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## ON THE REPRESENTATION OF CERTAIN FUNCTIONALS BY MEASURES ON THE CHOQUET BOUNDARY

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### 1. Introduction.

M. HERVÉ [6] has recently published a simple proof of Choquet's theorem on the representation of the points of a compact convex metrizable subset of a locally convex real linear topological space as barycentres of measures carried by the extreme points of the set. F. F. BONSALL [5] has shown that, by a use of the Hahn-Banach theorem, the discussion can be made still more simple and that a restatement of the problem then allows the convexity condition to be dropped. The present paper shows that further pursuit of these ideas provides new information about the *Choquet boundary*, as defined by BISHOP and DE LEEUW [4]. It is then possible to give a simple direct proof of a result of these authors: that, in the presence of a separability condition (stated in § 4), the Choquet boundary is a  $G_\delta$  set and every probability Radon measure admits *balayage* onto it. These methods are also shown, in § 5, to lead to a proof of one of Bauer's main theorems in his theory [3] of an abstract Dirichlet problem. The effect of an additional equicontinuity condition is also considered in § 5.

We consider here only real-valued functions, remarking that the passage to the complex case is known [4] to be a straightforward matter.

I am indebted to Professor F. F. BONSALL for showing me his work before publication.

## 2. Construction of functionals; the Choquet theorem.

We consider a compact Hausdorff space  $X$ , the set  $C$  of all real continuous functions on  $X$ , and a linear manifold  $L$  of  $C$  that contains the constant functions. We denote by  $M$ ,  $M^+$ , and  $P$  respectively the spaces of Radon measures, non-negative Radon measures, and probability Radon measures on  $X$ , and by  $R$  the set of real numbers. For each  $x \in X$  we define the set

$$M_x \equiv M_x(L) = \{\mu \in M^+ | \mu(g) = g(x) \text{ for all } g \in L\}.$$

The unit atomic measure at  $x$ , denoted by  $\varepsilon_x$ , belongs to  $M_x$ . Also since  $1 \in L$  we have  $\mu(1) = 1$  for all  $\mu \in M_x$ , so that  $M_x \subseteq P$ .

For each  $f \in C$  and each  $\sigma \in P$  we define

$$f^*(\sigma) = \inf \{\sigma(g) | g \in L, g \geq f\}.$$

Each  $f \in C$  is bounded,  $L$  contains the constants, and so  $f^*(\sigma)$  is well-defined and finite. Evidently

$$(1) \quad f^*(\sigma) \leq \max_{y \in X} f(y) \leq \|f\|,$$

and

$$(2) \quad g^*(\sigma) = \sigma(g) \text{ whenever } g \in L.$$

We adopt the convenient abuse of notation of writing  $f^*(x)$  for  $f^*(\varepsilon_x)$ , so that  $x \rightarrow f^*(x)$  is precisely the upper semi-continuous real function on  $X$  defined by

$$f^*(x) = \inf \{g(x) | g \in L, g \geq f\}.$$

**LEMMA.** — For each  $\sigma \in P$  the map  $f \rightarrow f^*(\sigma)$  of  $C$  into  $R$  is a sublinear functional on  $C$ .

Let  $p(f) = f^*(\sigma)$  for all  $f \in C$ . Suppose  $f_1, f_2 \in C$ , let  $\varepsilon > 0$ , and choose  $g_1, g_2 \in L$  such that

$$g_r \geq f_r, \quad \sigma(g_r) < p(f_r) + \varepsilon \quad (r = 1, 2).$$

Then  $f_1 + f_2 \leq g_1 + g_2 \in L$  and so

$$p(f_1 + f_2) \leq \sigma(g_1 + g_2) = \sigma(g_1) + \sigma(g_2) < p(f_1) + p(f_2) + 2\varepsilon.$$

But  $\varepsilon > 0$  was arbitrary and so

$$p(f_1 + f_2) \leq p(f_1) + p(f_2)$$

for all  $f_1, f_2 \in C$ . One proves similarly that

$$p(\lambda f) = \lambda p(f)$$

for all real  $\lambda \geq 0$  and  $f \in C$ .

Now choose  $h \in C$  and let  $W_h$  be the set of all points  $x \in X$  that satisfy the condition that for at least one  $\nu \in M_x(L)$  we have

$$\nu(h) > h(x).$$

**THEOREM 1.** — *For each  $h \in C$ .*

$$(3) \quad W_h = \{x \in X | h^*(x) > h(x)\};$$

*the set  $W_h$  is consequently an  $F_\sigma$  set. Moreover, given  $\tau \in P$ , we can find  $\mu \in P$  (depending on  $h$  and  $\tau$ ) such that*

$$\begin{aligned} (i) \quad & \mu(g) = \tau(g) \quad \text{for all } g \in L; \\ (ii) \quad & \mu(W_h) = 0. \end{aligned}$$

Let  $x \in W_h$  and let  $\nu \in M_x$  be such that  $\nu(h) > h(x)$ . Then if  $g \in L$ ,  $g \geq h$  we have

$$g(x) = \nu(g) \geq \nu(h) > h(x),$$

so that  $g(x) - h(x) \geq \nu(h) - h(x) > 0$  and hence  $h^*(x) > h(x)$ . This proves that

$$(4) \quad W_h \subseteq \{x \in X | h^*(x) > h(x)\}.$$

Now take a measure  $\sigma \in P$  and write  $p(f) = f^*(\sigma)$ , for all  $f \in C$ . Then by the lemma and the Hahn-Banach theorem there exists a linear functional  $\nu \equiv \nu_\sigma$  on  $C$  that satisfies

$$\nu(f) \leq p(f) \quad \text{for all } f \in C,$$

and

$$\nu(h) = p(h).$$

By (1) the functional  $\nu$  is continuous. For  $g \in L$  we have, by (2),  $\nu(g) \leq p(g) = \sigma(g)$  and also  $-g \in L$ , so that  $-\nu(g) = \nu(-g) \leq \sigma(-g) = -\sigma(g)$ , whence in fact

$$(5) \quad \nu(g) = \sigma(g) \quad \text{for all } g \in L.$$

Next, (1) implies that for  $f \in C$  with  $f \leq 0$  we have  $v(f) \leq 0$  and hence  $v(-f) \geq 0$ , so that  $v \geq 0$ , and thus  $v \in M^+$ .

Now take  $x \in X$  with  $h^*(x) > h(x)$  and let  $\sigma = \varepsilon_x$  in the above construction, so that now  $v \in M_x$ , and  $p(f) = f^*(x)$  for all  $f \in C$ . Then  $v(h) = p(h) = h^*(x) > h(x)$  and therefore  $x \in W_h$ . So we have

$$\{x \in X | h^*(x) > h(x)\} \subseteq W_h,$$

which with (4) establishes (3).

Next  $h^*$ , and hence  $(h^* - h)$ , is upper semi-continuous and hence

$$F_n = \left\{x \in X | h^*(x) - h(x) \geq \frac{1}{n}\right\}$$

is closed. Therefore  $W_h = \bigcup_{n=1}^{\infty} F_n$  is an  $F_\sigma$  set.

For the last part let  $\mu = v_\tau$  as above. Then  $\mu \geq 0$ , and (5) provides the proof of relation (i) of theorem 1 and in particular the fact that  $\mu(1) = \tau(1)$ , so that  $\mu \in P$ .

To prove that  $\mu(W_h) = 0$  it is enough to show that  $\mu(F_n) = 0$  for all  $n \geq 1$ . Suppose there is an exceptional  $n$  with

$$\mu(F_n) = \delta > 0.$$

Then if  $g \geq h$ ,  $g \in L$  we have  $g \geq h^*$  and consequently

$$(6) \quad \tau(g) - \mu(h) = \mu(g) - \mu(h) \geq \int_{F_n} (g - h) d\mu \geq \frac{\delta}{n}.$$

On the other hand

$$\mu(h) = h^*(\tau) = \inf \{\tau(g) | g \geq h, g \in L\},$$

which contradicts (6) and completes the proof that  $\mu(W_h) = 0$ .

**COROLLARY (CHOQUET).** — *Let  $X$  be a compact convex metrizable set in a locally convex real linear topological space. Then the set  $E$  of extreme points of  $X$  is a  $G_\delta$  set. Moreover, for each  $a \in X$  there exists a probability Radon measure  $\mu$  on  $X$  such that*

$$(j) \quad \mu(g) = g(a), \quad \text{for all } g \in L,$$

where  $L$  is now the set of restrictions to  $X$  of real continuous affine functions, and,

$$(jj) \quad \mu(\int E) = 0.$$

For the proof we take  $h$  to be the strictly convex real continuous function on  $X$  constructed by HERVÉ [6]. Then it is clear that  $W_h \cap E = \emptyset$  (see § 3). But HERVÉ shows that if  $x \in E$  then  $h^*(x) = h(x)$  so that, by (3), we have  $W_h = \int E$  for this  $h$ . On taking  $\tau = \varepsilon_a$  in theorem 1 we obtain therefore a  $\mu \in P$  satisfying (j) and (jj). In § 4 we present a generalization of this argument.

### 3. Characterizations of the Choquet boundary.

Now let  $A(L)$  denote the smallest uniformly closed subalgebra of  $C$  that contains  $L$ . Evidently

$$M_x(L) \supseteq M_x(A(L)) \quad \text{for all } x \in X.$$

The *Choquet boundary* of the space  $X$  for the class of functions  $L$  is by definition the set

$$\partial_L X = \{x \in X | M_x(L) = M_x(A(L))\}.$$

The Weierstrass-Stone theorem, together with a simple measure-theoretic argument like that used to prove proposition 1 below, implies that this definition is equivalent to the slightly different one given by BISHOP and DE LEEUW [4]. If  $L$  separates the points of  $X$  then  $A(L) = C$  and so, in this case,

$$\partial_L X = \{x \in X | M_x(L) = (\varepsilon_x)\}.$$

**PROPOSITION 1.** — *For each linear subspace  $L$  of  $C$  that contains the constants, we have*

$$(7) \quad \partial_L X = \bigcap_{h \in A(L)} \int W_h = \bigcap_{g \in L} \int W_{|g|}.$$

We emphasize here that  $W_f$ , for  $f \in C$ , depends on  $f$  and on  $L$ .

Suppose  $h \in A(L)$ ,  $x \in W_h$ . Then there is a  $\nu \in M_x(L)$  with  $\nu(h) > h(x)$ , so that  $\nu \notin M_x(A(L))$  and hence  $x \notin \partial_L X$ . This shows that

$$(8) \quad W_h \cap \partial_L X = \emptyset \quad \text{for all } h \in A(L).$$

Conversely suppose that  $a \in \partial_L X$ , let  $\nu \in M_a(L) \setminus M_a(A(L))$ , and let  $\text{supp } \nu$  denote the support of  $\nu$ . Then we can find  $b \in \text{supp } \nu$ , with  $b \neq a$ , together with a function  $g_1 \in L$  such that  $g_1(b) \neq g_1(a)$ . For otherwise we should have

$$g(x) = g(a) \quad \text{for all } x \in \text{supp } \nu, \quad g \in L,$$

which would imply

$$h(x) = h(a) \quad \text{for all } x \in \text{supp } \nu, \quad h \in A(L),$$

and hence  $\nu \in M_a(A(L))$ , contrary to hypothesis.

Now define

$$g(x) = g_1(x) - g_1(a) \quad (x \in X),$$

so that  $g \in L$ . Then the continuous non-negative function  $h = |g|$  is strictly positive at the point  $b \in \text{supp } \nu$  and so

$$\nu(h) > 0 = h(a),$$

so that  $a \in W_h = W_{|g|}$ . So we have proved that

$$\partial_L X \subseteq \bigcup_{g \in L} W_{|g|}$$

which with (8) yields the desired formula (7).

By theorem 1 we now have.

**COROLLARY 1.** — *Under the same conditions*

$$(9) \quad \begin{aligned} \partial_L X &= \{x \in X \mid h^*(x) = h(x) \text{ for all } h \in A(L)\} \\ &= \{x \in X \mid g^*(x) = |g(x)| \text{ for all } g \in L\}. \end{aligned}$$

Now write  $F = \overline{\partial_L X}$  and consider the restriction map

$$g \rightarrow \tilde{g} \equiv g|_F$$

from  $L$  into the space  $R(F)$  of real continuous functions on  $F$ , letting  $\tilde{L} = \{\tilde{g} \mid g \in L\}$ .

**COROLLARY 2.** — *If  $L$  separates the point of  $X$  then for each  $u \in R(F)$ ,  $x \in \partial_L X$ , we have*

$$(10) \quad \begin{aligned} u(x) &= \inf\{\nu(x) \mid \nu \in \tilde{L}, \nu \geq u\} \\ &= \sup\{\varpi(x) \mid \varpi \in \tilde{L}, \varpi \leq u\}. \end{aligned}$$

The first equality follows from the proof of (9), applied to the pair  $(F, \hat{L})$  in place of  $(X, L)$ , and the obvious fact that  $\partial_{\hat{L}}F \supseteq \partial_L X$ . The same reasoning applied to  $-u$  then yields the second part.

**4. Measures on the boundary for separable L.**

In this section we suppose that  $L$  is separable.

**PROPOSITION 2.** — *If  $L$  is a separable linear subspace of  $C$  that contains the constants then there exists a function  $h \in A(L)$  such that*

$$(11) \quad \partial_L X = \int W_h.$$

Let  $(g_m)_{m \geq 1}$  be a countable dense set in  $L$ , and let  $(r_n)_{n \geq 1}$  be an enumeration of the rationals, and let

$$h = \sum_{m, n \geq 1} \frac{1}{2^{m+n}} \frac{h_{mn}}{1 + \|h_{mn}\|},$$

where  $h_{mn}(x) = |g_m(x) - r_n|$  ( $m, n \geq 1; x \in X$ ),

so that  $h \in A(L)$ . We show that this  $h$  satisfies (11).

First if  $a \in X, \nu \in M_a(L), g \in L, r \in \mathbf{R}$  then

$$\nu(|g - r|) = \int |g(x) - r| \nu(dx) \geq \left| \int (g(x) - r) \nu(dx) \right| = |g(a) - r|,$$

and hence in particular

$$(12) \quad \nu(h_{mn}) \geq h_{mn}(a) \quad (m, n \geq 1).$$

Now suppose  $a \in \partial_L X$  and let  $\nu \in M_a(L) \setminus M_a(A(L))$ . Then as in the proof of proposition 1 we can find  $b \in \text{supp } \nu$ , with  $b \neq a$ , and  $p \geq 1$  such that  $g_p(b) \neq g_p(a)$ . We therefore have

$$\int |g_p(x) - g_p(a)| \nu(dx) = \delta > 0.$$

But we can find a rational  $r_q$  such that

$$h_{pq}(a) = |g_p(a) - r_q| < \frac{1}{2} \delta.$$

Then

$$\begin{aligned} \nu(h_{pq}) &= \int |g_p(x) - r_q| \nu(dx) \\ &\geq \int (|g_p(x) - g_p(a)| - |g_p(a) - r_q|) \nu(dx) > \delta - \frac{1}{2} \delta = \frac{1}{2} \delta. \end{aligned}$$



Hence  $v(h_{pq}) > h_{pq}(a)$  which together with (12) shows that  $v(h) > h(a)$ , so that  $a \in W_h$ . We have thus shown that  $\int \partial_L X \subseteq W_h$ . But  $W_h \cap \partial_L X = \emptyset$  and so (11) is proved.

By theorem 1 we now have the

**COROLLARY (BISHOP and DE LEEUW).** — *If  $L$  is a separable linear subspace of  $C$  that contains the constants then the Choquet boundary  $\partial_L X$  is a  $G_\delta$  set. Moreover, for each  $\tau \in P$  we can find  $\mu \in P$  such that*

- (i)  $\mu(g) = \tau(g)$  for all  $g \in L$ ;
- (ii)  $\mu(\int \partial_L X)$

### 5. The boundary when is lattice.

We shall not require  $L$  to be separable in this section.

In his paper [3] Bauer has shown that the theory of the Choquet boundary becomes specially satisfactory when  $L$  is a lattice. We show here that corollary 2 to proposition 1 makes possible a direct proof of one of Bauer's results, and then consider the effect of an additional equicontinuity condition.

**THEOREM 2 (BAUER).** — *If  $L$  is a linear subspace of  $C$  that contains the constants, separates the points of  $X$ , and is a lattice for the natural partial ordering, then  $\partial_L X$  is a closed set and the restriction map  $f \rightarrow \tilde{f} \equiv f|_{\partial_L X}$  from  $L$  into  $R(\partial_L X)$  is an isometric linear and lattice isomorphism onto a dense subset of  $R(\partial_L X)$  (and hence actually onto  $R(\partial_L X)$  if  $L$  is complete). Moreover, given  $\tau \in M$ , we can find a unique  $\mu \equiv \mu_\tau \in M$  satisfying*

- (i)  $\mu(g) = \tau(g)$  for all  $g \in L$ ;
- (ii)  $\text{supp } \mu \subseteq \partial_L X$ .

*The map  $\tau \rightarrow \mu_\tau$  in  $M$  is linear and it maps  $M^+$  isometrically into itself.*

For this we use Bauer's maximum principle [2], which we need only in the following weak form : *if  $L$  is a linear subspace of*

*C that contains the constants and separates the points of X then for each  $f \in L$  there is a point  $a \in \partial_L X$  such that*

$$f(a) = \max_{x \in X} f(x).$$

Now let  $F = \overline{\partial_L X}$  and consider the restriction map  $f \rightarrow \tilde{f} \equiv f|_F$  from  $L$  into  $R(F)$ . This is linear and order-preserving. The maximum principle applied to  $f$  and to  $-f$  shows that it is also an isometry. Now if also  $L$  is a lattice for the natural partial ordering then the restriction map preserves the lattice structure. For let  $f, g \in L, h = f \wedge g$ , and let  $\tilde{h}$  and  $u \in R(F)$  be compared, where

$$u(x) \equiv \min(f(x), g(x)) \quad (x \in F).$$

Following e.g. KADISON [7], we have  $h \leq f, h \leq g$  and hence  $\tilde{h} \leq u$ . If for some  $x \in \partial_L X$  we have  $h(x) < u(x)$  then by corollary 2 to proposition 1 we can find  $k \in L$  such that  $\tilde{k} \leq u$  and  $h(x) < k(x) \leq u(x)$ . Then  $\tilde{k} \leq \tilde{f}, \tilde{k} \leq \tilde{g}$  and the maximum principle implies that  $k \leq f, k \leq g$ ; whence  $k \leq f \wedge g = h$ , which contradicts the inequality  $h(x) < k(x)$ . Since  $\overline{\partial_L X} = F$  we must therefore have  $\tilde{h} = u$ ; that is, the restriction of  $f \wedge g$  to  $F$  is equal to  $\min(\tilde{f}, \tilde{g})$ . Likewise the restriction of  $f \vee g$  to  $F$  is  $\max(\tilde{f}, \tilde{g})$ .

The set  $\tilde{L}$  is thus a linear sublattice of  $R(F)$  that contains the constants and separates points and hence, by the Weierstrass-Stone theorem, it lies densely in  $R(F)$ . Any continuous linear functional on  $\tilde{L}$  is therefore representable by a unique Radon measure on  $F$ . The map  $\tilde{f} \rightarrow \tau(f)$  is such a functional, and so we find  $\mu \equiv \mu_\tau \in M$  to satisfy (i) and (ii). The remaining properties of the map  $\tau \rightarrow \mu_\tau$  are immediate, if we assume that  $F = \partial_L X$ .

We complete the proof by showing that  $F = \partial_L X$ . For this let  $x \in F, \nu \in M_x(L), g \in L$  and let  $H = \{y \in X | g(y) \leq g(x)\}$ . Adapting a construction of BISHOP and DE LEEUW we write, for any Borel set  $E, \tau(E) = \nu(E \cap H), \sigma(E) = \nu(E \setminus H)$ , so that  $\tau, \sigma \in M^+, \tau + \sigma = \nu$ . Then  $\mu_\tau + \mu_\sigma = \mu_\nu$ , and we have:  $\mu_\nu = \varepsilon_x$  because  $L$  is dense in  $R(F)$ ,  $\mu_\tau \geq 0, \mu_\sigma \geq 0$ , and consequently  $\mu_\tau = \tau(1)\varepsilon_x, \mu_\sigma = \sigma(1)\varepsilon_x$ . Therefore

$$\tau(g) = \tau(1)g(x), \quad \sigma(g) = \sigma(1)g(x),$$

which implies that  $g(y) = g(x)$  in  $\text{supp } \tau \cup \text{supp } \sigma = \text{supp } \nu$ . Thus every  $g \in L$  takes the constant value  $g(x)$  on  $\text{supp } \nu$ ; but  $L$  separates points, and hence  $\text{supp } \nu = x$ ,  $\nu = \varepsilon_x$ ,  $x \in \partial_L X$ , and the proof is complete.

Now suppose that  $L$  is complete and meets the conditions of theorem 2 and let  $\mu_x$  denote the measure constructed in that theorem for the special case  $\tau = \varepsilon_x$ , where  $x \in X$ . Suppose further that the functions  $f \in L$  with  $\|f\| \leq 1$  are equicontinuous at each point of  $\int \partial_L X$  and let  $\emptyset \neq K \subseteq \int \partial_L X$  with  $K$  compact. For each  $u \in R(\partial_L X)$  the map  $x \rightarrow \mu_x(u)$  from  $X$  into  $R$  is, by theorem 2, the unique function  $\bar{u}$  in  $L$  whose restriction to  $\partial_L X$  is  $u$ . If  $\mu^K(u)$  denotes the restriction of  $\bar{u}$  to  $K$  then, by the maximum principle,

$$\|\mu^K(u)\|_{R(K)} \leq \|u\|_{R(\partial_L X)}$$

and hence by Ascoli's theorem the map  $u \rightarrow \mu^K(u)$  from  $R(\partial_L X)$  into  $R(K)$  is a compact linear operator. If now  $E \in B$  ( $=$  the class of Borel subsets of  $\partial_L X$ ) then by a theorem of BARTLE, DUNFORD and SCHWARTZ [1] the map  $x \rightarrow \mu_x(E)$  restricted to  $K$  is an element  $\mu^K(E)$ , of  $R(K)$ . Moreover the map

$$\mu^K : B \rightarrow R(K)$$

is a vector-valued regular Borel measure with conditionally compact range and we have

$$\mu^K(u) = \int_{\partial_L X} u(x) \mu^K(dx) \quad \text{for all } u \in R(\partial_L X)$$

where the integral exists as a strong integral in the sense of [1].

*Note added in proof, 7 December 1962.*

Mokobodzki and Choquet (see Séminaire Brelot-Choquet-Deny (Théorie du Potentiel) 6<sup>e</sup> année, 1962, n<sup>o</sup> 12) have shown that further improvements in the use of the Hahn-Banach theorem to study barycentres are possible: If in the present context  $L$  separates points and  $\hat{L}$  denotes the set of all  $\nu \in C$  of the form

$$\nu = \inf (g_1, g_2, \dots, g_n),$$

where all the  $g_r$  are in  $L$ , and if for  $\sigma, \tau \in P$  we write  $\sigma \prec \tau$  whenever  $\sigma(\nu) \geq \tau(\nu)$  for all  $\nu \in \hat{L}$  then  $\sigma \prec \tau$  implies that

$\sigma(g) = \tau(g)$  for all  $g \in L$ . The relation  $\preccurlyeq$  is a partial ordering, and by Zorn's lemma each element of  $P$  is dominated by a maximal element of  $P$ . A modification of the construction in theorem 1 that uses  $\widehat{f}(\sigma) \equiv \inf \{ \nu(\sigma) \mid \nu \in \widehat{L}, \nu \geq f \}$  in place of  $f^*(\sigma)$  provides for each  $\tau \in P$  and  $h \in C$  a  $\mu \succcurlyeq \tau$  with  $\mu(W_h) = 0$ . It follows that the maximal elements of  $P$  are precisely those  $\mu \in P$  for which  $\mu(W_h) = 0$  for all  $h \in C$ .

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