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FINITE SUMS AND PRODUCTS OF COMMUTATORS IN INDUCTIVE LIMIT C*-ALGEBRAS

by Klaus THOMSEN

0. Introduction.

The purpose of this paper is to extend some methods and results, which were developed by Thierry Fack [9] and de la Harpe and Skandalis [13], [14], from the framework of AF-algebras to a larger class of inductive limit C^*-algebras; a class which contains the irrational rotation C^*-algebras, for example. The building blocks for the inductive limits we want to handle are of the form

$$C(X_1) \otimes M_{n_1} \oplus C(X_2) \otimes M_{n_2} \oplus \cdots \oplus C(X_m) \otimes M_{n_m} \ ,$$

where M_k denotes the C^*-algebra of complex k by k matrices and the X_i's are compact connected Hausdorff spaces. If there is a uniform bound on the covering dimension, $\dim(X)$, of the compact spaces involved (see for example [7] for the definition of the covering dimension), we are able to extend the result [9], Theorem 3.1, of Fack; in fact even slightly beyond the case where the inductive limit is simple. The right condition for the proof to work is that the inductive limit C^*-algebra A should satisfy that $K_0(A)$ has large denominators; *i.e.* for any $a \geq 0$ in $K_0(A)$ and any $n \in \mathbb{N}$ there should be a $b \in K_0(A)$ and $m \in \mathbb{N}$ such that $nb \leq a \leq mb$, see [15], Definition 2.2. This condition is always satisfied when A is simple and not finite dimensional (as we prove in Remark 1.9 below), but it

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occurs more generally, even among AF-algebras; for example it always holds when $A = B \otimes C$, where B is any AF-algebra and C is a simple infinite dimensional AF-algebra.

In order to extend the results of de la Harpe and Skandalis we have to restrict the class of compact spaces further. Specifically, we need to assume that the involved spaces X have $\dim(X) \leq 2$ and trivial second integral (Čech-)cohomology; *i.e.* that $H^2(X, \mathbb{Z}) = 0$. With these restrictions we are able to generalize first [13], Propositions 6.1 and 6.7, which characterize the kernel of the universal de la Harpe-Skandalis determinant, and then [14], Théorème 8.7 and Théorème 9.1, proving the essential simplicity of the commutator subgroup in the group of invertibles and in the group of unitaries. In order to summarize our main results we have to establish the following notation. When A is a unital C^*-algebra, we denote the group of invertible elements by $\mathrm{Gl}(A)$, the unitary group by $U(A)$ and their connected component of the identity by $\mathrm{Gl}(A)_0$ and $U(A)_0$, respectively. The universal de la Harpe-Skandalis determinant, introduced in [12], will be denoted by Δ_T and the commutator subgroup of any group G will be denoted by DG.

MAIN RESULTS. — *Let $A = \lim_{k\to\infty} (A_k, \phi_k)$ be a unital inductive limit of C^*-algebras, where each A_i is of the form $A_i = C(X_{i1}) \otimes M_{t(i,1)} \oplus C(X_{i2}) \otimes M_{t(i,2)} \oplus \cdots \oplus C(X_{in_i}) \otimes M_{t(i,n_i)}$, each X_{ik} being a compact connected Hausdorff space.*

i) *Assume there is some $d \in \mathbb{N} \cup \{0\}$ such that $\dim(X_{ik}) \leq d$ for all $i \geq 1$, $k \in \{1, 2, \ldots, n_i\}$ and that $K_0(A)$ has large denominators. If $a = a^* \in A$ and $\theta(a) = 0$ for all tracial states θ on A, then there are $d+7$ elements x_i, $i = 1, 2, \ldots, d+7$, in A such that $a = \sum_{i=1}^{d+7} [x_i, x_i^*]$.*

ii) *Assume $\dim(X_{ik}) \leq 2$ and $H^2(X_{ik}, \mathbb{Z}) = 0$ for all $i \geq 1$, $k \in \{1, 2, \ldots, n_i\}$, and that $K_0(A)$ has large denominators. Then $D\,\mathrm{Gl}(A)_0 = \{x \in \mathrm{Gl}(A)_0 : \Delta_T(x) = 0\}$ and $DU(A)_0 = \{u \in U(A)_0 : \Delta_T(u) = 0\}$.*

iii) *Assume $\dim(X_{ik}) \leq 2$ and $H^2(X_{ik}, \mathbb{Z}) = 0$ for all $i \geq 1$, $k \in \{1, 2, \ldots, n_i\}$, and that A is simple. If G is a subgroup of $U(A)$ which is normalized by $DU(A_0)$ and is not contained in the center of $U(A)$, then $DU(A) = DU(A)_0 \subseteq G$.* □

We also prove the version of the last mentioned result, iii), for the group of invertibles in place of the unitaries. The method of proof for these results are the same as the original ones for AF-algebras and the main contribution here consists in obtaining the appropriate substitutes for trivial homogeneous C^*-algebras of facts about matrix algebras which are crucial for the proofs to carry over. Consequently, we refrain from repeating the arguments of de la Harpe and Skandalis and limit part of the exposition to indications of how their proof should be rearranged in order to work in the more general setting.

The authors motivation for pursuing these generalizations of results of Fack, de la Harpe and Skandalis is twofold. One purpose is to demonstrate that our present insight into the structure of inductive limits of homogeneous C^*-algebras is now detailed enough to allow some very technical arguments from the theory of AF-algebras to extend to a larger class. Undoubtedly this is possible also with other methods originally developed to handle AF-algebras. And of course it is important to know that the conclusions about the structure in simple AF-algebras obtained by Fack, de la Harpe and Skandalis extend to C^*-algebras such as the irrational rotation C^*-algebras, and in fact even beyond the class of C^*-algebras which are topologically spanned by their projections. We remind the reader that a series of C^*-algebras which were originally introduced by other means have been shown to be inductive limits of finite direct sums of circle algebras $C(\mathbb{T}) \otimes M_n$. This is the case of the Bunce-Deddens algebras [10], the crossed product C^*-algebras arising from a minimal homeomorphism of the Cantor set [17], [5], Remark 4.3, and quite recently also the irrational rotation C^*-algebras [8]. Thus all these C^*-algebras are covered by the above theorem.

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1. The common kernel of the tracial states.

LEMMA 1.1 (*cf.* [9], Lemma 3.2). — *Let A be unital C^*-algebra and $e_1, e_2, \ldots, e_n$ orthogonal projections in A with $\sum_{i=1}^{n} e_i = 1$. If $a = a^* \in A$,* *then there is an element $u \in A$ such that $a - \sum_{i=1}^{n} e_i a e_i = [u, u^*]$.*

Proof. — Let λ_i, $i = 1, 2, \ldots, n$, be pairwise distinct real numbers. For $i, j = 1, 2, \ldots, n$ set $\beta(i,j) = (4(\lambda_i - \lambda_j))^{-1}$ if $i \neq j$ and $\beta(i,j) = 0$ if $i = j$. Then $(\lambda_j - \lambda_j)(\beta(i,j) - \beta(j,i)) = 1/2$ for all $i \neq j$. Set

$$u = \sum_{i=1}^{n} \lambda_i e_i - \sum_{i,j=1}^{n} \beta(i,j)(e_i a e_j - e_j a e_i) \ . \qquad \square$$

LEMMA 1.2 (*cf.* [9], Lemma 3.3). — *Let A be a C^*-algebra and $e_1, e_2, e_3, \ldots, e_n$ orthogonal projections in A such that $e_1 \preceq e_2 \preceq e_3 \preceq \cdots \preceq e_n$. If $a = a^* \in A$, there is an element $u \in A$ and an element $y \in A$, such that $a = [u, u^*] + y$ and $e_i y e_i = 0$, $i = 1, 2, \ldots, n-1$.*

Proof. — For each $i = 1, 2, \ldots, n-1$, choose a partial isometry $v_i \in A$ such that $v_i v_i^* = e_i$ and $v_i^* v_i \leq e_{i+1}$. Set $x_1 = e_1 a e_1$ and set

$$x_i = e_i a e_i + v_{i-1}^* a v_{i-1} + v_{i-1}^* v_{i-2}^* a v_{i-2} v_{i-1} + \cdots$$
$$+ v_{i-1}^* v_{i-2}^* \cdots v_1^* a v_1 v_2 \cdots v_{i-2} v_{i-1} \ ,$$

$i = 2, \ldots, n-1$. Let $u = \sum_{i=1}^{n-1} \left(\sqrt{(x_i)_+} v_i + v_i^* \sqrt{(x_i)_-} \right)$, where $(x_i)_+$ and $(x_i)_-$ denote the positive and negative part of x_i, respectively. Then $y = a - [u, u^*]$ will have the right properties. $\square$

In the following Tr will denote the usual trace on M_n obtained by adding the diagonal entries.

PROPOSITION 1.3. — *Let X be a compact Hausdorff space and $n \in \mathbb{N}$. If $a = a^* \in C(X) \otimes M_n$ and $\mathrm{Tr}(a(x)) = 0$ for all $x \in X$, we have elements $u, v \in C(X) \otimes M_n$ such that $a = [u, u^*] + [v, v^*]$.*

Proof. — Let $p_1, p_2, \ldots, p_n$ be orthogonal non-zero projections in M_n with sum 1 and set $e_i = 1 \otimes p_i$, $i = 1, 2, \ldots, n$. By Lemma 1.2 there are elements $v, y \in C(X) \otimes M_n$ such that $a = [v, v^*] + y$ and $e_i y e_i = 0$, $i = 1, 2, 3, \ldots, n-1$. Applying Lemma 1.1 to y, we find $u \in C(X) \otimes M_n$ such that $y = [u, u^*] + e_n y e_n$. Thus $a = [u, u^*] + [v, v^*] + e_n y e_n$. In particular, $\mathrm{Tr}(e_n y e_n(x)) = 0$ for all $x \in X$ which implies that $e_n y e_n = 0$. $\square$

While Proposition 1.3 is a quite satisfying result as far as trivial homogeneous C^*-algebras are concerned, it is of little use when dealing with inductive limits of homogeneous C^*-algebras because the norms of u and v increase with the size of the matrix algebra. However, it follows rather painlessly from Proposition 1.3 that, when A is the inductive limit of a sequence of finite direct sums of trivial homogeneous C^*-algebras and $a = a^* \in A$ is a selfadjoint element such that $\omega(a) = 0$ for all bounded traces ω on A, then a can be approximated arbitrarily closely by two selfadjoint commutators. But to obtain qualitatively better results for such inductive limits we have to control the size of the elements in the commutators.

LEMMA 1.4. — *Let X be a compact Hausdorff space of covering dimension $\leq d$, $d \in \mathbb{N} \cup \{0\}$. Let $a = a^* \in C(X) \otimes M_n$ such that $\mathrm{Tr}(a(x)) = 0$ for all $x \in X$ and let $\varepsilon > 0$. Let $p \in C(X) \otimes M_n$ be a projection such that $pap = a$.*

Then there are elements $v_1, v_2, \ldots, v_{d+1} \in pC(X) \otimes M_n p$ such that $\|v_j\| \leq \sqrt{2}\|a\|^{1/2}$ for all j and

$$\left\| a - \sum_{j=1}^{d+1} [v_j, v_j^*] \right\| < \varepsilon .$$

Proof. — By [9], Lemma 3.5, we can find for each $x \in X$ an open neighbourhood V of x and an element $c \in pC(X) \otimes M_n p$ such that $\|c\| \leq \sqrt{2}\|a\|^{1/2}$ and $\|a(y) - [c(y), c(y)^*]\| < \varepsilon$ for all $y \in V$. Thus we get by compactness a finite open cover $\{V_i : i \in I\}$ of X and elements $c_i \in pC(X) \otimes M_n p$ such that $\|c_i\| \leq \sqrt{2}\|a\|^{1/2}$ and $\|a(y) - [c_i(y), c_i(y)^*]\| < \varepsilon$ for all $y \in V_i$, $i \in I$. By Ostrand's theorem, *cf.* [7], Theorem 3.2.4, we may suppose that

$$I = I_1 \cup I_2 \cup I_3 \cup \cdots \cup I_{d+1}$$

and that $V_i \cap V_j = \emptyset$ for $i, j \in I_k$, $i \neq j$, $k = 1, 2, \ldots, d+1$. Let $\{f_i : i \in I\}$ be a partition of unity subordinate to $\{V_i : i \in I\}$. Then set $v_j(x) = \sum_{i \in I_j} \sqrt{f_i(x)} c_i(x)$, $x \in X$, $j = 1, 2, \ldots, d+1$. □

LEMMA 1.5. — *Let A be a unital C^*-algebra. Let $p \in A$ be a projection and $k \in \mathbb{N}$ an integer such that* $\mathrm{diag}(1, 0, 0, \ldots, 0) \preceq \mathrm{diag}(p, p, p, \ldots, p)$ *in $M_k(A)$. Then every trace state of pAp extends to a bounded trace on A.*

Proof. — By assumption there are elements $v_{i,j} \in A$, $i, j = 1, 2, \ldots, k$ such that $\sum_{i,j} v_{i,j} v_{i,j}^* = 1$ and $\sum_{i,j} v_{i,j}^* v_{i,j} = kp$. It follows that if ϕ is a trace state on pAp, and $\tilde{\phi}$ is the lower semi-continuous trace on A extending ϕ, *cf.* [16], 5.2.7, then $\tilde{\phi}(1) = k$; *i.e.* $\tilde{\phi}$ is bounded. $\square$

Now we assume that A is the inductive limit C^*-algebra, $A = \lim\limits_{k\to\infty}(A_k, \phi_k)$, with each A_i a finite direct sum $A_i = C(X_{i1}) \otimes M_{t(i,1)} \oplus C(X_{i2}) \otimes M_{t(i,2)} \oplus \cdots \oplus C(X_{in_i}) \otimes M_{t(i,n_i)}$ of trivial homogeneous C^*-algebras. Assume that $d \in \{0, 1, 2, \ldots\}$ and that each X_{ik} is a compact connected Hausdorff space of (covering dimension) $\dim X_{ik} \leq d$ for all i. Furthermore, we assume that A is unital, and can therefore assume that each connecting $*$-homomorphism ϕ_k is unital. In the following we need an additional assumption on the sequence (A_k, ϕ_k) which ensures that the methods from Thierry Fack's proof of [9], Theorem 3.1, can be adopted in our setting. This condition is described in the following lemma and has the nice property that it can be described both as a condition on the sequence (A_k, ϕ_k), so that it is easy to realize in examples, or alternatively as a condition on $K_0(A)$, so that it can be checked in some situations where the sequence building up A is not completely specified. We adopt now the notation from [11]. In particular $e_{i\ell}$ is the unit of $C(X_{i\ell}) \otimes M_{t(i,\ell)} \subseteq A_i$ and $\mu_i : A_i \to A$ the canonical $*$-homomorphism. We will assume that $\phi_{ji}(e_{i\ell}) \neq 0$ for $j \geq i$, $\ell = 1, 2, \ldots, n_i$. This is no restriction because, if it was not the case, we could simply omit the direct summands of A_i for which $\mu_i(e_{i\ell}) = 0$.

LEMMA 1.6. — *In the above setting, the following conditions are equivalent.*

a) *For all $i \in \mathbb{N}$ and all minimal non-zero central projections $e_{i\ell} \in A_i$, we have*

$$\lim_{j\to\infty}(\min\{\operatorname{rank}\phi_{ji}(e_{i\ell})_k : k = 1, 2, \ldots, n_j,\ \operatorname{rank}\phi_{ji}(e_{i\ell})_k \neq 0\}) = \infty .$$

b) *$K_0(A)$ has large denominators in the sense of Nistor [15], Definition 2.2.*

Proof. — a) $\Longrightarrow$ b) : let $p \in A$ be a projection and let $n \in \mathbb{N}$. It suffices to show that there is a projection $q \in A$ such that $n[q] \leq [p] \leq m[q]$ in $K_0(A)$ for some $m \in \mathbb{N}$. Thus we can assume that $p = \mu_k(e)$ for some projection $e \in A_k$. Write $e = (e_1, e_2, \ldots, e_{n_k})$, where $e_i \in C(X_{ki}) \otimes M_{t(k,i)}$. Then $[p] = \sum\limits_{i=1}^{n_k}[\mu_k(e_i)]$, so we can assume that $e = e_\ell \in C(X_{k\ell}) \otimes M_{t(k,\ell)}$. Choose $j \geq k$ such that

$$\min\{\operatorname{rank}\phi_{jk}(e)_i : i = 1, 2, \ldots, n_j,\ \phi_{jk}(e)_i \neq 0\} \geq n + d/2 .$$

For each $i \in \{1,2,3,\ldots,n_j\}$ with the property that $\phi_{jk}(e)_i \neq 0$, let $q_i \in C(X_{ji}) \otimes M_{t(j,i)}$ be a projection of rank 1. Then $n[q_i] \leq [\phi_{jk}(e)_i] \leq (t(j,i)+d/2)[q_i]$ in $K_0(A_j)$ by [11], Theorem 2.5, (c). So if we set $q = \sum_i \mu_j(q_i)$ (where we sum over i with $\phi_{jk}(e)_i \neq 0$) and $m = \max\{t(j,i)+ d/2 : \phi_{jk}(e)_i \neq 0\}$, we have $n[q] \leq [p] \leq m[q]$ as desired.

b) $\Longrightarrow$ a) : it follows from [11], Lemma 2.4, that

$$\min\{\operatorname{rank}\phi_{ji}(e_{i\ell})_k : k = 1,2,\ldots,n_j,\ \operatorname{rank}\phi_{ji}(e_{i\ell})_k \neq 0\}$$

increases with j. Let $n \in \mathbb{N}$. Since $K_0(A)$ has large denominators, there is a $j \geq i$ and a projection $p \in A_j$ such that $n[p] \leq [\phi_{ji}(e_{i\ell})] \leq m[p]$ in $K_0(A_j)$ for some $m \in \mathbb{N}$. By taking traces one finds that

$$\min\{\operatorname{rank}\phi_{ji}(e_{i\ell})_k : k = 1,2,\ldots,n_j,\ \operatorname{rank}\phi_{ji}(e_{i\ell})_k \neq 0\} \geq n\ . \qquad \square$$

So now we assume that the two equivalent conditions of Lemma 1.6 are also satisfied. The next lemma generalizes [9], Lemma 3.6.

LEMMA 1.7. — *In the above setting, there are sequences, $\{p_n\}$, $\{q_n\}$ and $\{r_n\}$, of projections in A such that*

(i) $p_1+q_1+r_1 = 1$,

(ii) $p_n \preceq q_n \preceq r_n$, $\quad (n \geq 1)$

(iii) *the r_n's are mutually orthogonal,*

(iv) $r_{n-1} = p_n+q_n \quad (n \geq 2)$.

Proof. — From condition a) of Lemma 1.6 and the assumption that the connecting $*$-homomorphisms are unital, it follows that $\lim_{k\to\infty} \min\{t(k,i): i = 1,2,\ldots,n_k\} = \infty$. As soon as $\min\{t(k,i) : i = 1,2,\ldots,n_k\} \geq 5$, there are orthogonal projections p_1', q_1', $r_1' \in A_k$ such that $p_1' \preceq q_1' \preceq r_1'$ in A_k, $2\operatorname{rank} r_i' < \operatorname{rank} e_{ki}$ for all $i = 1,2,\ldots,n_k$, and $p_1'+q_1'+r_1' = 1$. Set $p_1 = \mu_k(p_1')$, $q_1 = \mu_k(q_1')$ and $r_1 = \mu_k(r_1')$. Now assume that p_i, q_i and r_i have been constructed for $i \leq n-1$ and that $r_i = \mu_k(r_i')$ for some k and some mutually orthogonal projections $r_i' \in A_k$ with $2^i \operatorname{rank}(r_i')_j < \operatorname{rank} e_{kj}$ for all $j = 1,2,\ldots,n_k$, $i = 1,2,\ldots,n-1$. Since $2^{n-1}\operatorname{rank}(r_{n-1}')_i < e_{ki}$ for all $i = 1,2,\ldots,n_k$, it follows from condition a) of Lemma 1.6 and [11], Lemma 2.4, that

$$\min\left\{\operatorname{rank} e_{ji} - 2^{n-1}\operatorname{rank}\phi_{jk}(r_{n-1}')_i : i = 1,2,\ldots,n_j\right\}$$

becomes arbitrarily large with j. Choose j so large that the interval

$$]2^{n-1}\operatorname{rank}\phi_{jk}(r'_{n-1})_i + 2^{n-1}d/2, \operatorname{rank} e_{ji}[$$

contains an element from $\mathbb{N}2^n$ for all $i = 1,2,\dots,n_j$. There is then a projection $f \in A_j$ such that $2^n \operatorname{rank} f_i \in]2^{n-1}\operatorname{rank}\phi_{jk}(r'_{n-1})_i + 2^{n-1}d/2, \operatorname{rank} e_{ji}[$ for all $i = 1,2,\dots,n_j$. To simplify notation, set $a_\ell = \phi_{jk}(r'_\ell)$, $\ell = 1,2,\dots,n-1$. It follows from [11], Lemma 2.4, and the assumption that the inequalities hold "at level k", that

$$2^\ell \operatorname{rank}(a_\ell)_i < \operatorname{rank} e_{ji}$$

for all $i = 1,2,\dots,n_j$, $\ell = 1,2,\dots,n-1$. Thus $\sum_{\ell=1}^{n-1}\operatorname{rank}(a_\ell)_i + \operatorname{rank} f_i < \operatorname{rank} e_{ji}$ for all i. We can therefore assume, by increasing j further and employing condition a) of Lemma 1.6 and [11], Lemma 2.4, in the same way as above, that $\sum_{\ell=1}^{n-1}\operatorname{rank}(a_\ell)_i + \operatorname{rank} f_i + d/2 < \operatorname{rank} e_{ji}$. Then

$$\operatorname{rank} f_i + d/2 < \operatorname{rank}\left(1 - \sum_{\ell=1}^{n-1} a_\ell\right)_i$$

for all $i = 1,2,\dots,n_j$. By [11], Theorem 2.5, there is then a projection $r'_n \leq 1 - \sum_{\ell=1}^{n-1} a_\ell$ in A_j equivalent to f. Note that $2\operatorname{rank}(r'_n)_i = 2\operatorname{rank} f_i > \operatorname{rank}\phi_{jk}(r'_{n-1})_i + d/2$ for all i. By increasing j even further we can assume that there is an even number ξ_i in each of the intersections

$$\Big]\operatorname{rank}(\phi_{jk}(r'_{n-1}))_i + d/2, 2\operatorname{rank}(\phi_{jk}(r'_{n-1}))_i - d\Big[\bigcap \Big]\operatorname{rank}(\phi_{jk}(r'_{n-1}))_i + d/2, 2\operatorname{rank}(r'_n)_i - d\Big[,$$

$i = 1,2,\dots,n_j$. Let e be a projection in A_j with $\operatorname{rank} e_i = 1/2\ \xi_i$ for all i. Then $e \preceq \phi_{jk}(r'_{n-1})$ in A_j by [11], Theorem 2.5. Let $q'_n \leq \phi_{jk}(r'_{n-1})$ be a projection equivalent to e in A_j. Set $q_n = \mu_j(q'_n)$ and $p_n = \mu_k(r'_{n-1}) - q_n$. Since $2\operatorname{rank}(q'_n)_i = 2\operatorname{rank} e_i \geq \operatorname{rank}\phi_{jk}(r'_{n-1})_i + d/2$ for all i, we see that $\operatorname{rank}\,(\phi_{jk}(r'_{n-1}) - q'_n)_i + d/2 \leq \operatorname{rank}(q'_n)_i$ for all i. Thus $\phi_{jk}(r'_{n-1}) - q'_n \preceq q'_n$ in A_j, again by [11], Theorem 2.5. It follows that $p_n \preceq q_n$ in A. Furthermore, since $\operatorname{rank} e_i < \operatorname{rank}(r'_n)_i - d/2$ for all i, we have that $q'_n \sim e \preceq r'_n$ in A_j. Set $r_n = \mu_j(r'_n)$. Then $p_n \preceq q_n \preceq r_n$, $r_{n-1} = \mu_k(r'_{n-1}) = p_n + q_n$ and $\sum_{i=1}^{n} r_i < 1$. Since also $2^n \operatorname{rank}(r'_n)_i < e_{ji}$ for all $i = 1,2,\dots,n_j$, it follows that we can construct the desired sequences by induction. $\square$

THEOREM 1.8. — *Let A be the inductive limit C^*-algebra of a sequence of finite direct sums of trivial homogeneous C^*-algebras, each* *of which has a primitive ideal spectrum which is a compact connected Hausdorff space of covering dimension $\leq d$. Assume that A is unital and that $K_0(A)$ has large denominators.*

If $a = a^* \in A$ and $\theta(a) = 0$ for all tracial states θ on A, then there are $d+7$ elements x_i, $i = 1, 2, \ldots, d+7$, in A such that $a = \sum_{i=1}^{d+7} [x_i, x_i^*]$.

Proof. — The proof is a repetition of Thierry Fack's proof of [9], Theorem 3.1, with 2-3 minor modifications. Since it does not consume more paper we give the complete proof here rather that just indicate the necessary changes. We use the notation established above and as in [9] we denote by B_0 the set of selfadjoint elements in the C^*-algebra B that are annihilated by all tracial states of B. The proof starts by choosing sequences of projections $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ meeting the conditions of Lemma 1.7. We can assume that $\|a\| \leq 1$. By [9], Lemma 3.4, there are elements $u, v \in A$ such that $a = [u, u^*]+[v, v^*]+a_1$, where $a_1 \in r_1 A r_1$ and $\|a_1\| \leq 3$. Clearly, $a_1 \in A_0$, so $a_1 \in (r_1 A r_1)_0$ by Lemma 1.5. We will construct by induction sequences of elements $u(i,n) \in r_n A r_n$, $i = 1, 2, 3, \ldots, d+1$, $a_n = (r_n A r_n)_0$, $v_n, w_n \in (r_n+r_{n+1})A(r_n+r_{n+1})$, such that $a_n = \sum_{i=1}^{d+1} [u(i,n), u(i,n)^*]+[v_n, v_n^*]+[w_n, w_n^*]+a_{n+1}$, $\|a_n\| \leq 3/n$, $\|u(i,n)\| \leq 2\sqrt{3/n}$, $i = 1, 2, \ldots, d+1$, $\|v_n\| \leq 2^{-n}$, $\|w_n\| \leq 2^{-n}$, $n \in \mathbb{N}$. Suppose $(a_1, a_2, \ldots, a_n), (u(i,1), u(i,2), \ldots, u(i,n-1))$, $i = 1, 2, \ldots, d+1$, $(v_1, v_2, \ldots, v_{n-1})$ and $(w_1, w_2, \ldots, w_{n-1})$ have been constructed. Note that there is a unitary $c \in A$ and a projection $r \in A_k$ for some k such that $c r_n c^* = \mu_k(r)$. Set $b = c a_n c^*$. Then $b \in (\mu_k(r) A \mu_k(r))_0$ and $\mu_k(r) A \mu_k(r)$ is the inductive limit C^*-algebra of the sequence

$$rA_k r \xrightarrow{\phi_k} \phi_k(r) A_{k+1} \phi_k(r) \xrightarrow{\phi_{k+1}} \phi_{k+2,k}(r) A_{k+2} \phi_{k+2,k}(r) \xrightarrow{\phi_{k+2}} \cdots .$$

Let $\delta > 0$ be so small that $13\sqrt{2\delta} < 2^{-n}$. Since, by [3], there is a sequence $\{c_i\} \subseteq \mu_k(r) A \mu_k(r)$ such that $b = \sum_i [c_i, c_i^*]$, we can find $j \geq k$ and $y \in (\phi_{j,k}(r) A_j \phi_{j,k}(r))_0$ such that $\|y\| \leq 2\|a_n\|$ and $\|\mu_j(y) - b\| \leq \delta$. By Lemma 1.4 we can find $s_1, s_2, \ldots, s_{d+1} \in \phi_{jk}(r) A_j \phi_{jk}(r)$ such that $\|s_i\| \leq \sqrt{2\|y\|}$ for all i and

$$\left\| y - \sum_{i=1}^{d+1} [s_i, s_i^*] \right\| \leq \delta .$$

Set $u(i,n) = c^* \mu_j(s_i) c$ and $z = a_n - \sum_{i=1}^{d+1} [u(i,n), u(i,n)^*]$. Then $z \in (r_n A r_n)_0$ and $\|z\| \leq 2\delta$. By [9], Lemma 3.4, there are elements $v_n, w_n \in (r_n +$

$r_{n+1})A(r_n+r_{n+1})$ such that $\|v_n\| \le 3\sqrt{\|z\|}$, $\|w_n\| \le 13\sqrt{\|z\|}$ and $a_{n+1} = z-[v_n, v_n^*]-[w_n, w_n^*] \in r_{n+1}Ar_{n+1}$, $\|a_{n+1}\| \le 3\|z\|$. Since $(r_nAr_n)_0 \subseteq ((r_n+r_{n+1})A(r_n+r_{n+1}))_0$, we see that $a_{n+1} \in ((r_n+r_{n+1})A(r_n+r_{n+1}))_0$. But the conditions of Lemma 1.7 implies that we can use Lemma 1.5 to conclude that each tracial state of $r_{n+1}Ar_{n+1}$ extends to a positive bounded trace on $(r_n+r_{n+1})A(r_n+r_{n+1})$. Thus $a_{n+1} \in (r_{n+1}Ar_{n+1})_0$. By the choice of $\delta > 0$, $\|u(i,n)\| \le 2\sqrt{3/n}$, $i = 1,2,\ldots,d+1$, $\|a_{n+1}\| \le 3/(n+1)$ and $\|v_n\|, \|w_n\| \le 2^{-n}$. Thus we can construct the desired sequences by induction. Set $x_1 = u$, $x_2 = v$, $x_i = \sum\limits_{n=1}^{\infty} u(i-2,n)$, $i = 3,4,\ldots,d+3$, $x_{d+4} = \sum\limits_{i \text{ even}} v_i$, $x_{d+5} = \sum\limits_{i \text{ odd}} v_i$, $x_{d+6} = \sum\limits_{i \text{ even}} w_i$ and $w_{d+7} = \sum\limits_{i \text{ odd}} w_i$. □

Remark 1.9. — If A is an inductive limit C^*-algebra of the type considered in this section and if A is simple then $K_0(A)$ has large denominators unless A is finite dimensional. Since this is an important point for potential applications of our results, we include a proof here. On the other hand we shall not need the fact in the following and since all arguments are fairly standard we only sketch them.

Assume first that there is no $N \in \mathbb{N}$ such that $n_i = 1$ for all $i \ge N$. By compressing the given sequence of trivial homogeneous C^*-algebras we can then assume that $n_i \ge 2$ for all i. For fixed $i \in \mathbb{N}$ and $\ell \in \{1,2,\ldots,n_i\}$, the projection $\mu_i(e_{i\ell})$ is non-zero in A because we have deleted redundant summands of A_i. Since A is algebraically simple there is a finite set of elements x_k, y_k in A such that $\sum\limits_k x_k\mu_i(e_{i\ell})y_k = 1$. By a standard approximation argument this gives us $j \ge i$ such that the ideal in A_j generated by $\phi_{ji}(e_{i\ell})$ is all of A_j. Thus rank $\phi_{ji}(e_{i\ell})_k \ne 0$ for all $k \in \{1,2,\ldots,n_j\}$. This shows that we can compress the sequence even further and obtain that rank $\phi_{i+1}(e_{i\ell})_k \ne 0$ for all $i \in \mathbb{N}$, $\ell \in \{1,2,\ldots,n_i\}$, $k \in \{1,2,\ldots,n_{i+1}\}$. Since we still have $n_i \ge 2$ for all i, it follows that $\min\{\operatorname{rank}\phi_{ji}(e_{i\ell})_k : k = 1,2,\ldots,n_k\} \ge 2^{j-i-1}$ for all $j > i$, $\ell \in \{1,2,\ldots,n_i\}$ in the compressed sequence. Hence $K_0(A)$ has large denominators by Lemma 1.6. In the remaining case we can assume that $n_i = 1$ for all i. By Lemma 1.6 all we have to show is that $t(i,1)$ tends to infinity when i does. Assume not. Then A is the inductive limit of a sequence with $A_i = C(X_i) \otimes M_N$ for all i, where X_i is a compact Hausdorff space and $N \in \mathbb{N}$ is fixed. Since $\mu_1(1 \otimes M_N)$ is a full matrix algebra in A, it follows that $A \simeq M_N(B)$ where B is the relative commutant of $\mu_1(1 \otimes M_N)$ in A. The elements of B can be approximated by elements from $\mu_k(C(X_k) \otimes 1)$, $k \in \mathbb{N}$, and thus B must be abelian. But then the simplicity of A implies that $B = \mathbb{C}$, which is impossible because A was assumed not to be finite dimensional. This contradiction shows that $t(1,i) \to \infty$. $\square$

2. The commutator subgroup of $\mathrm{Gl}(C(X) \otimes M_n)$.

For a unital C^*-algebra A, we let $\mathrm{Gl}_n(A)$ and $U_n(A)$ denote the group of invertibles and unitaries, respectively, in $M_n \otimes A$. To simplify notation, we set $\mathrm{Gl}_1(A) = \mathrm{Gl}(A)$ and $U_1(A) = U(A)$.

LEMMA 2.1. — *Let A be a unital C^*-algebra and $x_1, x_2, \dots, x_n \in \mathrm{Gl}(A)$ such that $x_n x_{n-1} x_{n-2} \cdots x_2 x_1 = 1$. Set $d = \mathrm{diag}(x_1, x_2, \dots, x_n) \in \mathrm{Gl}_n(A)$. Then $d = (x, y)$ for some $x \in U_n(A)_0$, $y \in \mathrm{Gl}_n(A)$. If $x_i \in U(A)$, $\mathrm{Gl}(A)_0$ or $U(A)_0$ for all i, we can choose y in $U_n(A)$, $\mathrm{Gl}_n(A)_0$ and $U_n(A)_0$, respectively.*

Proof. — Set $d_1 = \mathrm{diag}(1, x_1, x_2 x_1, x_3 x_2 x_1, \dots, x_{n-1} x_{n-2} \cdots x_1)$. Then

$$dd_1 = \mathrm{diag}(x_1, x_2 x_1, x_3 x_2 x_1, \dots, x_{n-1} x_{n-2} \cdots x_1, 1) .$$

For a suitably chosen permutation unitary $v \in M_n$ we have $(1 \otimes v) d d_1 (1 \otimes v)^* = d_1$, i.e. $d = u^{-1} d_1 u d_1^{-1}$ where $u = 1 \otimes v$. The lemma then follows from the fact that the unitary group of M_n is connected. $\square$

LEMMA 2.2. — *Let X be a compact Hausdorff space and $A = C(X) \otimes M_n$. There is an $\varepsilon = \varepsilon(n) > 0$ such that every unitary $u \in A$ with $\|u - 1\| < \varepsilon$ and $\det(u(x)) = 1$ for all $x \in X$ is the product $u = \prod_{i=1}^{n} (v_i, w_i)$ for some $v_i, w_i \in U(A)_0$, $i = 1, 2, \dots, n$.*

Proof. — Let $\varepsilon < 1/4^n 3$. After $n-1$ applications of [13], Lemme 5.16, we get $v_i, w_i \in U(A)_0$, $i = 1, 2, \dots, n-1$, and unitaries $s_i \in C(X)$, $\|s_i - 1\| < 1/3$, $i = 1, 2, \dots, n$, such that $u = \prod_{i=1}^{n-1} (v_i, w_i) \, \mathrm{diag}(s_1, s_2, \dots, s_n)$. Since $\det(u(x)) = 1$ for all $x \in X$, $s_1 s_2 \cdots s_n = 1$ and we can apply Lemma 2.1 to $\mathrm{diag}(s_1, s_2, \dots, s_n)$. $\square$

LEMMA 2.3. — *Let X be a compact Hausdorff space. There is an $\varepsilon = \varepsilon(n) > 0$ such that every invertible $y \in A = C(X) \otimes M_n$ with*

$\|y-1\| < \varepsilon$ *and* $\det(y(x)) = 1$, $x \in X$, *is the product* $y = \prod_{i=1}^{2n-2} (v_i, w_i)$ *for some* $v_i, w_i \in \mathrm{Gl}_1(A)_0$, $i = 1, 2, \ldots, 2n-1$.

Proof. — If we set $\varepsilon(n) = 1/4^n 10$, the lemma follows from $n-1$ applications of [13], Proposition 5.12. □

PROPOSITION 2.4. — *Let X be a compact Hausdorff space and $A = C(X) \otimes M_n$. Then*

$$DU(A)_0 = \{u \in U(A)_0 : \det(u(x)) = 1, \ x \in X\}$$

and

$$D\,\mathrm{Gl}(A)_0 = \{z \in \mathrm{Gl}(A)_0 : \det(z(x)) = 1, \ x \in X\} .$$

Proof. — Let $z \in \mathrm{Gl}(A)_0$ such that $\det(z(x)) = 1$ for all $x \in X$. Since $z \in \mathrm{Gl}(A)_0$, $z = e^{a_1} e^{a_2} \cdots e^{a_m}$ for some $a_i \in A$. Then $\mathrm{Tr}\left(\sum_{i=1}^{m} a_i(x)\right) \in 2\pi i \mathbb{Z}$ for all $x \in X$. Set $f_i = \frac{1}{n}\mathrm{Tr}(a_i(\cdot)) \in C(X)$ and $b_i = a_i - f_i \otimes 1 \in A$, $i = 1, 2, \ldots, m$. Then

$$z = \mathrm{diag}\left(e^{h/n}, e^{h/n}, \ldots, e^{h/n}\right) e^{b_1} e^{b_2} \cdots e^{b_m} ,$$

where $h(x) = \mathrm{Tr}\left(\sum_{i=1}^{m} a_i(x)\right)$, $x \in X$. By Lemma 2.1 $\mathrm{diag}\left(e^{h/n}, e^{h/n}, \ldots, e^{h/n}\right)$ is a commutator of two elements of $\mathrm{Gl}(A)_0$ so it suffices to show that $e^{b_i} \in D\,\mathrm{Gl}(A)_0$ for each i. Choose $k \in \mathbb{N}$ so large that $\left\| \exp b_i/k - 1 \right\| < \varepsilon$ where $\varepsilon = \varepsilon(n)$ is the ε of Lemma 2.3. Since $\mathrm{Tr}(b_i(x)) = 0$ for all $x \in X$, Lemma 2.3 implies that

$$e^{b_i} = \left(e^{b_i/k}\right)^k \in D\,\mathrm{Gl}(A)_0 .$$

The unitary case follows in the same way by using Lemma 2.2 instead of Lemma 2.3. □

For the purpose of studying inductive limits of trivial homogeneous C^*-algebras, Proposition 2.4 suffers from the same weakness as Proposition 1.3 when compared with Lemma 1.4. We have too poor control over the norms of the invertibles building up the commutators, or their distance to 1 to be precise. Furthermore, we have lost information about how many commutators are involved. These weaknesses are removed by the following lemmas which, however, forces us to restrict attention to a particular class of compact spaces. But first we make the following simple and general observation.

LEMMA 2.5. — *Let X a compact Hausdorff space and set $A = C(X) \otimes M_n$. There exists $0 < \varepsilon < 1$ such that whenever $z \in \mathrm{Gl}(A)$, $\det z(x) = 1$, $x \in X$, and $\|z-1\| < \varepsilon$, then $\theta(\mathrm{Log}(z)) = 0$ for all tracial states of A.*

Proof. — If $\varepsilon > 0$ is small enough, $\|z-1\| < \varepsilon$ implies that $z = e^a$ with $|\operatorname{Tr}(a(x))| < 2\pi$ for all $x \in X$. But $\det z(x) = 1 \Longrightarrow \operatorname{Tr}(a(x)) \in 2\pi i\mathbb{Z}$, so $\operatorname{Tr}(a(x)) = 0$ for all $x \in X$ in this case. □

LEMMA 2.6. — *Let X be a compact Hausdorff space and $f : X \to \mathbb{R}$ a continuous function. Set $d = \begin{bmatrix} f & 0 \\ 0 & -f \end{bmatrix} \in C(X) \otimes M_2$. There there are elements $y, z \in \mathrm{Gl}(C(X) \otimes M_2)_0$ such that $e^d = (y,z)$ and $\|y-1\|^2 \leq 3\|e^d-1\|$, $\|z-1\|^2 \leq 3\|e^d-1\|$. Furthermore, $d(x) = 0 \Longrightarrow z(x) = y(x) = 1$, $x \in X$.*

Proof. — Write $f = g-h$ where $g, h \geq 0$ and $gh = 0$. Set

$$y = \begin{bmatrix} e^{\frac{1}{2}g} & \sqrt{e^g-1} \\ 0 & e^{-\frac{1}{2}g} \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{2}h} & 0 \\ \sqrt{e^h-1} & e^{\frac{1}{2}h} \end{bmatrix}$$

and

$$z = \begin{bmatrix} e^{\frac{1}{2}h} & \sqrt{e^h-1} \\ 0 & e^{-\frac{1}{2}h} \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{2}g} & 0 \\ \sqrt{e^g-1} & e^{\frac{1}{2}g} \end{bmatrix}.$$

It is then straightforward to check that $e^d = (y,z)$. To get the norm estimates, set $\lambda = e^{\frac{1}{2}g}$ and $t = \sqrt{e^g-1}$. Since $\lambda \geq 1$ we have $\|\lambda^{-1}-1\| \leq \|\lambda-1\|$. Furthermore, $(\lambda-1)^2 \leq (\lambda+1)(\lambda-1) \leq \lambda^2-1 = t^2$ and hence $\|\lambda-1\| \leq \|t\|$. It follows that

$$\left\| \begin{bmatrix} e^{\frac{1}{2}g} & \sqrt{e^g-1} \\ 0 & e^{-\frac{1}{2}g} \end{bmatrix} - 1 \right\| = \left\| \begin{bmatrix} \lambda-1 & t \\ 0 & \lambda^{-1}-1 \end{bmatrix} \right\| \leq \sqrt{3}\|t\| = \sqrt{3}\sqrt{\|e^g-1\|}\ .$$

By dealing with the other factor in y in a similar way and by using that $gh = 0$ we get $\|y-1\| \leq \sqrt{3}\max\left\{\sqrt{\|e^g-1\|},\ \sqrt{\|e^h-1\|}\right\}$. The same argument applies to get

$$\|z-1\| \leq \sqrt{3}\max\left\{\sqrt{\|e^g-1\|}, \sqrt{\|e^h-1\|}\right\}.$$

The last quantity is exactly $\sqrt{3}\sqrt{\|e^d-1\|}$. □

LEMMA 2.7. — *Let $A = C(X) \otimes M_n$ where X is a compact Hausdorff space of covering dimension $\dim X = d \leq 2$ and with $H^2(X, \mathbb{Z}) = 0$. Let v be a positive invertible element of A and assume that $\omega(\mathrm{Log}(v)) = 0$ for all tracial states ω of A. For any $0 < \varepsilon < 1$ there are $2(d+1)$ pairs, y_i, $z_i \in \mathrm{Gl}(A)_0$, $i = 1, 2, \ldots, 2(d+1)$, and $v_1 \in \mathrm{Gl}(A)_0$, such that*

$$v = \prod_{i=1}^{2(d+1)} (y_i, z_i) v_1 \ ,$$

$$\|y_i - 1\| \leq 2(\|v-1\|)^{1/2}, \|z_i - 1\| \leq 2(\|v-1\|)^{1/2}, \ i = 1, 2, \ldots, 2(d+1) \ ,$$

$$\|v_1 - 1\| < \varepsilon \ ,$$

and

$$\theta(\mathrm{Log}(v_1)) = 0 \text{ for all tracial states of } A \ .$$

Proof. — By Lemma 2.5 it suffices (by taking a smaller ε if necessary) to produce $2(d+1)$ pairs, y_i, z_i, $i = 1, 2, \ldots, 2(d+1)$, of elements in $\mathrm{Gl}(A)$ with distance to 1 dominated by $2\|v-1\|^{1/2}$ such that

$$\left\| v - \prod_{i=1}^{2(d+1)} (y_i, z_i) \right\| < \varepsilon \ .$$

We may assume that $\|v-1\| > 0$. Choose $\delta > 0$ so small that $3(e^\delta \|v-1\| + |e^\delta - 1|) < 4\|v-1\|$. Set $a = \mathrm{Log}(v)$. Since a is selfadjoint we can choose continuous functions $t_i : X \to [-\|a\|, \|a\|]$, $i = 1, 2, 3, \ldots, n$, such that $t_1(x) \leq t_2(x) \leq \cdots \leq t_n(x)$ are the eigenvalues (counting multiplicities) of $a(x)$, $x \in X$. Set $b = \mathrm{diag}(t_1, t_2, \ldots, t_n) \in A$. For fixed $x \in X$, $\sum_{i=1}^{n} t_i(x) = 0$, so there is a permutation $\sigma \in \Sigma_n$ (depending on x) such that

$$t_1(x) \leq \sum_{i=1}^{k} t_{\sigma(i)}(x) \leq t_n(x), \ k = 1, 2, 3, \ldots, n \ .$$

By using the compactness of X we can find a finite open cover $\{U_i\}_{i \in J}$ of X and for each U_i a constant permutation unitary $W_i \in A$ and a permutation $\sigma_i \in \Sigma_n$ such that

$$W_i^* b W_i(y) = \mathrm{diag}(t_{\sigma_i(1)}(y), t_{\sigma_i(2)}(y), \cdots, t_{\sigma_i(n)}(y)) \ ,$$

and $t_1(y) - \delta \leq \sum_{j=1}^{k} t_{\sigma_i(j)}(y) \leq t_n(y) + \delta$, $y \in U_i$, $k = 1, 2, 3, \ldots, n$, $i \in J$. By Ornstein's theorem, *cf.* [7], Theorem 3.2.4, we may assume that J is partitioned, $J = J_1 \cup J_2 \cup \cdots \cup J_{d+1}$, such that $U_i \cap U_j = \emptyset$ when $i \neq j$

come from the same J_m. Let ϕ_i, $i \in J$, be a partition of unity subordinate to $\{U_i\}_{i\in J}$. Fix $i \in J$. Set

$$b_i(x) = \phi_i(x) W_i \operatorname{diag}(t_{\sigma_i(1)}(x), t_{\sigma_i(2)}(x), \ldots, t_{\sigma_i(n)}(x)) W_i^*, \ x \in X ,$$

and $g_k(x) = \phi_i(x) \sum_{j=1}^{k} t_{\sigma_i(j)}(x)$, $x \in X$, $k = 1, 2, \ldots, n$. Then

$$W_i^* b_i W_i = \operatorname{diag}(g_1, -g_1, g_3, -g_3, \ldots) + \operatorname{diag}(0, g_2, -g_2, g_4, -g_4, \ldots) ,$$

where the first diagonal ends with a 0 if n is odd and the second with a 0 if n is even. Thus we can apply Lemma 2.6 to get invertibles $r_i^k \in A$, $k = 1, 2, 3, 4$, such that

(1) $r_i^k(x) = 1, \quad x \notin U_i,$

(2) $\exp(b_i) = (r_i^1, r_i^2)(r_i^3, r_i^4)$,

and $\|r_i^k - 1\|^2$ is dominated by $3\sup\{|e^{t_i(x)\pm\delta} - 1| : x \in X,\ i = 1, n\} \leq 3(e^\delta \|e^b - 1\| + |e^\delta - 1|) = 3(e^\delta \|v - 1\| + |e^\delta - 1|)$. By the choice of δ

(3) $\|r_i^k - 1\| \leq 2(\|v - 1\|)^{1/2}$,

$i = 1, 2, 3, 4$. Since $b = \sum_i b_i$ and the b_i's commute, we have that $e^b = \prod_{j\in J} e^{b_j}$. Set $r(m, k) = \prod_{i\in J_m} r_i^k$, $m = 1, 2, \ldots, d+1$, $k = 1, 2, 3, 4$. Since $U_i \cap U_j = \emptyset$ for $i \neq j$ in J_m it follows from (1) and (3) that $\|r(m, k) - 1\| \leq 2(\|v - 1\|)^{1/2}$. Furthermore, $e^b = \prod_{m=1}^{d+1} (r(m, 1), r(m, 2))(r(m, 3), r(m, 4))$ by (2). Finally, note that by [22], Theorem 1.2, there is a sequence $\{u_n\}$ of unitaries in A such that $\lim_{n\to\infty} u_n b u_n^* = a$. But then $\lim_{n\to\infty} u_n e^b u_n^* = v$ and the proof is complete. □

LEMMA 2.8. — *Let $A = C(X) \otimes M_n$ where X is a compact Hausdorff space of covering dimension $\dim X = d \leq 2$ and with $H^2(X, \mathbb{Z}) = 0$. Let $v \in U(A)$ with $\|v - 1\| < 1$ and assume that $\omega(\operatorname{Log}(v)) = 0$ for all tracial states ω of A. For any $0 < \varepsilon < 1$ there are $2(d+1)$ pairs, $y_i z_i \in U(A)_0$, $i = 1, 2, \ldots, 2(d+1)$, and $v_1 \in U(A)_0$, such that*

$$v = \prod_{i=1}^{2(d+1)} (y_i, z_i) v_1$$

and

$$\|y_i - 1\| \leq \sqrt{2}(\|v - 1\|)^{1/2}, \quad \|z_i - 1\| \leq \sqrt{2}(\|v - 1\|)^{1/2}, \quad i = 1, 2, \ldots, 2(d+1) ,$$

$$\|v_1 - 1\| < \varepsilon ,$$

and

$$\theta(\mathrm{Log}(v_1)) = 0 \text{ for all tracial states of } A .$$

Proof. — The proof is essentially the same as the proof of Lemma 2.7, but with [13], Lemme 5.13, substituting for Lemma 2.6. We omit the details. □

In analogy with Proposition 5.5 of [13] the unitary and positive case dealt with in the preceding lemmas can be combined by using the polar decomposition to yield the following.

LEMMA 2.9. — *Let $A = C(X) \otimes M_n$ where X is a compact Hausdorff space of covering dimension $d \leq 2$ and with $H^2(X, \mathbb{Z}) = 0$. Let $v \in A$ and assume that $\|v-1\| < 1/4$ and $\mathrm{Tr}(\mathrm{Log}(v(x)) = 0$ for all $x \in X$. Then, for any $0 < \varepsilon < 1$, there are $4(d+1)$ pairs, $y_i, z_i \in \mathrm{Gl}(A)_0$, $i = 1, 2, 3, \ldots, 4(d+1)$, and $v_1 \in \mathrm{Gl}(A)_0$ such that*

$$v = \prod_{i=1}^{4(d+1)} (y_i, z_i) v_1 ,$$

$$\|y_i - 1\| \leq 2\|v-1\|^{1/2}, \quad \|z_i - 1\| \leq 2\|v-1\|^{1/2}, \quad i = 1, 2, \ldots, 4(d+1) ,$$

$$\|v_1 - 1\| < \varepsilon ,$$

and

$$\theta(\mathrm{Log}(v_1)) = 0 \text{ for all tracial states } \theta \text{ of } A . \qquad \square$$

We conclude this section with a lemma which will be used several times below.

LEMMA 2.10. — *Let e, f be projections in $C(X) \otimes M_k$, where X is a compact connected Hausdorff space with $\dim X \leq 2$ and $H^2(X, \mathbb{Z}) = 0$.*

Then e and f are unitarily equivalent in $C(X) \otimes M_k$ if and only if $\mathrm{rank}(e(x)) = \mathrm{rank}(f(x))$ for some (and hence all) $x \in X$.

Proof. — This a special case of [22], Theorem 1.2. □

3. The kernel of the de la Harpe-Skandalis determinant.

Now we consider an inductive limit C^*-algebra $A = \lim_{k\to\infty} (a_k, \phi_k)$ with each A_i a finite direct sum $A_i = C(X_{i1}) \otimes M_{t(i,1)} \oplus C(X_{i2}) \otimes M_{t(i,2)} \oplus \cdots \oplus C(X_{in_i}) \otimes M_{t(i,n_i)}$ and each X_{ik} a compact connected Hausdorff space of (covering dimension) $\dim X_{ik} \leq d \leq 2$ with $H^2(X_{ik}, \mathbb{Z}) = 0$ for all i and k. Furthermore, we assume that A is unital, and can therefore assume that each connecting $*$-homomorphism ϕ_k is unital. Let $\mu_k : A_k \to A$ be the canonical $*$-homomorphisms. Also recall that we assume that redundant summands have been deleted so that for each i and $\ell \in \{1, 2, \ldots, n_i\}$, we have that $\mu_i(e_{i\ell}) \neq 0$ where $e_{i\ell}$ is the unit of $C(X_{ij}) \otimes M_{t(i,\ell)} \subseteq A_i$.

We denote by Δ_T *the universal determinant* as introduced by de la Harpe and Skandalis in [12]. The following two proofs present only slight alterations of the corresponding arguments in [13].

LEMMA 3.1. — *Let $x \in \mathrm{Gl}(A)_0$ and assume that $\Delta_T(x) = 1$. Let $\varepsilon > 0$. Then there is a $k \in \mathbb{N}$ and a finite set $a_1, a_2, \ldots, a_N \in A_k$ and $b \in A$ such that $x = \mu_k(e^{a_1} e^{a_2} \cdots e^{a_N}) e^b$, $\|b\| < \varepsilon$ and*

$$\theta(\mu_k(a_1 + a_2 + \cdots + a_N) + b) = 0$$

for all tracial states θ on A.

Proof. — There is a $k \in \mathbb{N}$ and a $y \in \mathrm{Gl}(A_k)$ such that $\|x - \mu_k(y)\|$ is as small as we want. Since $x \in \mathrm{Gl}(A)_0$ we can choose $y \in \mathrm{Gl}(A_k)_0$, *i.e.* $y = e^{a_2} e^{a_3} \cdots e^{a_N}$ for some $a_i \in A_k$ and if $\|x - \mu_k(y)\|$ is small enough, $x = \mu_k(y) e^b$ for some $b \in A$ with $\|b\| < \varepsilon$. Since $\Delta_T(x) = 1$, there is an element $\xi \in K_0(A)$ such that $(2\pi i)^{-1} \theta(\mu_k(a_2 + a_3 + \cdots + a_N) + b) = \underline{\theta}(\xi)$ for all tracial states θ of A. By increasing k we may assume that $\xi = [\mu_k(e)] - [\mu_k(f)]$ for projections $e, f \in M_n(A_k)$. By using Lemma 2.10 we see that there are integers $k_1, k_2, \ldots, k_n \in \mathbb{Z}$ and commuting projections $e_1, e_2, \ldots, e_n \in A_k$ such that $[e] - [f] = \sum_{i=1}^{n} k_i [e_i]$ in $K_0(A_k)$. Set $a_1 = -2\pi i \sum_{j=1}^{n} k_j e_j \in A_k$. Then $e^{a_1} = 1$ and $\theta(\mu_k(a_1 + a_2 + \cdots + a_n) + b) = 0$ for all tracial states of A. Since $x = \mu_k(e^{a_1} e^{a_2} \cdots e^{a_N}) e^b$, the proof is complete. □

LEMMA 3.2. — *Let $x \in \mathrm{Gl}(A)_0$ and assume that $\Delta_T(x) = 0$. For any $0 < \varepsilon < 1$ there is a $y \in D\,\mathrm{Gl}(A)_0$ and a $y_1 \in \mathrm{Gl}(A)_0$ such that $x = yy_1$, $\|y_1 - 1\| < \varepsilon$ and $\theta(\mathrm{Log}(y_1)) = 0$ for all tracial states θ of A.*

Proof. — Let $0 < \delta < 1$ be so small that when s_1, s_2 are elements in a C^*-algebra such that $\|s_i\| < \delta$ for $= 1,2$, then $\|e^{s_i} - 1\| < 1$ for $i = 1,2$, $(1 - \|e^{s_1} - 1\|)(1 - \|e^{s_2} - 1\|) > 1/2$ and $\|e^{s_1}e^{s_2} - 1\| < \varepsilon$. By Lemma 3.1 there is an integer $k \in \mathbb{N}$ and elements $a_1, a_2, \dots, a_N \in A_k$ such that $x = e^{b_1}e^{b_2} \cdots e^{b_N}e^{b}$ where $b_i = \mu_k(a_i)$, $\|b\| < \delta/2$ and $\theta(b_1 + b_2 + \cdots b_N + b) = 0$ for all tracial states θ of A. By using [3], Theorem 2.9, we can find a $k_1 > k \in \mathbb{N}$ and a finite set of elements $u_j, v_j \in A_{k_1}$ such that

$$\Big\| b_1 + b_2 + \cdots + b_N + b - \mu_{k_1}\Big(\sum_{j=1}^{n} [u_j, v_j]\Big)\Big\| < \delta/2 \,.$$

Set $b'_i = \phi_{k_1,k}(a_i)$, $c = -b'_1 - b'_2 - \cdots - b'_N + \sum_{j=1}^{n} [u_j, v_j] \in A_{k_1}$ and $r = e^{b'_1}e^{b'_2} \cdots e^{b'_N}e^{c} \in A_{k_1}$. By increasing k_1 we may assume that $\|c\| < \delta$. In any irreducible representation of A_{k_1} the image of r has determinant 1, so $r \in D\,\mathrm{Gl}(A_{k_1})_0$ by Proposition 2.4. We have that $x = \mu_{k_1}(r)e^{-\mu_{k_1}(c)}e^{b}$, $\mu_{k_1}(r) \in D\,\mathrm{Gl}(A)_0$ and $\|\mu_{k_1}(c)\| < \delta$, $\|b\| < \delta$. By our choice of δ we can apply [12], Lemme 3 b), to conclude that

$$e^{-\mu_{k_1}(c)}e^{b} = e^{s}$$

for some $s \in A$ with $\theta(s) = \theta(b - \mu_{k_1}(c))$ for all tracial states θ on A. But $\theta(\mu_{k_1}(c)) = \theta(b)$, so $\theta(s) = 0$. Set $y = \mu_{k_1}(r)$ and $y_1 = e^{-\mu_{k_1}(c)}e^{b}$. □

LEMMA 3.3. — *Let $x \in \mathrm{Gl}(A)$ such that $\|x - 1\| < 1/4$ and $\theta(\mathrm{Log}(x)) = 0$ for all tracial states θ of A. For any $0 < \varepsilon < 1$, there are $4(d+1)$ pairs $y_i, z_i \in \mathrm{Gl}(A)_0$, $i = 1,2,3,\dots,4(d+1)$, and $v_1 \in \mathrm{Gl}(A)_0$ such that*

$$x = \prod_{i=1}^{4(d+1)} (y_i, z_i)v_1 \,,$$

$$\|y_i - 1\| \le 4\|v - 1\|^{1/2}, \quad \|z_i - 1\| \le 4\|v - 1\|^{1/2}, \quad i = 1,2,\dots,4(d+1) \,,$$

$$\|v_1 - 1\| < \varepsilon \,,$$

and

$$\theta(\mathrm{Log}(v_1)) = 0 \text{ for all tracial states } \theta \text{ of } A \,.$$

Proof. — The proof is the same as de la Harpe and Skandalis's proof of Proposition 5.7 (b) of [13], with our Lemma 2.9 substituting for their Proposition 5.5. We omit the repetition. □

THEOREM 3.4. — *Let A be an inductive limit C^*-algebra $A = \lim_{k\to\infty}(A_k, \phi_k)$ with each A_i a finite direct sum $A_i = C(X_{i1}) \otimes M_{t(i,1)} \oplus C(X_{i2}) \otimes M_{t(i,2)} \oplus \cdots \oplus C(X_{in_i}) \otimes M_{t(i,n_i)}$, where X_{ik} is a compact connected space of (covering dimension) $\dim X_{ik} \leq 2$ and $H^2(X_{ik}, \mathbb{Z}) = 0$ for all i and k. Furthermore, assume that A is unital and that $K_0(A)$ has large denominators.*

Then $D\,\mathrm{Gl}(A)_0 = \{x \in \mathrm{Gl}(A)_0 : \Delta_T(x) = 0\}$ and $DU(A)_0 = \{u \in U(A)_0 : \Delta_T(u) = 0\}$.

Proof. — In the case of $\mathrm{Gl}(A)$ the proof is the same as that of de la Harpe and Skandalis for Proposition 6.1 of [13], with the obvious changes. Our Lemma 1.7 substitutes for the lemma of Fack and Lemma 3.2 and Lemma 3.3 for Proposition 5.7 (a) and (b) of [13], respectively. In the case of unitaries we have in Lemma 2.8 the analog of Lemma 2.9 and using that as a substitute the proof carries over with only the obvious changes to the unitary case. (This is exactly as Proposition 5.14 of [13] substitutes for Proposition 5.5 of that paper.) □

COROLLARY 3.5. — *Let A be as in Theorem 3.4. Then $D\,\mathrm{Gl}(A) = D\,\mathrm{Gl}(A)_0$ and $DU(A) = DU(A)_0$.*

Proof. — The idea is to show that $D\,\mathrm{Gl}(A) \subseteq \mathrm{Gl}(A)_0$ and then show that Δ_T annihilates $D\,\mathrm{Gl}(A)$. Taking the inclusion $D\,\mathrm{Gl}(A) \subseteq \mathrm{Gl}(A)_0$ for granted, we can argue as follows : if $x, y \in \mathrm{Gl}(A)$, we have that $\mathrm{diag}((x,y),1,1) = (\mathrm{diag}(x,x^{-1},1), \mathrm{diag}(y,1,y^{-1})) \in D\,\mathrm{Gl}_3(A)_0$, so the de la Harpe-Skandalis determinant certainly annihilates $D\,\mathrm{Gl}(A)$. Hence $D\,\mathrm{Gl}(A) = D\,\mathrm{Gl}(A)_0$ by Theorem 3.4. The unitary case follows in the same way. Thus all we have to show is that $D\,\mathrm{Gl}(A) \subseteq \mathrm{Gl}(A)_0$ so that Δ_T can be applied to commutators. This means that we must show that $\pi_0(\mathrm{Gl}(A))$ is abelian, a thing which is not automatic for C^*-algebras. To do this we show that the natural map $\pi_0(\mathrm{Gl}(A)) \to K_1(A)$ is an isomorphism under the present assumptions on A. In fact, we show that A is K-stable in the sense of [20]. Since the class of K-stable C^*-algebras is closed under direct sums and inductive limits, Lemma 1.6 shows that the only thing we need to prove is that $C(X) \otimes M_k$ is K-stable when $k \geq 2$ and X is a compact

Hausdorff space of dimension ≤ 2. But this follows from Theorem 2.10 of [19] since $\operatorname{tsr}(C(X)) \leq 2$ by Proposition 1.7 of [18]. $\square$

4. Essential simplicity of the commutator subgroup in $\mathrm{Gl}(A)$.

THEOREM 4.1. — *Let A be an inductive limit C^*-algebra, $A = \lim_{k\to\infty}(A_k, \phi_k)$, with each A_i a finite direct sum $A_i = C(X_{i1}) \otimes M_{t(i,1)} \oplus C(X_{i2}) \otimes M_{t(i,2)} \oplus \cdots \oplus C(X_{in_i}) \otimes M_{t(i,n_i)}$, such that each X_{ik} is a compact connected Hausdorff space of (covering dimension) $\dim X_{ik} \leq 2$ and $H^2(X_{ik}, \mathbb{Z}) = 0$ for all i and k. Furthermore, assume that A is unital and simple.*

If G is a subgroup of $\mathrm{Gl}(A)$ which is normalized by $D\,\mathrm{Gl}(A)_0$ and not contained in the center of $\mathrm{Gl}(A)$, then $D\,\mathrm{Gl}(A) = D\,\mathrm{Gl}(A)_0 \subseteq G$.

Proof. — It is an almost immediate consequence of Lemma 2.10 that $K_0(A)$ is a simple dimension group and therefore has large denominators by [15], Proposition 2.3, unless we are in the trivial case where A is finite dimensional. Thus the equality $D\,\mathrm{Gl}(A) = D\,\mathrm{Gl}(A)_0$ follows from Corollary 3.5. The proof of Théorème 8.7 from [14] can be recycled to prove that $D\,\mathrm{Gl}(A) \subseteq G$, with Theorem 3.4 replacing the Proposition 6.1 referred to in [14]. The only things which need comments are the existence of four non-zero projections $p_1, p_2, p_3, p_4 \in A$ such that $\sum_{i=1}^{4} p_i = 1$, $p_3 + p_4 \precsim p_1 + p_2$ and $p_i \precsim p_j + p_k$ for all i and $j \neq k$. Note that in

$$C(X_{j1}) \otimes M_{t(j,1)} \oplus C(X_{j2}) \otimes M_{t(j,2)} \oplus \cdots \oplus C(X_{jn_j}) \otimes M_{t(j,n_j)}$$

there are 5 projections q_i, $i = 1,2,3,4,5$, such that $q_5 \precsim q_1 \sim q_2 \sim q_3 \sim q_4$ and $\sum_{i=1}^{5} q_i = 1$ if $\max_k t(j,k) \geq 4$. But since A is simple, $\lim_{j\to\infty}(\min_k t(j,k)) = \infty$ as shown by Goodearl in [11], 2.2, (or use that $K_0(A)$ has large denominators together with Lemma 1.6), so we can certainly find such q_i's in some A_j for j large enough. Therefore, we can use $p_1 = \mu_j(q_1 + q_5)$ and $p_i = \mu_j(q_i)$, $i = 2,3,4$. Also, for the application of Theorem 3.4 in the proof, we must show that if $q \in A$ is a projection, then qAq is still a C^*-algebra in the class we consider. For this purpose we may assume, by a wellknown approximation argument, that $q = \mu_N(p)$ for some projection $p \in A_N$. Then qAq is the inductive limit of the sequence

$$pA_Np \xrightarrow{\phi_N} \phi_N(p)A_{N+1}\phi_{N+1}(p) \xrightarrow{\phi_{N+1}} \phi_{N+2,N}(p)A_{N+2}\phi_{N+2,N}(p) \xrightarrow{\phi_{N+2}} \cdots$$

If $f \in C(X) \otimes M_n$ is a projection, X is connected, $\dim X \leq 2$ and $H^2(X, \mathbb{Z}) = 0$, then there is a unitary $u \in C(X) \otimes M_n$ such that ufu^* is constant over X. This follows from Lemma 2.10. But if f is constant over X, $fC(X) \otimes M_n f \simeq C(X) \otimes M_k$, where $k = \operatorname{rank}(f)$. Thus qAq is an inductive limit C^*-algebra of the same type as A. □

The analog of Theorem 4.1 for the unitary group is also true, and the proof of [14], Théorème 9.1, does carry over, but the following substitute for the Proposition 9.10 of de la Harpe and Skandalis [14] requires some elaboration.

LEMMA 4.2. — *Let A be as in Theorem 4.1 and not finite dimensional. Let G be a non central subgroup of $U(A)$ which is normalized by $DU(A)_0$. Then there is a finite-dimensional unital C^*-subalgebra F of A such that $F \simeq M_{k_1} \oplus M_{k_2} \oplus \cdots \oplus M_{k_m}$ where $\min_i k_i \geq 9$ and $DU(F) \subseteq G$.*

Proof. — It is a standard fact from topology that if X is a compact Hausdorff space of covering dimension ≤ 2 and $H^2(X, \mathbb{Z}) = 0$, then any closed subset of X has the same two properties. Hence we can write $A = \lim_{n\to\infty} (B_n, \phi_n)$ where the connecting ϕ_n's are unital *and* injective and

$$B_i = C(Y_{i1}) \otimes M_{t(i,1)} \oplus C(Y_{i2}) \otimes M_{t(i,2)} \oplus \cdots \oplus C(Y_{in_i}) \otimes M_{t(i,n_i)}$$

where each Y_{ik} is of dimension ≤ 2 and has $H^2(Y_{ik}, \mathbb{Z}) = 0$. (To simplify notation we retain the notation ϕ_n and μ_n for the connecting maps and the canonical maps, respectively.) After we have passed to quotients in this way to get the connecting $*$-homomorphisms injective, the spaces Y_{ik} need no longer be connected. This will not affect us in the following, but we need the observation that $\min_i t(j,i)$ tends to infinity with j because this was the case in our original sequence.

By Lemme 9.3 of [14] there is a non central element $x \in G$ with $\|x-1\| < 1/2$. Then $x = e^{ia}$ for some non scalar selfadjoint a. Let $t_1 = \max\{s : s \in \sigma(a)\}$, $t_2 = \min\{s : s \in \sigma(a)\}$, $\lambda_1 = e^{it_1}$ and $\lambda_2 = e^{it_2}$. As in the proof of Lemme 9.2 of [14] we choose $\varepsilon > 0$ such that $0 < 48\varepsilon < (|\lambda_1 - \lambda_2| - 2\varepsilon)^3$. Let $\delta > 0$ be so small that $\|e^{id} - i^{ia}\| < \varepsilon$ whenever $d = d^* \in A$ and $\|d-a\| < 2\delta$. Choose $j \in \mathbb{N}$ and $b = b^* \in B_j$ such that $\|\mu_j(b) - a\| < \delta$. Let

$$B_j = C(Y_{i1}) \otimes M_{t(j,1)} \oplus C(Y_{j2}) \otimes M_{t(j,2)} \oplus \cdots \oplus C(Y_{jn_j}) \otimes M_{t(j,n_j)} ,$$

and $b = (b_1, b_2, \ldots, b_{n_j})$ the corresponding decomposition of b. Then there are $k_1, k_2 \in \{1, 2, 3, \ldots, n_j\}$ and points $x_1 \in Y_{jk_1}$, $x_2 \in Y_{jk_2}$ such that

$$\sigma(b_{k_i}(x_i)) \cap]t_i - \delta, t_i + \delta[\neq \emptyset, \quad i = 1, 2 .$$

Since A is simple and the connecting $*$-homomorphism are now injective, it follows from a simplicity condition which has been observed by several authors, *cf.* [2], p. 84, proof of Theorem 10 or [4], Proposition 2.1, that we may map b further out in the sequence and in this way, by increasing j, reach the situation where

$$\sigma(b_i(x)) \cap]t_1-\delta, t_1+\delta[\neq \emptyset \text{ and } \sigma(b_i(x)) \cap]t_2-\delta, t_2+\delta[\neq \emptyset ,$$

for all $x \in Y_{ji}$ and all $i = 1, 2, \dots, n_j$. By [2], Theorem 4, we may furthermore assume that $b_i(x)$ has $t(j,i)$ distinct eigenvalues for all i and all x. Thus there are continuous realvalued functions $\psi_i^1 \leq \psi_i^2 \leq \cdots \leq \psi_i^{t(j,i)}$ on Y_{ji} and orthogonal sets $q_i^1, q_i^2, \dots, q_i^{t(j,i)}$ of projections of sum 1 in $C(Y_{ji}) \otimes M_{t(j,i)}$ such that

$$b_i(x) = \sum_{r=1}^{t(j,i)} \psi_i^r(x) q_i^r(x), \quad x \in X_{ji} ,$$

$i = 1, 2, \dots, n_j$. Note that $\psi_i^1(x) \in]t_1-\delta, t_1+\delta[$ and $\psi_i^{t(j,i)}(x) \in]t_2-\delta, t_2+\delta[$ for all $x \in Y_{ji}$ and all i, so if we set

$$c_i(x) = \sum_{r=2}^{t(j,i)-1} \psi_i^r(x) q_i^r(x) + t_1 q_i^1(x) + t_2 q_i^{t(j,i)}(x) ,$$

and $c = (c_1, c_2, \dots, c_{n_j}) \in B_j$, then $\|\mu_j(c) - a\| < 2\delta$. Note that we may assume that $t(j,i) \geq 9$ for all $i = 1, 2, \dots, n_j$. By Lemma 2.10 we can find matrix units $\{f_{k\ell}^i : k, \ell = 1, 2, \dots, t(m,i),\ i = 1, 2, \dots, n_m\}$ in A_m such that $\sum_{i,j} f_{ii}^j = 1$, $f_{11}^i = q_i^1$ and $f_{22}^i = q_i^{t(j,i)}$, $i = 1, 2, \dots, n_j$. We let F be the finite dimensional C^*-algebra spanned by $\{\mu_j(f_{k\ell}^i)\}_{i,k,\ell}$ and set $p_1 = \mu_j\left(\sum_{i=1}^{n_j} f_{11}^i\right)$, $p_2 = \mu_j\left(\sum_{i=1}^{n_j} f_{22}^i\right) \in F$, and $y = \exp(i\mu_j(c)) \in U(A)$. Since $\|\mu_j(c) - a\| < 2\delta$, we have that $\|y - x\| < \varepsilon$. Furthermore, note that $yp_i = \lambda_i p_i$, $i = 1, 2$, and that $p_1 + p_2$ is equivalent to a subprojection of $p_3 = 1 - p_1 - p_2$. It follows that the arguments used by de la Harpe and Skandalis in their proof of Proposition 9.6 of [14], with our Theorem 3.4 substituting for their Proposition 6.7, now give us an element $z \in U(p_1 A p_1)$ such that $\sigma(z) \cap \{-1, 1\} = \emptyset$ and

$$\begin{bmatrix} z^* & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & p_3 \end{bmatrix} \in G ,$$

where the matrix representation is given relative to $1 = p_1 + p_2 + p_3$. Since $\sigma(z) \cap \{-1, 1\} = \emptyset$ there is a $\mu \in \mathbb{T}$ such that $|\operatorname{Im} \lambda| > |\operatorname{Im} \mu|$ for all

$\lambda \in \sigma(z)$ and $\mu^4 \neq 1$. By Proposition 2.4 above, every continuous function $\sigma(z) \to SU(2)$ represents an element of $DU(C(\sigma(z)) \otimes M_2)_0$, so Lemme 9.7 of [14] applies to show that $\mu^{-2}p_1+\mu^2p_2+p_3 \in G$. But then we see that $DU(F) \subseteq G$ because the groups $SU(t(j,i))$, $i = 1, 2, \ldots, n_j$, do not contain any non trivial non central normal subgroups. □

The proof of Théorème 9.1 of [14] can now be used ad verbatim to get the following unitary version of Theorem 4.1.

THEOREM 4.3. — *Let A be an inductive limit C^*-algebra, $A = \lim_{k\to\infty}(A_k, \phi_k)$ with each A_i a finite direct sum $A_i = C(X_{i1}) \otimes M_{t(i,1)} \oplus C(X_{i2}) \otimes M_{t(i,2)} \oplus \cdots \oplus C(X_{in_i}) \otimes M_{t(i,n_i)}$, such that each X_{ik} is a compact connected Hausdorff space of (covering dimension) $\dim X_{ik} \leq 2$ and $H^2(X_{ik}, \mathbb{Z}) = 0$ for all i and k. Furthermore, we assume that A is unital and simple.*

If G is a subgroup of $U(A)$ which is normalized by $DU(A)_0$ and is not contained in the center of $U(A) = DU(A)_0 \subseteq G$. □

Remark 4.4. — It follows from the preceding result that $DU(A) = DU(A)_0$ modulo its center is a simple group (for the C^*-algebras in question). In this remark we want to point out that $U(A)_0$ modulo its center, which is $\mathbb{T}$, is not even topologically simple unless A has real rank 0. In fact $DU(A)$ is not dense in $U(A)_0$ unless A has real rank 0. So see this, let S be the tracial state space of A where A is as in Theorem 4.3. The de la Harpe-Skandalis determinant gives a surjective homomorphism $\Delta_T : U(A)_0/\overline{DU(A)_0} \longrightarrow \mathrm{Aff}(S)/\mathcal{P}$, where $\mathrm{Aff}(S)$ denotes the real continuous affine functions on S and $\mathcal{P}$ the closed span of the functions represented by projections in A. Thus $U(A)_0 = \overline{DU(A)_0}$ implies that $\mathcal{P} = \mathrm{Aff}(S)$, *i.e.* that the projections of A separate the tracial states. Hence the real rank of A must be 0 by [1]. It is easy to see that $\mathbb{T} \subseteq \overline{DU(A)_0}$, so $U(A)_0/\mathbb{T}$ is certainly not topologically simple unless the real rank of A is 0. In the reverse direction it follows from a recent result of Elliott and Rørdam, *cf.* Theorem 2.3 of [6], that $U(A)_0/\mathbb{T}$ is topologically simple if A does have real rank 0.

To get an idea of how rich a structure of closed normal subgroups $U(A)_0$ may have when A does not have real rank 0, observe that by [21] we can realize any metrizable Choquet simplex S as the tracial state space of a simple unital inductive limit C^*-algebra B of a sequence of interval algebras $C[0,1] \otimes M_n$ in such a way that the corresponding subspace $\mathcal{P}$ only consists of the constants $\mathbb{R} \subseteq \mathrm{Aff}(S)$. Since the map $\Delta_T : U(B)_0/\overline{DU(B)_0} \longrightarrow \mathrm{Aff}(S)/\mathcal{P}$ is always continuous, $U(B)_0$ has at least as many distinct closed normal subgroups as there are distinct closed subspaces of $\mathrm{Aff}(S)/\mathbb{R}$. $\square$

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