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EXTENSIONS OF CERTAIN REAL RANK ZERO C^*-ALGEBRAS

by M. DADARLAT(*), AND A. LORING(**)

Introduction.

In this paper we study extensions of C^*-algebras

$$0 \to A \to E \to B \to 0$$

where A and B are what we call $\mathcal{AD}$-algebras. These $\mathcal{AD}$-algebras are the C^*-algebras studied by Elliott in the first of his recent classification papers [Ell]. Under the assumption that A and B have torsion-free K-theory, Lin and Rørdam [LR] showed that a necessary and sufficient condition for E to again be an $\mathcal{AD}$-algebra is that it has real rank zero and stable rank one. This generalized an earlier result of Brown [Br1] involving AF algebras where these conditions were automatically fulfilled. For a C*-algebra A let $RR(A)$ denote its real rank (see [BP]) and let $\operatorname{tsr}(A)$ denote its topological stable rank (see [R]).

We exhibit extensions where E is *not* an $\mathcal{AD}$-algebra even though $RR(E) = 0$, $\operatorname{tsr}(E) = 1$ and A and B are $\mathcal{AD}$-algebras. In fact, E is not even a limit of homogeneous algebras [BD], so is not covered by the more general classification schemes now emerging. Despite this, $U \otimes E$ is an AF algebra when U is an appropriate UHF algebra.

The obstruction that prevents E from being an $\mathcal{AD}$-algebra is an element of $\operatorname{Ext}(T, K_1(A))$ where T denotes the torsion subgroup of $K_1(B)$

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(see Propositions 4.2 and 4.4). This is a K_1 analog of the obstruction to quasidiagonality discovered by L. G. Brown [Br2] and will be further explored in [BD]. See also [Sa].

We conjecture that the vanishing of this obstruction implies that E is an $A\mathcal{D}$-algebra. We were able to prove this under stronger assumptions— namely if $K_1(A)$ is trivial or $K_1(B)$ is torsion free— (see Theorem 3.1).

Let us now describe explicitly the C^*-algebras we will be working with. The so-called building blocks are the scalars, $\mathbb{C}$; the circle algebra, $C(\mathbb{T})$; the unital dimension-drop interval, defined below; and all C^*-algebras arising from these by forming matrix algebras and taking finite direct sums. The collection of all building blocks will be denoted $\mathcal{D}$. A C^*-algebra will be called an $A\mathcal{D}$-algebra if it is isomorphic to an inductive limit of elements of $\mathcal{D}$. If only circle algebras are involved, we call such a limit an $A\mathbb{T}$-algebra.

By the non-unital dimension-drop interval we mean

$$\mathbb{I}_n = \{f \in C([0,1], M_n) \mid f(0) = 0,\ f(1) \text{ is scalar}\}.$$

Let $\widetilde{\mathbb{I}}_n$ denote the unitization of $\mathbb{I}_n$;

$$\widetilde{\mathbb{I}}_n = \{f \in C([0,1], M_n) \mid f(0), f(1) \text{ are scalars}\}.$$

It is automatic that an $A\mathcal{D}$-algebra has topological stable rank equal to one. The real rank must be either zero or one.

Elliott showed that ordered K-theory is a complete isomorphism invariant for the simple $A\mathcal{D}$-algebras of real rank zero. In particular the $A\mathcal{D}$-algebras that have torsion free K_1-groups are exactly the $A\mathbb{T}$-algebras and those that have vanishing K_1-groups are exactly the AF-algebras.

The importance of $\widetilde{\mathbb{I}}_n$ is that it introduces torsion in K_1, a feature lacking in $A\mathbb{T}$-algebras. If one wishes to use a commutative building block that has K_1 equal to $\mathbb{Z}/n$, one can use the mapping cone of a degree-n $*$-homomorphism $C_0(\mathbb{R}^2) \to C_0(\mathbb{R}^2)$. This will be $C_0(Y)$ for some three-dimensional CW complex. Unfortunately, a commutative C^*-algebra behaves rather badly with respect to inductive limits if the dimension of its spectrum is greater than one (see [DL1]).

A nice description of $\mathbb{I}_n$ is that it is the mapping cone of the unital $*$-homomorphism $\mathbb{C} \to M_n(\mathbb{C})$. From this, (or from the exact sequence involving the ideal SM_n) one sees that

$$K_0(\mathbb{I}_n) = 0,\ K_1(\mathbb{I}_n) = \mathbb{Z}/n$$

and thus

$$K_0(\widetilde{\mathbb{I}}_n) = \mathbb{Z},\ K_1(\widetilde{\mathbb{I}}_n) = \mathbb{Z}/n.$$

It is quite reasonable to think of $\mathbb{I}_n \otimes \mathcal{K}$ as a deformed version of $C_0(Y) \otimes \mathcal{K}$. (We argue this point in [DL3]).

The advantage of $\mathbb{I}_n$ over $C_0(Y)$ is that it is generated by stable relations (cf [Lo1], [Lo3]) and so is very well behaved with respect to inductive limits. For example, see Proposition 1.2. This gives an alternate characterization of $\mathcal{AD}$-algebras that does not mention inductive limits. The test is to approximate a finite set of elements by an appropriate subalgebra with no regard to whether the various subalgebras are nested.

These building blocks also seem to be key from the point of view of K-theory. Since $C(S^1)$ is universally generated by a unitary, it is essentially by definition that there is a natural isomorphism

$$[C_0(\mathbb{R}), A \otimes \mathcal{K}] \xrightarrow{\cong} K_1(A)$$

where we use $[-,-]$ to denote homotopy classes of $*$-homomorphisms. We have shown [DL2], using a "suspension theorem" in E-theory that there is an analogous isomorphism

$$[\mathbb{I}_n, A \otimes \mathcal{K}] \xrightarrow{\cong} K_1(A;\mathbb{Z}/n) = KK(\mathbb{I}_n, A).$$

That this involves actual $*$-homomorphisms rather than asymptotic morphism is a consequence of stable relations. In contrast, all that can be said (cf [DL2]) using $C_0(Y)$ is

$$[[C_0(Y), A \otimes \mathcal{K}]] \xrightarrow{\cong} K_1(A;\mathbb{Z}/n).$$

Here $[[-,-]]$ denotes homotopy classes of asymptotic morphisms [CH].

The proof of our main theorem requires that we generalize several results regarding unitaries in a C^*-algebra A, i.e. maps from $C(S^1)$ to A, to the context of $*$-homomorphisms from $\mathbb{I}_n$ to A. For example, we extended a result of Lin on finite-spectrum unitaries [Li1] to show that when $RR(A) = 0$, any $\varphi : \mathbb{I}_n \to A$ that is zero on K-theory can be perturbed slightly to have finite-dimensional range.

We also need a result, Theorem 2.3, regarding lifting homomorphisms from $\operatorname{Hom}(\mathbb{I}_n, E/A)$ to $\operatorname{Hom}(\mathbb{I}_n, E)$ when E has stable rank zero and real rank one. This is much more involved than the analogous result about lifting unitaries.

1. Perturbation results.

Recall from the introduction that $\mathcal{D}$ denotes the class of C^*-algebras isomorphic to C^*-algebras of the form $\bigoplus_{i=1}^{r} M_{m(i)}(D_i)$ where $D_i = \widetilde{\mathbb{I}}_{n(i)}$ or $D_i = C(\mathbb{T})$, for some $r, m(i), n(i) \in \mathbb{N}$. It was proven in [Lo3] that any C^*-algebra $D \in \mathcal{D}$ is isomorphic to some universal C^*-algebra $C^*\langle G, R\rangle$ with finitely many generators $G = \{g_1, \ldots, g_N\}$ and exactly stable relations $R = \{p_1, \ldots, p_M\}$ with p_j polynomials in g_i, g_i^*. See [Lo1] for the definitions regarding stable relations. Using results in [Lo1] one easily derives the following perturbation result.

THEOREM 1.1. — *Let $D \cong C^*\langle G, R\rangle$ be a C^*-algebra with exactly stable relations. Then for any $\eta > 0$ there is $\delta > 0$ such that for any C^*-algebras E, B and $*$-homomorphisms $\sigma : D \to B$, $\pi : E \to B$, with π surjective and for any $y_1, \ldots y_N \in E$ satisfying*

$$\begin{aligned} ||p_j(y_1, \ldots, y_N)|| &< \delta \quad j = 1, \ldots, M \\ ||y_i|| &< ||g_i|| + \delta \quad i = 1, \ldots, N \\ ||\pi(y_i) - \sigma(g_i)|| &< \delta \quad i = 1, \ldots, N \end{aligned}$$

there is a $*$-homomorphism $\varphi : D \to E$ such that $\pi\varphi = \sigma$ and

$$||\varphi(g_i) - y_i|| < \eta, \ i = 1, \ldots, N.$$

Proof. — Since π is onto there are $z_1, \ldots z_N \in E$ with $||z_i - y_i|| < \delta$ and $\pi(z_i) = \sigma(g_i)$ for $i = 1, \ldots, N$. The relations p_j are polynomials. Thus there is a real continuous function $\alpha = \alpha(\delta), \alpha(0) = 0$ such that

$$\begin{aligned} ||p_j(z_1, \ldots, z_N)|| &\leqslant \alpha(\delta) \quad j = 1, \ldots M \\ ||z_i|| &\leqslant ||g_i|| + \alpha(\delta) \quad i = 1, \ldots, N. \end{aligned}$$

For the rest of the proof we will write α for $\alpha(\delta)$. Let $D_\alpha = C^*_\alpha\langle G, R\rangle$ denote the universal C^*-algebra generated by $g_1^\alpha, \ldots, g_N^\alpha$ subject to

$$||g_i^\alpha|| \leqslant ||g_i|| + \alpha, \ ||p_j(g_1^\alpha, \ldots, g_n^\alpha)|| \leqslant \alpha.$$

By the universality of D_α the map $g_i^\alpha \mapsto z_i$ extends to a well defined $*$-homomorphism $\varphi_\alpha : D_\alpha \to E$. If $p_\alpha : D_\alpha \to D$, $p(g_i^\alpha) = g_i$ is the canonical surjection it is clear that $\pi\varphi_\alpha = \sigma p_\alpha$. By hypothesis D has exactly stable relations. This means that if $\alpha = \alpha(\delta)$ is small enough, then there is a $*$-homomorphism $\sigma_\alpha : D \to D_\alpha$ such that $p_\alpha \sigma_\alpha = \mathrm{id}_D$ and $\|\sigma_\alpha(g_i) - g_i^\alpha\| \leqslant \mu(\alpha)$ with $\mu(\alpha) \to 0$ as $\alpha \to 0$. Setting $\varphi = \varphi_\alpha \sigma_\alpha : D \to E$ we have $\pi\varphi = \pi\varphi_\alpha\sigma_\alpha = \sigma p_\alpha \sigma_\alpha = \sigma$ and

$$\begin{aligned} \|\varphi(g_i) - y_i\| &= \|\varphi_\alpha\sigma_\alpha(g_i) - \varphi_\alpha(g_i^\alpha) + z_i - y_i\| \\ &\leqslant \mu(\alpha(\delta)) + \delta < \eta \end{aligned}$$

for small enough δ. □

The following proposition gives a characterization of $\mathcal{AD}$-algebras.

Proposition 1.2. — *A C^*-algebra E is an $\mathcal{AD}$-algebra if and only if for any finite subset F of E and $\epsilon > 0$ there is a C^*-algebra $D \in \mathcal{D}$ and a $*$-homomorphism $\varphi : D \to E$ such that $\operatorname{dist}(x, \varphi(D)) < \epsilon$ for all $x \in F$.*

Proof. — We omit the proof which is based on Theorem 1.1 and is very similar to the proof of Theorem 3.8 in [Lo1]. □

Proposition 1.3. — *If A is an $\mathcal{AD}$-algebra of real rank zero then every hereditary C^*-subalgebra of A is an $\mathcal{AD}$-algebra.*

Proof. — The proof is very similar to the proof of Proposition 3 in [LR] and is omitted. □

Theorem 1.4. — *Let $D \in \mathcal{D}$ and let $\varphi : D \to B$ be a $*$-homomorphism to a real rank zero C^*-algebra. Suppose that $\varphi_* : K_1(D) \to K_1(B)$ is the zero map. Then for any finite subset $G \subset D$ and $\epsilon > 0$ there exists a $*$-homomorphism $\psi : \mathcal{D} \to B$ with finite dimensional image such that*

$$\|\varphi(a) - \psi(a)\| < \epsilon \text{ for all } a \in G.$$

The proof of Theorem 1.4 is divided into a number of lemmas.

Lemma 1.5. — *Suppose that D is a C^*-algebra with exact stable relations and B is a real rank zero C^*-algebra. Suppose that $\varphi : M_n(D) \to B$ is a $*$-homomorphism. For any $\epsilon > 0$ and any finite set $G \subset D$ there exist matrix units $p_{ij} \in B$ and $\varphi_0 : D \to p_{11}Bp_{11}$ such that the $*$-homomorphism $\varphi_0 \otimes \mathrm{id}_{M_n} : D \to M_n(p_{11}Bp_{11}) \subset B$ approximates φ within ϵ on G.*

Proof. — Let $D = C^*\langle G_0, R\rangle$ for exactly stable relations R. Clearly we may assume that $G_0 = G$. The lemma is trivial if D has a unit. If not, we may still apply Propositions 3.1 and 3.2 of [Lo2] to conclude that φ has a factorization $\varphi = \gamma \circ (\varphi_1 \otimes \mathrm{id}_{M_n})$ where $\varphi_1 : D \to B_0$ is a $*$-homomorphism to some hereditary subalgebra B_0 of B and $\gamma : M_n(B_0) \to B$ is such that $\gamma(b \otimes e_{11}) = b$ for $b \in B_0$. By Theorem 2.6 in [BP], B_0 has an approximate unit (q_λ) consisting of projections. Choosing λ_0 large, we may assume that $q_{\lambda_0}\varphi_1(a)q_{\lambda_0}$ is close to $\varphi_1(a)$ for all $a \in G$. The map $a \to q_{\lambda_0}\varphi_1(a)q_{\lambda_0}$ is an approximate representation of D in $q_{\lambda_0}Bq_{\lambda_0}$ which is close to φ_1 on G. By Theorem 1.1 this implies that there exists $\varphi_0 : A \to q_{\lambda_0}Bq_{\lambda_0}$ with $||\varphi_0(a) - \varphi_1(a)|| < \epsilon$ for all $a \in G$. Let $p_{ij} = \gamma(q_{\lambda_0} \otimes e_{ij})$. Clearly $\gamma \circ (\varphi_0 \otimes \mathrm{id})$ factors through $M_n(p_{11}Bp_{11}))$ is in the desired fashion. □

LEMMA 1.6. — *Suppose that B is a real rank zero C^*-algebra and $\varphi : M_n(C_0(0,1)) \to B$ induces the zero map on K_1. Then for any finite set $G \subset C_0(0,1)$ and $\epsilon > 0$, there exists $\psi : M_n(C_0(0,1)) \to B$ with finite dimensional image and $||\varphi(a) - \psi(a)|| < \epsilon$ for all a in B.*

Proof. — For $n = 1$, this is basically a result of Lin (see [Li2]). In the general case the previous lemma allows us to assume, without loss of generality that φ factors as

$$A \xrightarrow{\varphi_0 \otimes \mathrm{id}} M_n(B_0) \overset{i}{\hookrightarrow} B$$

where B_0 is a corner in B. Also $M_n(B_0)$ is a corner. Using Lemma 2.3 in [Li1] one can show that i induces an injective map on K_1. Therefore φ_0 must be injective on K_1 and we have reduced the problem to the (proven) case $n = 1$. □

We now proceed to the proof of Theorem 1.4. Arguing as in the proof of Lemma 1.6 we reduce the proof to the cases $D = \widetilde{\mathbb{I}}_n$ and $D = C(\mathbb{T})$. To be specific, let e be a central projection in D and set $f = \varphi(e)$. By Theorem 2.5 in [BP], fBf has real rank zero. Using Lemma 2.3 of [Li1] it is easily seen that the restriction of φ to eDe induces the zero map $K_1(eDe) \to K_1(fBf)$. Thus we may assume that D is of the form $M_m(\widetilde{\mathbb{I}}_n)$ or $D = M_m(C(\mathbb{T}))$ and φ is unital. Using matrix units one can put φ in the form $\varphi = \psi \otimes \mathrm{id}_{M_m}$ where ψ maps $\widetilde{\mathbb{I}}_n$ (or $C(\mathbb{T})$) into the commutant of $\varphi(M_m)$ in B. This commutant is a real rank zero C^*-algebra and ψ induces the zero map on K_1. Since the case $D = C(\mathbb{T})$ is covered by [Li2], it suffices to consider the case when $D = \widetilde{\mathbb{I}}_n$ and φ is unital. Equivalently it is enough to consider the restriction of φ to $\mathbb{I}_n$. Write $\mathbb{I}_n = C^*\langle G, R\rangle$ where $G = \{a_2, \ldots, a_n, x_1, \ldots, x_n, h\}$ is the canonical set of generators with $a_j, x_i \in M_n(C_0(0,1))$ and h corresponding to $t \otimes 1_{\mathbb{I}_n}$ (see Proposition 2.3 in [Lo1]. There $v = e^{ih}$ was used rather than h.) The restriction of φ to $M_n(C_0(0,1))$ is zero on K_1. Lemma 1.6 implies that there is a finite dimensional subalgebra $B_0 \subseteq B$ and a $*$-homomorphism $\psi_0 : M_n(C_0(0,1)) \to B_0$ such that $\psi_0(a_j), \psi_0(x_i)$ are close to $\varphi(a_j), \varphi(x_i)$. Our basic strategy is now to apply the "stable relation operation" given in the proof of Theorem 6.2 of [Lo1]. With a small modification (discussed below) this operation yields elements A_j, X_i, H in B that are close to $\psi_0(a_j), \psi_0(x_i), \varphi(h)$, are an exact representation for the relations of $\mathbb{I}_n$ and also $A_j, X_i \in B_0$. This last condition forces H to have finite spectrum since $e^{2\pi i H} - 1 \in B_0$. The map $\psi : \mathbb{I}_n \to B$ determined by A_j, X_i, H must therefore have finite-dimensional image, so we are done modulo the following. The required modification of the proof of Theorem 6.2 is to replace the map β in the proof by

$$\beta(t) = \begin{cases} 0 & t \in [-\delta, \delta] \\ \dfrac{t-\delta}{1-2\delta} & t \in [\delta, 1-\delta] \\ 1 & t \in [1-\delta, 1+\delta]. \end{cases}$$

The construction now keeps the first $2n-1$ generators inside any given subalgebra. □

LEMMA 1.7. — *Let A be a C^*-algebra and let $\varphi : \mathbb{I}_n \to A$ be a $*$-homomorphism. Then* $\operatorname{diag}(\varphi, \varphi, \ldots, \varphi) : \mathbb{I}_n \to M_n(A)$ *is null homotopic.*

Proof. — Let $\alpha : \mathbb{I}_n \to M_n(\mathbb{I}_n)$ be given by $\alpha(a) = (a, \ldots, a)$. Since $\operatorname{diag}(\varphi, \ldots, \varphi)$ factors through α it is enough to prove that α is null homotopic. Let 1_n denote the unit of M_n. There is a unitary $u \in M_n \otimes M_n$ such that $u^*\ (1_n \otimes M_n)\ u = M_n \otimes 1_n$. Then

$$\gamma_s : \mathbb{I}_n \to M_n(\mathbb{I}_n), \quad (\gamma_s)_{s \in [0,1]}$$
$$\gamma_s(a)(t) = u \operatorname{diag}(a(ts), \ldots, a(ts))\ u^*$$

defines a homotopy of $*$-homomorphisms connecting the null map to $\operatorname{ad}(u) \circ \alpha$. Let u_s be a continuous path of unitaries connecting u to the identity. Then $\operatorname{ad}(u_s) \circ \alpha$ is a homotopy joining $\operatorname{ad}(u) \circ \alpha$ with α. □

PROPOSITION 1.8. — *Let A be a unital C^*-algebra of real rank zero and let $\Phi_t : \mathbb{I}_n \to A$, $(\Phi_t)_{t \in [0,1]}$, be a homotopy of $*$-homomorphisms. Fix $\epsilon > 0$ and let G be a finite subset of $\mathbb{I}_n$. Then for any $r \geqslant 1$ there is a $*$-homomorphism $\tau : \mathbb{I}_n \to M_{nr}(A)$ with finite dimensional image and there* *is a unitary* $u \in M_{nr+1}(\mathbb{C}1_A)$ *such that*

$$\begin{aligned}&||\mathrm{diag}(\Phi_0(a),\tau(a)) - u\ \mathrm{diag}(\Phi_1(a),\tau(a)))\ u^*|| \\ &\qquad \leqslant \epsilon + 2\max_{b\in G}\sup_{|s-t|\leqslant\frac{1}{r}} ||\Phi_s(b)-\Phi_t(b)|| \quad \textit{for all } a \in G.\end{aligned}$$

Proof. — Let $m = nr$ and for $0 \leqslant k < \ell \leqslant m+1$ define $\Psi_{[k,\ell)} : \mathbb{I}_n \to M_{\ell-k}(A)$ by $\Psi_{[k,\ell)} = \mathrm{diag}\left(\Phi_{\frac{k}{m}}, \Phi_{\frac{k+1}{m}}, \ldots, \Phi_{\frac{\ell-1}{m}}\right)$. We also need to consider $\Gamma : \mathbb{I}_n \to M_{nr}(A)$, $\Gamma = \mathrm{diag}\,(\Gamma_0, \Gamma_1, \ldots, \Gamma_{r-1})$ where $\Gamma_k : \mathbb{I}_n \to M_n(A)$ is given by $\Gamma_k = \mathrm{diag}\left(\Phi_{\frac{k}{r}}, \Phi_{\frac{k}{r}}, \ldots, \Phi_{\frac{k}{r}}\right)$. Let $\alpha(G,r) = \max_{b\in G}\sup_{|s-t|\leqslant\frac{1}{r}} ||\Phi_s(b)-\Phi_t(b)||$. One can check easily that

$$\max\{||\Psi_{[1,m+1)}(a)-\Gamma(a)||, ||\Psi_{[0,m)}(a)-\Gamma(a)||\} \leqslant \alpha(G,r)$$

for all $a \in G$. Since

$$\Psi_{[0,m+1)} = \mathrm{diag}(\Phi_0, \Psi_{[1,m+1)}) = \mathrm{diag}(\Psi_{[0,m)}, \Phi_1)$$

it follows that

$$\begin{aligned}||\Psi_{[0,m+1)}(a) - \mathrm{diag}(\Phi_0(a),\Gamma(a))|| \leqslant \alpha(G,r)\\ ||\Psi_{[0,m+1)}(a) - \mathrm{diag}(\Gamma(a),\Phi_1(a))|| \leqslant \alpha(G,r)\end{aligned}$$

whence

$$||\mathrm{diag}(\Phi_0(a),\Gamma(a)) - u\ \mathrm{diag}(\Phi_1(a),\Gamma(a))\ u^*|| \leqslant 2\alpha(G,r)$$

for a suitable permutation unitary $u \in U_{nr+1}(\mathbb{C})$.

Γ is null homotopic by Lemma 1.7 hence it induces the zero map on K-theory. Using Theorem 1.4 we can perturb Γ to a $*$-homomorphism τ with finite dimensional image. $\square$

2. A lifting result.

LEMMA 2.1. — *Let A be a closed ideal of a real rank zero algebra E. If F is a finite dimensional C^*-subalgebra of E, then there is an increasing sequence of projections $(p_k)_k$, which forms an approximate unit for A such that each p_k commutes with F.*

Proof. — Let $F = F_1 \oplus \ldots \oplus F_r$ be the decomposition of F into a direct sum of factors. If f_i is the unit of F_i then f_iEf_i has real rank zero (see [BP]) and f_iAf_i is a closed ideal in f_iEf_i. Thus it will suffice to find an increasing sequence of projections $(p_k^{(i)})_k$ which form an approximate unit for f_iAf_i and which commute with $F_i \subset f_iEf_i$. Indeed it is clear that $p_k = p_k^{(i)} + \ldots + p_k^{(r)}$ will have the desired properties. Thus we have reduced the proof of the lemma to the case $F \cong M_m$.

Let (e_{ij}) be matrix units in M_m. Then both $e_{11}Ee_{11}$ and its closed ideal $e_{11}Ae_{11}$ have real rank zero. By [BP], Theorem 2.9 or [Z] Theorem 6, $e_{11}Ae_{11}$ has an increasing sequence of projections $(q_k)_k$ which form an approximate unit. Then $p_k = \sum_{i=1}^{m} e_{i1}q_ke_{1i}$ defines an approximate unit having the desired properties. □

LEMMA 2.2. — *Let*

$$0 \to A \to E \xrightarrow{\pi} B \to 0$$

be an extension of C^*-algebras. Suppose that A, B and E have real rank zero and stable rank one. Then for any $*$-homomorphism $\tau : D \to B$ with finite dimensional image there is a $*$-homomorphism $\tau' : D \to E$ with finite dimensional image such that $\pi\tau' = \tau$. If E, B are unital and τ is unit preserving then one can arrange that τ' is unit preserving.

Proof. — The proof is based on results in [BP] and [Zh] and is formally similar to the proof in the case of AF-algebras (see [Eff]). One needs to lift matrix units in B to matrix units in E. The whole argument is essentially contained in the proof of Lemma 6 in [LR]. □

THEOREM 2.3. — *Let*

$$0 \to A \to E \xrightarrow{\pi} B \to 0$$

be an extension of separable C^*-algebras. Suppose that A, B and E have real rank zero and stable rank one. Suppose that E and B have units and let $\sigma : M_m(\widetilde{\mathbb{I}}_n) \to B$ be a unital $*$-homomorphism. If the map $\sigma_* : K_1(\widetilde{\mathbb{I}}_n) \to K_1(B)$ lifts to a morphism of groups $K_1(\widetilde{\mathbb{I}}_n) \to K_1(E)$ then there is a $*$-homomorphism $\varphi : M_m(\widetilde{\mathbb{I}}_n) \to M_R(E)$, for some $R \geqslant 1$, such that $(\pi \otimes \mathrm{id}_{M_R}) \circ \varphi = \mathrm{diag}(\sigma, 0, \ldots, 0)$.

Proof. — First we prove the theorem for $m = 1$. For most of the proof we deal with the restriction of σ to $\mathbb{I}_n$ which is denoted by σ too. For the sake of clarity we divide the proof in several stages.

a) Lifting σ at the level of KK-theory.

By [K] there is an exact sequence of KK-groups

$$KK(\mathbb{I}_n, E) \xrightarrow{\pi_*} KK(\mathbb{I}_n, B) \xrightarrow{\delta_E} KK_1(\mathbb{I}_n, A).$$

Thus $[\sigma] \in KK(\mathbb{I}_n, B)$ lifts to an element in $KK(\mathbb{I}_n, E)$ if and only if $\delta_E[\sigma] = 0$. By the universal coefficient theorem of [RS]

$$KK_1(\mathbb{I}_n, A) \cong \mathrm{Hom}(K_1(\mathbb{I}_n), K_0(A)) \oplus \mathrm{Ext}(K_1(\mathbb{I}_n), K_1(A)).$$

According to this decomposition $\delta_E[\sigma]$ has two components $h \in \mathrm{Hom}(K_1(\mathbb{I}_n), K_0(A))$ and $e \in \mathrm{Ext}(K_1(\mathbb{I}_n), K_1(A))$. The first component h is equal to $\delta_1 \circ \sigma_*$ where $\delta_1 : K_1(B) \to K_0(A)$ is the index map. Since A, B and E have real rank zero and stable rank one it follows from Proposition 4 in [LR] that both index maps δ_1 and $\delta_0 : K_0(B) \to K_1(A)$ are the zero maps. In particular this shows that $h = 0$. The second component of $\delta_E[\sigma]$ is given by the isomorphism class $e \in \mathrm{Ext}(K_1(\mathbb{I}_n), K_1(A))$ of the pullback by $\sigma_* : K_1(\mathbb{I}_n) \to K_1(B)$ of the extension

$$0 \to K_1(A) \to K_1(E) \to K_1(B) \to 0.$$

Note that the above extension occurs since the index maps δ_0 and δ_1 are vanishing. Since σ_* lifts to a morphism $K_1(\mathbb{I}_n) \to K_1(E)$ it follows that e is the isomorphism class of a split extension hence $e = 0$. This proves that $[\sigma] \in \pi_* KK(\mathbb{I}_n, E)$.

b) Lifting σ up to a homotopy.

By Corollary 7.1 in [DL2] for any C^*-algebra D

$$KK(\mathbb{I}_n, D) = [\mathbb{I}_n, D \otimes \mathcal{K}] = \lim_r [\mathbb{I}_n, M_r(D)].$$

In view of a) this implies that there is a $*$-homomorphism $\psi : \mathbb{I}_n \to M_r(E)$ such that $\Phi_1 \overset{\text{def}}{=} \pi_r \psi$ is homotopic to $\Phi_0 \overset{\text{def}}{=} \mathrm{diag}(\sigma, 0, \ldots, 0)$ via a homotopy $\Phi_t : \mathbb{I}_n \to M_r(B)$, $t \in [0,1]$. Here π_r stands for the $*$-homomorphism $\pi \otimes \mathrm{id}_{M_r} : M_r(E) \to M_r(B)$.

c) Producing approximate liftings.

Let $\delta > 0$ and let G be a finite subset of $\mathbb{I}_n$. Using Proposition 1.8 we find a $*$-homomorphism $\tau : \mathbb{I}_n \to M_{r'}(B)$ with finite dimensional image and a unitary $u \in M_{r+r'}(\mathbb{C}1_B)$ such that

$$||\mathrm{diag}(\Phi_0(a), \tau(a)) - u \; \mathrm{diag}(\Phi_1(a), \tau(a)) \; u^*|| < \delta$$

for all $a \in G$. Using Lemma 2.2 we lift τ to a $*$-homomorphism $\tau' : \mathbb{I}_n \to M_{r'}(E)$ with finite dimensional image. Let $V \in M_{r+r'}(\mathbb{C}1_E)$ be the obvious lifting of u. Then the formula

$$\varphi_\delta(a) = V \operatorname{diag}(\psi(a), \tau'(a))\, V^*$$

defines a $*$-homomorphism $\varphi_\delta : \mathbb{I}_n \to M_{r+r'}(E)$ such that

$$||\operatorname{diag}(\sigma(a), 0_{r-1}, \tau(a)) - \pi_{r+r'}\varphi_\delta(a)|| < \delta$$

for all $a \in G$.

d) Perturbing approximate liftings to exact liftings.

Choosing $\delta > 0$ and G in step c) to be as in Theorem 1.1, one can perturb φ_δ to a $*$-homomorphism $\varphi_0 : \mathbb{I}_n \to M_{r+r'}(E)$ with

$$\pi_{r+r'}\varphi_0 = \operatorname{diag}(\sigma, 0_{r-1}, \tau).$$

Next we modify φ_0 in order to get a lifting of $\operatorname{diag}(\sigma, 0_{r+r'-1})$. By Lemma 2.1 there is an approximate unit of projections $(q_k)_k$ in $M_{r+r'-1}(A)$, which commutes with $\operatorname{diag}(0_{r-1}, \tau'(a))$ for all $a \in \mathbb{I}_n$. Since for all $a \in \mathbb{I}_n$

$$\pi_{r+r'}(\varphi_0(a) - \operatorname{diag}(0_r, \tau'(a))) = \operatorname{diag}(\sigma(a), 0_{r+r'-1})$$

it follows that all the entries, but the $(1,1)$ entry of $\varphi_0(a) - \operatorname{diag}(0_r, \tau'(a))$ lie in A. If $f_k = \operatorname{diag}(1_E, q_k)$ this is easily seen to imply that

$$\lim_{k\to\infty} ||\varphi_0(a) f_k - f_k \varphi_0(a)|| = 0$$

for all $a \in \mathbb{I}_n$.

Let $E_k = f_k M_{r+r'}(E) f_k$. By compressing φ_0 by the projections f_k we get a sequence of linear self-adjoint maps $\varphi_k : \mathbb{I}_n \to E_k$, $\varphi_k(a) = f_k\varphi_0(a) f_k$ such that $||\varphi_k(ab) - \varphi_k(a)\varphi_k(b)|| \to 0$ for all $a, b \in \mathbb{I}_n$ and $\pi_{r+r'}\varphi_k = \operatorname{diag}(\sigma, 0_{r+r'-1})$. We conclude from Theorem 1.1 that if k is big enough then φ_k can be perturbed to a $*$-homomorphism $\varphi : \mathbb{I}_n \to E_k \subset M_{r+r'}(E)$ such that $\pi_{r+r'}\varphi(a) = \operatorname{diag}(\sigma(a), 0_{r+r'-1})$ for all $a \in \mathbb{I}_n$. Finally we extend φ to a unital $*$-homomorphism $\varphi : \widetilde{\mathbb{I}}_n \to E_k$ by setting $\varphi(1) = f_k$. The proof for the case $m = 1$ is complete. However, we choose to perturb φ once more in order to arrange that $\varphi(1)$ is of the form $\operatorname{diag}(1_E, p, \ldots, p) \in M_{r+r'}(E)$ for some projections $p \in A$. Let $(p_k)_k$ be an approximate unit of A consisting of projections and let $g_k = \operatorname{diag}(1_E, p_k, \ldots, p_k)$. Then g_k commutes asymptotically with φ and $\pi(g_k) = (1_B, 0_{r+r'-1})$. As above we can perturb $g_k\varphi(\cdot)g_k$ to a $*$-homomorphism $\mathbb{I}_n \to g_k M_{r+r'}(E) g_k$ which lifts

$\mathrm{diag}(\sigma, 0_{r+r'-1})$. Then we extend this $*$-homomorphism to a unital one that sends 1 to g_k.

e) The general case $m \geqslant 1$.

Let (e^0_{ij}) be a system of matrix units for $M_m \subset M_m(\widetilde{\mathbb{I}}_n)$ and let $e_{ij} = \sigma(e^0_{ij})$. Lemma 2.2 shows that there are matrix units $(\overline{e}_{ij})$ in E such that $\pi(\overline{e}_{ij}) = e_{ij}$ and $\overline{e}_{11} + \ldots + \overline{e}_{mm} = 1_E$. Consider the extension

$$0 \to \overline{e}_{11} A \overline{e}_{11} \to \overline{e}_{11} E \overline{e}_{11} \to e_{11} B e_{11} \to 0$$

and the $*$-homomorphism $\sigma_0 : \widetilde{\mathbb{I}}_n \to e_{11}Be_{11}$ given by $\sigma_0(d) = \sigma(d \otimes e^0_{11})$. This extension and σ_0 do satisfy all the hypotheses of the Theorem in the case $m = 1$. Indeed by [R] and [BP] full corners of a given C^*-algebra C have the same stable rank and real rank as C. Thus by the first part of the proof we find a $*$-homomorphism $\varphi_0 : \widetilde{\mathbb{I}}_n \to M_R(E)$ such that $\pi_R \varphi_0 = \mathrm{diag}(\sigma_0, 0_{R-1})$ and $\varphi_0(1) = \mathrm{diag}(\overline{e}_{11}, p, \ldots, p)$ for some projection $p \in \overline{e}_{11} A \overline{e}_{11}$. Setting $a_{ij} = \overline{e}_{i1} p \overline{e}_{1j} \in A$ we define $\varphi : M_m(\widetilde{\mathbb{I}}_n) \to M_R(E)$ by

$$\varphi(d \otimes e^0_{ij}) = g_{i1} \varphi_0(d) g_{1j}$$

where $g_{ij} \stackrel{\mathrm{def}}{=} \mathrm{diag}(\overline{e}_{ij}, a_{ij}, \ldots, a_{ij}) \in M_R(E)$. It is clear then that φ is a lifting of σ. □

3. A result on extensions.

THEOREM 3.1. — *Let*

$$0 \to A \to E \xrightarrow{\pi} B \to 0$$

be an extension of C^*-algebras where A and B are $\mathcal{AD}$-algebras of real rank zero. Suppose that either $K_1(A) = 0$ or $K_1(B)$ is torsion free. Then the following are equivalent :

(i) *E is an $\mathcal{AD}$-algebra of real rank zero.*

(ii) *E has real rank zero and stable rank one.*

(iii) *The index maps $\delta_0 : K_0(B) \to K_1(A)$ and $\delta_1 : K_1(B) \to K_0(A)$ are zero.*

Proof. — (ii) $\Leftrightarrow$ (iii) follows from Proposition 4 in [LR] while (i) $\Rightarrow$ (ii) since all $\mathcal{AD}$-algebras have stable rank one. It remains to prove that

(ii) $\Rightarrow$ (i). Consider the extension

$$0 \to A \otimes \mathcal{K} \to E \otimes \mathcal{K} \xrightarrow{\pi \otimes \mathrm{id}_{\mathcal{K}}} B \otimes \mathcal{K} \to 0$$

and let

$$E_\infty = \{y \in E \otimes \mathcal{K} : \pi \otimes \mathrm{id}_{\mathcal{K}}(y) \in B \otimes e_{11}\}.$$

We form the extension

$$0 \to A \otimes \mathcal{K} \to E_\infty \xrightarrow{\pi_\infty} B \to 0$$

where π_∞ is the restriction of $\pi \otimes \mathrm{id}_{\mathcal{K}}$ to E_∞. Note that $E \cong E \otimes e_{11}$ is a hereditary subalgebra of E_∞. Therefore by Proposition 1.3 it suffices to prove that E_∞ is an $A\mathcal{D}$-algebra. For the sake of clarity we divide the proof into several stages.

a) We prove that for any finite subset F of E_∞ and any $\epsilon > 0$, there is a C^*-algebra $D \in \mathcal{D}$ and a $*$-homomorphism $\varphi : D \to E_\infty$ such that $\mathrm{dist}(x, \varphi(D) + A \otimes \mathcal{K}) < \epsilon$ for all $x \in F$.

Since $E_\infty = E \otimes e_{11} + A \otimes \mathcal{K}$ it is clear that we may assume that $F \subset E \otimes e_{11} \cong E$. Since B is an inductive limit of C^*-algebras in $\mathcal{D}$, after a small perturbation of F, we may assume that $\pi(F)$ is contained in the image of some $*$-homomorphism $\sigma : D \to B$ with $D \in \mathcal{D}$. Decompose D into a direct sum $D = D_1 \oplus \ldots \oplus D_r$ with $D_i \cong M_{m(i)}(\widetilde{\mathbb{I}}_{n(i)})$ or $D_i = M_{m(i)}(C(\mathbb{T}))$. Let f_i^0 be the unit of D_i and let $f_i = \varphi(f_i^0)$. Using Lemma 2.2 we lift $f_1, \ldots f_r$ to mutual orthogonal projections $\overline{f}_1, \ldots, \overline{f}_r \in E$. All the C^*-algebras in the extension

$$0 \to \overline{f}_i A \overline{f}_i \to \overline{f}_i E \overline{f}_i \xrightarrow{\pi} f_i B f_i \to 0$$

have real rank zero (see [BP]) and stable rank one (see [R]). In particular, the index maps $\delta_* : K_*(f_i B f_i) \to K_{*+1}(\overline{f}_i A \overline{f}_i)$ are zero by Proposition 4 in [LR]. Consider now the case $K_1(A) = 0$. By Proposition 2.3 in [Li1] $K_1(\overline{f}_i A \overline{f}_i)$ is isomorphic to a subgroup of $K_1(A)$ hence is zero. Consequently

$$\pi_* : K_1(\overline{f}_i E \overline{f}_i) \to K_1(f_i B f_i)$$

is an isomorphism. Thus if $D_i \cong M_{m(i)}(\widetilde{\mathbb{I}}_{n(i)})$ we can use Theorem 3.3 in order to produce $*$-homomorphisms $\varphi_i : D_i \to M_R(\overline{f}_i E \overline{f}_i)$ such that $\pi\varphi_i = \mathrm{diag}(\sigma_i, 0_{R-1})$ where σ_i denotes the restriction of σ to D_i. In the case $D_i = M_{m(i)}(C(\mathbb{T}))$ we use Lemma 6 in [LR] to get a lifting φ_i for σ_i. Note that Lemma 6 in [LR] is valid for an arbitrary short exact sequence that involves only C*-algebras with real rank zero and stable rank one - with the same proof. Consider now the case when $K_1(B)$ is torsion free. By Elliott's classification theorem [Ell] this implies that B is an $A\mathbb{T}$-algebra. Thus we may assume that each D_i is a matrix algebra over $C(\mathbb{T})$. As above Lemma 6 in [LR] will provide the desired lifting of σ_i.

It is clear then that the formula

$$\varphi(d_1, \ldots, d_r) = \varphi_1(d_1) + \ldots + \varphi_r(d_r)$$

defines a $*$-homomorphism $\varphi : D \to M_R(E)$ such that $\pi_R\varphi = \mathrm{diag}(\sigma, 0_{R-1})$. We conclude that $\pi(F) \subset \varphi(D) + A \otimes \mathcal{K}$.

b) Let $\varphi : D \to M_R(E) \cap E_\infty$ be as above. We prove that for any $\mu > 0$ and any finite subset G of D there is an increasing sequence of projections $(p_k)_k$ which form an approximate unit of A and such that

$$(*) \qquad ||\varphi(a)p_k - p_k\varphi(a)|| < \mu \text{ for all } a \in G.$$

The case when $K_1(B)$ is torsion free is solved by Lemma 10 in [LR] which is valid for any C*-algebra A of real rank zero and stable rank one. Thus we need to consider only the case $K_1(A) = 0$. For the beginning we assume that $A \otimes \mathcal{K}$ is an essential ideal of E_∞. Thus we can regard E_∞ as a subalgebra of the multiplier algebra $M(A \otimes \mathcal{K})$. Since $K_1(A) = 0$ it follows by a result of Lin [Li2] that $M(A \otimes \mathcal{K})$ has real rank zero. On the other hand $K_1(M(A \otimes \mathcal{K})) = 0$ (see [Bl]). But then Theorem 1.4 shows that we can perturb φ to a $*$-homomorphism $\psi : D \to M(A \otimes \mathcal{K})$ with finite dimensional image and $||\varphi(a) - \psi(a)|| < \mu$ for all $a \in G$. Lemma 2.1 gives an increasing sequence of projections $(p_k)_k$ which form an approximate unit of $A \otimes \mathcal{K}$ commuting with the image of ψ. It is immediate that each p_k satisfies $(*)$.

The general case when $A \otimes \mathcal{K}$ is not necessarily an essential ideal of E_∞ follows from the special case when $A \otimes \mathcal{K}$ is an essential ideal. One can argue as in the proof of Lemma 9 in [LR].

c) Let $F = \{x_1, \ldots, x_\ell\} \subset E_\infty$ and let $\epsilon > 0$. We prove that there is $C \in \mathcal{D}$ and there is a $*$-homomorphism $\psi : C \to E_\infty$ such that $\mathrm{dist}(x_i, \psi(C)) < 3\epsilon$ for all x_i in F. By Proposition 1.3 this will imply that E_∞ (and therefore E) is an $A\mathcal{D}$-algebra.

Let D and $\varphi : D \to E_\infty$ be given by a). It follows that there are $d_1, \ldots, d_\ell \in D$ and $a_1, \ldots, a_\ell \in A \otimes \mathcal{K}$ such that $||x_i - \varphi(d_i) - a_i|| < \epsilon$ for

$i = 1, \dots, \ell$. Let G be a set of generators of D such that $\{d_1, \dots, d_\ell\} \subset G$. Using b) we find an approximate unit of projections $(q_k)_k$ for A such that

$$||q_k\varphi(d) - \varphi(d)q_k|| \to 0 \text{ for all } d \in D.$$

For big enough k

$$||x_i - (1 - q_k)\varphi(d_i)(1 - q_k) - q_k(\varphi(d_i) + a_i)q_k|| < \epsilon$$

for $i = 1, \dots, \ell$. The sequence of linear selfadjoint maps $\varphi_k : D \to (1 - q_k)E_\infty(1 - q_k), \varphi_k(d) = (1 - q_k)\varphi(d)(1 - q_k)$ satisfies : $||\varphi_k(dc) - \varphi_k(d)\varphi_k(c)|| \to 0$ for all $d, c \in D$. Theorem 1.1 shows that if k is big enough, then we find a $*$-homomorphism $\psi_1 : D \to (1 - q_k)E_\infty(1 - q_k)$ such that $||\psi_1(d_i) - \varphi_k(d_i)|| < \epsilon$ for $i = 1, \dots, \ell$. Since A is an $A\mathcal{D}$-algebra, there exist $D' \in \mathcal{D}$ and a $*$-homomorphism $\psi_2 : D' \to q_kE_\infty q_k$ such that $\mathrm{dist}(q_k(\psi(d_i) + a_i)q_k,\ \psi_2(D')) < \epsilon$ for all $i = 1, \dots, \ell$. Setting $\psi(d \oplus d') = \psi_1(d) + \psi_2(d')$ we obtain a $*$-homomorphism $\psi : D \oplus D' \to E_\infty$ such that $\mathrm{dist}(x_i, \psi(D \oplus D')) < 3\epsilon$ for $i = 1, \dots, \ell$. □

4. Examples.

The purpose of this section is to produce examples of extensions

$$0 \to A \to E \to B \to 0$$

where A, B, E are C^*-algebras of real rank zero and stable rank one and such that A and B are $A\mathcal{D}$-algebras but E is not an $A\mathcal{D}$-algebra.

LEMMA 4.1. — *The K_1-group of any proper quotient of $M_m(\widetilde{\mathbb{I}}_n)$ or $M_m(C(\mathbb{T}))$ is zero.*

Proof. — It suffices to consider the case $m = 1$. We give the proof only for dimension-drop interval algebras. The proof for circle algebras is similar and easier. Let $\pi : \widetilde{\mathbb{I}}_n \to B$ be a surjective $*$-homomorphism with nonzero kernel $J \neq \widetilde{\mathbb{I}}_n$. It follows that

$$J = \{f \in \widetilde{\mathbb{I}}_n \mid \quad f|_F = 0\}$$

for some closed proper subset F of $[0, 1]$. Write $F = \bigcap_{i=1}^{\infty} F_i$ where $F_1 \supset F_2 \supset \dots$ and each F_i is a finite union of closed subintervals of $[0, 1]$. Setting

$$J_i = \{f \in \widetilde{\mathbb{I}}_n \mid \quad f|_{F_i} = 0\}$$

it is clear that $J_1 \subset J_2 \subset \ldots$ and $\overline{\cup J_i} = J$. It follows that $B = \overline{\cup B_i}$ where $B_i \cong \widetilde{\mathbb{I}}_n / J_i$. Since K_1 is a continuous functor we have $K_1(B) = \varinjlim K_1(B_i)$ hence it suffices to prove that $K_1(B_i) = 0$ for all large enough i. But for large i, B_i is isomorphic to a direct sum of C^*-algebras, each of which is isomorphic either to $C([0,1], M_n)$ or to

$$\{f \in C([0,1], M_n) | f(1) \in \mathbb{C}1_n\}.$$

Since the K_1-group of these algebras is zero we conclude that $K_1(B_i) = 0$. □

PROPOSITION 4.2. — *Let*

$$0 \to A \to E \xrightarrow{\pi} B \to 0$$

be an extension of separable C^*-algebras. If E is an $A\mathcal{D}$-algebra then $\pi_* : K_1(E) \to K_1(B)$ is surjective and any torsion element of $K_1(B)$ lifts to a torsion element of $K_1(E)$.

Proof. — By assumption E is the inductive limit of a sequence

$$D_1 \to D_2 \to \ldots$$

of C^*-algebras $D_i \in \mathcal{D}$. This gives canonical maps $\varphi_i : D_i \to E$ with $E = \overline{\bigcup_i \varphi_i(D_i)}$. Setting $B_i = \pi\varphi_i(D_i)$ one has $B = \overline{\bigcup_i B_i}$. This shows that the extension in the statement of the proposition is the inductive limit of the extensions

$$(**) \qquad 0 \to J_i \to D_i \xrightarrow{\pi\varphi_i} B_i \to 0.$$

Since the K_1 functor is continuous it suffices to prove the proposition for the extensions $(**)$. Write $D_i \simeq E_1 \oplus \ldots \oplus E_r$ with $E_j \simeq M_{m(j)}(\widetilde{I}_{n(j)})$. Accordingly B_i decomposes as $B_i = C_1 \oplus \ldots \oplus C_r$ where $C_j \simeq E_j / J_i \cap E_j$ and $\pi\varphi_i = \mathrm{diag}(\psi_1, \ldots, \psi_r)$ with $\psi_j : E_j \to C_j$ surjective. By Lemma 4.1 we conclude, for each j, that either $K_1(C_j)$ is zero or ψ_j (hence $K_1(\psi_j)$) is an isomorphism. This proves that $K_1(\pi\varphi_i)$ has a right inverse. □

Remark 4.3. — One can conclude from Proposition 4.2 that we can replace (i) in Theorem 3.1 by

(i′) E is an $A\mathcal{D}$-algebra.

Indeed if E is an $A\mathcal{D}$-algebra then π_* is surjective hence the index map $\delta_1 : K_1(B) \to K_0(A)$ is zero. Thus (i′) $\Rightarrow$ (iii) of Theorem 3.1.

THEOREM 4.4. — *Let A, B be $\mathcal{AD}$-algebras of real rank zero and let $e \in \operatorname{Ext}(K_1(B), K_1(A))$. Suppose that $K_1(B)$ has a finite subgroup H such that the image of e under the restriction map*

$$\operatorname{Ext}(K_1(B), K_1(A)) \to \operatorname{Ext}(H, K_1(A))$$

is nonzero. Then there is an extension

$$0 \to A \otimes \mathcal{K} \to E \to B \to 0$$

representing $e \in KK_1(B, A)$ such that E has real rank zero, stable rank one and E is not an $\mathcal{AD}$-algebra.

Proof. — The universal coefficient theorem of [RS] gives an exact sequence

$$0 \to \operatorname{Ext}(K_*(B), K_*(A)) \xrightarrow{j} KK_1(B, A) \xrightarrow{\gamma} \operatorname{Hom}(K_*(B), K_*(A)) \to 0.$$

The map γ has degree zero and the map j has degree 1. Each $e \in \operatorname{Ext}(K_1(B), K_1(A))$ gives an element $j(e) \in KK_1(B, A)$ represented by some extension

$$0 \to A \otimes \mathcal{K} \to E \to B \to 0.$$

The index maps in the long exact sequence in K-theory associated with this extension are given by

$$(\delta_0, \delta_1) = \gamma j(e) = 0$$

hence are vanishing. By Proposition 4 in [LR] this implies that E has real rank zero and stable rank one whenever A and B have real rank zero and stable rank one. On the other hand e corresponds to the isomorphism class of the extension

$$0 \to K_1(A) \to K_1(E) \to K_1(B) \to 0.$$

If e is chosen such that its image in $\operatorname{Ext}(H, K_1(A))$ is non-zero then we conclude from Proposition 4.2 that E is not an $\mathcal{AD}$-algebra. □

Example 4.5. — Let A be a real rank zero $\mathcal{AD}$-algebra with $K_0(A) = \mathbb{Z}\left[\frac{1}{2}\right]$ and $K_1(A) = \mathbb{Z}/2$. By a result of Elliott [Ell] such a C^*-algebra exists and is unique up to an isomorphism. Let e be the generator of $\operatorname{Ext}(K_1(A), K_1(A)) = \mathbb{Z}/2$. Theorem 4.4 shows that there is an extension

$$0 \to A \otimes \mathcal{K} \to E \to A \to 0$$

such that E has real rank zero, stable rank one and E is not an $\mathcal{AD}$-algebra. Futhermore it can be shown that E is not isomorphic to an inductive limit of homogeneous C*-algebras [BD].

However $U \otimes E$ is an AF-algebra if U is a suitable UHF-algebra (cf. [EK]). To be specific, let U be a UHF-algebra with $K_0(U) = \mathbb{Z}\left[\frac{1}{2}\right]$. Then

$$0 \to U \otimes A \otimes \mathcal{K} \to U \otimes E \to U \otimes A \to 0$$

is an extension where $U \otimes A$ is an $\mathcal{AD}$-algebra with $K_1(U \otimes A) = 0$. Elliott's classification theorem [Ell] implies that $U \otimes A$ is an AF-algebra. We conclude from Brown's theorem on extensions of AF-algebras [Br1] that $U \otimes E$ is an AF-algebra.

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