(cf.[15], Chapter.16, Proposition.16.1.1), we find

$$\begin{aligned}
 \Xi(s, f, L) &= \frac{\Gamma_{\Omega}\left(\frac{N}{2m}\right)}{\pi^{\frac{N}{2}}} \zeta_L(s) \int_E f(\xi) [\det(Q(\xi))]^{s - \frac{N}{2m}} d\xi \\
 &= \frac{\Gamma_{\Omega}\left(\frac{N}{2m}\right)}{\pi^{\frac{N}{2}}} \zeta_L(s) \cdot Z\left(f, s - \frac{N}{2m}\right).
 \end{aligned}$$

Recall the following lemma :

LEMMA 2.4.2. — *Let $f \in \mathcal{S}(E)$ be a radial function, namely, there exists a function F defined on $\bar{\Omega}$, such that $f(\xi) = F(Q(\xi))$, then the Fourier transform \hat{f} of f is radial (cf.[15], Chapter 16, Proposition 16.2.5).*

Moreover, we can find a radial function f such that f and its Fourier transform \hat{f} vanish on the set

$$\{\xi \in E \mid \det(Q(\xi)) = 0\}.$$

PROPOSITION 2.4.3. — *Let $F \in \mathcal{S}(\bar{\Omega})$, K -invariant, and $f(\xi) = F(Q(\xi))$. If f and \hat{f} vanish on the set $\{\xi \in E \mid \det(Q(\xi)) = 0\}$, then $\Xi(s, f, L)$ is an analytic function on its convergence domain, it admits an analytic continuation as analytic function on the whole \mathbf{C} , and it satisfies*

to the functional equation

$$\Xi\left(\frac{N}{2m} - s, \hat{f}, L^*\right) = \text{vol}(L)\Xi(s, f, L).$$

Proof. — Set

$$\Omega_+ = \{x \in \Omega \mid \det(x) \geq 1\}$$

$$\Omega_- = \{x \in \Omega \mid \det(x) \leq 1\}$$

and we define

$$\begin{aligned}\Xi_+(s, f, L) &= \int_{\Gamma/\Omega_+} \Theta(x, f, L) \det(x)^s d^*x \\ \Xi_-(s, f, L) &= \int_{\Gamma/\Omega_-} \Theta(x, f, L) \det(x)^s d^*x.\end{aligned}$$

The integral defining $\Xi_+(s, f, L)$ converges for each $s \in \mathbf{C}$ and is analytic on the whole \mathbf{C} . Indeed, for each positive constant A, B there exists a positive constant C such that

$$|F(y)| \leq C \det(y)^{-A} (1 + \text{tr}(y))^{-B},$$

and

$$|F'(x, a)| \leq C \det(x)^{-A} \det(a)^{-A} (1 + (a \mid x))^{-B}.$$

If $\text{Re}(s) \leq A$, and $\det(x) \geq 1$, then $|\det(x)^s| \leq \det(x)^A$,

and

$$|F'(x, a) \det(x)^{s - \frac{n}{2m}}| \leq C \det(a)^{-A} (1 + (a \mid x))^{-B}.$$

Otherwise,

$$\int_{\Omega_+} (1 + (a \mid x))^{-B} dx \leq \det(a)^{-\frac{n}{2m}} \int_{\Omega} (1 + \text{tr}(y))^{-B} dy.$$

So, if $A > \frac{N}{2m}$, then

$$\sum_{a \in \Gamma \backslash Q(L')} \mu(a) \det(a)^{-A} \int_{\Omega_+} (1 + (a \mid x))^{-B} dx < \infty.$$

On an other side, as

$$\Theta(x^{-1}, f, L) = \text{vol}(L)^{-1} \det(x)^{\frac{n}{2m}} \Theta(x, \hat{f}, L^*),$$

then if $\Xi_-(s, f, L)$ converges, i.e. if $\Xi(s, f, L)$ converges, then

$$\begin{aligned}\Xi_-(s, f, L) &= \int_{\Gamma/\Omega_+} \Theta(x^{-1}, f, L) \det(x)^{-s} d^*x \\ &= \text{vol}(L)^{-1} \int_{\Gamma/\Omega_+} \Theta(x, \hat{f}, L^*) \det(x)^{\frac{N}{2m}-s} d^*x \\ &= \text{vol}(L)^{-1} \Xi_+ \left(\frac{N}{2m} - s, \hat{f}, L^* \right)\end{aligned}$$

i.e.

$$\Xi_-(s, f, L) = \text{vol}(L)^{-1} \Xi_+ \left(\frac{N}{2m} - s, \hat{f}, L^* \right).$$

We deduce that $\Xi_-(s, f, L)$ is analytic on its convergence domain and as $\Xi_+(s, f, L)$ is analytic on \mathbf{C} , then the above equation gives the analytic continuation of $\Xi_-(s, f, L)$ as analytic function on \mathbf{C} . It is also the same for the Mellin transform $\Xi(s, f, L)$ which is given by

$$\begin{aligned}\Xi(s, f, L) &= \Xi_+(s, f, L) + \Xi_-(s, f, L) \\ &= \Xi_+(s, f, L) + \text{vol}(L)^{-1} \Xi_+ \left(\frac{N}{2m} - s, \hat{f}, L^* \right).\end{aligned}$$

Moreover, it satisfies to the functional equation

$$\Xi \left(\frac{N}{2m} - s, \hat{f}, L^* \right) = \text{vol}(L) \Xi(s, f, L). \quad \square$$

2.5. Analytic continuation and functional equation.

From the above, we deduce

$$\begin{aligned}\Xi \left(\frac{N}{2m} - s, f, L \right) &= \frac{\Gamma_\Omega \left(\frac{N}{2m} \right)}{\pi^{\frac{N}{2}}} \zeta_L \left(\frac{N}{2m} - s \right) Z(f, -s) = \text{vol}(L) \Xi(s, \hat{f}, L^*) \\ &= \text{vol}(L) \frac{\Gamma_\Omega \left(\frac{N}{2m} \right)}{\pi^{\frac{N}{2}}} \zeta_{L^*}(s) Z \left(\hat{f}, s - \frac{N}{2m} \right) \\ &= \text{vol}(L) \frac{\Gamma_\Omega \left(\frac{N}{2m} \right)}{\pi^{\frac{N}{2}}} \zeta_{L^*}(s) \pi^{\frac{N}{2}-2ms} \frac{\Gamma_\Omega(s)}{\Gamma_\Omega \left(\frac{N}{2m} - s \right)} Z(f, -s)\end{aligned}$$

and finally, we have the theorem :

THEOREM 2.5.1. — *Under the assumptions :*

(H₃) $N > m(m-1)d$,

(H₄) *the arithmetic subgroup Γ_\circ is self-adjoint,*

the zeta function $\zeta_L(s)$ admits an analytic continuation as a meromorphic function on the whole \mathbf{C} and satisfies to the functional equation

$$\zeta_L\left(\frac{N}{2m} - s\right) = \text{vol}(L)\pi^{\frac{N}{2}-2ms} \frac{\Gamma_\Omega(s)}{\Gamma_\Omega\left(\frac{N}{2m} - s\right)} \zeta_{L^*}(s).$$

Remark. — If $\tilde{\Gamma}_\circ$ is a finite-index subgroup of Γ_\circ , then the zeta series defined by

$$\tilde{\zeta}_L(s) = \sum_{l \in \tilde{\Gamma}_\circ \backslash L'} \det(Q(l))^{-s}$$

has the same properties than $\zeta_L(s)$.

3. Examples.

In this section we look at some examples of zeta functions.

3.1. Case of symmetric real matrices.

Let $V = \text{Sym}(m, \mathbf{R})$ be the Jordan algebra with the product $A \circ B = \frac{1}{2}(AB + BA)$, then the symmetric associated cone is the cone Ω of positive definite symmetric real matrices. Let $E = M(m, n, \mathbf{R})$ (with $n \geq m$), and ϕ the representation

$$\phi : V \rightarrow \text{Sym}(E), x \mapsto \phi(x) : \xi \mapsto x\xi.$$

The associated quadratic form Q is given by $Q(\xi) = \xi\xi', \forall \xi \in E$. (ξ' is the adjoint of ξ .) Let $V_{\mathbf{Q}} = \text{Sym}(m, \mathbf{Q})$, then $V_{\mathbf{Q}}$ is a split \mathbf{Q} -structure of V , and let $E_{\mathbf{Q}} = M(m, n, \mathbf{Q})$ and L the lattice $L = M(m, n, \mathbf{Z})$. It is clear that ϕ is defined over \mathbf{Q} , moreover, the arithmetic group $GL(m, \mathbf{Z})$ is a finite-index subgroup of Γ_\circ , where

$$\Gamma_\circ = \{f \in F(\phi) \mid f(L) = L\},$$

and the zeta series is

$$\zeta_L(s) = \sum_{A \in GL(m, \mathbf{Z}) \setminus M(m, n, \mathbf{Z}), \text{rank}(A)=m} \text{Det}(AA')^{-s},$$

which is the classical Kœcher zeta function.

3.2. Case of Hermitian complex matrices.

Let $V = \text{Herm}(m, \mathbf{C})$ be the Jordan algebra with the above product “ \circ ”, $E = M(m, n, \mathbf{C})$ (with $n \geq m$), and ϕ the representation

$$\phi(x)\xi = x\xi, \forall x \in V, \xi \in E.$$

The associated quadratic form is given by $Q(\xi) = \xi\bar{\xi}'$.

Let K be an imaginary quadratic field and \mathcal{O} its ring of integers. Then $V_{\mathbf{Q}} = \text{Herm}(m, K)$ is a split \mathbf{Q} -structure of V and the space $E_{\mathbf{Q}} = M(m, n, K)$ is a \mathbf{Q} -structure of E . Let L be the lattice $L = M(m, n, \mathcal{O})$, then it is clear that the representation ϕ is defined over \mathbf{Q} , and the group $GL(m, \mathcal{O})$ is of finite index in Γ_{\circ} , and we obtain the zeta series

$$\zeta_L(s) = \sum_{A \in GL(m, \mathcal{O}) \setminus M(m, n, \mathcal{O}), \text{rank}(A)=m} [\text{Det}(A\bar{A}')]^{-s},$$

and this case gives a new example of zeta function.

3.3. Case of Hermitian quaternionic matrices.

Let $V = \text{Herm}(m, \mathbf{H})$ with the Jordan product, $E = M(m, n, \mathbf{H})$ and ϕ the representation $\phi(x)\xi = x\xi, \forall x \in V, \xi \in E$. If \mathcal{O} denotes the ring of Hurwitz integers, then $L = M(m, n, \mathcal{O})$ is a lattice in E and the group $GL(m, \mathcal{O})$ is of finite-index in the arithmetic subgroup Γ_{\circ} associated to ϕ . The zeta series is the one studied by A. Krieg in [15], and is given by

$$\zeta_L(s) = \sum_{A \in GL(m, \mathcal{O}) \setminus M(m, n, \mathcal{O}), \text{rank}(A)=m} \text{Det}(A\bar{A}')^{-s}.$$

The case of zeta functions of representations of rank 2-Jordan algebras gives new examples of zeta functions and constitutes for itself an other article (cf. [1]).

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BIBLIOGRAPHY

- [1] D. ACHAB, Représentations des algèbres de Jordan de rang 2 et fonctions zeta associées, Annales de l'Institut Fourier, Grenoble, 45-2 (1995), 437-451.
- [2] T. ARAKAWA, Dirichlet Series Corresponding to Siegel's Modular Forms, Math. Ann., 238 (1978), 157-173.
- [3] A. ASH, D. MUMFORD, M. RAPOPPORT et Y.S. TAI, Smooth compactifications of locally symmetric varieties. Lie groups : History, frontiers and applications, Math. Sci. Press, 1975.
- [4] A. BOREL, Introduction aux groupes arithmétiques, Hermann, Paris, 1969.
- [5] A. BOREL, Linear algebraic groups. Second Enlarged Edition, Springer-Verlag, 1991.
- [6] A. BOREL, Density and maximality of arithmetic subgroups, J. Reine Angw. Math., 224 (1966), 78-89.
- [7] BRAUN et KOECHER, Jordan-Algebren, Springer, 1966.
- [8] J.L. CLERC, Représentations d'une algèbre de Jordan, polynômes invariants et harmoniques de Stiefel, J. Reine Angw. Math., 423 (1992), 47-71.
- [9] J. FARAUT et A. KORANYI, Analysis on symmetric cones, Oxford University Press, 1994.
- [10] M. KOECHER, Über Dirichlet-Reihen mit Funktionalgleichung, J. Reine Angw. Math., 192 (1953), 1-23.
- [11] A. KRIEG, Koecher-Maass- Series for Modular Forms of Quaternions, Manuscripta Math., 66 (1990), 431-451.
- [12] A. KRIEG, Modular Forms on Half-Spaces of Quaternions, Springer-Verlag, 1980.
- [13] H. MAASS, Siegel's Modular Forms and Dirichlet Series, Lect. Notes in Math. 216, Springer-Verlag, 1971.
- [14] M. SATO et T.SHINTANI, On zeta functions associated with prehomogeneous vector spaces, Ann. of Math., 100 (1974), 131-170.
- [15] I. SATAKE, private communication.

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