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# CONSTRUCTIVE INVARIANT THEORY FOR TORI

by David L. WEHLAU

## Introduction.

Let $\rho : G \to GL(V)$ be a rational representation of a reductive algebraic group over the algebraically closed field $\mathbf{k}$. The action of $G$ on $V$ induces an action of $G$ on $\mathbf{k}[V]$, the algebra of polynomial functions on $V$, via $(g \cdot f)(v) = f(\rho(g^{-1})v)$ for $g \in G$, $f \in \mathbf{k}[V]$ and $v \in V$. The functions which are fixed by this action form a finitely generated subalgebra, $\mathbf{k}[V]^G$, the ring of invariants. The problem of constructive invariant theory is to give an algorithm which in a finite number of steps will explicitly construct a minimal set of homogeneous generators for the $\mathbf{k}$-algebra, $\mathbf{k}[V]^G$.

Now if $\{f_1, \ldots, f_p\}$ is such a set with $\deg f_1 \geq \deg f_2 \geq \ldots \geq \deg f_p$ then although the $f_i$ are not uniquely determined the $p$-tuple of degrees $(\deg f_1, \ldots, \deg f_p)$ is unique. The number $N_{V,G} = \deg f_1$ is of special interest. It is the minimal integer $N$ such that $\mathbf{k}[V]^G$ is generated by the subspace $\bigoplus_{m=0}^{N} \mathbf{k}[V]^G_m$ of invariants of degree at most $N$. Clearly an algorithm which constructs $\{f_1, \ldots, f_p\}$ also produces $N_{V,G} = \max\{\deg f_i \mid 1 \leq i \leq p\}$. For many groups, $G$, (e.g. if char $k = 0$ and $G$ is reductive) the converse is also true : given $N_{V,G}$ there is a finite algorithm which constructs $\{f_1, \ldots, f_p\}$ (cf. [K], [P]).

If $G$ is a finite group and the characteristic of $\mathbf{k}$ does not divide $|G|$, then by a celebrated theorem of Emmy Noether's, $N_{V,G} \leq |G|$ (see [N1]),

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[N2]). Recently Schmid has considered the question of whether this bound is sharp ([S]). She has shown that $N_{V,G} < |G|$ if $G$ is not cyclic and has determined $N_{V,G}$ for various groups of small order including all abelian groups of order less than 30.

If $G$ is semi-simple and the characteristic of $\mathbf{k}$ is zero and the representation $\rho$ is almost faithful, then Popov has given in [P] an upper bound for $N_{V,G}$. Following the methods of Popov, Kempf ([K]) derived an upper bound for $N_{V,G}$ in the case that $G$ is a torus and the characteristic of $\mathbf{k}$ is zero. Kempf also observed that these three bounds (for $G$ finite, $G$ semi-simple and $G$ a torus) could be combined (by multiplying them) to obtain a bound for the general reductive group in characteristic zero.

The bounds for infinite groups are very large. In this paper we will consider the case $G = T$ is a torus and give better bounds for $N_{V,T}$. In addition we will construct certain distinguished elements of a minimal generating set for $\mathbf{k}[V]^T$.

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## Diagonalization.

Let $\mathbf{k}$ be an algebraically closed field of any characteristic. Let $T$ be a torus, i.e., $T$ is an algebraic group which is (abstractly) isomorphic to $(\mathbf{k}^*)^r$ and suppose that $\rho : T \to GL(V)$ is a rational representation of $V$. Let $X^*(T)$ denote the lattice of characters of $T$. Then $X^*(T)$ is (abstractly) isomorphic to $\mathbb{Z}^r$. From now on we will assume that we have chosen a fixed basis of $V$ consisting of eigenvectors, $\{v_1, \ldots, v_n\}$, and that $\{x_1, \ldots, x_n\}$, is the corresponding dual basis of $V^*$. Furthermore we will denote the weight of $v_i$ by $\omega_i$. Then $\rho$ induces an action of $T$ on $V^* \subset \mathbf{k}[V]$ which in terms of weights is given by $t \cdot x_i = -\omega_i(t)x_i$. The action on all of $\mathbf{k}[V] \cong \mathbf{k}[x_1, \ldots, x_n]$ is obtained from the action on $V^*$ by the two requirements $t \cdot (fg) = (t \cdot f)(t \cdot g)$ and $t \cdot (f + g) = t \cdot f + t \cdot g$ for $t \in T$ and $f, g \in \mathbf{k}[V]$.

We will consider monomials $X^A = x_1^{a_1} x_2^{a_2} \cdot \ldots \cdot x_n^{a_n}$ where $A = (a_1, \ldots, a_n) \in \mathbb{N}^n$. Clearly $T$ acts on $X^A$ by $t \cdot X^A = \chi(t)X^A$ where $\chi$ is the character $\chi = -(a_1\omega_1 + \ldots + a_n\omega_n)$. We will denote $\chi$ by $\mathrm{wt}(X^A)$. The invariant monomials are in one-to-one correspondence with the semi-group, $S := \{A \in \mathbb{N}^n \mid X^A \in \mathbf{k}[V]^T\} = \{A \in \mathbb{N}^n \mid a_1\omega_1 + \ldots + a_n\omega_n = \mathbf{o}\}$ where $\mathbf{o}$ is the trivial character in $X^*(T)$. This semi-group was first studied by Gordan. He used it to show that $\mathbf{k}[V]^T$ is a finitely generated algebra by showing that $S$ is finitely generated as a semi-group (see [Go]).

Recall that a representation $\rho : G \to GL(V)$ is called *stable* if the union of the closed $G$-orbits in $V$ contains an open dense subset of $V$. It is sufficient to consider only faithful stable torus representations, (cf. [W], Lemma 2). From now on we will suppose that $\rho$ is both faithful and stable.

## Kempf's bound.

Choosing an explicit isomorphism $\psi : T \longrightarrow (k^*)^r$ induces an explicit isomorphism $\psi^* : X^*(T) \longrightarrow \mathbb{Z}^r$. The isomorphism $\psi$ is determined only up to Aut $(T) \cong GL(r, \mathbb{Z})$. Having fixed a choice for $\psi$ we may write out the weights of $V$ as $r$-tuples: $\omega_i = (\omega_{i,1}, \ldots, \omega_{i,r}) \in \mathbb{Z}^r$ for $1 \leq i \leq n$. Then we may define $w := \max\{ |\omega_{i,j}| : 1 \leq i \leq n,\ 1 \leq j \leq r\}$. Kempf showed in [K] that $N_{V,T} \leq n\, C(n\, r!\, w^r)$ where $C(m)$ is the least common multiple of the integers $1, 2, \ldots, m$. This bound has the disadvantage of being dependent on $w$ which depends on the choice of $\psi$.

*Example 1.* — Let $T \cong (\mathbf{k}^*)^2$ and let $V$ be the 4 dimensional representation of $T$ with weights $(2,2)$, $(-1,0)$, $(0,-5)$ and $(2,-1)$. It is fairly simple, for example using the iterative method of the next section, to compute a homogeneous minimal system of generators for $\mathbf{k}[V]^T$. We find that $\mathbf{k}[V]^T = \mathbf{k}[X^{R_1}, X^{R_2}, X^A]$ where $R_1 = (5, 10, 2, 0)$, $R_2 = (1, 6, 0, 2)$ and $A = (3, 8, 1, 1)$. Therefore $N_{V,T} = \deg R_1 = 17$. Here $r = 2$, $n = 4$ and $w = 5$. Hence for this example Kempf's bound gives $N_{V,T} \leq 4\, C(4 \cdot 2! \cdot 5^2) = 4\, C(200) > 4(3 \times 10^{89}) > 10^{90}$.

## An iterative method.

Consider first the case $r = 1$. Here the isomorphism of $T$ with $\mathbf{k}^*$ is determined up to $GL(1, \mathbb{Z}) \cong \{\pm 1\}$ and thus $w$ is completely determined in this case. Fixing one of the two choices $\psi : T \longrightarrow k^*$ we may write the weights of $V$ as integers : $\omega_1, \omega_2, \ldots, \omega_n \in \mathbb{Z}$. Set $w_- := \min\{\omega_i | 1 \leq i \leq n\}$ and $w_+ := \max\{\omega_i | 1 \leq i \leq n\}$. Our assumptions that $\rho$ is stable and faithful together imply that $w_- < 0$ and $w_+ > 0$.

THEOREM 1. — *Let $V$ be a representation of $\mathbf{k}^*$ with weights $\omega_1 \geq \omega_2 \geq \ldots \geq \omega_n$ and set $B := \omega_1 - \omega_n$. Then $N_{V,\mathbf{k}^*} \leq B$.*

*Proof.* — Suppose $X^A \in \mathbf{k}[V]^T$ has degree $d$. We will construct a sequence of $d$ monomials: $h_1, h_2, \ldots, h_d$ with $\omega_n \le \mathrm{wt}(h_i) \le \omega_1 - 1$ for all $1 \le i \le d$ as follows. Choose $j$ such that $\omega_j < 0$ and define $h_1 := x_j$. If $\mathrm{wt}(h_m) \ge 0$ then we choose $j$ such that $x_j$ divides $X^A/h_m$ and $\omega_j \le 0$. Similarly if $\mathrm{wt}(h_m) < 0$ then we choose $j$ such that $x_j$ divides $X^A/h_m$ and $\omega_j > 0$. In either case we define $h_{m+1} := x_j\, h_m$. If $d > B$ then by the pigeon hole principle, two of the monomials have the same weight : $\mathrm{wt}(h_i) = \mathrm{wt}(h_j)$ where we may assume $i < j$. But then $h := h_j/h_i \in k[V]^T$ divides $X^A$ and so we see that $X^A$ is not irreducible. □

*Remark 1.* — If $\gcd(\omega_1, \omega_n) = 1$ then the invariant $x_1^{-\omega_n} x_n^{\omega_1}$ is irreducible and has degree $B = N_{V,\mathbf{k}^*}$.

*Remark 2.* — Note that $w = \max\{\omega_1, -\omega_n\}$ and therefore $N_{V,\mathbf{k}^*} \le 2w$.

THEOREM 2. — $N_{V,T} \le (2w)^{2^r - 1}$

*Proof.* — We proceed by induction on $r$. The theorem is true for the case $r = 1$ by Remark 2. For higher values of $r$ we consider the coordinate decomposition of $T$ induced by the isomorphism $\psi$, i.e., $T \cong T_1 \times \ldots \times T_r$ where $T_j \cong \mathbf{k}^*$ and the weight of $x_i$ with respect to $T_j$ is $\omega_{i,j}$. Set $T' = T_2 \times \ldots \times T_r$ so that $T = T_1 \times T'$. By induction, there exist monomial generators $h_1, \ldots, h_p$ of $\mathbf{k}[V]^{T'}$ with $\deg h_i \le (2w)^{(2^{r-1}-1)}$ for all $1 \le i \le p$. Write $h_i = X^A$ and set $\nu_i := \mathrm{wt}(h_i) \in X^*(T_1) \cong \mathbb{Z}$. Then $\nu_i = a_1\omega_{1,1} + \ldots + a_n\omega_{n,1}$. Hence $|\nu_i| \le a_1 w + \ldots + a_n w = (\deg h_i) w \le (2w)^{(2^{r-1}-1)}\, w$.

Let $V_1$ be a $p$ dimensional $\mathbf{k}$-vector space and suppose that $T_1$ acts on $V_1$ by the weights $-\nu_1, \ldots, -\nu_p$. Then we have a $T_1$-equivariant surjection $\mathbf{k}[V_1] \twoheadrightarrow \mathbf{k}[V]^{T'} = \mathbf{k}[h_1, \ldots, h_p]$. In particular we have the surjection $\mathbf{k}[V_1]^{T_1} \twoheadrightarrow (\mathbf{k}[V]^{T'})^{T_1} = \mathbf{k}[V]^T$. Hence $N_{V,T} \le N_{V,T'} \cdot N_{V_1,T_1} \le (2w)^{(2^{r-1}-1)} \cdot 2(2w)^{(2^{r-1}-1)}\, w = (2w)^{2^r-1}$. □

For the representation described in Example 1 (for which $N_{V,T} = 17$) this theorem gives the bound $N_{V,T} \le 1000$. This is a better bound than Kempf's for this example but this is only because $r$ is so small in the example. As a function of $r$ the bound given in Theorem 2 grows much much faster than Kempf's bound. This new bound is, however, distinguished by the fact that it is independent of $n = \dim V$.

## Geometric bounds.

In this section we will construct a set of distinguished monomials which is a subset of a minimal generating set for $\mathbf{k}[V]^T$. We begin with some notation and definitions. We will use $\mathbf{o}$ to denote the origin in $X^*(T)\otimes\mathbb{Q}\cong\mathbb{Q}^n$. If $Z=(z_1,\ldots,z_n)\in\mathbb{Q}^n$ define $\deg Z := z_1+\ldots+z_n$. We also define $\operatorname{supp}(Z) := \{i \mid 1\le i\le n,\, z_i\ne 0\}$ and the length of $Z$, $\ell(Z) := \#\operatorname{supp}(Z)-1$. If $\{Z_1,\ldots,Z_d\}\subset\mathbb{Q}^n$ then $\mathcal{H}(Z_1,\ldots,Z_d)$ denotes the convex hull of the points $Z_1,\ldots,Z_d$ and $\mathcal{P}(Z_1,\ldots,Z_d)$ denotes the convex set $\left\{\sum_{i=1}^{d}\alpha_i Z_i \mid \alpha_i\in[0,1] \text{ for } i=1,\ldots,d\right\}$. Notice that if $\{Z_1,\ldots,Z_d\}$ is linearly independent then $\mathcal{P}(Z_1,\ldots,Z_d)$ is a $d$-dimensional parallelepiped.

By a polytope we will mean a compact convex set having finitely many vertices. The vertices of a polytope $P$ are characterized by the property that $Y$ is a vertex of $P$ if and only if the set $P\setminus\{Y\}$ is a convex set. A $d$ dimensional polytope having $d+1$ vertices is a simplex. We will often consider the case of a $d$ dimensional polytope $P\subset\mathbb{Q}^m$ with $m\ge d$. In this case when we refer to the volume of $P$ we mean the (positive) $d$ dimensional volume of $P$ obtained by considering $P$ as a subset of the $d$ dimensional affine space, $\mathbb{A}^d$, spanned by $P$. If we wish to consider the $m$ dimensional volume of $P$ (which is zero if $d<m$) we will write $\operatorname{vol}_m(P)$. Similarly the relative interior of $P$ refers to the interior of $P$ defined by the subspace topology induced by $P\subset\mathbb{A}^d$.

The monomial generators of $\mathbf{k}[V]^T$ correspond to generators of the semi-group $S$. Gordan showed how to find the generators of $S$ (see for example [O], Proposition 1.1 (ii)). Consider the pointed (half) cone $\Gamma\subset(\mathbb{Q}^+)^n$ generated by $S$: $\Gamma := (\mathbb{Q}^+\cdot S)$ where $\mathbb{Q}^+=\{q\in\mathbb{Q}\mid q\ge 0\}$. This cone, $\Gamma$, is just the set of solutions $(z_1,\ldots,z_n)\in(\mathbb{Q}^+)^n$ of the system of equations :

$$(*)\qquad z_1\omega_1+\ldots+z_n\omega_n=\mathbf{o}.$$

If $\mathcal{L}$ is an extremal ray of $\Gamma$ then $\mathcal{L}\cap S$ is a semigroup isomorphic to $\mathbb{N}$. Let $R_{\mathcal{L}}$ denote the unique generator of this semigroup. Write $\{R_1,\ldots,R_s\}=\{R_{\mathcal{L}}\mid \mathcal{L}\text{ an extremal ray of } C\}$. The intersection $\mathcal{P}(R_1,\ldots,R_s)\cap S$ is a finite generating set for $S$. Following Stanley ([St]), we call these $R_j$ *completely fundamental generators* of $S$. These are characterized by the fact that if $mR_j=A+B$ for some $m\in\mathbb{N}$ and some $A,B\in S$ then $A=kR_j$ and $B=(m-k)R_j$ for some integer $k\le m$ ([St], p. 36). The elements

$X^{R_1}, \ldots, X^{R_s}$ are the distinguished monomial generators we referred to earlier.

Now we are ready to begin our construction of the completely fundamental generators.

LEMMA 1. — *There exists $A \in S$ with* $\mathrm{supp}(A) = \Omega$ *if and only if* $\mathbf{o}$ *lies in the relative interior of* $\mathcal{H}(\omega_i \mid i \in \Omega)$.

*Proof.* — Suppose $0 \neq A \in S$ and $\mathrm{supp}(A) = \Omega$. Then we have $\mathbf{o} = \sum_{i=1}^{n} a_i\omega_i = \sum_{i\in\Omega} a_i\omega_i = \sum_{i\in\Omega} (a_i/\deg A)\,\omega_i$. Since $a_i \geq 0$ for all $i$ and $\sum_{i\in\Omega} a_i = \deg A$ we see that $\mathbf{o} \in \mathcal{H}(\omega_i \mid i \in \Omega)$. Furthermore, since the coefficient $a_i/\deg A$ is non-zero for each $i \in \Omega$, $\mathbf{o}$ is an interior point of $\mathcal{H}(\omega_i \mid i \in \Omega)$.

Conversely, suppose that $\mathbf{o}$ lies in the relative interior of $\mathcal{H}(\omega_i \mid i \in \Omega)$. Then there exist rational numbers $p_i/q$ where $p_i, q \in \mathbb{N}$ with $1 \leq p_i \leq q$ such that $\sum_{i\in\Omega} (p_i/q)\,\omega_i = \mathbf{o}$ and $\sum_{i\in\Omega} p_i/q = 1$. Hence if we define $p_i = 0$ if $i \notin \Omega$ we have $\sum_{i=1}^{n} p_i\omega_i = \mathbf{o}$ and $A := (p_1, \ldots, p_n) \in S$ with $\mathrm{supp}(A) = \Omega$.

□

Define a partial order on $\Gamma \setminus \{\mathbf{o}\}$ by inclusion of supports, i.e., if $Y_1, Y_2 \in \Gamma \setminus \{\mathbf{o}\}$ with $\mathrm{supp}(Y_1) \subseteq \mathrm{supp}(Y_2)$ then $Y_1 \preceq Y_2$. Also given $Y \in \Gamma$, define $\sigma(Y) := \mathcal{H}(\omega_i \mid i \in \mathrm{supp}(Y))$.

PROPOSITION 1. — *Let $\mathbf{o} \neq Y \in S$ with $Y/m \notin S$ for all $m \geq 2$. Then the following are all equivalent :*

(1) *$Y$ is minimal in $\Gamma$.*

(2) *$\sigma(Y)$ is an $\ell(Y)$ dimensional simplex with $\mathbf{o}$ in its relative interior.*

(3) *$Y$ is a completely fundamental generator of $S$.*

*Proof.* — The proof that (1) $\Longrightarrow$ (2) follows from Lemma 1. Let $Y$ be an element of $S$ which is minimal with respect to the partial order. Then by Lemma 1, $\mathbf{o}$ lies in the relative interior of $\sigma(Y)$. Therefore $\sigma(Y)$ is an $\ell(Y)$ dimensional simplex with $\mathbf{o}$ in its relative interior. For if this were not true, by Carathéodory's theorem (see for example [B], Corollary 2.4 or [O], Theorem A.3), we could find a proper subset $\Omega \subsetneq \mathrm{supp}(Y)$ such that $\mathbf{o} \in \mathcal{H}(\omega_i \mid i \in \Omega)$. But this would contradict the minimality of $Y$.

In particular, this implies that any proper subset of $\{\omega_i \mid i \in \operatorname{supp}(Y)\}$ is linearly independent.

Now to see that (2) $\Longrightarrow$ (3), suppose (2) holds and that there exists $n \in \mathbb{N}$ and $A, B \in S$ with $nY = A + B$. Since $\sigma(Y)$ is a simplex, $\mathbf{o}$ can be expressed *uniquely* as a convex linear combination of $\{\, \omega_i \mid i \in \operatorname{supp}(Y) \,\}$: $\sum_{i \in \operatorname{supp}(Y)} \alpha_i \omega_i = \mathbf{o}$ where $\alpha_i \in [0,1]$ and $\sum_{i \in \operatorname{supp}(Y)} \alpha_i = 1$. Now $\sum_i a_i \omega_i = \mathbf{o}$ and $a_i = 0$ if $i \notin \operatorname{supp}(Y)$. Hence, by the uniqueness, we have $a_i / \deg(A) = \alpha_i = y_i / \deg(Y)$. Therefore $A = (\deg A / \deg Y) Y$ from which it follows that $Y$ is completely fundamental.

Finally, we prove that (3) $\Longrightarrow$ (1). Suppose $Y$ is a completely fundamental generator of $S$ and $Z \in \Gamma$ with $Z \preceq Y$. Clearly, clearing denominators, we may suppose that $Z \in S$. Since $Z \preceq Y$, for $m \in \mathbb{N}$ sufficiently large we have $my_i \geq z_i$ for all $1 \leq i \leq n$. Hence $mY$ decomposes within $S$ as $mY = Z + (mY - Z)$. Since $Y$ is completely fundamental, this implies that $Z = kY$ for some $k \leq m$. Hence $\operatorname{supp}(Y) = \operatorname{supp}(Z)$ and $Y \preceq Z$. □

Thus to each minimal element $Y$ of $\Gamma$ we have an associated $\ell(Y)$ dimensional simplex, $\sigma(Y) := \mathcal{H}(\omega_i \mid i \in \operatorname{supp}(Y))$. Given $\operatorname{supp}(Y)$ we can recover $Y$ since every point in a simplex can be written *uniquely* as a convex linear combination of the vertices of the simplex. Therefore the map $Y \mapsto \operatorname{supp}(Y)$ is one-to-one. Moreover, if $Y \in \Gamma$ is minimal then $\{\omega_i \mid i \in \operatorname{supp}(Y)\}$ is a minimal linearly dependent subset of $\{\omega_1, \ldots, \omega_n\}$.

Note that the map $Y \mapsto \sigma(Y)$ is not necessarily one-to-one. More precisely, $\operatorname{supp}(Y) \mapsto \sigma(Y)$ is one-to-one if and only if the weights of $V$ are distinct. If $V_1$ and $V_2$ are two representations of $T$ having the same weights (except for multiplicities) then clearly, $N_{V_1,T} = N_{V_2,T}$ and thus it would suffice to consider only representations whose weights were distinct.

THEOREM 3. — *If the $R_j$ are ordered so that* $\deg R_1 \geq \deg R_2 \geq \ldots \geq \deg R_s$ *then* $N_{V,T} \leq \sum_{j=1}^{n-r} \deg R_j \leq (n-r) \deg R_1$.

*Proof.* — Suppose $\mathbf{o} \neq A \in S$. By Carathéodory's theorem we may write

$$A = \alpha_1 R_{j_1} + \ldots + \alpha_{n-r} R_{j_{n-r}}$$

where each $\alpha_j \geq 0$. If $\alpha_j > 1$ then we may decompose $A$ within $S$ as $A = (A - R_{j_i}) + R_{j_i}$. Hence if $A$ is a generator of $S$ then each $\alpha_i \leq 1$. But then $\deg A = \alpha_1 \deg R_{j_1} + \ldots + \alpha_{n-r} \deg R_{j_{n-r}} \leq \deg R_{j_1} + \ldots + \deg R_{j_{n-r}} \leq \deg R_1 + \ldots + \deg R_{n-r}$. □

*Remark 3.* — Applying these two bounds to the representation of Example 1 we get $N_{V,T} \leq 17 + 9 = 26$ and $N_{V,T} \leq 2 \cdot 17 = 34$.

A theorem of Ewald and Wessels ([EW], Theorem 2) allows us to improve the preceding theorem. Specifically, (using the notation of Theorem 3) they show that if $\alpha_1 + \ldots + \alpha_{n-r} > n - r - 1 \geq 1$ then $A$ is decomposable within $S$. Thus we have the following corollary.

COROLLARY 1. — *If $n - r \geq 2$ then $N_{V,T} \leq (n - r - 1) \deg R_1$.*

*Remark 4.* — If we apply this result to Example 1 we find that $N_{V,T} \leq (4 - 2 - 1) \cdot 17 = 17$.

The following proposition shows how the completely fundamental solutions are distinguished among the elements of a monomial minimal generating set.

PROPOSITION 2 (Stanley [St], Theorem 3.7). — *Suppose $\{X^{A_1}, \ldots, X^{A_q}\}$ is any minimal set of monomials such that $\mathbf{k}[V]^T$ is integral over $\mathbf{k}[X^{A_1}, \ldots, X^{A_q}]$. Then $q = s$ and there exists a permutation $\pi$ of $\{1, \ldots, s\}$ such that $\operatorname{supp}(R_j) = \operatorname{supp}(A_{\pi(j)})$. In fact, there exist positive integers $m_1, \ldots, m_s$ such that $A_{\pi(j)} = m_j \cdot R_j$.*

*Remark 5.* — Kempf ([K]) also constructed the elements $R_1, \ldots, R_s$. His method of construction is somewhat less direct than that which we will give in the next section and consequently the bound he gave for $\deg R_j$ is larger than the one we will give.

## Computing the completely fundamental generators.

In this section we will give an algorithm for finding the completely fundamental generators. Suppose $\Omega$ is a minimal linearly dependent subset of $\{\omega_1, \ldots, \omega_n\}$ with $\mathbf{o} \in \mathcal{H}(\omega \in \Omega)$. Then $\Omega = \{\, \omega_i \mid i \in \operatorname{supp}(R_j) \,\}$ for some $j$. We want to compute $R_j$. Set $d := \ell(R_j) \leq r$. Then without loss of generality we may suppose that $\operatorname{supp}(R_j) = \{1, 2, \ldots, d+1\}$. Consider the system of $r$ linear equations in $d$ unknowns :

$$(\dagger) \qquad y_1\omega_1 + \ldots + y_d\omega_d = -\omega_{d+1}.$$

These $r$ equations impose only $d$ conditions and so in order to solve this system we take the $r \times d$ matrix of rank $d$, $M := (\omega_1\ \omega_2\ \ldots\ \omega_d)$ and choose a $d \times d$ non-singular submatrix $M'$. If $M'$ consists of the rows $j_1, \ldots, j_d$ of $M$ then the $i$th column of $M'$ is $\omega'_i := (\omega_{i,j_1}, \ldots, \omega_{i,j_d})$ for $1 \le i \le d$. Also define $\omega'_{d+1} := (\omega_{d+1,j_1}, \ldots, \omega_{d+1,j_d})$. Then solving (†) is equivalent to solving

$$(\dagger\dagger) \qquad y_1\omega'_1 + \ldots + y_d\omega'_d = -\omega'_{d+1}.$$

But we may solve (††) by Cramer's rule :

$$y_1 = \frac{|\omega'_{d+1}, \omega'_2, \ldots, \omega'_d|}{|\omega'_1, \omega'_2, \ldots, \omega'_d|}, \quad \ldots, \quad y_d = \frac{|\omega'_1, \ldots, \omega'_{d-1}, \omega'_{d+1}|}{|\omega'_1, \omega'_2, \ldots, \omega'_d|}.$$

Then if we define

$$\begin{aligned} q_i &= y_i|\omega'_1, \ldots, \omega'_d| \\ &= |\omega'_1, \ldots, \omega'_{i-1}, \omega'_{d+1}, \omega'_{i+1}, \ldots, \omega'_d| \text{ for } 1 \le i \le d \\ \text{and } q_{d+1} &= -|\omega'_1, \omega'_2, \ldots, \omega'_d| \end{aligned}$$

we have

$$q_1\omega_1 + \ldots + q_{d+1}\omega_{d+1} = \mathbf{o}$$

where each $q_i \in \mathbb{Z}$. This solution is unique up to scalar multiplication by an element of $\mathbb{Q}$. Since $\mathbf{o} \in \mathcal{H}(\omega_1, \ldots, \omega_{d+1})$ all the $q_i$ must have the same sign and, multiplying by $-1$ if necessary, we get each $q_i \in \mathbb{N}$. If we define $q_i = 0$ for all $i \notin \{1, \ldots, d+1\}(= \mathrm{supp}(R_j))$ and $Q_j := (q_1, \ldots, q_n)$ then $R_j = Q_j/m$ where $m$ is the greatest common divisor of the integers $q_1, \ldots, q_{d+1}$.

Thus to construct $\{R_1, \ldots, R_s\}$ we consider each minimal linearly dependent subset, $\Omega$, of the weights $\{\omega_1, \ldots, \omega_n\}$. For each such $\Omega$ we compute the determinants $q_1, \ldots, q_{d+1}$. If any two of these determinants have opposite signs then $\Omega$ does not correspond to any invariant. If however, all the $q_i$ have the same sign then $(q_1/m, \ldots, q_n/m)$ is one of the completely fundamental generators.

## Degrees as volumes.

In this section we will continue to study the fixed $R_j$ of the previous section. We will obtain bounds on $\deg R_j$ and thus on $N_{V,T}$ in terms of volumes of certain polytopes.

THEOREM 4. — *Let* $\sigma_j$ *be the simplex* $\sigma_j = \mathcal{H}(\omega_i \mid i \in \operatorname{supp}(R_j))$. *Then* $\deg R_j \leq d! \operatorname{vol}(\sigma_j)$.

*Proof.* — Let $\Delta$ denote the perpendicular (coordinate) projection :

$$\Delta : X^*(T) \otimes \mathbb{Q} \cong \mathbb{Q}^r \to \mathbb{Q}^d \text{ given by } \Delta(u_1, \ldots, u_r) = (u_{j_1}, \ldots, u_{j_d}).$$

Then $\Delta(\omega_i) = \omega_i'$. Define $\sigma_j(i) := \mathcal{H}(\mathbf{o}, \omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_{d+1})$, $\sigma_j' := \Delta(\sigma_j)$ and $\sigma_j'(i) := \Delta(\sigma_j(i))$. Notice that $q_i$ is the $d$ dimensional volume of the parallelepiped $\mathcal{P}(\omega_1', \ldots, \omega_{i-1}', \omega_{i+1}', \ldots, \omega_{d+1}')$. Hence $q_i = d! \operatorname{vol}(\sigma_j'(i))$.

Now $\sigma_j' = \sigma_j'(1) \cup \ldots \cup \sigma'(d+1)$ is a triangulation of $\sigma_j'$ by $d$-simplices since $\mathbf{o}$ lies in the relative interior of $\sigma_j'$. Thus $\deg Q_j = q_1 + \ldots + q_{d+1} = d! \operatorname{vol}(\sigma_j')$. Therefore $\deg R_j \leq \deg Q_j = d! \operatorname{vol}(\sigma_j') \leq d! \operatorname{vol}(\sigma_j)$ where the last inequality follows for example from [Ga], (30) p. 253. □

Let $\mathcal{W} := \mathcal{H}(\omega_1, \ldots, \omega_n)$, the convex hull of the weights in $X^*(T) \otimes \mathbb{Q} \cong \mathbb{Q}^r$.

THEOREM 5. — $\deg R_j \leq r! \operatorname{vol}(\mathcal{W})$.

*Proof.* — It is not true in general that $d! \operatorname{vol}(\sigma_j') \leq r! \operatorname{vol}(\mathcal{W})$ when $d < r$. Hence to prove this theorem we consider a slightly different construction of $R_j$ (when $d < r$). Recall that we have assumed that $\operatorname{supp}(R_j) = \{1, \ldots, d+1\}$. Without loss of generality we may assume that $\Sigma := \mathcal{H}(\omega_1, \ldots, \omega_{d+1}, \ldots, \omega_{r+1})$ is an $r$ dimensional simplex. To construct $R_j$ we solve the system of $r$ linearly independent equations in $r$ unknowns :

$$y_2\omega_2 + \ldots + y_{r+1}\omega_{r+1} = -\omega_1.$$

As before we apply Cramer's rule to solve this system and so find $(a_1, \ldots, a_{r+1}) \in \mathbb{N}^{r+1}$ with

$$a_1\omega_1 + \ldots + a_{r+1}\omega_{r+1} = \mathbf{o}$$
$$\text{and } a_i = r! \operatorname{vol}_r(\mathcal{H}(\mathbf{o}, \omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_{r+1})).$$

Again we set $a_{r+2} = \ldots = a_n = 0$ and $A = (a_1, \ldots, a_n)$. Notice that $a_{d+2} = \ldots = a_{r+1} = 0$ and that $A$ is a multiple of $R_j$. Hence $\deg R_j \leq \deg A = a_1 + \ldots + a_n = r! \operatorname{vol}(\Sigma) \leq r! \operatorname{vol}(\mathcal{W})$. □

COROLLARY 2. — *If* $n - r \geq 2$ *then* $N_{V,T} \leq (n - r - 1)\, r! \operatorname{vol}(\mathcal{W})$. *If* $1 \leq n - r \leq 2$ *then* $N_{V,T} \leq r! \operatorname{vol}(\mathcal{W})$.

*Remark 6.* — This bound is invariant under the action of $\operatorname{Aut}(T) \cong GL(r, \mathbb{Z})$ and thus is independent of the choice of $\psi$.

*Remark 7.* — For the representation of Example 1, $\mathcal{W}$ is a quadrilateral of area 23/2. Hence we get the bound $N_{V,T} \leq 2! \cdot (23/2) = 23$.

It seems likely that the factor $n - r - 1$ is unnecessary in the first statement of Corollary 2. I know of no examples of representations where $N_{V,T} > r!\,\mathrm{vol}(\mathcal{W})$. Conversely for all values of $n$ and $r$ there exist faithful stable $n$ dimensional representations, $V$, of $T \cong (\mathbf{k}^*)^r$ such that $N_{V,T} = r!\,\mathrm{vol}(\mathcal{W})$ – for example this often occurs when $\mathcal{W}$ is itself a simplex.

CONJECTURE. — *There is a (small) constant $c \in \mathbb{R}$ such that $N_{V,T} \leq c\,r!\,\mathrm{vol}\,(\mathcal{W})$.*

## Bounds in terms of $w$.

Next we bound $\deg R_j$ in terms of $w := \max\{|\omega_{i,m}| : 1 \leq i \leq n, 1 \leq m \leq r\}$.

THEOREM 6. — $\deg R_j \leq \lfloor w^d\,(d+1)^{(d+1)/2} \rfloor$.

*Proof.* — We have $\deg R_j \leq d!\,\mathrm{vol}(\sigma'_j)$ where $\sigma'_j = \mathcal{H}(\omega'_1, \ldots, \omega'_{d+1}) \subset [-w, w]^d \subset \mathbb{Q}^d$. Define $\widetilde{\sigma'_j} := \mathcal{H}(\omega'_1/2w, \ldots, \omega'_{d+1}/2w) + (1/2, \ldots, 1/2)$. Then $\widetilde{\sigma'_j}$ is a $d$ dimensional simplex contained in $[0,1]^d$ with $\mathrm{vol}(\sigma'_j) = (2w)^d \mathrm{vol}(\widetilde{\sigma'_j})$.

Thus we now seek to bound the value $B := \max\{\mathrm{vol}(\tau) \mid \tau \subset [0,1]^d$ is a $d$ dimensional simplex$\}$. By linear programming it is clear that the value $B$ is attained by a simplex $\mu$ all of whose vertices are also vertices of the cube $[0,1]^d$. Without loss of generality we may assume that $(0, \ldots, 0)$ is one of the vertices of $\mu$. Let $\nu_1, \ldots, \nu_d$ be the other vertices of $\mu$. Then $\mathrm{vol}(\mu) = |\det(M)|/d!$ where $M = (\nu_1\ \ldots\ \nu_d)$ is a $d \times d$ matrix all of whose entries are either 0 or 1. But then by a theorem of Ryser (see [R], Equation (11)) we have

$$|\det(M)| \leq 2\left(\frac{\sqrt{d+1}}{2}\right)^{d+1}.$$

Thus we get the bound $\deg R_j \leq w^d(d+1)^{(d+1)/2} \leq w^r(r+1)^{(r+1)/2}$. □

COROLLARY 3. — *If $n-r \geq 2$ then $N_{V,T} \leq (n-r-1)\lfloor w^r(r+1)^{(r+1)/2} \rfloor$. If $1 \leq n-r \leq 2$ then $N_{V,T} \leq \lfloor w^r(r+1)^{(r+1)/2} \rfloor$.*

*Remark 8.* — In Example 1 we had $n = 4$, $r = 2$ and $w = 5$. Thus Corollary 3 gives $N_{V,T} \leq \lfloor 5^2 \cdot (2+1)^{(2+1)/2} \rfloor = \lfloor 25 \cdot 3^{3/2} \rfloor = 129$.

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*Note added in proof* : The construction of the completely fundamental generators given here was also pointed out by B. Sturmfels in "Gröbner bases of toric varieties", Tôhoku Math. J., second series, vol. 43, no. 2 (1991).

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