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S. FRANKEL

MICHAEL KATZ

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THE MORSE LANDSCAPE OF A RIEMANNIAN DISK

by S. FRANKEL & M. KATZ

Consider Riemannian metrics g on the two-dimensional disk D such that the length of the boundary S^1 is less than 1, and the Riemannian distance from every point of D to the boundary is less than 1.

QUESTION 1. — *Does there exist a constant $C > 0$ such that for every metric g as above, the boundary circle can be homotoped to a point through a family of curves each of which has length less than C?*

The answer is negative.

THEOREM 2. — *One can construct a sequence of metrics g_n on D and a function $f(n)$ tending to infinity with n, such that every homotopy of S^1 to a point in (D, g_n) contains an intermediate curve of length bigger than $f(n)$.*

In terms of the length functional on the loop space of D, one can say that the boundary loop is separated from the constant loop by a mountain range of height at least $f(n)$. Hence the term "Morse landscape" used in [1].

The above Question 1 was asked by M. Gromov in his talk at Luminy, July 1992. It is related to questions in geometric group theory (see [1]). A different counterexample was constructed independently by Jean-Marc Schlenker.

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Note that a counterexample cannot be obtained by attaching spikes to the standard disk, because the curve can pass over them one at a time. In our example the basic construction is not a spike but a "wall" (see below).

The proof uses a topological property of trees in the disk. If the tree is sufficiently complicated, then any homotopy of the boundary to a point will contain an intermediate curve that crosses several edges. If we now build a wall over the tree, the curve will have to climb over it several times and become very long. To prevent the curves from traveling along an entire edge, we make the edges of the tree long. This can be done either by redefining the Riemannian metric in coordinates adapted to the tree by making it large in the direction of the edges, or simply imbedding the edges in the standard disk by means of a sufficiently wiggly zig-zag contained in a thin neighborhood of the edge. It is conceptually simpler to reformulate this construction in terms of hyperbolic geometry as follows.

PROPOSITION $2\frac{1}{2}$. — *Let $T \subset D$ be a tree. There exists a metric h on the disk with the following properties :*

(i) *h is hyperbolic on the complement of T.*

(ii) *the circumference equals 1 and the Riemannian distance from every point of D to the boundary is ≤ 1.*

(iii) *the distance between any two non-adjacent edges of T is at least 1.*

Proof. — Let $U = U_c$ be the unit ball in the hyperbolic plane of curvature $-c^2$. Let $q > 0$ and consider a regular $2q$-gon with geodesic sides inscribed in the unit ball. As $c \to \infty$, the length of the side tends to 2. Choose c suffficiently large so the sides have length 1.

Every tree T with q edges can be thought of as a circle subdivided into $2q$ arcs with a suitable identification among pairs of arcs (compare with the projection of the boundary of an ϵ-neighborhood of $T \subset \mathbf{R}^2$ to T). Here two neighboring arcs are identified if and only if their common point projects to a terminal vertex of T.

We partition the circle ∂U by the vertices of the regular $2q$-gon. Let $f : \partial U \to T$ be the (locally isometric) projection. Then $T \cup_f U$ is homeomorphic to the sphere S^2. Let $\epsilon > 0$ be chosen so that the concentric ball $B_\epsilon \subset U$ is of circumference 1. We now set $(D, h) = (T \cup_f U) \setminus B_\epsilon$. Then the metric is hyperbolic on $D \setminus T$, and has infinite positive curvature along the edges of T. Property (iii) is satisfied by construction.

Proof of Theorem 2. — We perturb the homotopy so as to satisfy the hypothesis of Proposition 5 below. The perturbation does not substantially increase the length of the curves, and does not increase the number of the (closed!) edges crossed by each curve. Now Proposition 5 provides a curve γ meeting a large number of edges. Since the valence at each vertex is bounded (by 3), we can choose a long sequence of non-neighboring edges crossed consecutively by γ. By Proposition $2\frac{1}{2}$, the length of the sequence is a lower bound for the length of γ, proving the theorem.

We give an inductive definition of a binary tree.

DEFINITION 3. — *The binary tree T_1 of height 1 is a point, which is its own root. The binary tree T_{n+1} of height $n+1$ is constructed by choosing its root and joining it by two edges to the roots of two binary trees of height n.*

LEMMA 4. — *The tree T_n has $2^n - 1$ vertices. Removing an edge separates T_n into two connected components, one of which contains the root, and the other is another binary tree $T_{n'}$ of height $n' < n$.*

The proof is obvious.

PROPOSITION 5. — *Imbed the binary tree T_n in the disk D. Consider a homotopy of the boundary S^1 to a point such that each intermediate curve passes through at most one vertex of T_n. Then some intermediate curve meets at least $O\left(\frac{n}{\ln n}\right)$ edges of T_n.*

Proof. — Given a curve $\gamma \subset D$ not passing through any of the vertices of T_n, we may consider the number of vertices of T_n with respect to which γ has *even* winding number. Such vertices will be called even (with respect to γ), and the others odd.

Every integer between 1 and $2^n - 1$ must occur as the number of even vertices of some intermediate curve. Indeed, at the beginning of the homotopy this number is 0, at the end it is $2^n - 1$, and the vertices are passed one at a time, hence change their parity one at a time.

The idea of the proof is to find a number "poorly representable" as a sum (with $\pm$ signs) of numbers of the form $2^n - 1$. It is technically easier to show that such a number exists than to exhibit the number.

LEMMA 6. — *Let $T_n \subset D$ be a binary tree and let $\gamma \subset D$ be a* *curve not passing through any vertex of T_n. Suppose γ meets T_n in k edges $e_1, \ldots, e_k$. Let n_i be the height of the binary tree T_{n_i} separated by removing e_i (see Lemma 4), and a_i the winding number of γ with respect to the root of T_{n_i}. Then the number of even vertices of T_n can be written in the form*

$$p_0 + \sum_{i \in S} (-1)^{a_i} (2^{n_i} - 1).$$

Here p_0 is 0 if the root of T_n is odd, and $2^n - 1$ if it is even, while $S \subset \{1, \ldots, k\}$ is a subset.

Proof. — The proof is in the spirit of de Morgan's law in set theory. The binary trees T_{n_i} are partially ordered by inclusion. The unique maximal element is T_n itself. Trees nearest for the partial order will be called neighbors. Clearly each T_{n_i} has a unique bigger neighbor.

First we suppose that the curve meets an edge of T_n no more than once, and that the intersection is transverse. Then the roots of a pair of neighbors always have opposite parity, and $S = \{1, \ldots, k\}$.

Assume that the root of T_n is odd. Then the set of even vertices is the disjoint union of the smaller neighbors of T_n, minus the disjoint union of the smaller neighbors of these neighbors, plus the third neighbors, etc. This proves the formula in the case of odd root. Otherwise we have to start with T_n itself and so get the extra summand $p_0 = 2^n - 1$.

Without the assumption of unique transversal intersection, the parity of the roots of two neighbors may be the same. We delete from the partial order those trees whose bigger neighbor has a root of the same parity. The remaining trees define the set $S \subset \{1, \ldots, k\}$ of the lemma.

Returning to the proof of Proposition 5, we note that each cut e_i gives rise to $2n$ possibilities (the height of the separated tree T_{n_i} and the parity of the root of T_{n_i}). Thus there are at most $(2n)^{k+1}$ possible numbers of even vertices that can be represented by k cuts. Since we need to represent $2^n - 1$ of them, we must have the inequality $(2n)^{k+1} > 2^n - 1$, which implies that of Proposition 5.

With an additional assumption on the "monotonicity" of the homotopy, we can provide a linear lower bound for the number of crossings. If the curve sweeps out the disk without coming back upon itself, we can think of the curves as level sets of a real valued function.

PROPOSTION 7. — *Every continuous real-valued function on a binary tree of height n has a level set with at least $n/2$ points.*

Proof. — We argue by induction on the height n of the tree. For $n = 1$ and 2 the statement is immediate from the fact that the complement of a point has nontrivial fundamental group.

Define numbers $a, b \in \mathbf{R}$ by $a = \inf f(T_n)$, $b = \sup f(T_n)$, where $f : T_n \to \mathbf{R}$ is the function. Choose preimages $\hat{a}, \hat{b} \in T_n$. Consider the imbedded path $\alpha \subset T_n$ joining $\hat{a}$ with $\hat{b}$. The complement $T_n \setminus \alpha$ contains a binary tree B of height at least $n-2$. Indeed, if α does not pass through the root of T_n then α stays away from a branch of height $n-1$. If α passes through the root, then at the next vertex it has to go either right or left, and the remaining edge leads to a tree of height $n-2$. By the inductive hypothesis applied to B, there is a point $x \in \mathbf{R}$ with at least $\dfrac{n-2}{2} = \dfrac{n}{2} - 1$ preimages in B. Since $a \leq x \leq b$, the point x has an additional preimage on α, proving the proposition.

QUESTION 8. — *Consider a binary tree T_n of height n imbedded in the 2-disk D, and a homotopy shrinking the boundary to a point. Does there exist an intermediate curve meeting T_n in at least $n-1$ points?*

QUESTION 9. — *What is the answer to the original question of Gromov with the additional bound on the area of the Riemannian metric on the disk?*

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S. FRANKEL,
Columbia University and Université de Nantes
&
M. KATZ,
URA 750 (CNRS)
Département de Mathématiques
Université de Nancy I
B. P. 239
54506 Vandoeuvre (France).