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ON ALGEBRAIC SETS INVARIANT BY ONE-DIMENSIONAL FOLIATIONS ON $\mathbf{C}P(3)$

by Marcio G. SOARES

1. Introduction and statement of results.

In this work we consider the problem of extending the result of J.P. Jouanolou [J] on the density of singular holomorphic foliations on $\mathbf{C}P(2)$ without algebraic solutions to the case of foliations by curves on $\mathbf{C}P(3)$.

If $\mathcal{F}$ is an one-dimensional foliation on $\mathbf{C}P(3)$ with singular set $\text{sing}(\mathcal{F})$ and Γ is an irreducible algebraic curve, we say that Γ is an algebraic solution of $\mathcal{F}$ if $\Gamma\backslash\text{sing}(\mathcal{F})$ is a leaf of the foliation. In what follows, by *invariant algebraic set of* $\mathcal{F}$ we mean either an algebraic solution or an algebraic surface $\mathcal{S} \subset \mathbf{C}P(3)$ invariant by the foliation.

One-dimensional holomorphic foliations on $\mathbf{C}P(3)$ are represented, in an affine coordinate system (x, y, z), by a vector field of the form

$$X = gR + \sum_{\ell=0}^{d} X_\ell$$

where g is a homogeneous polynomial of degree d, R is the radial vector field $x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} + z\dfrac{\partial}{\partial z}$ and X_ℓ is a vector field whose components are homogeneous polynomials of degree ℓ, $0 \leq \ell \leq d$. If $g \not\equiv 0$ or if $g \equiv 0$ and X_d cannot be written as hR where h is homogeneous of degree $d-1$, then X has a pole of order $d-1$ at infinity. We call d the *degree* of the foliation [GM] [OB]. We prove the following

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THEOREM 1. — *Let X_μ , $\mu \in \mathbf{C}$, be the vector field*

$$X_\mu = (\mu x + y^d - x^{d+1})\frac{\partial}{\partial x} + (\mu y + z^d - yx^d)\frac{\partial}{\partial y} + (\mu z + 1 - zx^d)\frac{\partial}{\partial z}$$

and let $\mathcal{F}_\mu$ be the foliation on $\mathbf{C}P(3)$ represented by X_μ. Then, for $0 <\mid \mu \mid << 1$ and $d \geq 2$, $\mathcal{F}_\mu$ has no invariant algebraic set.

THEOREM 2. — *Let $\aleph_d$ denote the space of one-dimensional foliations of degree d on $\mathbf{C}P(3)$. For each $d \geq 2$ there is a dense subset $\Im_d \subset \aleph_d$ such that any $\mathcal{F} \in \Im_d$ has no invariant algebraic set.*

In [J] Jouanolou proved both theorems for $\mathbf{C}P(2)$. Later A. Lins Neto [LN1] improved theorem 2 and showed that the set $\Im_d$ is open as well. He also gave new proofs of these results based on residues associated to foliations. The proofs we give follow this line of argument and we make use of a result of D. Lehmann [L] on residues which is of the kind of those of Baum and Bott [BB], [C], Camacho and Sad [CS], Brasselet and Lins Neto [LN2]. The plan of this work is as follows. In section 2 we quote the result on residues which will be used and prove some auxiliary lemmas. In sections 3, 4 and 5 we prove theorem 1 and in section 6 we prove theorem 2.

2. Auxiliary results.

Let $\mathcal{W}$ be an n-dimensional complex manifold, $\mathcal{F}$ an one-dimensional singular holomorphic foliation on $\mathcal{W}$ with sing($\mathcal{F}$) a discrete set of points and $V \subset \mathcal{W}$ a complex submanifold invariant by $\mathcal{F}$ with $\dim_{\mathbf{C}} V = m$. For each point $p \in \mathrm{sing}(\mathcal{F})$ take a coordinate domain $\mathcal{U}$ around p with $\mathcal{U} \cap \mathrm{sing}(\mathcal{F}) = \{p\}$ and such that $U = V \cap \mathcal{U}$ is given by $y_1 = \ldots = y_q = 0$ where $(x_1, \ldots, x_m, y_1, \ldots, y_q)$ are coordinates in $\mathcal{U}$ and $m + q = n$. Let the foliation $\mathcal{F}$ be represented in $\mathcal{U}$ by the vector field

$$X = \sum_{i=1}^{m} A_i(x, y)\frac{\partial}{\partial x_i} + \sum_{j=1}^{q} B_j(x, y)\frac{\partial}{\partial y_j}$$

where $B_j(x, 0) = 0$ for $1 \leq j \leq q$. If $\varphi \in \mathbf{R}[c_1, \ldots, c_q]$ is a characteristic class of dimension $2m$, $\mathcal{J}(x)$ is the matrix $\left(\frac{\partial B_i}{\partial y_j}(x, 0)\right)$, $1 \leq i,j \leq q$, and if we define

$$\mathrm{Res}_{\mathcal{F}}(\varphi, V, p) = \begin{bmatrix} \varphi(\mathcal{J}(x))\; dx_1 \wedge \ldots \wedge dx_m \\ A_1(x, 0), \ldots, A_m(x, 0) \end{bmatrix}$$

where [ ] denotes the Grothendieck residue symbol then we have, provided V is compact, the following

THEOREM 2.1 [L].

$$\int_V \varphi(\nu_{V/\mathcal{W}}) = \sum_{p \in \operatorname{sing}\mathcal{F} \cap V} \operatorname{Res}_{\mathcal{F}}(\varphi, V, p)$$

where the integral is over the fundamental class of V and $\nu_{V/\mathcal{W}}$ is the normal bundle of V in $\mathcal{W}$. □

Remark 2.2. — If a vector field X has non-degenerated linear part at a singular point p, $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the linear part of X at this point and if V is one-dimensional, invariant by X and tangent at p to the direction associated to λ_i then, by taking $\varphi = c_1$ we have

$$\operatorname{Res}_X(c_1, V, p) = \begin{bmatrix} c_1(\mathcal{J}(x))\ dx_i \\ A_i(\ x,\ 0\ ) \end{bmatrix} = \frac{\sum_{i \neq j} \lambda_j}{\lambda_i}.$$

In case V is two-dimensional, invariant by X and tangent at p to the plane determined by eigenvectors associated to the eigenvalues λ_i and λ_j then, by taking $\varphi = c_1^2$ we get

$$\operatorname{Res}_X(c_1^2, V, p) = \begin{bmatrix} c_1^2(\mathcal{J}(x))\ dx_i \wedge dx_j \\ A_i(\ x,\ 0\ ), A_j(\ x,\ 0\ ) \end{bmatrix} = \frac{(\sum_{k \neq i, k \neq j} \lambda_k)^2}{\lambda_i \lambda_j}$$

(see [BB] or [GH] pg 658).

We will also need the

LEMMA 2.3. — *Let $\Gamma \subset \mathbf{C}P(n)$ be an irreducible algebraic curve whose singularities, in case they exist, are such that Γ has only transverse smooth analytic branches through each of them. Suppose* $\operatorname{sing}(\Gamma) \subset \{p_1, \dots, p_m\}$ *and consider the sequence of blow-ups*

$$\mathbf{C}P(n) := M_0 \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} M_2 \dots \xleftarrow{\pi_m} M_m := \mathcal{M}$$

where M_i is obtained by blowing-up M_{i-1} at $\pi_{i-1}^{-1} \circ \dots \circ \pi_1^{-1}(p_i)$. Let $\Gamma^\star \subset \mathcal{M}$ be the proper transform of Γ. Then

$$\int_{\Gamma^\star} c_1(\nu_{\Gamma^\star/\mathcal{M}}) = (n+1)d^0(\Gamma) - \chi(\Gamma^\star) - (n-1)\sum_{i=1}^{m} \ell(p_i)$$

where $d^0(\Gamma)$ is the degree of Γ, $\chi(\Gamma^\star)$ is the Euler characteristic of $\Gamma^\star$ and $\ell(p_i)$ is the number of analytic branches of Γ through p_i.

Proof. — If M is an n-dimensional complex manifold then $c_1(K_M) = -c_1(M)$ where K_M is the canonical bundle, i.e., $K_M = \bigwedge^n T^*M$ with T^*M the holomorphic cotangent bundle. If $\mathcal{M}$ is obtained by blowing-up M at p then $K_{\mathcal{M}} = \pi^* K_M + (n-1)E$ where E is the line bundle associated to the exceptional divisor $\mathcal{E}$ [GH]. Applying this to the sequence above and noticing that the blow-up's do not take place at points which lie on an exceptional divisor we arrive at

$$K_{\mathcal{M}} = (\pi_1 \circ \ldots \circ \pi_m)^* K_{\mathbf{C}P(n)} + (n-1)(\pi_2 \circ \ldots \circ \pi_m)^* E_1 + \ldots$$

$$\ldots + (n-1)\pi_m{}^* E_{m-1} + (n-1)E_m$$

and hence

$$c_1(K_{\mathcal{M}}) = c_1((\pi_1 \circ \ldots \circ \pi_m)^* K_{\mathbf{C}P(n)}) \\ + (n-1)c_1((\pi_2 \circ \ldots \circ \pi_m)^* E_1) + \ldots + (n-1)c_1(E_m).$$

Let us evaluate $\int_{\Gamma^\star} c_1(K_{\mathcal{M}|\Gamma^\star})$. To do this we must compute $\int_{\Gamma^\star} c_1((\pi^* K_{\mathbf{C}P(n)})_{|\Gamma^\star})$ and $\int_{\Gamma^\star} c_1(((\pi_i \circ \ldots \circ \pi_m)^* E_{i-1})_{|\Gamma^\star})$ where $\pi = \pi_1 \circ \ldots \circ \pi_m$. Now, $K_{\mathbf{C}P(n)} = [-(n+1)H]$ where $[H]$ is the hyperplane bundle. By choosing a hyperplane H such that $H \cap \mathrm{sing}(\Gamma) = \varnothing$ and H is transverse to Γ we have

$$\int_{\Gamma^\star} c_1((\pi^* K_{\mathbf{C}P(n)})_{|\Gamma^\star}) = \int_\Gamma c_1(K_{\mathbf{C}P(n)})_{|\Gamma} \\ = -(n+1)\int_\Gamma \eta_H = -(n+1)[\Gamma, H] = -(n+1)d^0(\Gamma)$$

where η_H is the Poincaré dual of H and $[\Gamma, H]$ is the intersection number of Γ and H which is just the degree of Γ. Now for $\int_{\Gamma^\star} c_1(((\pi_i \circ \ldots \circ \pi_m)^* E_{i-1})_{|\Gamma^\star})$. Since the exceptional divisors $\mathcal{E}_i$ in $\mathcal{M}$ are two by two disjoint we can consider each one separately and as $\Gamma^\star$ intersects $\mathcal{E}_i$ at precisely $\ell(p_i)$ distinct points we get

$$\int_{\Gamma^\star} c_1(((\pi_i \circ \ldots \circ \pi_m)^* E_{i-1})_{|\Gamma^\star}) = \int_{\Gamma^\star} \eta_{\mathcal{E}_{i-1}} = [\Gamma^\star, \mathcal{E}_{i-1}] = \ell(p_{i-1}).$$

Hence

$$\int_{\Gamma^\star} c_1(K_{\mathcal{M}|\Gamma^\star}) = -(n+1)d^0(\Gamma) + (n-1)\sum_{i=1}^m \ell(p_i).$$

Since

$$c_1(\mathcal{M})_{|\Gamma^\star} = c_1(\Gamma^\star) + c_1(\nu_{\Gamma^\star/\mathcal{M}})$$

so that

$$\int_{\Gamma^\star} c_1(\nu_{\Gamma^\star/\mathcal{M}}) = \int_{\Gamma^\star} c_1(\mathcal{M})_{|\Gamma^\star} - \int_{\Gamma^\star} c_1(\Gamma^\star)$$

and as

$$\int_{\Gamma^\star} c_1(\mathcal{M})_{|\Gamma^\star} = -\int_{\Gamma^\star} c_1(K_{\mathcal{M}|\Gamma^\star})$$

we finally arrive at

$$\int_{\Gamma^\star} c_1(\nu_{\Gamma^\star/\mathcal{M}}) = (n+1)d^0(\Gamma) - (n-1)\sum_{i=1}^{m} \ell(p_i) - \chi(\Gamma^\star). \qquad \square$$

Let us consider now an one-dimensional singular holomorphic foliation $\mathcal{F}$ on $\mathbf{C}P(n)$ with $\mathrm{sing}(\mathcal{F})$ a finite set of points and such that if X_p is a vector field representing $\mathcal{F}$ in a neighborhood of $p \in \mathrm{sing}(\mathcal{F})$ then p is a non-degenerated singularity of X_p and further, the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathbf{D}X_p(p)$ are all distinct.

Let $\Gamma \subset \mathbf{C}P(n)$ be as in Lemma 2.3 and suppose Γ is invariant by $\mathcal{F}$. For each $p \in \mathrm{sing}(\mathcal{F}) \cap \Gamma$ let $\mathcal{B}_p$ denote the set of analytic branches of Γ through p and note that since Γ is invariant by $\mathcal{F}$, if $p \in \mathrm{sing}(\Gamma)$ then $p \in \mathrm{sing}(\mathcal{F})$. We have the

LEMMA 2.4.

$$\sum_{p \in \mathrm{sing}(\mathcal{F}) \cap \Gamma} \; \sum_{B \in \mathcal{B}_p} \mathrm{Res}_{\mathcal{F}}(c_1, B, p) = (n+1)d^0(\Gamma) - \chi(\Gamma^\star).$$

Proof. — First note that if Γ is invariant by $\mathcal{F}$ and $p \in \mathrm{sing}(\mathcal{F}) \cap \Gamma$ then each branch of Γ through p is necessarily tangent to exactly one direction associated to an eigenvalue of $\mathbf{D}X_p(p)$ and that two different branches cannot be tangent to the same direction [PM]. Write $\mathrm{sing}(\mathcal{F}) \cap \Gamma = \{p_1, \ldots, p_m\}$ and let $\pi := \pi_1 \circ \ldots \circ \pi_m : \mathcal{M} \longrightarrow \mathbf{C}P(n)$ be as in Lemma 2.3. Given $p_i \in \mathrm{sing}(\mathcal{F}) \cap \Gamma$ let $\lambda_1^i, \ldots, \lambda_n^i$ be the eigenvalues of $\mathbf{D}X_{p_i}(p_i)$ and $B_1^i, \ldots, B_{\ell(p_i)}^i$ be the branches of Γ through p_i (note that $\ell(p_i) \leq n$). After a renumbering of the eigenvalues we may assume that B_j^i is tangent to the direction associated to λ_j^i. Now let $X_{p_i}^*$ be a lifting of X_{p_i} to a neighborhood of the exceptional divisor $\mathcal{E}_i$ in $\mathcal{M}$. Then $X_{p_i}^*$ has precisely n singularities on $\mathcal{E}_i$, say $q_1^i, \ldots, q_n^i$ and the eigenvalues of $\mathbf{D}X_{p_i}^*(q_j^i)$ are $\lambda_1^i - \lambda_j^i, \ldots, \lambda_j^i, \ldots, \lambda_n^i - \lambda_j^i$ and since B_j^i is tangent to the direction associated to λ_j^i, $1 \leq j \leq \ell(p_i)$, the proper transform of B_j^i is transverse to $\mathcal{E}_i$ at the point q_j^i and is tangent to the direction associated to the eigenvalue λ_j^i of $\mathbf{D}X_{p_i}^*(q_j^i)$ for $1 \leq j \leq \ell(p_i)$. By Remark 2.2 we have that, denoting by $\mathcal{F}^*$ the foliation induced by $\mathcal{F}$ on $\mathcal{M}$ via π

$$\begin{aligned}\mathrm{Res}_{\mathcal{F}^*}(c_1, \Gamma^\star, q_j^i) = \frac{\sum_{k\neq j}(\lambda_k^i - \lambda_j^i)}{\lambda_j^i} &= \sum_{k\neq j} \frac{\lambda_k^i}{\lambda_j^i} - (n-1)\\ &= \mathrm{Res}_{\mathcal{F}}(c_1, B_j^i, p_i) - (n-1).\end{aligned}$$

Summing over the $\ell(p_i)$ branches of Γ through p_i we get

$$\sum_{j=1}^{\ell(p_i)} \mathrm{Res}_{\mathcal{F}^*}(c_1, \Gamma^\star, q_j^i) = \sum_{j=1}^{\ell(p_i)} \mathrm{Res}_{\mathcal{F}}(c_1, B_j^i, p_i) - (n-1)\ell(p_i)$$

and summing over $p_i \in \mathrm{sing}(\mathcal{F}) \cap \Gamma$

$$\sum_{i=1}^{m}\sum_{j=1}^{\ell(p_i)} \mathrm{Res}_{\mathcal{F}^*}(c_1, \Gamma^\star, q_j^i) = \sum_{i=1}^{m}\sum_{j=1}^{\ell(p_i)} \mathrm{Res}_{\mathcal{F}}(c_1, B_j^i, p_i) - (n-1)\ell(p_i).$$

By Theorem 2.1

$$\sum_{i=1}^{m}\sum_{j=1}^{\ell(p_i)} \mathrm{Res}_{\mathcal{F}^*}(c_1, \Gamma^\star, q_j^i) = \int_{\Gamma^\star} c_1(\nu_{\Gamma^\star/\mathcal{M}})$$

and by Lemma 2.3

$$\int_{\Gamma^\star} (\nu_{\Gamma^\star/\mathcal{M}}) = (n+1)d^0(\Gamma) - \chi(\Gamma^\star) - (n-1)\sum_{i=1}^{m} \ell(p_i)$$

hence

$$\sum_{i=1}^{m}\sum_{j=1}^{\ell(p_i)} \mathrm{Res}_{\mathcal{F}}(c_1, B_j^i, p_i) = (n+1)d^0(\Gamma) - \chi(\Gamma^\star)$$

thus proving the lemma. □

Remark 2.5. — Since $d^0(\Gamma) \geq 1$ and $-\chi(\Gamma^\star) = 2g - 2 \geq -2$ we have

$$\sum_{p\in\mathrm{sing}(\mathcal{F})\cap\Gamma} \sum_{B\in\mathcal{B}_p} \mathrm{Res}_{\mathcal{F}}(c_1, B, p) \geq n-1$$

and is an integer (compare with Theorem A of [LN1]).

3. Residues associated to $\mathcal{F}_\mu$.

Recall the family of vector fields X_μ, $\mu \in \mathbf{C}$, given by

$$X_\mu = (\mu x + y^d - x^{d+1})\frac{\partial}{\partial x} + (\mu y + z^d - yx^d)\frac{\partial}{\partial y} + (\mu z + 1 - zx^d)\frac{\partial}{\partial z}.$$

The foliation $\mathcal{F}_\mu$, induced by X_μ on $\mathbf{C}P(3)$, has no singularities at infinity, as can be easily verified and the singular set $\operatorname{sing}(\mathcal{F}_\mu)$ consists of $D = d^3 + d^2 + d + 1$ points $p_{\ell,\mu} = (\ x_{\ell,\mu},\ y_{\ell,\mu},\ z_{\ell,\mu})$, $1 \le \ell \le D$. In fact, the singularities are given by the roots of

$$x(x^d - \mu)^{d^2+d+1} = 1 \tag{3.1}$$

with y and z given by $y = (x^d - \mu)^{-d-1}$ and $z = (x^d - \mu)^{-1}$. For $\mu = 0$ we have $\operatorname{sing}(\mathcal{F}_0) = \{(\ \xi^\ell,\ \xi^{-\ell(d^2+d)},\ \xi^{-\ell d})\ :\ 1 \le \ell \le D\}$ where ξ is a primitive root of unity of order D. Hence, for μ sufficiently close to 0, (3.1) has D distinct roots and it is immediate that these depend analytically on μ. The characteristic polynomial of $\mathbf{D}X_\mu$ at $p_{\ell,\mu}$ is given by

$$P_{\ell,\mu}(\lambda) = \lambda^3 + \sigma_{1,\ell}(\mu)\lambda^2 + \sigma_{2,\ell}(\mu)\lambda + \sigma_{3,\ell}(\mu)$$

where

$$\sigma_{1,\ell}(\mu) = dx^d_{\ell,\mu} + 3(x^d_{\ell,\mu} - \mu)$$

$$\sigma_{2,\ell}(\mu) = (d^2 + 2d)x^d_{\ell,\mu}(x^d_{\ell,\mu} - \mu) + 3(x^d_{\ell,\mu} - \mu)^2$$

$$\sigma_{3,\ell}(\mu) = (D - 1)x^d_{\ell,\mu}(x^d_{\ell,\mu} - \mu)^2 + (x^d_{\ell,\mu} - \mu)^3$$

and for $\mu = 0$ the eigenvalues are $\lambda_{1,\ell,0} = (-1+id)x^d_{\ell,0}$, $\lambda_{2,\ell,0} = (-1-d)x^d_{\ell,0}$, $\lambda_{3,\ell,0} = (-1-id)x^d_{\ell,0}$. Note that all singularities of $\mathcal{F}_0$ are of Poincaré type and non-resonant. Put

$$\alpha_\ell^{(i,j,k)}(\mu) = \frac{\lambda_{j,\ell,\mu} + \lambda_{k,\ell,\mu}}{\lambda_{i,\ell,\mu}}$$

and

$$\beta_\ell^{(i,j,k)}(\mu) = \frac{\lambda^2_{i,\ell,\mu}}{\lambda_{j,\ell,\mu}\lambda_{k,\ell,\mu}}.$$

Then

$$(3.2)\begin{cases} \alpha_\ell^{(1,2,3)}(0) = \frac{2+d-d^2+i(d^2+3d)}{1+d^2} \\ \alpha_\ell^{(2,1,3)}(0) = \frac{2}{1+d} \\ \alpha_\ell^{(3,1,2)}(0) = \frac{2+d-d^2-i(d^2+3d)}{1+d^2} \\ \beta_\ell^{(1,2,3)}(0) = \frac{1-3d^2-i(3d-d^3)}{D} \\ \beta_\ell^{(2,1,3)}(0) = \frac{d^2+2d+1}{d^2+1} \\ \beta_\ell^{(3,1,2)}(0) = \frac{1-3d^2+i(3d-d^3)}{D} \end{cases}$$

and, by differentiating $P_{\ell,\mu}(\lambda)$, $\sigma_{1,\ell}(\mu)$, $\sigma_{2,\ell}(\mu)$, $\sigma_{3,\ell}(\mu)$ and x as a function of μ in (3.1) we get

$$(3.3)\begin{cases} \frac{\mathbf{d}\alpha_\ell^{(1,2,3)}}{\mathbf{d}\mu}(0) = \frac{-d^5+d-3i(d^5-d)}{4(1-id)^2D} x_{\ell,0}^{-d} \\ \frac{\mathbf{d}\alpha_\ell^{(2,1,3)}}{\mathbf{d}\mu}(0) = \frac{d^5-d}{2(1+d)^2D} x_{\ell,0}^{-d} \\ \frac{\mathbf{d}\alpha_\ell^{(3,1,2)}}{\mathbf{d}\mu}(0) = \frac{-d^5+d+3i(d^5-d)}{4(1+id)^2D} x_{\ell,0}^{-d} \\ \frac{\mathbf{d}\beta_\ell^{(1,2,3)}}{\mathbf{d}\mu}(0) = \frac{d^7+d^6-2d^5+2d^3+d^2+6d-6+i(d^7+4d^6+4d^5-3d^4-4d^3-7d^2+3d)}{4(1+d)^2(1+id)^2D} x_{\ell,0}^{-d} \\ \frac{\mathbf{d}\beta_\ell^{(2,1,3)}}{\mathbf{d}\mu}(0) = \frac{-d^7-8d^6-13d^5-6d^4+9d^3+8d^2-d-2}{2(1+d^2)^2D} x_{\ell,0}^{-d} \\ \frac{\mathbf{d}\beta_\ell^{(3,1,2)}}{\mathbf{d}\mu}(0) = \frac{d^7+d^6-2d^5+2d^3+d^2+6d-6-i(d^7+4d^6+4d^5-3d^4-4d^3-7d^2+3d)}{4(1+d)^2(1-id)^2D} x_{\ell,0}^{-d} \end{cases}$$

Note that all derivatives are non-zero for $d \geq 2$.

4. First part of the proof of Theorem 1.

4.1 Non-existence of algebraic solutions having only smooth analytic branches.

Suppose $\Gamma_\mu \subset \mathbf{C}P(3)$ is an irreducible algebraic curve whose singularities, in case they exist, are such that Γ_μ has only smooth analytic branches through each of them.

Assume Γ_μ is invariant by $\mathcal{F}_\mu$. Then $\text{sing}(\mathcal{F}_\mu) \cap \Gamma_\mu \neq \varnothing$ (see [J]) and moreover, if $p_{\ell,\mu} \in \text{sing}(\mathcal{F}_\mu) \cap \text{sing}(\Gamma_\mu)$ then the branches of Γ_μ through $p_{\ell,\mu}$ are transverse to each other. Blow-up $\mathbf{C}P(3)$ at each $p_{\ell,\mu} \in \text{sing}(\mathcal{F}_\mu) \cap \Gamma_\mu$ to obtain a manifold $CP(3) \xleftarrow{\pi} \mathcal{M}$ and let $\Gamma_\mu^\star \subset \mathcal{M}$ be the proper transform of Γ_μ. By Lemma 2.4 applied to $n = 3$ we have that

$$\sum_{p_{\ell,\mu} \in \text{sing}(\mathcal{F}_\mu) \cap \Gamma_\mu} \sum_{B \in \mathcal{B}_{p_{\ell,\mu}}} \text{Res}_{\mathcal{F}_\mu}(c_1, B, p_{\ell,\mu}) = 4d^0(\Gamma_\mu) - \chi(\Gamma_\mu^\star).$$

On the other hand

$$\mathrm{Res}_{\mathcal{F}_\mu}(c_1, B, p_{\ell,\mu}) = \alpha_\ell^{(i,j,k)}(\mu)$$

for some

$$(i,j,k) \in \{(1,2,3),(2,1,3),(3,1,2)\}$$

and the sum of residues reads

$$\sum_{p_{\ell,\mu}\in\Gamma_\mu\cap\mathrm{sing}(\mathcal{F}_\mu)} \sum_{B\in\mathcal{B}_{p_{\ell,\mu}}} \alpha_\ell^{(i,j,k)}(\mu).$$

By using formulae (3.2) and (3.3) we conclude that, with three exceptions, these sums are either numbers which are not positive integers or have non-zero derivative for the parameter value $\mu = 0$. Therefore, apart from the three exceptions, for $0 <\mid \mu \mid << 1$, Γ_μ cannot exist since such a sum cannot be an integer greater than or equal to 2, as required by Remark 2.5. The three exceptions are the following :

Case 1.

Let $\mathcal{G}$ be a non-zero subgroup of $\mathbf{Z}/D\mathbf{Z}$ and let

$$\Theta_{\mathcal{G}}(\mu) = \sum_{\{\ell: x^d_{\ell,0}\in\mathcal{G}\}} \alpha_\ell^{(2,1,3)}(\mu).$$

Note that $x^d_{\ell,0} = \xi^{\ell d}$ where ξ is a primitive $D-th$ root of unity and d is prime with D and hence, if $x^d_{\ell,0}$ is a generator of such a $\mathcal{G}$ then $\Theta_{\mathcal{G}}(0) = \dfrac{2}{1+d} \mid \mathcal{G} \mid$ can be a positive integer provided $\mid \mathcal{G} \mid$ (the order of $\mathcal{G}$) is a multiple of $d+1$. Also $\sum_{\{\ell: x^d_{\ell,0}\in\mathcal{G}\}} \dfrac{\mathrm{d}\alpha_\ell^{(2,1,3)}}{\mathrm{d}\mu}(0) = 0$ since $\sum_{\{\ell: x^d_{\ell,0}\in\mathcal{G}\}} x^{-d}_{\ell,0} = 0$ for $x^d_{\ell,0}$ is a generator of $\mathcal{G}$. Therefore we must show that $\Theta_{\mathcal{G}}(\mu)$ is not constant as a function of μ for all non-zero $\mathcal{G} \lhd \mathbf{Z}/D\mathbf{Z}$. The proof of this fact is based on the following :

LEMMA. — *Let $\xi \neq 1$ be an $m-th$ root of unity. Then*

$$\sum_{\ell=1}^{m} \frac{\xi^\ell - X}{\xi^\ell - Y} = \frac{m(-1+XY^{m-1})}{Y^m - 1}.$$

Proof. — Formally we have

$$\frac{1}{Y-\xi^\ell}=\frac{1}{Y}\sum_{q=0}^{\infty}\frac{\xi^{\ell q}}{Y^q}.$$

Hence

$$\frac{X-\xi^\ell}{Y-\xi^\ell}=\frac{X}{Y}\sum_{q=0}^{\infty}\frac{\xi^{\ell q}}{Y^q}-\sum_{q=0}^{\infty}\frac{\xi^{\ell(q+1)}}{Y^{q+1}}$$

and

$$\begin{aligned}\sum_{\ell=1}^{m}\frac{\xi^\ell-X}{\xi^\ell-Y}&=\frac{X}{Y}\sum_{\ell=1}^{m}\sum_{q=0}^{\infty}\frac{\xi^{\ell q}}{Y^q}-\sum_{\ell=1}^{m}\sum_{q=0}^{\infty}\frac{\xi^{\ell(q+1)}}{Y^{q+1}}\\ &=\frac{X}{Y}\sum_{j=0}^{\infty}\frac{m}{Y^{jm}}-\sum_{j=1}^{\infty}\frac{m}{Y^{jm}}\\ &=\frac{mX}{Y}\left(\frac{Y^m}{Y^m-1}\right)-m\left(\frac{Y^m}{Y^m-1}-1\right)\\ &=\frac{m(-1+XY^{m-1})}{Y^m-1}.\end{aligned}$$

Remark that if $X=f(t)$ and $Y=g(t)$ where f and g are germs of holomorphic functions at $0\in\mathbf{C}$ such that $f(0)=g(0)=0$ then the conclusion of the Lemma holds. □

Now,

$$\alpha_\ell^{(2,1,3)}(\mu)=\frac{\lambda_{1,\ell,\mu}+\lambda_{3,\ell,\mu}}{\lambda_{2,\ell,\mu}}=-\frac{\sigma_{1,\ell}(\mu)}{\lambda_{2,\ell,\mu}}-1$$

where $\sigma_{1,\ell}(\mu)$ is as in page 7 and note that, by (3.1),

$$\sigma_{1,\ell}(\mu)=\sum_{i\geq 0}f_i(d,\xi^\ell)\mu^i$$

$$\lambda_{2,\ell,\mu}=\sum_{i\geq 0}g_i(d,\xi^\ell)\mu^i$$

where f_i,g_i are polynomials in ξ^ℓ with coefficents rational functions of d. Differentiating implicitly (3.1) we get

$$\sigma_{1,\ell}(\mu)=(d+3)\xi^{\ell d}+\frac{d(D-1)-3}{D}\mu+\dots$$

$$\lambda_{2,\ell,\mu}=(-1-d)\xi^{\ell d}+\frac{-d(D-1)+d+2}{2D}\mu+\dots$$

where the dots indicate higher order terms and observe that the coefficients of μ do not involve $\xi^{\ell d}$. Hence

$$-\frac{\sigma_{1,\ell}(\mu)}{\lambda_{2,\ell,\mu}}-1=\left(\frac{d+3}{d+1}\right)\frac{\xi^{\ell d}-\left(-\frac{d(D-1)-3}{(d+3)D}\mu+\dots\right)}{\xi^{\ell d}-\left(-\frac{d(D-1)-d-2}{2(d+1)D}\mu+\dots\right)}-1$$
$$=\left(\frac{d+3}{d+1}\right)\frac{\xi^{\ell d}-X_\ell(\mu)}{\xi^{\ell d}-Y_\ell(\mu)}-1$$

so that

$$\Theta_{\mathcal{G}}(\mu)=\left(\frac{d+3}{d+1}\right)\left[\sum_{\{\ell:\xi^{\ell d}\in\mathcal{G}\}}\frac{\xi^{\ell d}-X_\ell(\mu)}{\xi^{\ell d}-Y_\ell(\mu)}\right]-\mid\mathcal{G}\mid .$$

Write $X_\ell = X + X_{2,\ell} + \dots$, $Y_\ell = Y + Y_{2,\ell} + \dots$ where $X_{i,\ell}$,$Y_{i,\ell}$ are the terms of order μ^i, $i \geq 2$. By applying the same reasoning as in the lemma above we conclude that

$$\sum_{\{\ell:x^d_{\ell,0}\in\mathcal{G}\}}\frac{\xi^{\ell d}-X_\ell(\mu)}{\xi^{\ell d}-Y_\ell(\mu)}=\frac{\mid\mathcal{G}\mid(-1+XY^{|\mathcal{G}|-1})+\dots}{-1+Y^{|\mathcal{G}|}+\dots}$$

where the dots indicate higher powers of μ. Hence

$$\Theta_{\mathcal{G}}(\mu)$$
$$=\left(\frac{d+3}{d+1}\right)\frac{\mid\mathcal{G}\mid\left[-1+\left(-\frac{d(D-1)-3}{(d+3)D}\left(-\frac{d(D-1)-d-2}{2(d+1)D}\right)^{|\mathcal{G}|-1}\right)\mu^{|\mathcal{G}|}+\dots\right]}{-1+\left(-\frac{d(D-1)-d-2}{2(d+1)D}\right)^{|\mathcal{G}|}\mu^{|\mathcal{G}|}+\dots}-|\mathcal{G}|.$$

This shows $\Theta_{\mathcal{G}}(\mu)$ is not constant as a function of μ.

Case 2.
$\sum_{\ell=1}^{D}\left[\alpha_\ell^{(1,2,3)}(\mu)+\alpha_\ell^{(2,1,3)}(\mu)+\alpha_\ell^{(3,1,2)}(\mu)\right].$

This corresponds to a curve Γ_μ with $\mathrm{sing}(\Gamma_\mu) = \mathrm{sing}(\mathcal{F}_\mu)$ and at each $p_{\ell,\mu}$, Γ_μ has three branches B_1,B_2 and B_3 such that B_i is tangent to the direction associated to $\lambda_{i,\ell,\mu}$, $1 \leq i \leq 3$. Since $\alpha_\ell^{(1,2,3)}(\mu) + \alpha_\ell^{(2,1,3)}(\mu) + \alpha_\ell^{(3,1,2)}(\mu)$ is a symmetric function of the eigenvalues of $\mathbf{D}X_\mu(p_{\ell,\mu})$ and the summation extends over all points in $\mathrm{sing}(\mathcal{F}_\mu)$, it follows from Baum-Bott's theorem (see[**C**]) that this sum is independent of μ and equals $-2d^3 + 2d^2 + 6d + 6$, as can be seen by calculating it for $\mu = 0$. By Remark 2.5 we must have $-2d^3 + 2d^2 + 6d + 6 \geq 2$. Now, if $d \geq 3$ then

$-2d^3+2d^2+6d+6<0$ and it remains to consider the case $d=2$ in which we get the sum equal to 10 and $D=2^3+2^2+2+1=15$. By Lemma 2.4 we have $10=4d^0(\Gamma_\mu)-\chi(\Gamma_\mu^\star)$ or $\chi(\Gamma_\mu^\star)=4d^0(\Gamma_\mu)-10$ and since Γ_μ is a curve in $\mathbf{C}P(3)$ with 15 triple points we must have $d^0(\Gamma_\mu)>3$ which gives $\chi(\Gamma_\mu^\star)>2$, an absurd.

Case 3.

As in Case 2 but now with $d=2$ and the summation extends over the subgroup $\mathcal{G}\lhd\mathbf{Z}/15\mathbf{Z}$ (note that $D=15$) of order 3, for then we have the sum equal to $3.\frac{10}{15}=2$. By Lemma 2.4 we have $2=4d^0(\Gamma_\mu)-\chi(\Gamma_\mu^\star)$ or $\chi(\Gamma_\mu^\star)=4d^0(\Gamma_\mu)-2$ and since Γ_μ is a curve with 3 triple points then $d^0(\Gamma_\mu)>3$ which gives $\chi(\Gamma_\mu^\star)>10$, an absurd.

4.2 Non-existence of invariant smooth algebraic surfaces.

Suppose $\mathcal{S}_\mu\subset\mathbf{C}P(3)$ is a smooth algebraic surface invariant by $\mathcal{F}_\mu$. By the Vanishing Theorem of [L] we must have $\mathcal{S}_\mu\cap\operatorname{sing}(\mathcal{F}_\mu)\neq\emptyset$. From Remark 2.2 it follows that

$$\operatorname{Res}_{\mathcal{F}_\mu}(c_1^2,\mathcal{S}_\mu,p_{\ell,\mu})=\beta_\ell^{(i,j,k)}(\mu)$$

where (i,j,k) is one of the triples $(1,2,3)$, $(2,1,3)$ or $(3,1,2)$. Hence

$$\sum_{p_{\ell,\mu}\in\mathcal{S}_\mu\cap\operatorname{sing}(\mathcal{F}_\mu)}\operatorname{Res}_{\mathcal{F}_\mu}(c_1^2,\mathcal{S}_\mu,p_{\ell,\mu})=\sum_{p_{\ell,\mu}\in\mathcal{S}_\mu\cap\operatorname{sing}(\mathcal{F}_\mu)}\beta_\ell^{(i,j,k)}(\mu)$$

and by Theorem 2.1 this sum equals

$$\int_{\mathcal{S}_\mu}c_1^2\left(\nu_{\mathcal{S}_\mu/\mathbf{C}P(3)}\right).$$

Now, this integral is just the degree of the surface $\mathcal{S}_\mu$ for $c_1(\nu_{\mathcal{S}_\mu/\mathbf{C}P(3)})$ equals the Poincaré dual $\eta_{\mathcal{S}_\mu}$ of the cycle $[\mathcal{S}_\mu]$ and so the integral is just the intersection number of $\mathcal{S}_\mu$ with a generic line in $\mathbf{C}P(3)$ and this is the degree of $\mathcal{S}_\mu$ (see [GH]). In particular, the sum of residues is a positive integer. By using formulae (3.2) and (3.3) we conclude that, with one exception, the sums above are either numbers which are not positive integers or have non-zero derivative for the parameter value $\mu=0$. Hence, for $0<|\mu|<<1$, $\mathcal{S}_\mu$ cannot exist but for this exceptional case. The exception is

$$\sum_{\{\ell:x_{\ell,0}^d\in\mathcal{G}\}}\beta_\ell^{(2,1,3)}(\mu)\ \textit{where } \mathcal{G} \textit{ is a non-zero subgroup of } \mathbf{Z}/D\mathbf{Z}.$$

That is because since $\beta_\ell^{(2,1,3)}(0) = \dfrac{d^2+2d+1}{d^2+1}$ the sum above could be a positive integer provided the order $\mid \mathcal{G} \mid$ of $\mathcal{G}$ is a multiple of d^2+1 and, as in (4.1), the sum of derivatives (see (3.3)) is zero in case $x_{\ell,0}^d$ is a generator of $\mathcal{G}$. Put

$$\Theta_{\mathcal{G}}(\mu) = \sum_{\{\ell : x_{\ell,0}^d \in \mathcal{G}\}} \beta_\ell^{(2,1,3)}(\mu)$$

and let us use the lemma in (4.1) to show $\Theta_{\mathcal{G}}(\mu)$ is not constant for μ sufficiently small. Now, $\beta_\ell^{(2,1,3)}(\mu) = -\dfrac{\lambda_{2,\ell,\mu}^{\ 3}}{\sigma_{3,\ell}(\mu)}$ where $\sigma_{3,\ell}(\mu)$ is the independent term of the characteristic polynomial $P_{\ell,\mu}(\lambda)$ (see page 7). Differentiating implicitly (3.1) we get

$$\lambda_{2,\ell,\mu}^3 = -(d+1)^3\xi^{3\ell d} + 3(d^2+1)\left(\frac{-d(D-1)+d+2}{2D}\right)\xi^{2\ell d}\mu + \ldots$$

$$\sigma_{3,\ell}(\mu) = D\xi^{3\ell d} + (D-4)\xi^{2\ell d}\mu + \ldots$$

where the dots indicate higher order terms. Hence

$$-\frac{\lambda_{2,\ell,\mu}^3}{\sigma_{3,\ell}(\mu)} = \frac{(d+1)\xi^{\ell d} + 3(d^2+1)\left(\frac{d(D-1)-d-2}{2D}\right)\mu + \ldots}{D\xi^{\ell d} + (D-4)\mu + \ldots}$$

$$= \left(\frac{d+1}{D}\right)\frac{\xi^{\ell d} + 3\left(\frac{d^2+1}{d+1}\right)\left(\frac{d(D-1)-d-2}{2D}\right)\mu + \ldots}{\xi^{\ell d} + \left(\frac{D-4}{D}\right)\mu + \ldots}$$

and applying the lemma and the same reasoning as in (4.1) we get

$$\Theta_{\mathcal{G}}(\mu)$$

$$= \left(\frac{d+1}{D}\right)\frac{\mid \mathcal{G} \mid \left[-1 + 3\left(\frac{d^2+1}{d+1}\right)\left(\frac{d(D-1)-d-2}{2D}\right)\left(\frac{D-4}{D}\right)^{|\mathcal{G}|-1}\mu^{|\mathcal{G}|} + \ldots\right]}{1 - \left(\frac{D-4}{D}\right)^{|\mathcal{G}|}\mu^{|\mathcal{G}|} + \ldots}$$

thus proving that $\Theta_{\mathcal{G}}(\mu)$ is not constant.

5. Conclusion of the proof of Theorem 1.

Throughout this section X denotes the germ at $0 \in \mathbf{C}^n$ of a holomorphic vector field with an isolated singularity at $0 \in \mathbf{C}^n$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{D}X(0)$ and consider the following :

Condition 1 : $\lambda_i \notin \mathbf{Q}^+\lambda_j, \quad i \neq j, \quad i,j = 1,\ldots,n.$

Condition 2 : *There is no resonance relation among the* $\lambda_i, \quad i = 1,\ldots,n.$

LEMMA 5.1. — *If X satisfies condition 1 then no germ at $0 \in \mathbf{C}^n$ of a singular irreducible analytic curve can be invariant by X.*

Proof. — By a formal change of coordinates we have that X is expressed as (see [A], Poincaré-Dulac's theorem) $X = \sum_{i=1}^{n}(\lambda_i x_i + \varphi_i(x_1,\ldots,x_n))\frac{\partial}{\partial x_i}$ where φ_i is a formal power series starting with terms of degree ≥ 2. Suppose on the contrary that Γ is the germ at $0 \in \mathbf{C}^n$ of a singular irreducible analytic curve invariant by X. Let $p : \mathbf{D}_\epsilon \longrightarrow \mathbf{C}^n$ be a parametrization of Γ where $\mathbf{D}_\epsilon = \{z \in \mathbf{C} : \mid z \mid < \epsilon\}$. Under the formal change of coordinates to p there corresponds a formal parametrization p^* which we may assume to be of the form $p^*(t) = (t^{\ell_1}, a_2 t^{\ell_2} + \ldots, \ldots, a_n t^{\ell_n} + \ldots)$ where $1 < \ell_1 \leq \ell_i$ for $i = 2,\ldots,n$. Since Γ is invariant by X we have $\frac{\mathbf{d}p^*}{\mathbf{d}t} = \nu(t)X(p^*(t))$ with $\nu(t)$ not identically zero and this reads

$$\ell_1 t^{\ell_1} = \nu(t)(\lambda_1 t^{\ell_1} + \ldots)$$

$$a_2\ell_2 t^{\ell_2-1} + \ldots = \nu(t)(\lambda_2 a_2 t^{\ell_2} + \ldots)$$

$$\ldots$$

$$a_n\ell_n t^{\ell_n-1} + \ldots = \nu(t)(\lambda_n a_n t^{\ell_n} + \ldots)$$

where the dots indicate higher order terms. Eliminate ν from the first two equations to get

$$\frac{\ell_1 t^{\ell_1-1}}{\lambda_1 t^{\ell_1} + \ldots} = \frac{a_2\ell_2 t^{\ell_2-1} + \ldots}{a_2\lambda_2 t^{\ell_2} + \ldots}$$

and then

$$a_2\lambda_2\ell_1 t^{\ell_1+\ell_2-1} + \ldots = a_2\lambda_1\ell_2 t^{\ell_1+\ell_2-1} + \ldots$$

which gives

$$a_2\ell_1\lambda_2 = a_2\ell_2\lambda_1 \quad \textit{or} \quad \frac{\lambda_1}{\lambda_2} = \frac{\ell_1}{\ell_2} \in \mathbf{Q}^+$$

and this contradicts condition 1. $\square$

LEMMA 5.2. — *If X satisfies both conditions* 1 *and* 2 *then no germ V at $0 \in \mathbf{C}^n$ of a complex hypersurface with an isolated singular point can be invariant by X.*

Proof. — Suppose on the contrary that V is invariant by X and let $G(x_1, \ldots, x_n) = 0$ be a defining equation of V. By Poincaré's theorem [A] X is formally equivalent to

$$Z = \lambda_1 x_1 \frac{\partial}{\partial x_1} + \ldots + \lambda_n x_n \frac{\partial}{\partial x_n}$$

and under this formal change of coordinates G becomes $F(x_1, \ldots, x_n)$ say. Since V is invariant by X we have $Z(F) = \nu F$. Write $F = F_m + F_{m+1} + \ldots$ and $\nu = \nu_0 + \nu_1 + \ldots$, decompositions into homogeneous polynomials. $Z(F) = \nu F$ gives

$$\lambda_1 x_1 \frac{\partial F_m}{\partial x_1} + \ldots + \lambda_n x_n \frac{\partial F_m}{\partial x_m} = \nu_0 F_m.$$

If $\nu_0 = 0$ then $\lambda_1 x_1 \frac{\partial F_m}{\partial x_1} + \ldots + \lambda_n x_n \frac{\partial F_m}{\partial x_n} \equiv 0$ and this gives resonance relations among the eigenvalues or implies that at least one of them is zero, depending on whether F_m has a pure power of some x_i or not. But this contradicts our hypothesis on X. Hence $\nu_0 \neq 0$ and ν is a unit. This implies $\nu^{-1} Z(F) = F$, i.e., $F \in JF$, its Jacobian ideal. By a theorem of Saito [S], in suitable coordinates $(y_1, \ldots, y_n)$, F is a quasi-homogeneous function and there exist positive rational numbers $a_1, \ldots, a_n$ such that $F = a_1 y_1 \frac{\partial F}{\partial y_1} + \ldots + a_n y_n \frac{\partial F}{\partial y_n}$. Let X be written in these coordinates as $Y_1 \frac{\partial}{\partial y_1} + \ldots + Y_n \frac{\partial}{\partial y_n}$ so that

$$Y_1 \frac{\partial F}{\partial y_1} + \ldots + Y_n \frac{\partial F}{\partial y_n} = \nu^* \left(a_1 y_1 \frac{\partial F}{\partial y_1} + \ldots + a_n y_n \frac{\partial F}{\partial y_n} \right)$$

where ν^* is a unit since ν is. Now, since $0 \in \mathbf{C}^n$ is an isolated singular point of V, $\left\{ \frac{\partial F}{\partial y_1}, \ldots, \frac{\partial F}{\partial y_n} \right\}$ is a regular sequence and therefore

$$Y_i = \nu^* a_i y_i \qquad i = 1, \ldots, n.$$

Writing $\nu^* = \nu_0^* + \nu_1^* + \ldots$ we get

$$Y_i = \nu_0^* a_i y_i + \ldots \qquad i = 1, \ldots, n$$

which gives $\frac{\lambda_i}{\lambda_j} = \frac{a_i}{a_j} \in \mathbf{Q}^+$ contradicting condition 1. $\square$

We can now finish the proof of Theorem 1. Note that for $0 \leq |\mu| << 1$, X_μ satisfies both conditions 1 and 2. By (4.1) and Lemma 5.1 $\mathcal{F}_\mu$ has no algebraic solution and by (4.2) and Lemma 5.2 $\mathcal{F}_\mu$ has no invariant normal surface. Now, if we had an invariant singular surface with non-isolated singularities, its singular set would contain an invariant curve and this is forbidden by (4.1) and Lemma 5.1. Theorem 1 is proved.

6. Proof of Theorem 2.

Let $\aleph_d$ denote the space of one-dimensional foliations on $\mathbf{C}P(3)$ of degree $d \geq 2$ and let $\Xi_d \subset \aleph_d$ be the set of non-degenerated foliations of degree d, i.e., foliations with non-zero eigenvalues at each of its $D = d^3 + d^2 + d + 1$ singularities (see [C]). The next lemma is a 3-dimensional version of Lemma 5 of [LN1] and we repeat the proof here for the sake of clarifying the arguments that follow.

LEMMA 6.1. — *Ξ_d is open, dense and connected in $\aleph_d$. Moreover, given $\mathcal{F}_0 \in \Xi_d$ with $\operatorname{sing}(\mathcal{F}_0) = \{p_1, \ldots, p_D\}$ there are neighborhoods $\mathcal{U}_0$ of $\mathcal{F}_0$ in $\aleph_d$, V_j of p_j in $\mathbf{C}P(3)$ and analytic functions $\psi_j : \mathcal{U}_0 \longrightarrow V_j$, $j = 1, \ldots, D$ such that $V_i \cap V_j = \emptyset$, $i \neq j$, and for any $\mathcal{F} \in \mathcal{U}_0$, $\psi_j(\mathcal{F})$ is the unique singularity of $\mathcal{F}$ in V_j.*

Proof. — Given $\mathcal{F}_0 \in \Xi_d$ with $\operatorname{sing}(\mathcal{F}_0) = \{p_1, \ldots, p_D\}$ consider an affine coordinate system (x, y, z) such that $\operatorname{sing}(\mathcal{F}_0) \cap H_\infty = \emptyset$ where H_∞ is the plane at infinity for this system. Suppose that $\mathcal{F}_0$ is represented in this coordinate system by the vector field

$$[P_0(x,y,z) + xg_0(x,y,z)]\frac{\partial}{\partial x} \\ +[Q_0(x,y,z) + yg_0(x,y,z)]\frac{\partial}{\partial y} + [R_0(x,y,z) + zg_0(x,y,z)]\frac{\partial}{\partial z}$$

where P_0, Q_0, R_0 and g_0 are polynomials of degree d with g_0 homogeneous. Since this representation of $\mathcal{F}_0$ is defined up to multiplication by a non-zero complex number let us fix the coefficient of a monomial of P_0, Q_0, R_0 or g_0 to be 1. Under this condition there is a neighborhood $\mathcal{V}$ of $\mathcal{F}_0$ such that any $\mathcal{F} \in \mathcal{V}$ has no singularities at H_∞ and has an unique representation in the coordinates (x, y, z) of the form

$$[P(x,y,z)+xg(x,y,z)]\frac{\partial}{\partial x}+[Q(x,y,z)+yg(x,y,z)]\frac{\partial}{\partial y}$$
$$+[R(x,y,z)+zg(x,y,z)]\frac{\partial}{\partial z}$$

where P, Q, R and g are polynomials with g homogeneous and the same coefficient of P, Q, R or g is 1. Define maps

$$F : \mathcal{V} \times \mathbf{C}^3 \longrightarrow \mathbf{C}^3 \qquad G : \mathcal{V} \times \mathbf{C}^3 \longrightarrow \mathbf{C}$$

by

$$F(\mathcal{F}, x, y, z) = (P + xg, Q + yg, R + zg)$$

$$G(\mathcal{F}, x, y, z) = \det \partial_2 F(\mathcal{F}, x, y, z) \equiv \operatorname{tr} \wedge^3 \partial_2 F(\mathcal{F}, x, y, z)$$

where ∂_2 means derivative with respect to the variables in $\mathbf{C}^3$, tr denotes the trace and $\wedge^i$ denotes the $i-th$ exterior product. Now, $p_j = (x_j, y_j, z_j)$ is a non-degenerated singularity of $\mathcal{F}_0$ if and only if $F(\mathcal{F}_0, x_j, y_j, z_j) = 0$ and $G(\mathcal{F}_0, x_j, y_j, z_j) \neq 0$. By the Implicit Function Theorem applied to F at $p_1, \ldots, p_D$ we have the functions $\psi_j : \mathcal{U}_0 \longrightarrow V_j$, $j = 1, \ldots, D$. Note that this shows that Ξ_d is open and that, if $\mathcal{V}$ is as above, then $\mathcal{F} \in \mathcal{V} \backslash \Xi_d$ if and only if there exists a point $(x, y, z) \in \mathbf{C}^3$ with $F(\mathcal{F}, x, y, z) = 0$ and $G(\mathcal{F}, x, y, z) = 0$. This implies that $\mathcal{V} \backslash \Xi_d$ is an analytic subset of $\mathcal{V}$ of codimension ≥ 1 since, by Theorem 1, Ξ_d is not empty. Hence Ξ_d is dense and connected. $\square$

Given $\mathcal{F}_0 \in \Xi_d$ let $\mathcal{U}_0$ and $\psi_j : \mathcal{U}_0 \longrightarrow V_j$ be as in Lemma 6.1 and consider the maps

$$\Psi_j : \mathcal{U}_0 \longrightarrow \mathbf{C}^3 \qquad j = 1, \ldots, D$$

defined by

$$\Psi_j(\mathcal{F}) = (\operatorname{tr} \partial_2 F(\mathcal{F}, \psi_j(\mathcal{F})), \operatorname{tr} \wedge^2 \partial_2 F(\mathcal{F}, \psi_j(\mathcal{F})), \operatorname{tr} \wedge^3 \partial_2 F(\mathcal{F}, \psi_j(\mathcal{F}))).$$

The components of Ψ_j are the elementary symmetric functions of the eigenvalues of the linear part of $\mathcal{F}$ at $\psi_j(\mathcal{F})$. If we let $\mathcal{D} \subset \mathbf{C}^3$ denote the discriminant variety of monic polynomials of degree 3 then the linear part of $\mathcal{F}$ at $\psi_j(\mathcal{F})$ has a repeated eigenvalue if and only if $\Psi_j(\mathcal{F}) \in \mathcal{D}$. Since, by Theorem 1, there exists $\mathcal{F} \in \Xi_d$ whose linear part at each singularity has distinct eigenvalues and Ξ_d is open and connected, we have that $\Psi_j^{-1}(\mathcal{D})$ is an analytic subset of $\mathcal{U}_0$ of codimension ≥ 1, $j = 1, \ldots, D$.

Hence, if $\Xi_d^* \subset \Xi_d$ is the subset consisting of foliations whose linear part at each singularity has distinct eigenvalues then $\Xi_d^* \subset \aleph_d$ is open, dense and connected. Let $\gamma : \mathbf{C}^3 \longrightarrow \mathbf{C}^3$ be defined by $\gamma(\lambda_1, \lambda_2, \lambda_3) = (\sigma_1, \sigma_2, \sigma_3)$ where $\sigma_i, \ \ i = 1, 2, 3$ are the elementary symmetric functions of $\lambda_1, \lambda_2, \lambda_3$. Then

$$\gamma_{|\mathbf{C}^3 \setminus \mathcal{R}} : \mathbf{C}^3 \setminus \mathcal{R} \longrightarrow \mathbf{C}^3 \setminus \mathcal{D}$$

is locally biholomorphic where $\mathcal{R} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbf{C}^3 : \lambda_i = \lambda_j, i \neq j\}$. Given $\mathcal{F}_0 \in \Xi_d^*$ choose a neighborhood $W_j \subset \mathbf{C}^3$ of $\Psi_j(\mathcal{F}_0)$ in which a local inverse δ_j of γ is defined and let $\mathcal{U}_0^* \subset \mathcal{U}_0 \cap \Xi_d^*$ be an open set such that $\Psi_j(\mathcal{U}_0^*) \subset W_j$ for $j = 1, \ldots, D$. Define

$$\Phi_j : \mathcal{U}_0^* \longrightarrow \mathbf{C}^3 \qquad \Phi_j = \delta_j \circ \Psi_j.$$

Then

$$\Phi_j(\mathcal{F}) = (\lambda_1(\psi_j(\mathcal{F})), \lambda_2(\psi_j(\mathcal{F})), \lambda_3(\psi_j(\mathcal{F})))$$

where $\lambda_i(\psi_j(\mathcal{F})), i = 1, 2, 3$, are the eigenvalues of the linear part of $\mathcal{F}$ at $\psi_j(\mathcal{F})$. To define $\Im_d$ it is enough to say what is $\Im_d \cap \mathcal{U}_0^*$. Put

$$\alpha_\ell^{(i,j,k)}(\mathcal{F}) = \frac{\lambda_j(\psi_\ell(\mathcal{F})) + \lambda_k(\psi_\ell(\mathcal{F}))}{\lambda_i(\psi_\ell(\mathcal{F}))}$$

$$\beta_\ell^{(i,j,k)}(\mathcal{F}) = \frac{[\lambda_i(\psi_\ell(\mathcal{F}))]^2}{\lambda_j(\psi_\ell(\mathcal{F}))\lambda_k(\psi_\ell(\mathcal{F}))}$$

where $(i, j, k) \in \{(1, 2, 3), (2, 1, 3), (3, 1, 2)\}$ and let $I_q = \{m \in \mathbf{N} : m \geq q\}$. Let us consider all conditions on the eigenvalues we have used, namely, $\lambda_i \notin \mathbf{Q}^+\lambda_j \ \ i \neq j$, all non-resonances and all sums of residues of the form

$$\sum \alpha_\ell^{(i,j,k)}(\mathcal{F}) \notin I_2$$

$$\sum \beta_\ell^{(i,j,k)}(\mathcal{F}) \notin I_1$$

$$\sum [\alpha_\ell^{(1,2,3)}(\mathcal{F}) + \alpha_\ell^{(2,1,3)}(\mathcal{F})] \notin I_2$$

$$\sum [\alpha_\ell^{(1,2,3)}(\mathcal{F}) + \alpha_\ell^{(3,1,2)}(\mathcal{F})] \notin I_2$$

$$\sum [\alpha_\ell^{(2,1,3)}(\mathcal{F}) + \alpha_\ell^{(3,1,2)}(\mathcal{F})] \notin I_2$$

$$\sum [\alpha_\ell^{(1,2,3)}(\mathcal{F}) + \alpha_\ell^{(2,1,3)}(\mathcal{F}) + \alpha_\ell^{(3,1,2)}(\mathcal{F})] \notin I_2$$

where in the last expression, because of Baum-Bott's theorem, the summation extends over proper subsets of $\mathrm{sing}(\mathcal{F})$. The denial of these conditions gives a denumerable set of analytic subsets of codimension 1 of $\mathcal{U}_0^*$ whose complement in $\mathcal{U}_0^*$ is $\Im_d \cap \mathcal{U}_0^*$. This proves Theorem 2 since $\Im_d$ is not empty.

$\square$

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CORRIGENDUM

In section 5, condition 2 must be included as a hypothesis of Lemma 5.1 for the result to hold. For in this case, by Poincaré's theorem, the vector field is diagonalisable and the proof of the lemma is correct. Also, for $\mu = 0$ and d odd we have the resonance

$$\lambda_{2,\ell,0} = \left(\frac{d+1}{2}\right)(\lambda_{1,\ell,0} + \lambda_{3,\ell,0})$$

but the derivative with respect to μ of this equation at $\mu = 0$ reads

$$\frac{-dD+2d+2}{2D} = \left(\frac{d+1}{2}\right)\left(\frac{-dD+4}{2D}\right)$$

which holds only for $d = 1$. Therefore, for $0 < |\mu| \ll 1$, $\mathcal{F}_\mu$ admits no resonances and all the results of section 5 hold. I wish to thank J.F. Mattei, J. Cano and E. Salem for pointing out these mistakes.