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SEMISTABLE REDUCTION AND TORSION SUBGROUPS OF ABELIAN VARIETIES

by A. SILVERBERG and Yu. G. ZARHIN

1. Introduction.

The main theorem of this paper is that if an abelian variety over a field F has a maximal isotropic subgroup of n-torsion points all of which are defined over F, and $n \geq 5$, then the abelian variety has semistable reduction away from n. We deduce bounds for the size of (certain) isotropic subgroups of torsion points defined over a field, for non-semistable abelian varieties. Our main theorem can be viewed as an extension of Raynaud's theorem that if an abelian variety and all its n-torsion points are defined over a field F, and $n \geq 3$, then the abelian variety has semistable reduction away from n.

The proofs of our results rely on certain generalizations of a theorem of Minkowski and a lemma of Serre (see §4). In §6 we give our main results pertaining to semistable reduction of abelian varieties (see Proposition 6.1, Theorem 6.2, and Corollaries 6.3 and 6.4). In §7 we give some information about the Néron models in the cases where $n = 2$, 3, and 4.

Earlier work in the same direction was done by Lenstra and Oort (see Theorem 1.13 of [8]) and by Lorenzini (see [9]); they found bounds on sizes of (certain) torsion subgroups of abelian varieties having purely additive reduction. Our results generalize to higher dimensional abelian varieties earlier work on elliptic curves due to Frey (see Theorem 2 of [5]) and to Flexor and Oesterlé (see [4]). We state some of their results in §6.

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2. Definitions and notation.

If F is a field, let F^s denote a separable closure, let $\bar{F}$ denote an algebraic closure, and let $\mathrm{char}(F)$ denote the characteristic of F. If L is a Galois extension of F, v is a discrete valuation on F, and w is an extension of v to L, let $\mathcal{I}(w/v)$ denote the inertia subgroup at w of $\mathrm{Gal}(L/F)$.

If X is an abelian variety over F, write X_n for the kernel of multiplication by n in $X(F^s)$. Polarizations on X will be viewed as isogenies from X onto its Picard variety. If X is an abelian variety defined over a field F, λ is a polarization on X, n is a positive integer not divisible by $\mathrm{char}(F)$, and $\boldsymbol{\mu}_n$ is the $\mathrm{Gal}(F^s/F)$-module of n-th roots of unity in F^s, then the e_n-pairing induced by the polarization λ,

$$e_{\lambda,n} : X_n \times X_n \to \boldsymbol{\mu}_n$$

(see §75 of [17]), is a skew-symmetric bilinear map which satisfies :

$$\sigma(e_{\lambda,n}(x_1, x_2)) = e_{\sigma(\lambda),n}(\sigma(x_1), \sigma(x_2))$$

for every $\sigma \in \mathrm{Gal}(F^s/F)$ and $x_1, x_2 \in X_n$. If n is relatively prime to the degree of the polarization λ, then the pairing $e_{\lambda,n}$ is nondegenerate.

If X is an abelian variety over F and ℓ is a prime number, let

$$T_\ell(X) = \varprojlim X_{\ell^r}$$

(the Tate module) and let $V_\ell(X) = T_\ell(X) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$. The Galois action on the torsion points gives rise to the ℓ-adic representation

$$\rho_\ell : \mathrm{Gal}(F^s/F) \to \mathrm{Aut}(T_\ell(X)) \cong \mathrm{GL}_{2d}(\mathbf{Z}_\ell),$$

where $d = \dim(X)$.

Write $M_g(\mathcal{O})$ for the ring of $g \times g$ matrices over $\mathcal{O}$, and write I_g (or I when it is unambiguous) for the $g \times g$ identity matrix. Write $F(\zeta_n)$ for the extension of F obtained by adjoining the n-th roots of unity.

3. Reduction of abelian varieties.

Suppose X is an abelian variety over a field F, and v is a discrete valuation on F with residue field k and valuation ring R. The Néron model

$\mathcal{X}$ of X at v is a smooth separated model of X over R such that for every smooth scheme $\mathcal{Y}$ over R and morphism $f : \mathcal{Y} \otimes_R F \to X$ over F there is a unique morphism $f_R : \mathcal{Y} \to \mathcal{X}$ over R which extends f. The generic fiber of $\mathcal{X}$ can be canonically identified with X, and $\mathcal{X}$ is a commutative group scheme over R whose group structure extends that of X.

Let $X_v = \mathcal{X} \otimes_R k$ and let X_v^0 denote the identity component of the special fiber X_v.

DEFINITION 3.1. — *If X is an abelian variety over a field F, and v is a discrete valuation on F, then X has*

(i) *good reduction at v if X_v^0 is an abelian variety (equivalently, X_v is an abelian variety; equivalently, $\mathcal{X}$ is an abelian scheme),*

(ii) *semistable reduction at v if X_v^0 is an extension of an abelian variety by an affine torus,*

(iii) *potential good reduction at v if X has good reduction at an extension of v to a finite algebraic extension of F.*

THEOREM 3.2 (Criterion of Néron-Ogg-Shafarevich). — *Suppose X is an abelian variety over a field F, v is a discrete valuation on F, $\bar{v}$ is an extension of v to a separable closure of F, and ℓ is a prime different from the residue characteristic of v. Then X has good reduction at v if and only if $\mathcal{I}(\bar{v}/v)$ acts as the identity on $T_\ell(X)$.*

Proof. — See Theorem 1 of [14] and Theorem 5 on p. 183 of [2]. □

THEOREM 3.3 (Galois criterion of semistable reduction-Grothendieck). — *Suppose X is an abelian variety over a field F, v is a discrete valuation on F, L is a Galois extension of F, w is an extension of v to L, X has semistable reduction at w, and ℓ is a prime different from the residue characteristic of v. Let $W = V_\ell(X)^{\mathcal{I}(w/v)}$, the subspace of $V_\ell(X)$ on which $\mathcal{I}(w/v)$ acts as the identity. Then X has semistable reduction at v if and only if $\mathcal{I}(w/v)$ acts as the identity on $V_\ell(X)/W$.*

Proof. — See Proposition 3.5 of [7] and Theorem 6 on p. 184 of [2].

□

THEOREM 3.4 (Semistable reduction theorem-Grothendieck). — *Suppose X is an abelian variety over a field F and v is a discrete valuation on*

F. Then there is a finite Galois extension L of F such that X has semistable reduction at the extensions of v to L.

Proof. — See Proposition 3.6 of [7]. □

Theorem 6.2 and Corollary 7.2 below extend the following result of Raynaud.

THEOREM 3.5 (Raynaud criterion of semistable reduction). — *Suppose X is an abelian variety over a field F with a discrete valuation v, n is a positive integer not divisible by the residue characteristic, and the points of X_n are defined over an extension of F which is unramified over v. If $n \geq 3$, then X has semistable reduction at v. If $n = 2$, the valuation ring is henselian, and the residue field is separably closed, then X acquires semistable reduction above v in a $(\mathbf{Z}/2\mathbf{Z})^r$-extension of F, for some r.*

Proof. — See Proposition 4.7 of [7]. □

4. Minkowski-Serre type results.

We now give some generalizations and variations of results of Minkowski and Serre. Minkowski [10] showed that an integral matrix of finite multiplicative order, which is congruent to the identity matrix modulo n, is the identity if $n \geq 3$. Serre (see Lemma 4.7.1 of [7] and p. 17–19 of [13]) proved analogous results for automorphisms of semi-abelian varieties.

THEOREM 4.1. — *Suppose n is a positive integer, $\mathcal{O}$ is an integral domain of characteristic zero such that no rational prime which divides n is a unit in $\mathcal{O}$, $\alpha \in \mathcal{O}$, α has finite multiplicative order, and $\alpha - 1 \in n\mathcal{O}$. If $n \geq 3$ then $\alpha = 1$, and if $n = 2$ then $\alpha^2 = 1$.*

Proof. — See p. 17–19 of [13] and p. 207 of [11]. This result also follows from Theorem 4.2 below, since if $\alpha - 1 \in n\mathcal{O}$ then $(\alpha - 1)^2 \in n^2\mathcal{O}$. □

THEOREM 4.2. — *Suppose n is a positive integer, $\mathcal{O}$ is an integral domain of characteristic zero such that no rational prime which divides n is a unit in $\mathcal{O}$, $\alpha \in \mathcal{O}$, α has finite multiplicative order, and $(\alpha - 1)^2 \in n\mathcal{O}$. If $n \geq 5$ then $\alpha = 1$, if $n = 4$ then $\alpha^2 = 1$, if $n = 3$ then $\alpha^3 = 1$, and if $n = 2$ then $\alpha^4 = 1$.*

Proof. — Let M be the exact multiplicative order of α. If $M = 1$, then $\alpha = 1$. Suppose $M \neq 1$ and let ℓ^r be a prime power which exactly divides M, with $r \geq 1$. Let $\zeta = \alpha^{M/\ell^r}$. Then $(\zeta - 1)^2 \in n\mathcal{O}$. If i is a positive integer less than ℓ^r and not divisible by ℓ, then the elements $\zeta^i - 1$ each generate the same ideal in $\mathbf{Z}[\zeta] \subseteq \mathcal{O}$, and therefore,

$$\ell^2 = (\Phi_{\ell^r}(1))^2 = \prod_{i \in (\mathbf{Z}/\ell^r\mathbf{Z})^\times} (1 - \zeta^i)^2 \in n^{\varphi(\ell^r)}\mathcal{O},$$

where φ is the Euler φ-function and Φ_{ℓ^r} is the ℓ^r-th cyclotomic polynomial. Thus, $\ell^2 n^{-\varphi(\ell^r)} \in \mathcal{O}$.

We will now show that $\mathbf{Z}[1/n] \cap \mathcal{O} = \mathbf{Z}$. Suppose $\beta \in \mathbf{Z}[1/n] \cap \mathcal{O}$. If $\beta \notin \mathbf{Z}$, then we can write $\beta = \dfrac{a}{pb}$ where a, $b \in \mathbf{Z}$ and p is a prime dividing n but not dividing a. Since p does not divide a, we have $\dfrac{1}{p} \in \mathbf{Z} + \mathbf{Z}\dfrac{a}{p} = \mathbf{Z} + \mathbf{Z}b\beta \subseteq \mathcal{O}$, contradicting the assumption that no rational prime which divides n is a unit in $\mathcal{O}$. Therefore $\beta \in \mathbf{Z}$.

Therefore, $\ell^2 n^{-\varphi(\ell^r)} \in \mathbf{Z}$. Thus, $n^{\varphi(\ell^r)}$ divides ℓ^2, so n is a prime power of the form ℓ^m with

$$2 \geq m\varphi(\ell^r) = m(\ell - 1)\ell^{r-1} \geq m(\ell - 1).$$

Therefore, $n \leq 4$. Further, n is a power of every prime which divides the order of α, so the order of α is a prime power ℓ^r, with $m(\ell - 1)\ell^{r-1} \leq 2$. This gives the desired result. □

LEMMA 4.3 (Lemma 3.4 of [15]). — *Suppose $A \in M_g(\mathbf{Z}_2)$ is a matrix of finite multiplicative order, $0 \leq a \leq g$, and b is an $a \times (g - a)$ matrix over $\mathbf{Z}_2$ such that*

$$A \in \begin{pmatrix} I_a & b \\ 0 & I_{g-a} \end{pmatrix} + 4M_g(\mathcal{O}).$$

Then $A = I$.

THEOREM 4.4. — *Suppose ℓ is a prime, M is a free $\mathbf{Z}_\ell$-module of finite rank, A is an automorphism of M, M^A is the submodule of A-invariants in M, and c is the corank of M^A in M. Suppose $A^\ell = 1$, and suppose either*

(a) *$\ell = 2$ and $A - 1 \in 2\mathrm{End}(M)$,*

(b) *$\ell = 2$ and $(A - 1)^2 \in 4\mathrm{End}(M)$, or*

(c) *$\ell = 3$ and $(A - 1)^2 \in 3\mathrm{End}(M)$.*

Then the torsion subgroup of $M/(A-1)M$ is a vector space over $\mathbf{Z}/\ell\mathbf{Z}$ of dimension $c/(\ell-1)$.

Proof. — Let $V = M \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$. Define an endomorphism p of V to be $(1+A)/2$ in cases (a) and (b) and $(1+A+A^2)/3$ in case (c). Since $A^\ell = 1$, we have $p^2 = p$. We have $p \in \mathrm{End}(M)$, since $p = 1-(1-A)/2$ in case (a), $p = 1-(1-A)^2/4$ in case (b), and $p = A+(1-A)^2/3$ in case (c). Let $M_1 = pM$ and $M_2 = (1-p)M$. Then M_1 and M_2 are A-invariant free $\mathbf{Z}_\ell$-submodules of M, $M = M_1 \oplus M_2$, and $(A-1)M_1 = 0$. Let $\mathcal{O} = \mathbf{Z}_2[x]/(x+1) = \mathbf{Z}_2$ in cases (a) and (b), and let $\mathcal{O} = \mathbf{Z}_3[x]/(x^2+x+1)$ in case (c). Clearly, $\mathcal{O}$ is a principal ideal domain, $\mathcal{O}/(x-1)\mathcal{O} \cong \mathbf{Z}/\ell\mathbf{Z}$, and $\mathcal{O}$ is a free $\mathbf{Z}_\ell$-module of rank $\ell-1$. Then M_2 carries a natural structure of a free $\mathcal{O}$-module, where x acts as A; let r be the $\mathcal{O}$-rank of M_2. Then c is equal to the $\mathbf{Z}_\ell$-rank of M_2, which is equal to $r(\ell-1)$. The torsion subgroup of $M/(A-1)M$ is isomorphic to

$$M_2/(A-1)M = M_2/(x-1)M_2 \cong (\mathcal{O}/(x-1)\mathcal{O})^r \cong (\mathbf{Z}/\ell\mathbf{Z})^r.$$

But $r = c/(\ell-1)$. □

THEOREM 4.5. — *Suppose G is a commutative group scheme over a field, G is an extension of an abelian variety by a torus, and α is an automorphism of G. If r is a positive integer, let $G[r]$ denote the scheme-theoretic kernel of multiplication by r. Suppose either*

(a) $\alpha^2 = 1$ *and* $\alpha - 1$ *is* 0 *on* $G[2]$,

(b) $\alpha^2 = 1$ *and* $(\alpha-1)^2$ *is* 0 *on* $G[4]$, *or*

(c) $\alpha^3 = 1$ *and* $(\alpha-1)^2$ *is* 0 *on* $G[3]$.

Then G is the direct sum of α-invariant connected subschemes G_1 and G_2 such that G_1 is the identity component of $\ker(1-\alpha)$, *G_2 is the identity component of* $\ker(1+\alpha)$ *in cases* (a) *and* (b), *and G_2 is the identity component of* $\ker(1+\alpha+\alpha^2)$ *in case* (c).

Proof. — Note that $\mathrm{End}(G)$ is torsion-free. We have $\alpha-1 \in 2\mathrm{End}(G)$ in case (a), $(\alpha-1)^2 \in 4\mathrm{End}(G)$ in case (b), and $(\alpha-1)^2 \in 3\mathrm{End}(G)$ in case (c). Define an element $p \in \mathrm{End}(G) \otimes \mathbf{Q}$, as in the proof of Theorem 4.4, by letting $p = (1+\alpha)/2$ in cases (a) and (b) and letting $p = (1+\alpha+\alpha^2)/3$ in case (c). Then $p^2 = p$, and $p \in \mathrm{End}(G)$. Let $G_1 = p(G) = \ker(1-p)$ and $G_2 = (1-p)(G) = \ker(p)$. Then G_1 and G_2 are α-invariant subschemes of G, $G = G_1 \oplus G_2$, and G_1 and G_2 are connected. Let $\ell = 2$ in cases (a) and

(b) and let $\ell = 3$ in case (c). Then

$$\ell\ker(1-\alpha) \subseteq G_1 \subseteq \ker(1-\alpha), \qquad \ell\ker(\ell p) \subseteq G_2 \subseteq \ker(\ell p).$$

Therefore, $\ker(1-\alpha)/G_1$ and $\ker(\ell p)/G_2$ are killed by ℓ, and so are finite. Therefore, G_1 is the identity component of $\ker(1-\alpha)$ and G_2 is the identity component of $\ker(\ell p)$. □

5. Preliminary lemmas.

LEMMA 5.1 (Corollaire 3.8 of [7]). — *Suppose X is an abelian variety over a field F, v is a discrete valuation on F with residue field k, and $\bar{v}$ is an extension of v to a separable closure of F. If $\mathcal{I}(\bar{v}/v)$ acts by unipotent operators on the Tate module $V_\ell(X)$ for some prime $\ell \neq \mathrm{char}(k)$, then X has semistable reduction at v.*

LEMMA 5.2. — *Suppose that d and n are positive integers, and for each prime ℓ which divides n we have a matrix $A_\ell \in M_{2d}(\mathbf{Z}_\ell)$ such that the characteristic polynomials of the A_ℓ have integral coefficients independent of ℓ, and such that $(A_\ell - I)^2 \in nM_{2d}(\mathbf{Z}_\ell)$. Then for every eigenvalue α of A_ℓ, $(\alpha-1)/\sqrt{n}$ satisfies a monic polynomial with integer coefficients.*

Proof. — If α is an eigenvalue of A_ℓ, then $(\alpha-1)^2/n$ is an eigenvalue of $n^{-1}(A_\ell - I)^2 \in M_{2d}(\mathbf{Z}_\ell)$. Thus $(\alpha-1)^2/n$, and therefore also $(\alpha-1)/\sqrt{n}$ and $(1-\alpha)/\sqrt{n}$, satisfy monic polynomials with coefficients in $\mathbf{Z}_\ell$. Let $f(x) = \det(A_\ell - Ix) \in \mathbf{Z}[x]$, the characteristic polynomial of A_ℓ. Let

$$g_1(x) = n^{-d}f(1+\sqrt{n}x) = \det((\sqrt{n})^{-1}(A_\ell - I) - Ix) \in \mathbf{Z}[1/\sqrt{n}][x],$$

let

$$g_2(x) = n^{-d}f(1-\sqrt{n}x) = \det((-\sqrt{n})^{-1}(A_\ell - I) - Ix) \in \mathbf{Z}[1/\sqrt{n}][x],$$

and let $h(x) = g_1(x)g_2(x)$. Then $h \in \mathbf{Q}[x]$; in fact, $h \in \mathbf{Z}[1/n][x]$. The roots of h are exactly the numbers $\pm(\alpha-1)/\sqrt{n}$ for eigenvalues α of A_ℓ. Therefore the coefficients of h satisfy monic polynomials with coefficients in $\mathbf{Z}_\ell$, but are rational numbers, and therefore must lie in $\mathbf{Q} \cap \mathbf{Z}_\ell$, and in fact in $\mathbf{Z}[1/n] \cap \mathbf{Z}_\ell$, for every prime ℓ which divides n. Since $\cap_{\ell|n}(\mathbf{Z}[1/n] \cap \mathbf{Z}_\ell) = \mathbf{Z}$, we have $h \in \mathbf{Z}[x]$. Therefore, for every eigenvalue α of A_ℓ, $(\alpha-1)/\sqrt{n}$ satisfies the monic polynomial $h \in \mathbf{Z}[x]$. □

In Lemma 5.5 we will give a condition under which an abelian variety acquires semistable reduction over a totally and tamely ramified extension of the ground field.

Remark 5.3. — Suppose v is a discrete valuation on a field F, and m is a positive integer not divisible by the residue characteristic. Then every degree m Galois extension of F totally ramified at v is cyclic (by reducing to the case where F is complete with respect to v, and applying Theorem 1 on p. 29 in §8 of [6]). If $F(\zeta_m) = F$, then F has a cyclic extension of degree m which is totally ramified at v. In particular, if the residue characteristic is not 2 then F has a quadratic extension which is (totally and tamely) ramified at v. If F is a local field or a global field, then by class field theory a degree m cyclic extension of F which is totally ramified at v exists if and only if m divides the order of the multiplicative group of the residue field. If the valuation ring is henselian and the residue field is separably closed, then $F = F(\zeta_m)$ and therefore F has a cyclic totally ramified extension of degree m. Note also that F has no non-trivial unramified extensions if and only if the valuation ring is henselian and the residue field is separably closed.

LEMMA 5.4. — *Suppose F is a field with a discrete valuation v, $\bar{v}$ is an extension of v to a separable closure F^s of F, and r is a positive integer not divisible by the residue characteristic of v. Suppose F coincides with its maximal extension E in F^s such that the restriction of $\bar{v}$ to E is unramified over v. Then F has a unique degree r Galois extension K in F^s, and the restriction of $\bar{v}$ to K is totally and tamely ramified over v.*

Proof. — Adjoining the r-th root of a uniformizing parameter gives a degree r Galois extension of F in F^s. Suppose K and L are two degree r Galois extensions of F in F^s. Then the restrictions of $\bar{v}$ to K and to L are totally and tamely ramified over v. Therefore the restriction of $\bar{v}$ to the compositum KL is totally and tamely ramified over v (by §8 of [6] the compositum of two totally and tamely ramified extensions is totally and tamely ramified), and $\mathrm{Gal}(KL/F)$ is a cyclic group which injects into a direct product of cyclic groups of order r, $\mathrm{Gal}(K/F) \times \mathrm{Gal}(L/F)$. Therefore, $\mathrm{Gal}(KL/F) = \mathrm{Gal}(K/F)$, so $K = L$. □

LEMMA 5.5. — *Suppose v is a discrete valuation on a field F with residue characteristic $p \geq 0$, m is a positive integer, ℓ is a prime, p does not divide $m\ell$, K is a degree m Galois extension of F which is totally*

ramified above v, and $\bar{v}$ is an extension of v to a separable closure K^s of K. Suppose that X is an abelian variety over F, and for every $\sigma \in \mathcal{I}(\bar{v}/v)$, all the eigenvalues of $\rho_\ell(\sigma)$ are m-th roots of unity. Then X has semistable reduction at the extension of v to K.

Proof. — If M is a field with $F \subseteq M \subseteq K^s$, let v_M denote the restriction of $\bar{v}$ to M. Let E be the maximal extension of F in $F^s = K^s$ such that v_E is unramified over v. Since v_{KE} is unramified over v_K, X has semistable reduction at v_K if and only if X has semistable reduction at v_{KE}, by Theorem 3.3. Replacing F by E and K by KE, we may assume that F coincides with its maximal extension in F^s such that the restriction of $\bar{v}$ is unramified over v; in particular, $\mathcal{I}(\bar{v}/v) = \mathrm{Gal}(F^s/F)$. Let $L = F(X_{\ell^2}) \subset K^s$ (i.e., L is the smallest extension of F in K^s over which all the ℓ^2-torsion points of X are defined). Then L is a totally ramified Galois extension of F. Let $V = V_\ell(X)$ and $T = T_\ell(X)$. By Raynaud's Criterion (Theorem 3.5) X has semistable reduction at v_L, since $X_{\ell^2} \subset X(L)$ and $\ell^2 > 3$.

Next, we will show that L is tamely ramified over F. Suppose $\sigma \in \mathcal{I}(\bar{v}/v)$. By our hypothesis, $\rho_\ell(\sigma)^m$ is a unipotent operator on V. Let $\eta = \rho_\ell(\sigma)^m - 1$. Then $\eta T \subseteq T$, and for some integer $t \geq 2$, we have $\eta^t = 0$. It follows that $(\rho_\ell(\sigma)^{m\ell^t} - 1)T \subseteq \ell^2 T$. In other words, $\sigma^{m\ell^t} = 1$ on $X_{\ell^2} \cong T/\ell^2 T$, so the image of σ in $\mathcal{I}(v_L/v)$ has order dividing $m\ell^t$. Consequently, $[L : F]$ is relatively prime to the residue characteristic, so L is totally and tamely ramified over F, and therefore L is cyclic over F.

Let $W = V^{\mathcal{I}(\bar{v}/v_L)}$. Then ρ_ℓ induces a homomorphism

$$\rho : \mathcal{I}(\bar{v}/v) \to \mathrm{Aut}(W) \times \mathrm{Aut}(V/W)$$

which factors through the finite cyclic group $\mathcal{I}(v_L/v) = \mathrm{Gal}(L/F)$. Therefore the image of ρ is a finite cyclic group, of order, say r, dividing $[L : F]$ and m. (To see that r divides m, let τ be a generator of $\mathcal{I}(v_L/v)$. Then τ^m acts on W and on V/W as a unipotent operator of finite order. Therefore, τ^m is in the kernel of ρ.)

Let K', respectively L', be the (unique, cyclic) degree r extension of F in K, respectively, in L. Then ρ factors through $\mathcal{I}(v_{L'}/v) = \mathrm{Gal}(L'/F)$, and the kernel of ρ is $\mathcal{I}(\bar{v}/v_{L'})$. By Theorem 3.3, X has semistable reduction at $v_{L'}$. By Lemma 5.4, $K' = L'$. Therefore X has semistable reduction at $v_{K'}$, and thus also at v_K. □

6. Main results.

PROPOSITION 6.1. — *Suppose X is an abelian variety over a field F, v is a discrete valuation on F with residue field k, λ is a polarization on X, n is a positive integer not divisible by* $\mathrm{char}(k)$, *$\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$, λ and the points of $\widetilde{X}_n$ are defined over an extension of F which is unramified over v, and L is a finite extension of F over which X has semistable reduction at an extension w of v.*

(i) *Suppose $\bar{v}$ is an extension of w to a separable closure of F, ℓ is a prime which divides n, $\sigma \in \mathcal{I}(\bar{v}/v)$, α is an eigenvalue of $\rho_\ell(\sigma)$, and d is the dimension of X. Then $(\rho_\ell(\sigma) - I)^2 \in nM_{2d}(\mathbf{Z}_\ell)$, and*

$$\alpha = 1 \text{ if } n \geq 5,$$

$$\alpha^2 = 1 \text{ if } n = 4,$$

$$\alpha^3 = 1 \text{ if } n = 3, \text{ and}$$

$$\alpha^4 = 1 \text{ if } n = 2.$$

(ii) *If n is relatively prime to $[L : F]$ then X has semistable reduction at v.*

Proof. — Suppose $\sigma \in \mathcal{I}(\bar{v}/v)$. Then $(\sigma - 1)\widetilde{X}_n = 0$. Since the extension $F(\zeta_n)$ over F is unramified outside n, and n is not divisible by $\mathrm{char}(k)$, $\mathcal{I}(\bar{v}/v)$ acts as the identity on the n-th roots of unity in F^s. Therefore, for $x \in X_n$ and $y \in \widetilde{X}_n$ we have

$$e_{\lambda,n}((\sigma - 1)x, y) = \frac{e_{\lambda,n}(\sigma(x), \sigma(y))}{e_{\lambda,n}(x, y)} = \frac{\sigma(e_{\lambda,n}(x, y))}{e_{\lambda,n}(x, y)} = 1.$$

Since $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n, we have $(\sigma - 1)X_n \subseteq \widetilde{X}_n$, and therefore

$$(\sigma - 1)^2 X_n = 0.$$

By Theorem 4.3 of [7], for primes $\ell \neq \mathrm{char}(k)$ the characteristic polynomial of $\rho_\ell(\sigma)$ has integer coefficients which are independent of ℓ. Let $m = [L : F]$. Then $\sigma^m \in \mathcal{I}(\bar{v}/w)$, so $(\sigma^m - 1)^2 = 0$ on $V_\ell(X)$ by

Theorem 3.3. If α is an eigenvalue of $\rho_\ell(\sigma)$, then $(\alpha^m - 1)^2 = 0$, so $\alpha^m = 1$. If ℓ^t exactly divides n, then

$$X_{\ell^t} \cong T_\ell(X)/nT_\ell(X) \cong \mathbf{Z}_\ell^{2d}/n\mathbf{Z}_\ell^{2d},$$

so $(\rho_\ell(\sigma) - I)^2 \in nM_{2d}(\mathbf{Z}_\ell)$. Let $\bar{\mathbf{Z}}$ denote the ring of algebraic integers in an algebraic closure of $\mathbf{Q}$. Lemma 5.2 implies that if α is an eigenvalue of $\rho_\ell(\sigma)$, then $(\alpha - 1)^2 \in n\bar{\mathbf{Z}}$. Part (i) now follows by applying Theorem 4.2 with $\mathcal{O} = \bar{\mathbf{Z}}$. Further, if $(m, n) = 1$, then the eigenvalues of the $\rho_\ell(\sigma)$ are all 1, so $\mathcal{I}(\bar{v}/v)$ acts by unipotent operators on $V_\ell(X)$. Part (ii) follows from Lemma 5.1. □

THEOREM 6.2. — *Suppose X is an abelian variety over a field F, v is a discrete valuation on F with residue field k, λ is a polarization on X, n is an integer greater than 4 and not divisible by* $\mathrm{char}(k)$, *$\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$, and λ and the points of $\widetilde{X}_n$ are defined over an extension of F which is unramified over v. Then X has semistable reduction at v.*

Proof. — The semistable reduction theorem (Theorem 3.4) says that X has semistable reduction over some finite extension of F. Theorem 6.2 now follows immediately from Proposition 6.1i and Lemma 5.1. □

COROLLARY 6.3. — *Suppose F is a field, v is a discrete valuation on F with residue field k, X is an abelian variety over F which does not have semistable reduction at v, λ is a polarization on X, n is a positive integer not divisible by* $\mathrm{char}(k)$, *$\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$, and λ and the points of $\widetilde{X}_n$ are defined over an extension of F which is unramified over v. Then $n \leq 4$.*

If we drop the hypothesis that n is not divisible by $\mathrm{char}(k)$ in Corollary 6.3 then we conclude that n is the product of a power of $\mathrm{char}(k)$ and an integer that is at most 4. We therefore obtain a stronger result if there are two discrete valuations, with different residue characteristics, at which the abelian variety does not have semistable reduction.

COROLLARY 6.4. — *Suppose F is a field, v and w are discrete valuations on F of different residue characteristics, X is an abelian variety over F which has semistable reduction at neither v nor w, λ is a polarization on X, n is a positive integer, $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$, and λ and the points of $\widetilde{X}_n$ are defined over an extension*

of F which is unramified over v and w. Then n divides 12. If further the residue characteristic of v or w is greater than 3, then $n \leq 4$.

In the case of elliptic curves, Corollaries 6.3 and 6.4 imply the following results.

COROLLARY 6.5. — *Suppose X is an elliptic curve over a field F and X has a point of order n defined over F. If X has additive reduction at a discrete valuation of F whose residue characteristic does not divide n, then $n \leq 4$.*

COROLLARY 6.6. — *Suppose X is an elliptic curve over a field F and X has a point of order n defined over F. If X has additive reduction at at least two discrete valuations of F with different residue characteristics then n divides 12, and if at least one of those residue characteristics is greater than 3 then $n \leq 4$.*

These results also follow from earlier work on elliptic curves due to Flexor-Oesterlé and Frey. Frey (see Theorem 2 of [5]) proved that if E is an elliptic curve over a global field F of characteristic not 2 or 3, q is a prime, $q \geq 5$, b is a positive integer, and $E(F)$ has a point of order q^b, then E has semistable reduction at all places of F of residue characteristic different from q. Among the results of Flexor and Oesterlé [4] is that if E is an elliptic curve defined over a field F which is complete with respect to a discrete valuation v of residue characteristic p with perfect residue field, and if E has additive reduction at v, then the torsion subgroup of $E(F)$ has order of the form $p^n m$ with $n \geq 0$ and $m \leq 4$. They also show that if E is an elliptic curve over a number field F, with additive reduction at at least two discrete valuations of F with different residue characteristics, then the torsion subgroup of $E(F)$ has order dividing 12. To show the latter result is sharp, they produced an example of an elliptic curve over $\mathbf{Q}(\sqrt{-3})$ with additive reduction at valuations of residue characteristics 2 and 3, whose Mordell-Weil group is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z}$ (but did not give an example with a point of order 4). We now give an example to show Corollary 6.5 is sharp.

Example 6.7. — Let X be the elliptic curve over $\mathbf{Q}$ defined by the minimal equation $y^2 = x^3 + 6x - 7$. Then X has additive reduction at 3 (and at 2, and good reduction elsewhere; the conductor of X is 72; see [1]), and $X(\mathbf{Q})$ is the cyclic group of order 4 generated by the point $(4, 9)$.

Remark 6.8. — Let

$$N(k) = \{\text{ prime powers } \ell^m : 0 \leq m(\ell - 1) \leq k\}.$$

If X is an abelian variety over a field F, v is a discrete valuation on F, $\bar{v}$ is an extension of v to a separable closure of F, k and n are positive integers, $n \notin N(k)$, n is not divisible by the characteristic of the residue field, and for every $\sigma \in \mathcal{I}(\bar{v}/v)$, $(\sigma - 1)^k$ is zero on X_n, then X has semistable reduction at v. The proof follows the proofs of Proposition 6.1 and Theorem 6.2 and a suitable generalization of Theorem 4.2.

7. The cases $n = 2, 3, 4$.

COROLLARY 7.1. — *Suppose X is an abelian variety over a field F, v is a discrete valuation on F with residue field k, λ is a polarization on X, n is a postive integer which is not divisible by* $\mathrm{char}(k)$, *$\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$, and λ and the points of $\widetilde{X}_n$ are defined over an extension of F which is unramified over v. If $n = 2$, 3, or 4, then X has semistable reduction above v over every totally ramified Galois (necessarily cyclic) extension of F of degree 4, 3, or 2, respectively.*

Proof. — The corollary follows from Proposition 6.1i and Lemma 5.5. See Remark 5.3 for criteria for the existence of a totally ramified cyclic extension of appropriate degree. □

The next result generalizes Raynaud's Criterion (Theorem 3.5) in the case $n = 2$.

COROLLARY 7.2. — *If X is an abelian variety over a field F with a discrete valuation v whose residue characteristic is not 2, and the points of X_2 are defined over an extension of F which is unramified over v, then X acquires semistable reduction above v in every quadratic extension of F ramified over v.*

Proof. — The result follows from Lemma 5.5 and Theorem 4.1, as in the proof of Proposition 6.1. □

THEOREM 7.3. — *Suppose X is an abelian variety with complex multiplication by an order $\mathcal{O}$, F is a field of definition for X and the endomorphisms induced by $\mathcal{O}$, v is a discrete valuation on F of finite*

residue characteristic $p \neq 3$, the number μ of roots of unity in $\mathcal{O}[1/p]$ is not divisible by 3, λ is a polarization on X, $\widetilde{X}_3$ is a maximal isotropic subgroup of X_3 with respect to $e_{\lambda,3}$, and λ and the points of $\widetilde{X}_3$ are defined over an extension of F which is unramified over v. Then X has good reduction at v.

Proof. — Let $\bar{v}$ be an extension of v to a separable closure of F. The group $\rho_3(\mathcal{I}(\bar{v}/v))$ is contained in the group of roots of unity in $\mathcal{O}[1/p]$ (see Theorem 6 of [14]). If $\sigma \in \mathcal{I}(\bar{v}/v)$ and α is an eigenvalue of $\rho_3(\sigma)$, then α is a μ-th root of unity. By Proposition 6.1i, α is a cube root of unity. Since μ is not divisible by 3, we have $\alpha = 1$. Therefore $\mathcal{I}(\bar{v}/v)$ acts by unipotent operators on $V_3(X)$, so X has semistable reduction at v. Since X has complex multiplication, X must have good reduction at v. □

THEOREM 7.4. — *Suppose n is a positive integer, $n \geq 4$, F is a field, v is a discrete valuation on F whose residue characteristic does not divide n, X is an abelian variety over F which has potential good reduction at v, λ is a polarization on X, $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$, λ and the points of $\widetilde{X}_n$ are defined over an extension of F which is unramified over v, and if $n = 4$ then $\widetilde{X}_4 \cong (\mathbf{Z}/4\mathbf{Z})^b$ for some non-negative integer b. Then X has good reduction at v.*

Proof. — Let $\bar{v}$ be an extension of v to a separable closure of F and let ℓ be a prime divisor of n. Since X has potential good reduction at v, $\rho_\ell(\mathcal{I}(\bar{v}/v))$ is finite by Theorem 2i of [14]. If $n = 4$, under our assumptions on $\widetilde{X}_4$, the proof of Proposition 6.1 shows that for every $\sigma \in \mathcal{I}(\bar{v}/v)$, $\rho_2(\sigma)$ is conjugate to a matrix in

$$\begin{pmatrix} I_b & \beta \\ 0 & I_{2d-b} \end{pmatrix} + 4M_{2d}(\mathbf{Z}_2),$$

with $d = \dim(X)$ and with β a $b \times (2d - b)$ matrix over $\mathbf{Z}_2$. Theorem 4.2, Lemma 4.3, and the proof of Proposition 6.1 imply that $\rho_\ell(\sigma) = I$ for every $\sigma \in \mathcal{I}(\bar{v}/v)$, so X has good reduction at v by Theorem 3.2. □

Suppose X is an abelian variety over a field F, v is a discrete valuation on F, the valuation ring is henselian, and the residue field k is algebraically closed. If X has potential good reduction at v, then X_v^0 is an extension of an abelian variety by a unipotent group whose dimension is called the *unipotent rank* of X at v (see Remark 1 on p. 500 of [14]). Let $\Phi = X_v/X_v^0$, a group scheme over k. The group $\Phi(k)$ is the (finite) group of connected components of X_v.

In [16], Silverman gives information on the group of connected components for abelian varieties with potential good reduction over a local field. We provide additional information in special cases.

THEOREM 7.5. — *Suppose v is a discrete valuation on a field F with henselian valuation ring and algebraically closed residue field, X is an abelian variety over F which has potential good reduction at v, and either*

(a) *$n = 2$ and the points of X_2 are defined over F, or*

(b) *$n = 3$ or 4, λ is a polarization on X, $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$, and λ and the points of $\widetilde{X}_n$ are defined over F.*

Suppose the residue characteristic p (≥ 0) of v does not divide n. Let c denote the unipotent rank of X at v, and let Φ' denote the prime-to-p part of the group of connected components of the special fiber of the Néron model of X at v (with Φ' the full group of components if $p = 0$). Then $\Phi' \cong (\mathbf{Z}/2\mathbf{Z})^{2c}$ if $n = 2$ or 4, and $\Phi' \cong (\mathbf{Z}/3\mathbf{Z})^c$ if $n = 3$.

Proof. — Let $\bar{v}$ be an extension of v to a separable closure of F, let $\mathcal{I} = \mathcal{I}(\bar{v}/v)$, let k be the residue field of v, and let $\mathcal{J}$ be the first ramification group (i.e., $\mathcal{J}$ is trivial if $p = 0$ and $\mathcal{J}$ is the pro-p-Sylow subgroup of $\mathcal{I}$ if $p > 0$). Suppose q is a prime not equal to p, and let Φ_q denote the q-part of Φ'. Since X has potential good reduction at v, $\rho_q(\sigma)$ has finite multiplicative order for every $\sigma \in \mathcal{I}$, and there is an exact sequence of commutative algebraic groups over k,

$$0 \to C \to X_v^0 \to B \to 0$$

where B is an abelian variety and C is a unipotent group of dimension c (see Remark 1 on p. 500 of [14]). Let τ be a lift to $\mathcal{I}$ of a generator of the pro-cyclic group $\mathcal{I}/\mathcal{J}$. By §11 of [7] (see Lemma 2.1 of [9]),

$$\Phi_q \text{ is isomorphic to the torsion subgroup of } T_q(X)^{\mathcal{J}}/(\rho_q(\tau) - I)T_q(X)^{\mathcal{J}}.$$

Let $\ell = 2$ if $n = 2$ or 4 and let $\ell = 3$ if $n = 3$. By Corollaries 7.1 and 7.2 and Remark 5.3, X has semistable reduction (and therefore good reduction) above v over a totally ramified Galois extension of F of degree ℓ. Therefore $\mathcal{I}$ acts on $T_q(X)$ through a cyclic quotient of order ℓ, so $\rho_q(\sigma)^\ell = I$ for every $\sigma \in \mathcal{I}$. Since $p \neq \ell$, we have $\rho_q(\sigma) = I$ for every $\sigma \in \mathcal{J}$. Therefore, $T_q(X)^{\mathcal{J}} = T_q(X)$. If $q \neq \ell$, then $T_q(X)/(\rho_q(\tau) - I)T_q(X)$ is torsion-free, so Φ_q is trivial. The reduction map induces an isomorphism

from $T_\ell(X)^{\mathcal{I}}$ onto $T_\ell(X_v)$ (see Lemma 2 on p. 495 of [14]), and we have $T_\ell(X_v) \cong T_\ell(X_v^0) \cong T_\ell(B)$ (see §2 of [7]), especially (2.1.8)). Therefore, $T_\ell(X)^\tau = T_\ell(X)^{\mathcal{I}} \cong T_\ell(B)$, so the $\mathbf{Z}_\ell$-corank of $T_\ell(X)^\tau$ in $T_\ell(X)$ is $2c$ (see also the Corollary and Remark 1 on p. 500 of [14]). Let d denote the dimension of the abelian variety X. In case (a), we have $\rho_2(\tau) - I \in 2M_{2d}(\mathbf{Z}_2)$, and in case (b) we have $(\rho_\ell(\tau) - I)^2 \in nM_{2d}(\mathbf{Z}_\ell)$ by Proposition 6.1i. Applying Theorem 4.4 with $M = \mathbf{Z}_\ell^{2d}$ and $A = \rho_\ell(\tau)$ shows that Φ_ℓ is a vector space over $\mathbf{Z}/\ell\mathbf{Z}$ whose dimension is $2c/(\ell - 1)$. □

Example 7.6. — Suppose F is a field with a discrete valuation v whose residue characteristic is not 2, the valuation ring is henselian, and the residue field is algebraically closed. Suppose X is the Jacobian variety of a hyperelliptic curve over F all of whose Weierstrass points are defined over F (i.e., a curve of the form $y^2 = f(x)$ where $f(x)$ is a product of linear factors over F). Then all the points of X_2 are defined over F (see §2 of Chapter IIIa of [12], especially Lemma 2.4 on p. 3.32). If X has potential good reduction at v, then X satisfies the hypotheses of Theorem 7.5a, so $\Phi' \cong (\mathbf{Z}/2\mathbf{Z})^{2c}$ where c is the unipotent rank of X at v.

Remark 7.7. — Under the hypotheses in Theorem 7.5, if we assume in addition that X has purely additive reduction at v (i.e., X_v^0 is a unipotent group), then the prime-to-p part of the torsion subgroup of $X(F)$ is isomorphic to Φ' (see Remark 1.3 of [9]). Therefore Theorem 7.5 gives restrictions on the torsion subgroup of $X(F)$.

In Theorem 7.8 we will apply Theorem 4.5 and a result of Edixhoven to obtain additional information on the connected component of the special fiber of the Néron model.

THEOREM 7.8. — *Suppose X is an abelian variety over a field F, v is a discrete valuation on F with henselian valuation ring and algebraically closed residue field, and either*

(a) *$n = 2$ and the points of X_2 are defined over F, or*

(b) *$n = 3$ or 4, λ is a polarization on X, $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$, and λ and the points of $\widetilde{X}_n$ are defined over F.*

Let $\ell = 2$ if $n = 2$ or 4 and let $\ell = 3$ if $n = 3$. Suppose K is a degree ℓ Galois extension of F which is totally ramified above v, let w be the extension of v to K, and let $H = \mathrm{Gal}(K/F)$. Suppose the residue characteristic of v is not ℓ. Let C denote the maximal unipotent subgroup

of X_v^0 and let D denote the identity component $((X_w^0)^H)^0$ of $(X_w^0)^H$. Then X has semistable reduction at w, $D = (X_w^H)^0$, D is a direct summand of X_w^0, and the base change map induces an isomorphism $X_v^0/C \cong D$.

Proof. — By Corollaries 7.1 and 7.2, X has semistable reduction at w. The Galois group $H = \mathrm{Gal}(K/F) = \mathcal{I}(w/v)$ is a cyclic group which acts on $X_K = X \otimes_F K$ through the second factor. This action induces the usual action of H on $X(K) = X_K(K)$. By the functoriality of Néron models, the action of H on X_K extends to an action on the Néron model of X at w, and therefore induces an action on X_w^0. Since X has semistable reduction at w, X_w^0 is an extension of an abelian variety by a torus. Let τ be a generator of H. Then τ has order ℓ, and τ acts on X_w^0 as an algebraic automorphism (since the inertia group acts trivially on the residue field — see p. 497 of [14] and the proof of Theorem 4.3 of [7]). We have $(\tau - 1)X_2 = 0$ in case (a) and $(\tau - 1)^2 X_n = 0$ in case (b). Letting $(X_w^0)_n$ denote the kernel of multiplication by n in the group scheme X_w^0, then $(\tau - 1)$ is zero on $(X_w^0)_2$ in case (a) and $(\tau - 1)^2$ is zero on $(X_w^0)_n$ in case (b). Applying Theorem 4.5, we have $X_w^0 = G_1 \oplus G_2$ where

$$G_1 = ((X_w^0)^\tau)^0 = ((X_w^0)^H)^0 = D.$$

Therefore, D is a direct summand of X_w^0. Clearly, $D = (X_w^H)^0$.

By Theorem 5.3 (see also Remark 5.4.1) of [3], the base change map induces an isomorphism $X_v/F^1X_v \cong (X_w)^H$, where F^1X_v is a connected unipotent closed subgroup scheme of X_v. Since F^1X_v is connected, we have $(X_v/F^1X_v)^0 \cong X_v^0/F^1X_v$. Since F^1X_v is unipotent and X has semistable reduction at w, we have $F^1X_v = C$. Therefore, the base change map induces an isomorphism $X_v^0/C \cong (X_w^H)^0 = D$, as desired. □

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