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# ON THE $K$ -THEORY AND HATTORI–STALLINGS TRACES OF MINIMAL PRIMITIVE FACTORS OF ENVELOPING ALGEBRAS OF SEMISIMPLE LIE ALGEBRAS : THE SINGULAR CASE

by Patrick POLO

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## Introduction.

Let  $G$  be a semisimple complex algebraic group, let  $\mathfrak{g} = \text{Lie}(G)$ , and let  $U = U(\mathfrak{g})$  be the enveloping algebra of  $\mathfrak{g}$ . Let  $X$  be the flag variety of  $G$ . Let  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{g}$  and let  $W$  be the Weyl group. For  $\mu \in \mathfrak{h}^*$ , let  $J_\mu$  be the corresponding minimal primitive ideal of  $U$ , let  $U_\mu = U/J_\mu$ , and let  $\mathcal{T}_{U_\mu} : K_0(U_\mu) \rightarrow \mathbb{C}$  be the Hattori–Stallings trace. One says that a weight  $\mu \in \mathfrak{h}^*$  is regular if its Weyl group stabilizer  $W_\mu$  is trivial, and singular otherwise. For a regular weight  $\mu$ , T.J. Hodges has shown [11] that  $K_0(U_\mu)$  is isomorphic to  $K_0(X)$  and is therefore generated by the classes corresponding to  $G$ -linearized line bundles on  $X$ .

Moreover, in [12], Hodges used the Hattori–Stallings trace to classify, in the case where  $\mathfrak{g} = \mathfrak{sl}_2$ , the  $\mathbb{C}$ -algebras  $U_\mu$  up to Morita equivalence and to obtain a short proof of Dixmier’s earlier description of the isomorphism classes. For an arbitrary semisimple  $\mathfrak{g}$  and a regular weight  $\mu$ , it was shown in [13], in a special case, and then in [21], in general, that the value of  $\mathcal{T}_{U_\mu}$  on the generators corresponding to  $G$ -linearized line bundles on  $X$  was given by Weyl’s dimension formula.

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In this paper, we obtain a similar description of the Hattori–Stallings trace for singular weights. This is done in three steps. Let  $G_0(U_\mu)$  be the Grothendieck group of the category of finitely generated left  $U_\mu$ -modules. Firstly, we show that  $G_0(U_\mu) \cong K_0(X)/\Sigma_s \operatorname{Im}(1-s)$ , where the sum runs over a set of simple reflections generating  $W_\mu$ . This fact, which results from [3], Quillen’s localization theorem, [19], and [11], was certainly known to some specialists, but we are not aware if it was known more widely. One deduces, in particular, that  $G_0(U_\mu)$  has rank  $|W/W_\mu|$ . Secondly, we prove that the Cartan map  $K_0(U_\mu) \rightarrow G_0(U_\mu)$  is an isomorphism up to torsion. It follows that  $K_0(U_\mu)_\mathbb{Q}$  is generated by the images of the  $G$ -linearized line bundles on  $X$ . Finally, as in [21], we show, using the Bernstein trace, that the value of  $\mathcal{T}_{U_\mu}$  on these generators is given by a certain polynomial formula. It is hoped that this will bring some information about the isomorphism and Morita equivalence classes of primitive factors.

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## 1. Preliminaries.

1.1. Throughout the paper, the base field  $k$  is algebraically closed and of characteristic zero. Let  $\mathfrak{g}$  be a semisimple Lie algebra over  $k$  and let  $U = U(\mathfrak{g})$  be its enveloping algebra. Let  $\mathfrak{h} \subset \mathfrak{b}$  be a Cartan subalgebra inside a Borel subalgebra of  $\mathfrak{g}$ , let  $W$  be the Weyl group of  $(\mathfrak{g}, \mathfrak{h})$ , let  $R^+$  be the set of roots of  $\mathfrak{h}$  in  $\mathfrak{b}$ , let  $\Delta$  be the corresponding set of simple roots, and let  $\rho$  be the half-sum of the elements of  $R^+$ . For  $\alpha \in R^+$ , let  $H_\alpha \in \mathfrak{h}$  be the corresponding coroot and let  $s_\alpha$  be the corresponding reflection in  $W$ .

For a weight  $\lambda \in \mathfrak{h}^*$ , let  $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_{\lambda-\rho}$  be the Verma module with highest weight  $\lambda - \rho$  and let  $J_\lambda = \operatorname{Ann} M(\lambda)$ . Let  $Z$  denote the centre of  $U$ . Via the Harish-Chandra isomorphism  $Z \xrightarrow{\sim} S(\mathfrak{h})^W$ , every  $\lambda \in \mathfrak{h}^*$  defines a central character  $\chi_\lambda$ . One has  $U(\operatorname{Ker} \chi_\lambda) = J_\lambda$  and  $J_\lambda = J_{\lambda'}$  if and only if  $\lambda$  and  $\lambda'$  are  $W$ -conjugate. Further,  $J_\lambda$  is a minimal primitive ideal of  $U$ . For all this, see [9], §§7.4, 8.4. Finally, let  $U_\lambda = U/J_\lambda$  and let  $U_\lambda\text{-Modf}$  denote the category of finitely generated left  $U_\lambda$ -modules.

Let  $\mu \in \mathfrak{h}^*$ . One says that  $\mu$  is antidominant if  $\mu(H_\alpha) \notin \mathbb{N}^+$ , for  $\alpha \in R^+$ . Let  $W_\mu$  denote the stabilizer of  $\mu$  in  $W$  and let  $\Delta_\mu = \{\alpha \in \Delta \mid s_\alpha \mu = \mu\}$ . Recall that  $\mu$  is said to be regular if  $W_\mu$  is trivial, and singular otherwise. Moreover, we shall say that  $\mu$  is tamely singular if  $W_\mu$  is generated by the simple reflections  $s_\alpha$ , for  $\alpha \in \Delta_\mu$ . By [7], Chap. V, n° 3.3, every weight  $\mu \in \mathfrak{h}^*$  is  $W$ -conjugate to at least one tamely singular, antidominant weight and hence, for the study of  $U_\mu$ , there is no loss of generality in assuming that  $\mu$  is antidominant and tamely singular.

1.2. Let  $G$  be a semisimple, connected and simply-connected, algebraic group over  $k$  such that  $\text{Lie}(G) = \mathfrak{g}$  and let  $T \subset B$  be the connected subgroups corresponding to  $\mathfrak{h} \subset \mathfrak{b}$ . Since  $G$  is simply-connected, the character group of  $T$  identifies with the lattice of integral weights  $\mathcal{P} := \{\nu \in \mathfrak{h}^* \mid \nu(H_\beta) \in \mathbb{Z}, \forall \beta \in R^+\}$ . For every rational  $B$ -module  $V$ , let  $\mathcal{L}(V)$  denote the associated  $G$ -equivariant vector bundle on  $G/B$ . For  $\nu \in \mathcal{P}$ , the line bundle associated with  $V = k_\nu$  will be denoted simply by  $\mathcal{L}(\nu)$ .

Let  $\mathbb{Z}\mathcal{P}$  be the group algebra of  $\mathcal{P}$ , with its natural basis  $e^\nu : \nu \in \mathcal{P}$ . By [20], §6, the map  $e^\nu \mapsto [\mathcal{L}(\nu)]$  induces an isomorphism  $\mathbb{Z}\mathcal{P}/I \xrightarrow{\sim} K_0(G/B)$ , where  $I$  denotes the ideal generated by the  $W$ -invariants in the augmentation ideal of  $\mathbb{Z}\mathcal{P}$ . Thus, via this isomorphism, the natural action of  $W$  on  $\mathbb{Z}\mathcal{P}$  induces a  $W$ -module structure on  $K_0(G/B)$ .

1.3. For  $\lambda \in \mathfrak{h}^*$ , let  $\mathcal{D}_\lambda$  denote the sheaf of twisted differential operators on  $G/B$  associated with  $\lambda + \rho$ . By [3], one has  $\Gamma(G/B, \mathcal{D}_\lambda) \cong U_\lambda$  (see also [19] 6.1–6.2). Let  $\mathcal{D}_\lambda\text{-Coh}$  denote the category of coherent left  $\mathcal{D}_\lambda$ -modules and let  $\Gamma_\lambda$  denote the restriction to  $\mathcal{D}_\lambda\text{-Coh}$  of the global sections functor  $\Gamma(G/B, -)$ . Suppose that  $\lambda$  is antidominant and regular. Then, by [3],  $\Gamma_\lambda$  induces an equivalence of categories  $\mathcal{D}_\lambda\text{-Coh} \xrightarrow{\sim} U_\lambda\text{-Modf}$ . By a spectral sequence argument as in [6], VI.1.10, this implies that  $U_\lambda$  has finite global dimension (see also [14] Theorem 3.9). Let us then recall the following theorem.

**THEOREM ([11], Theorem 2).** — *Let  $\lambda$  be a regular antidominant weight. Then the exact functor  $\mathcal{E} \mapsto \Gamma(G/B, \mathcal{D}_\lambda \otimes_{\mathcal{O}_{G/B}} \mathcal{E})$  induces isomorphisms  $K_n(G/B) \simeq K_n(U_\lambda)$ , for  $n \geq 0$ . In particular,  $K_0(U_\lambda)$  is a free  $\mathbb{Z}$ -module of rank  $|W|$ .*

1.4. For  $U$ -modules  $M$  and  $N$ , let  $L(M, N)$  denote the set of  $\mathfrak{g}$ -finite vectors in  $\text{Hom}_k(M, N)$ ; this is a  $U$ -sub-bimodule of  $\text{Hom}_k(M, N)$ ,

see [18], 1.2. By [17] Corollary 7.25, one has

$$(1.4.1) \quad U_\lambda \cong L(M(\lambda), M(\lambda)), \quad \forall \lambda \in \mathfrak{h}^*.$$

Further, one deduces from the proof of [23] Theorem 6, the following

**PROPOSITION.** — *For  $\lambda \in \mathfrak{h}^*$  and  $\nu \in \mathcal{P}$ , there is an isomorphism of  $U$ -bimodules  $\Gamma(G/B, \mathcal{D}_\lambda \otimes_{\mathcal{O}_{G/B}} \mathcal{L}(\nu)) \cong L(M(\lambda - \nu), M(\lambda))$ .*

Thus, one obtains the

**COROLLARY.** — *Let  $\lambda$  be a regular antidominant weight. Then, under the isomorphism  $K_0(G/B) \cong K_0(U_\lambda)$  of Theorem 1.3, the class of  $\mathcal{L}(\nu)$  corresponds to the class of  $L(M(\lambda - \nu), M(\lambda))$ , for  $\nu \in \mathcal{P}$ .*

## 2. On $G_0(U_\mu)$ and $K_0(U_\mu)$ for singular $\mu$ .

2.1. For the remainder of this paper, we fix a tamely singular, antidominant weight  $\mu$ . Let  $G_0(\mathcal{D}_\mu)$  denote the Grothendieck group of the category  $\mathcal{D}_\mu\text{-Coh}$  and let  $U_\mu\text{-Proj}$  denote the category of finitely generated projective left  $U_\mu$ -modules. By [3], the functor  $\Gamma_\mu$  is exact and takes  $\mathcal{D}_\mu\text{-Coh}$  to  $U_\mu\text{-Modf}$ . Thus, it induces a map  $\gamma_\mu : G_0(\mathcal{D}_\mu) \rightarrow G_0(U_\mu)$ . Also, let  $\phi_\mu : K_0(U_\mu) \rightarrow G_0(\mathcal{D}_\mu)$  be the map induced by the localization functor  $\Phi_\mu := \mathcal{D}_\mu \otimes_{U_\mu} -$ . Clearly,  $\Gamma_\mu \circ \Phi_\mu(U_\mu) \cong U_\mu$  and hence, by additivity,  $\Gamma_\mu \circ \Phi_\mu(P) \cong P$ , for every  $P \in U_\mu\text{-Proj}$ . Thus,  $\gamma_\mu \circ \phi_\mu$  equals the Cartan map  $c_\mu : K_0(U_\mu) \rightarrow G_0(U_\mu)$ . By [11] Theorem 1, the functor  $\mathcal{D}_\mu \otimes_{\mathcal{O}_{G/B}} -$  induces an isomorphism  $\psi_\mu : K_0(G/B) \xrightarrow{\sim} G_0(\mathcal{D}_\mu)$ . Thus, in particular,  $G_0(\mathcal{D}_\mu)$  is free and hence  $\phi_\mu$  factors through a map  $\bar{\phi}_\mu : K_0(U_\mu)/K_0(U_\mu)_{\text{tor}} \rightarrow G_0(\mathcal{D}_\mu)$ , where  $K_0(U_\mu)_{\text{tor}}$  denotes the torsion part of  $K_0(U_\mu)$ . Note that, by Proposition 1.4,

$$(2.1.1) \quad \gamma_\mu \circ \psi_\mu([\mathcal{L}(\nu)]) = [L(M(\mu - \nu), M(\mu))], \quad \forall \nu \in \mathcal{P}.$$

Then, one has the

**THEOREM.** — *There is a commuting diagram*

$$\begin{array}{ccccc} K_0(U_\mu)/K_0(U_\mu)_{\text{tor}} & \xrightarrow{\bar{\phi}_\mu} & G_0(\mathcal{D}_\mu) & \xrightarrow{\gamma_\mu} & G_0(U_\mu) \\ \cong \downarrow & & \cong \downarrow \psi_\mu^{-1} & & \cong \downarrow \\ K_0(G/B)^{W_\mu} & \xrightarrow{i} & K_0(G/B) & \xrightarrow{p} & \frac{K_0(G/B)}{\sum_{\alpha \in \Delta_\mu} \text{Im}(1 - s_\alpha)}, \end{array}$$

where  $i$  and  $p$  are the natural injection and projection. Both  $K_0(U_\mu)$  and  $G_0(U_\mu)$  have rank  $|W/W_\mu|$  and the kernel and cokernel of  $c_\mu$  are annihilated by  $|W_\mu|$ .

*Remark.* — The theorem gives a partial solution to the conjecture made in [11] that  $K_0(U_\mu)$  is a free  $\mathbb{Z}$ -module of rank  $|W/W_\mu|^{(1)}$ .

2.2. We shall prove the theorem in several steps. First, by [3],  $\Gamma_\mu$  is exact and induces an equivalence from the quotient category  $\mathcal{D}_\mu\text{-Coh} / \text{Ker } \Gamma_\mu$  to  $U_\mu\text{-Modf}$  and hence, by Quillen's localization theorem [22] §5, Theorem 5,  $G_0(U_\mu)$  is isomorphic to  $G_0(\mathcal{D}_\mu) / \text{Ker } \gamma_\mu$ . Thus, the commutativity of the right-hand square follows from the

PROPOSITION. — One has  $\psi_\mu^{-1}(\text{Ker } \gamma_\mu) = \sum_{\alpha \in \Delta_\mu} \text{Im}(1-s_\alpha)$  and hence  $\psi_\mu^{-1}$  induces an isomorphism  $G_0(U_\mu) \cong K_0(G/B) / \sum_{\alpha \in \Delta_\mu} \text{Im}(1-s_\alpha)$ .

*Proof.* — For  $\alpha \in \Delta$ , let  $P_\alpha \supset B$  be the corresponding parabolic subgroup and let  $\pi_\alpha$  denote the projection  $G/B \rightarrow G/P_\alpha$ . For  $\alpha \in \Delta_\mu$ , let  $\mathcal{A}_\mu^\alpha$  be the sheaf of twisted differential operators on  $G/P_\alpha$  associated with  $\mu$  (see, for instance, [19] 4.9.2). If  $\mathcal{N}$  is an  $\mathcal{A}_\mu^\alpha$ -module then, by [19] 8.1.1,  $\pi_\alpha^*(\mathcal{N})$  is a  $\mathcal{D}_{\mu-\rho}$ -module. Moreover, by [10] p. 328-329, there is an exact sequence of left  $\mathcal{D}_{\mu-\rho}$ -modules

$$0 \longrightarrow \mathcal{D}_{\mu-\rho} \otimes \mathcal{L}(-\alpha) \longrightarrow \mathcal{D}_{\mu-\rho} \longrightarrow \pi_\alpha^*(\mathcal{A}_\mu^\alpha) \longrightarrow 0.$$

Tensoring this exact sequence by  $\mathcal{L}(\rho)$  on the left, one obtains, using [10] A.3.1, an exact sequence of left  $\mathcal{D}_\mu$ -modules

$$(2.2.1) \quad 0 \longrightarrow \mathcal{D}_\mu \otimes \mathcal{L}(\rho - \alpha) \longrightarrow \mathcal{D}_\mu \otimes \mathcal{L}(\rho) \longrightarrow \mathcal{L}(\rho) \otimes \pi_\alpha^*(\mathcal{A}_\mu^\alpha) \longrightarrow 0.$$

In particular,  $\mathcal{L}(\rho) \otimes \pi_\alpha^*(\mathcal{A}_\mu^\alpha)$  is a coherent  $\mathcal{D}_\mu$ -module. One deduces that the exact functor  $\mathcal{N} \mapsto \mathcal{L}(\rho) \otimes \pi_\alpha^*(\mathcal{N})$  takes  $\mathcal{A}_\mu^\alpha\text{-Coh}$  to  $\mathcal{D}_\mu\text{-Coh}$  and hence induces a map  $f_\alpha : G_0(\mathcal{A}_\mu^\alpha) \rightarrow G_0(\mathcal{D}_\mu)$ .

Now, it follows from [19] Theorem 8.3.1, that

$$(2.2.2) \quad \text{Ker } \gamma_\mu = \sum_{\alpha \in \Delta_\mu} \text{Im } f_\alpha.$$

For  $\alpha \in \Delta_\mu$ , let us denote by  $\mathcal{L}_\alpha(V)$  the  $G$ -equivariant vector bundle on  $G/P_\alpha$  associated with a rational  $P_\alpha$ -module  $V$ . Then, it follows from

<sup>(1)</sup> The freeness of  $K_0(U_\mu)$  has now been obtained in collaboration with M. Holland, using different ideas and techniques [15].

[11] Theorem 1, applied to  $G/P_\alpha$ , together with [20] Proposition 6, that  $\text{Im } f_\alpha$  is generated by the classes of the objects

$$\mathcal{L}(\rho) \otimes \pi_\alpha^*(\mathcal{A}_\mu^\alpha \otimes \mathcal{L}_\alpha(V)) \cong \mathcal{L}(\rho) \otimes \pi_\alpha^*(\mathcal{A}_\mu^\alpha) \otimes \mathcal{L}(V),$$

for  $V$  an irreducible rational  $P_\alpha$ -module. But, tensoring (2.2.1) by  $\mathcal{L}(V)$  on the right, one obtains an exact sequence of left  $\mathcal{D}_\mu$ -modules

(2.2.3)

$$0 \longrightarrow \mathcal{D}_\mu \otimes \mathcal{L}(k_{\rho-\alpha} \otimes V) \longrightarrow \mathcal{D}_\mu \otimes \mathcal{L}(k_\rho \otimes V) \longrightarrow \mathcal{L}(\rho) \otimes \pi_\alpha^*(\mathcal{A}_\mu^\alpha) \otimes \mathcal{L}(V) \longrightarrow 0.$$

Let  $\mathcal{P}_\alpha^+ = \{\nu \in \mathcal{P} \mid \nu(H_\alpha) \geq 0\}$  and, for  $\nu \in \mathcal{P}_\alpha^+$ , let  $V_\alpha(\nu)$  denote the irreducible rational  $P_\alpha$ -module with highest weight  $\nu$ . Since the formal character of  $V_\alpha(\nu)$  equals  $e^\nu + e^{\nu-\alpha} + \dots + e^{s_\alpha \nu}$ , one deduces from (2.2.3) that

$$\begin{aligned} [\mathcal{L}(\rho) \otimes \pi_\alpha^*(\mathcal{A}_\mu^\alpha) \otimes \mathcal{L}(V_\alpha(\nu))] &= [\mathcal{D}_\mu \otimes \mathcal{L}(k_\rho \otimes V_\alpha(\nu))] \\ &\quad - [\mathcal{D}_\mu \otimes \mathcal{L}(k_{\rho-\alpha} \otimes V_\alpha(\nu))] \\ &= [\mathcal{D}_\mu \otimes \mathcal{L}(\rho + \nu)] - [\mathcal{D}_\mu \otimes \mathcal{L}(s_\alpha \rho + s_\alpha \nu)]. \end{aligned}$$

It follows that  $\psi_\mu^{-1}(\text{Ker } \gamma_\mu)$  is the  $\mathbb{Z}$ -submodule of  $K_0(G/B)$  spanned by  $[\mathcal{L}(\nu + \rho)] - [\mathcal{L}(s_\alpha(\nu + \rho))]: \alpha \in \Delta_\mu, \nu \in \mathcal{P}_\alpha^+$ . But, by [20] Proposition 4, this submodule is exactly  $\sum_{\alpha \in \Delta_\mu} (1 - s_\alpha)K_0(G/B)$ . This completes the proof of the proposition.

2.3. Let us briefly recall the definition of the translation functors (see [16], [5]). Let  $\mathcal{C}$  be the category of finitely generated left  $U$ -modules which are locally  $Z$ -finite. For  $M \in \mathcal{C}$  and  $\eta \in \mathfrak{h}^*$ , let  $\pi_\eta M = \{x \in M \mid (\text{Ker } \chi_\eta)^n x = 0, \text{ for } n \gg 0\}$ . Let  $\eta, \xi \in \mathfrak{h}^*$  such that  $\xi - \eta \in \mathcal{P}$  and let  $E$  denote the finite dimensional irreducible left  $U$ -module whose highest weight is  $W$ -conjugate to  $\xi - \eta$ . Then, the functor  $T_\eta^\xi$  is defined by  $T_\eta^\xi M = \pi_\xi(E \otimes \pi_\eta M)$ , for  $M \in \mathcal{C}$ . By [5] Corollary 2.6,  $T_\eta^\xi M$  belongs to  $\mathcal{C}$  and it follows from the definition that  $T_\eta^\xi$  and  $T_\xi^\eta$  are both left and right adjoint.

We shall also need the analogous functors for right modules. Let  $\mathcal{C}'$  be the category of finitely generated, locally  $Z$ -finite, right  $U$ -modules. For  $M' \in \mathcal{C}'$  and  $\eta \in \mathfrak{h}^*$ ,  $M'_\eta \pi$  is defined in the obvious way. If  $\eta, \xi$  and  $E$  are as above then  $E^*$  is in a natural way a right  $U$ -module and, for  $M' \in \mathcal{C}'$ , we set  $M' \overset{\xi}{T}_\eta = (M'_\eta \pi \otimes E^*)_\xi \pi$ . Then, for  $M, N \in \mathcal{C}$ , it is easily seen that  $L(M, N)$ , regarded as a left resp. right  $U$ -module, belongs to  $\mathcal{C}$  resp.  $\mathcal{C}'$  and that there are bimodule isomorphisms

$$(2.3.1) \quad T_\eta^\xi L(M, N) \cong L(M, T_\eta^\xi N) \quad \text{and} \quad L(M, N) \overset{\xi}{T}_\eta \cong L(T_\eta^\xi M, N).$$

Recall also the definition of the category  $\mathcal{O}$ , see, for instance, [17] Chap. 4. Let  $\mathcal{P}^+ = \{\nu \in \mathcal{P} \mid \nu(H_\beta) \geq 0, \forall \beta \in R^+\}$ . Then, one has the

LEMMA.

(a) Let  $N \subset M$  be objects in  $\mathcal{O}$  such that  $M/N \cong M(\xi)$ , for some  $\xi \in \mathfrak{h}^*$ , and let  $\kappa$  be an antidominant weight. Then the sequence  $0 \rightarrow L(M(\xi), M(\kappa)) \rightarrow L(M, M(\kappa)) \rightarrow L(N, M(\kappa)) \rightarrow 0$  is exact.

(b) Let  $\kappa, \zeta$  be antidominant weights and let  $\nu \in \mathcal{P}^+$ . Then  $L(T_\zeta^{\zeta-\nu} M(\zeta), M(\kappa))$  has a bimodule composition series with factors exactly the  $L(M(\xi), M(\kappa))$ , for  $\xi \in W_\zeta(\zeta-\nu)$ . Moreover, if  $W_\zeta \subseteq W_\kappa$  then these composition factors are all isomorphic to  $L(M(\zeta-\nu), M(\zeta))$ .

*Proof.* — Let  $\delta$  denote the duality functor in  $\mathcal{O}$ . For an antidominant weight  $\kappa$ ,  $M(\kappa)$  is simple (see [9] 7.6.24) and hence  $M(\kappa) \cong \delta M(\kappa)$ . Thus, assertion (a) follows from the proof of 6.9.(9) and Lemmas 4.7, 4.11 in [17]. Moreover, by [16] 2.9.c), 2.17,  $T_\zeta^{\zeta-\nu} M(\zeta)$  has a composition series with factors exactly the  $M(w(\zeta-\nu))$ , for  $w \in W_\zeta$  and hence the first part of assertion (b) follows from assertion (a). Finally, the last part follows from [17] Cor. 7.24 (see also the proof of [18] Prop. 4.19).

2.4. Let  $\lambda = \mu - \rho$ . Note that  $\lambda$  is antidominant and regular. Let  $U_\lambda\text{-Proj}$  denote the category of finitely generated projective left  $U_\lambda$ -modules. Let  $L = U_\lambda \overset{\mu}{\lambda} T$ . By the definition of  $\overset{\mu}{\lambda} T$ ,  $L$  belongs to  $U_\lambda\text{-Proj}$ . By (1.4.1), (2.3.1), and [16], 2.10.a), one has  $L \cong L(M(\mu), M(\lambda))$  and hence  $L$  is a right  $U_\mu$ -module. Thus, the functor  $L \otimes_{U_\mu} -$  takes  $U_\mu\text{-Proj}$  to  $U_\lambda\text{-Proj}$  and hence induces a map  $\theta^{\text{out}} : K_0(U_\mu) \rightarrow K_0(U_\lambda)$ .

Recall that the functor

$$\mathcal{L}^+ : \mathcal{M} \mapsto \mathcal{L}(\rho) \otimes_{\mathcal{O}_{G/B}} \mathcal{M}$$

induces an equivalence from  $\mathcal{D}_\lambda\text{-Coh}$  to  $\mathcal{D}_\mu\text{-Coh}$ , with inverse

$$\mathcal{L}^- : \mathcal{N} \mapsto \mathcal{L}(-\rho) \otimes_{\mathcal{O}_{G/B}} \mathcal{N},$$

see, for example, [10] A.3.1. Then one has the

LEMMA. — One has a commuting diagram

$$\begin{array}{ccc} K_0(U_\mu) & \xrightarrow{\phi_\mu} & G_0(\mathcal{D}_\mu) \\ \theta^{\text{out}} \downarrow & & \cong \uparrow \mathcal{L}^+ \circ \phi_\lambda \\ K_0(U_\lambda) & \xrightarrow{\cong} & G_0(U_\lambda). \end{array}$$



*Proof.* — Since  $\lambda$  is antidominant and regular then, by [3], the functors  $\Gamma_\lambda$  and  $\Phi_\lambda$  are mutually inverse. Thus, it suffices to show that the functor  $F := \Gamma_\lambda \circ \mathcal{L}^- \circ \Phi_\mu$  is isomorphic to  $L \otimes_{U_\mu} -$ . But  $F$  is exact and commutes with direct limits (since this is true for each of  $\Phi_\mu$ ,  $\mathcal{L}^-$  and  $\Gamma_\lambda$ ) and hence it is isomorphic to the functor  $F(U_\mu) \otimes_{U_\mu} -$ , by [5], Prop. 1.3. Further,  $\mathcal{L}^- \circ \Phi_\mu(U_\mu) \cong \mathcal{D}_\lambda \otimes_{\mathcal{O}_{G/B}} \mathcal{L}(-\rho)$  and hence  $F(U_\mu) \cong L$ , by Proposition 1.4. This proves the lemma.

2.5. Let  $L^\natural$  denote the  $(U_\mu, U)$ -bimodule  $U_\mu \overset{\lambda}{\mu} T$ . By the definition of  $\overset{\lambda}{\mu} T$ ,  $L^\natural$  is a finitely generated projective left  $U_\mu$ -module and its right annihilator contains a power  $J_\lambda^n$  of  $J_\lambda$ , for some  $n > 0$ . Therefore, the functor  $L^\natural \otimes_{(U/J_\lambda^n)} -$  induces a map  $\theta^\natural : K_0(U/J_\lambda^n) \rightarrow K_0(U_\mu)$ . Note also that  $L^\natural \cong L(T_\mu^\lambda M(\mu), M(\mu))$ , by (1.4.1) and (2.3.1).

Consider now the  $(U/J_\lambda^n, U)$ -bimodule  $\widehat{L} := (U/J_\lambda^n) \overset{\mu}{\lambda} T$ . By the definition of  $\overset{\mu}{\lambda} T$ , again,  $\widehat{L}$  is a finitely generated projective left  $(U/J_\lambda^n)$ -module and its right annihilator contains some power  $J_\mu^r$  of  $J_\mu$ . Therefore the functor  $\widehat{L} \otimes_{(U/J_\mu^r)} -$  induces a map  $\widehat{\theta}^{\text{out}} : K_0(U/J_\mu^r) \rightarrow K_0(U/J_\lambda^n)$ . Moreover, by [1] IX.1.3, the natural maps  $f : K_0(U/J_\lambda^n) \rightarrow K_0(U_\lambda)$  and  $g : K_0(U/J_\mu^r) \rightarrow K_0(U_\mu)$  are isomorphisms. Then, one has the

PROPOSITION. — *There is a commuting diagram*

$$\begin{array}{ccccc} K_0(U_\mu) & \xleftarrow[\cong]{g} & K_0(U/J_\mu^r) & \xrightarrow[\cong]{g} & K_0(U_\mu) \\ \theta^{\text{out}} \downarrow & & \widehat{\theta}^{\text{out}} \downarrow & & \downarrow |W_\mu| \text{ id} \\ K_0(U_\lambda) & \xleftarrow[\cong]{f} & K_0(U/J_\lambda^n) & \xrightarrow{\theta^\natural} & K_0(U_\mu). \end{array}$$

Thus,  $\theta^\natural \circ f^{-1} \circ \theta^{\text{out}} = |W_\mu| \text{ id}_{K_0(U_\mu)}$ .

*Proof.* — By [5] 1.3, one has  $U_\lambda \otimes_{U/J_\lambda^n} \widehat{L} \cong L$  and hence the left-hand square commutes. Further, by [5] 1.3, again, and (1.4.1), (2.3.1), one has

$$L^\natural \otimes_{U/J_\lambda^n} \widehat{L} \cong L^\natural \overset{\mu}{\lambda} T \cong L(T_\mu^\lambda M(\mu), M(\mu)).$$

Moreover, by [17] 4.7, 4.13(2),  $T_\mu^\lambda M(\mu)$  is isomorphic to a direct sum of  $|W_\mu|$  copies of  $M(\mu)$  and hence, by (1.4.1), again,  $L^\natural \otimes_{U/J_\lambda^n} \widehat{L}$  is isomorphic to a direct sum of  $|W_\mu|$  copies of  $U_\mu$ . This proves the commutativity of the right-hand square and the proposition follows.

2.6. For a subset  $J$  of  $\Delta$ , let  $W_J$  denote the subgroup of  $W$  generated

by  $s_\alpha : \alpha \in J$  and let  $\mathcal{P}_J^+ = \{\nu \in \mathcal{P} \mid \nu(H_\beta) \geq 0, \forall \beta \in J\}$ . For  $\nu \in \mathcal{P}_J^+$ , denote by  $\zeta_J(\nu)$  the element  $\sum_{\xi \in W_{J\nu}} e^\xi$  of  $\mathbb{Z}\mathcal{P}$ . We shall need the following

LEMMA. —  $K_0(G/B)^{W_J}$  is generated by the image of  $\zeta_J(\nu) : \nu \in \mathcal{P}^+$ .

*Proof.* — By [20] Prop. 6(a), it suffices to prove that  $(\mathbb{Z}\mathcal{P})^{W_J}$  is generated over  $(\mathbb{Z}\mathcal{P})^W$  by  $\zeta_J(\nu) : \nu \in \mathcal{P}^+$ . Clearly,  $\zeta_J(\nu) : \nu \in \mathcal{P}_J^+$  is a  $\mathbb{Z}$ -basis of  $(\mathbb{Z}\mathcal{P})^{W_J}$  and hence, being finitely generated over  $(\mathbb{Z}\mathcal{P})^W$ ,  $(\mathbb{Z}\mathcal{P})^{W_J}$  is generated over  $(\mathbb{Z}\mathcal{P})^W$  by a finite subset  $\zeta_J(\nu_1), \dots, \zeta_J(\nu_r)$ . Then one may pick  $\nu \in \mathcal{P}^+$  so that  $\nu(H_\alpha) = 0$ , for  $\alpha \in J$ , and  $\nu + \nu_i \in \mathcal{P}^+$ , for  $i = 1, \dots, r$ . Then one has  $\zeta_J(\nu + \nu_i) = e^\nu \zeta_J(\nu_i)$ , for  $i = 1, \dots, r$ , and these form another system of generators, since  $e^\nu$  is an invertible element of  $(\mathbb{Z}\mathcal{P})^{W_J}$ . This proves the lemma.

Then, one has the

PROPOSITION. — One has  $\text{Im}(\psi_\mu^{-1} \circ \phi_\mu) = K_0(G/B)^{W_\mu}$ .

*Proof.* — For  $\nu \in \mathcal{P}^+$ , let  $Q_\nu = L(\text{T}_\mu^{\mu-\nu} M(\mu), M(\mu))$ . By (1.4.1) and (2.3.1),  $Q_\nu$  is isomorphic to  $U_\mu^{\mu-\nu} T$  and belongs therefore to  $U_\mu\text{-Proj}$ . By Lemma 2.4, one has

$$\Phi_\mu(Q_\nu) \cong \mathcal{L}^+ \circ \Phi_\lambda(L \otimes_{U_\mu} Q_\nu).$$

Since  $L \cong L(M(\mu), M(\lambda))$  and  $Q_\nu \cong U_\mu^{\mu-\nu} T$  then, by [5] 1.3 and (2.3.1),

$$(2.6.1) \quad L \otimes_{U_\mu} Q_\nu \cong L(\text{T}_\mu^{\mu-\nu} M(\mu), M(\lambda)).$$

Then, using Lemma 2.3, Proposition 1.4 and the fact that  $\mathcal{L}^+ \circ \Phi_\lambda$  and  $\Gamma_\lambda \circ \mathcal{L}^-$  are mutually inverse, one deduces that

$$\phi_\mu([Q_\nu]) = \sum_{\xi \in W_{\mu\nu}} [\mathcal{D}_\mu \otimes_{\mathcal{O}_{G/B}} \mathcal{L}(\xi)].$$

Let  $J = \Delta_\mu$ . Then, one has

$$(2.6.2) \quad \psi_\mu^{-1} \circ \phi_\mu([Q_\nu]) = \zeta_J(\nu), \quad \forall \nu \in \mathcal{P}^+.$$

By the previous lemma, this implies that

$$(2.6.3) \quad \psi_\mu^{-1} \circ \phi_\mu(K_0(U_\mu)) \supseteq K_0(G/B)^{W_\mu}.$$

Further, for  $\nu \in \mathcal{P}^+$  let  $P_\nu = (U/J_\lambda^n)^{\lambda-\nu} T$ . Note that  $\lambda - \nu$  is antidominant and regular. By the definition of  ${}^{\lambda-\nu}_\lambda T$ ,  $P_\nu$  is a finitely

generated projective left  $(U/J_\lambda^n)$ -module and, using [5] 1.3, (1.4.1), (2.3.1), and [16] 2.10a), one obtains

$$(2.6.4) \quad U_\lambda \otimes_{U/J_\lambda^n} P_\nu \cong L(M(\lambda-\nu), M(\lambda))$$

$$(2.6.5) \quad L^\natural \otimes_{U/J_\lambda^n} P_\nu \cong L(T_\lambda^{\lambda-\nu} T_\mu^\lambda M(\mu), M(\lambda)).$$

Then, one deduces from (2.6.4), combined with [1] IX.1.3, Cor. 1.4, and the previous lemma, that  $K_0(U/J_\lambda^n)$  is generated by  $[P_\nu] : \nu \in \mathcal{P}^+$ . Further, by [5] 3.5, coupled with [17] 2.10.c), the functors  $T_\lambda^\mu T_{\lambda-\nu}^\lambda$  and  $T_{\lambda-\nu}^\mu$  are isomorphic and hence so are their adjoints  $T_\lambda^{\lambda-\nu} T_\mu^\lambda$  and  $T_\mu^{\lambda-\nu}$ . Coupled with (2.6.5), this gives

$$(2.6.6) \quad L^\natural \otimes_{U/J_\lambda^n} P_\nu \cong L(T_\mu^{\lambda-\nu} M(\mu), M(\lambda)) \cong Q_{\nu-\rho}.$$

Combining the previous paragraph with (2.6.2), one obtains  $\text{Im}(\psi_\mu^{-1} \circ \phi_\mu \circ \theta^\natural) \subseteq K_0(G/B)^{W_\mu}$  and hence, by Proposition 2.5, it follows that

$$(2.6.7) \quad |W_\mu| \text{Im}(\psi_\mu^{-1} \circ \phi_\mu) \subseteq K_0(G/B)^{W_\mu}.$$

Finally, by [20] Prop. 6,  $K_0(G/B)^{W_\mu}$  is a direct summand of  $K_0(G/B)$  and hence (2.6.3) and (2.6.7) together imply that  $\text{Im}(\psi_\mu^{-1} \circ \phi_\mu) = K_0(G/B)^{W_\mu}$ . This completes the proof of the proposition.

2.7. Now, to complete the proof of Theorem 2.1 it suffices, by virtue of Lemma 2.4 and Propositions 2.5, 2.6, to prove the following easy lemma.

LEMMA.

- (a) *The map  $p \circ i$  is injective and its cokernel is annihilated by  $|W_\mu|$ .*
- (b) *Both  $K_0(G/B)^{W_\mu}$  and  $K_0(G/B) / \sum_{\alpha \in \Delta_\mu} \text{Im}(1-s_\alpha)$  have rank  $|W/W_\mu|$ .*

*Proof.* — It is well-known that  $K_0(G/B)^{W_\mu}$  has rank  $|W/W_\mu|$ , see, for example, [20] Prop. 6b). Thus, the second assertion follows from the first, which we now prove. Let  $\sigma_\mu$  denote the operator  $\sum_{w \in W_\mu} w$ . For  $x \in K_0(G/B)$ , one has  $p(x) = p(s_\alpha x)$ , for  $\alpha \in \Delta_\mu$ , and hence  $p(x) = p(wx)$ , for  $w \in W_\mu$ . Thus,  $|W_\mu|p(x) = p(\sigma_\mu(x))$ . This shows that  $\text{Coker}(p \circ i)$  is annihilated by  $|W_\mu|$ .

Next, let  $x \in K_0(G/B)^{W_\mu} \cap \sum_{\alpha \in \Delta_\mu} \text{Im}(1-s_\alpha)$ . Then, on the one hand,  $\sigma_\mu(x) = |W_\mu|x$  and, on the other hand,  $\sigma_\mu(x) = 0$ , since  $\sigma_\mu \circ (1-s_\alpha) = 0$

for  $\alpha \in \Delta_\mu$ . This yields  $x = 0$ , since  $K_0(G/B)$  is torsion free. Thus, the lemma is proved and the proof of Theorem 2.1 is complete.

2.8. Let us then derive the following corollary.

COROLLARY. —  $K_0(U_\mu)/K_0(U_\mu)_{\text{tor}}$  is generated by the classes  $[L(T_\mu^{\mu-\nu} M(\mu), M(\mu))] : \nu \in \mathcal{P}^+$ .

*Proof.* — Let  $Q_\nu = L(T_\mu^{\mu-\nu} M(\mu), M(\mu))$ , for  $\nu \in \mathcal{P}^+$ . We saw in the proof of Proposition 2.6 that  $Q_\nu$  belongs to  $U_\mu\text{-Proj}$  and that  $\psi_\mu^{-1} \circ \phi_\mu([Q_\nu]) = \zeta_J(\nu)$ , where  $J = \Delta_\mu$ . But, by Theorem 2.1,  $\psi_\mu^{-1} \circ \phi_\mu$  induces an isomorphism from  $K_0(U_\mu)/K_0(U_\mu)_{\text{tor}}$  to  $K_0(G/B)^{W_\mu}$  and, by Lemma 2.6, the latter is generated by the image of  $\zeta_J(\nu) : \nu \in \mathcal{P}^+$ . The lemma follows.

### 3. Hattori-Stallings traces.

3.1. Let  $\mu$  be as in 2.1 and let  $\mathcal{T}_{U_\mu} : K_0(U_\mu) \rightarrow U_\mu/[U_\mu, U_\mu]$  denote the Hattori–Stallings trace, see, for example, [2] §2. It is well-known that  $U_\kappa/[U_\kappa, U_\kappa] = k$ , for  $\kappa \in \mathfrak{h}^*$  (see, for example, [9] 7.8.4) and hence  $\mathcal{T}_{U_\mu}$  takes values in  $k$ . Note also that  $\mathcal{T}_{U_\mu}$  factors through  $K_0(U_\mu)/K_0(U_\mu)_{\text{tor}}$ .

For  $\xi \in \mathfrak{h}^*$ , let  $\tau_\xi$  denote the translation operator on  $S(\mathfrak{h})$  defined by  $\tau_\xi F(\eta) = F(\eta + \xi)$ , for  $F \in S(\mathfrak{h})$ ,  $\eta \in \mathfrak{h}^*$ . Let  $R_\mu^+ = \{\alpha \in R^+ \mid \mu(H_\alpha) = 0\}$ . Let  $P = P_{R^+}$  denote the element  $\prod_{\alpha \in R^+} H_\alpha$  of  $S(\mathfrak{h})$  and let  $P_{R_\mu^+}$  and  $P_{R^+ \setminus R_\mu^+}$  be defined in the obvious way.

By Corollary 2.8, the classes of the projective modules  $L(T_\mu^{\mu-\nu} M(\mu), M(\mu)) : \nu \in \mathcal{P}^+$  generate  $K_0(U_\mu)/K_0(U_\mu)_{\text{tor}}$  and, similarly to [21] §2, one deduces from Bernstein's trace formula [4] §2 that the value of  $\mathcal{T}_{U_\mu}$  on these generators is given by the following proposition.

PROPOSITION. — For  $\nu \in \mathcal{P}^+$  one has  $\mathcal{T}_{U_\mu} L(T_\mu^{\mu-\nu} M(\mu), M(\mu)) = \left( \frac{\sum_{\xi \in W_\mu \nu} \tau_{-\xi} P}{P} \right) (\mu)$ .

Let us evaluate the right-hand side. Let  $a_\mu$  denote the operator  $\sum_{w \in W_\mu} \varepsilon(w)w$  and let  $w_\mu$  be the unique element of  $W_\mu$  such that  $w_\mu(\Delta_\mu) =$

$-\Delta_\mu$ . For  $w \in W$ , let  $D_w$  denote the corresponding Demazure operator on  $S(\mathfrak{h})$ , see [8] §4.

Let  $\nu \in \mathcal{P}^+$ . Since the stabilizer of  $\nu$  in  $W_\mu$  equals  $W_{\mu-\nu}$  and since  $wP = \varepsilon(w)P$ , for  $w \in W$ , one has

$$(3.1.1) \quad |W_{\mu-\nu}| \sum_{\xi \in W_{\mu\nu}} \tau_{-\xi} P = \sum_{w \in W_\mu} \tau_{-w\nu} P = a_\mu \tau_{-\nu} P.$$

By [8] Prop. 3(b), one has  $D_{w_\mu}(F) = a_\mu(F)/P_{R_\mu^+}$ , for  $F \in S(\mathfrak{h})$ . Thus, combining (3.1.1) with the previous proposition and noting that  $P_{R^+ \setminus R_\mu^+}(\mu) \neq 0$ , one obtains

$$(3.1.2) \quad \mathcal{T}_{U_\mu} L(\mathrm{T}_\mu^{\mu-\nu} M(\mu), M(\mu)) = |W_{\mu-\nu}|^{-1} \frac{(D_{w_\mu} \tau_{-\nu} P)(\mu)}{P_{R^+ \setminus R_\mu^+}(\mu)}.$$

Then, for  $\eta \in \mathfrak{h}^*$ , let  $\partial_\eta$  denote the corresponding derivation of  $S(\mathfrak{h})$ . For  $F \in S(\mathfrak{h})$ ,  $\alpha \in \Delta$  and  $\eta \in \mathfrak{h}^*$  such that  $\eta(H_\alpha) = 0$ , it is easily seen that  $(D_{s_\alpha} F)(\eta) = (\partial_\alpha F)(\eta)$ . Thus, denoting by  $\partial_{R_\mu^+}$  the differential operator  $\prod_{\alpha \in R_\mu^+} \partial_\alpha$ , one deduces that

$$(3.1.3) \quad (D_{w_\mu} \tau_{-\nu} P)(\mu) = (\partial_{R_\mu^+} \tau_{-\nu} P)(\mu) = (\partial_{R_\mu^+} P)(\mu - \nu).$$

Moreover, since  $P_{R_\mu^+}$  vanishes with multiplicity  $|R_\mu^+|$  at  $\mu$  and since  $(\partial_{R_\mu^+} P_{R_\mu^+})(\mu)$  equals  $(D_{w_\mu} P_{R_\mu^+})(\mu) = |W_\mu|$ , then one has  $(\partial_{R_\mu^+} P)(\mu) = |W_\mu| P_{R^+ \setminus R_\mu^+}(\mu)$ . Combined with (3.1.2) and (3.1.3), this gives the following

COROLLARY. — For  $\nu \in \mathcal{P}^+$  one has  $\mathcal{T}_{U_\mu} L(\mathrm{T}_\mu^{\mu-\nu} M(\mu), M(\mu)) = \frac{|W_\mu|}{|W_{\mu-\nu}|} \frac{(\partial_{R_\mu^+} P)(\mu - \nu)}{(\partial_{R_\mu^+} P)(\mu)}.$

3.2. Since the cokernel of the Cartan map  $K_0(U_\mu) \rightarrow G_0(U_\mu)$  is torsion, by Theorem 2.1, we can extend  $\mathcal{T}_{U_\mu}$  to a map  $\mathcal{T}'_{U_\mu} : G_0(U_\mu) \rightarrow k$ . Let  $\nu \in \mathcal{P}^+$ . By the last assertion of Lemma 2.3(b), one has in  $G_0(U_\mu)$  the equality

$$(3.2.1) \quad [L(\mathrm{T}_\mu^{\mu-\nu} M(\mu), M(\mu))] = |W_\mu / W_{\mu-\nu}| [L(M(\mu - \nu), M(\mu))].$$

Combined with Corollary 3.1, this yields

$$(3.2.2) \quad \mathcal{T}'_{U_\mu} [L(M(\mu - \nu), M(\mu))] = \frac{(\partial_{R_\mu^+} P)(\mu - \nu)}{(\partial_{R_\mu^+} P)(\mu)}, \quad \forall \nu \in \mathcal{P}^+.$$

Let  $F_\mu$  denote the polynomial  $\partial_{R_\mu^+} \tau_\mu P$ . Note that  $F_\mu$  is  $W_\mu$ -invariant. Moreover, it is well-known that  $P$  satisfies the difference equation

$$\sum_{\eta \in W\xi} P(\kappa + \eta) = |W\xi| P(\kappa), \quad \forall \kappa, \xi \in \mathfrak{h}^*,$$

and therefore so does  $F_\mu$ . Combining these facts, one obtains that the map  $\mathbb{Z}\mathcal{P} \rightarrow k$ ,  $e^\nu \mapsto F_\mu(-\nu)$  factors through  $K_0(G/B)/\sum_{\alpha \in \Delta_\mu} \text{Im}(1-s_\alpha)$  and hence, by Proposition 2.2, induces a  $\mathbb{Z}$ -linear map  $\varphi_\mu : G_0(U_\mu) \rightarrow k$ . Moreover, by Lemma 2.6, coupled with (2.1.1), the classes  $[L(M(\mu-\nu), M(\mu))]: \nu \in \mathcal{P}^+$  generate  $G_0(U_\mu)$  and hence (3.2.2) shows that  $T'_{U_\mu}$  and  $\varphi_\mu/((\partial_{R_\mu^+} P)(\mu))$  coincide on a generating set of  $G_0(U_\mu)$ . Therefore we obtain the following result, which generalizes [21] Th. 3.

**THEOREM.** — For every  $\nu \in \mathcal{P}$ , one has  $T'_{U_\mu}[L(M(\mu+\nu), M(\mu))] = \frac{(\partial_{R_\mu^+} P)(\mu+\nu)}{(\partial_{R_\mu^+} P)(\mu)}$ .

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