SYLVAIN KAHANE

On the complexity of sums of Dirichlet measures


<http://www.numdam.org/item?id=AIF_1993__43_1_111_0>
ON THE COMPLEXITY OF SUMS OF DIRICHLET MEASURES

by Sylvain KAHANE

1. Introduction.

Let \( E \) be a metrizable compact space. We denote by \( \mathcal{M}(E) \) (resp. \( \mathcal{M}_1(E) \)) the set of all non-negative (resp. probability) Borel measures on \( E \). Recall that \( \mathcal{M}_1(E) \) is a metrizable compact space for the weak* topology, which is the topology of the duality with the set \( C(E) \) of all continuous functions on \( E \); in the following, the topological complexity of a subset \( M \) of \( \mathcal{M}(E) \) actually means the topological complexity of \( M \cap \mathcal{M}_1(E) \). \( \mathcal{P}(E) \) (resp. \( \mathcal{K}(E) \)) denotes the set of all subsets (resp. compact subsets) of \( E \). Let \( C \) be a closed under countable intersections subset of \( \mathcal{P}(E) \). We denote by \( \mathcal{M}(C) \) the set of all non-negative Borel measures concentrated on an element of \( C : \mathcal{M}(C) = \bigcup_{X \in C} \mathcal{M}(X) \). Let \( C^\sigma \) (resp. \( C^\dagger \)) denote the set of all unions of sequences (resp. increasing sequences) of elements of \( C \). Note that \( \mathcal{M}(C^\dagger) \) is equal to the norm-closure \( \mathcal{M}(C) \) of \( \mathcal{M}(C) \) and that \( \mathcal{M}(C^\sigma) \) is the convex norm-closure of \( \mathcal{M}(C) \). We denote \( C^\perp \) the set of all measures which annihilate all elements of \( C \). We have the following algebraic decomposition : \( \mathcal{M}(E) = \mathcal{M}(C^\sigma) \oplus C^\perp \). Recall that \( \mathcal{K}(E) \) is a metrizable compact space in the Hausdorff topology. If \( C \) is a Borel subset of \( \mathcal{K}(E) \), then \( \mathcal{M}(C) \), \( \mathcal{M}(C^\dagger) \) and \( \mathcal{M}(C^\sigma) \) are analytic subsets of \( \mathcal{M}_1(E) \) and \( C^\perp \) is a coanalytic subset of \( \mathcal{M}_1(E) \).

Let \( T \) be the unit circle \( \mathbb{R}/\mathbb{Z} \). We are interested in the four following subsets of \( \mathcal{K}(T) \).

Key words : Analytic sets – Dirichlet measures – Singular measures – Sums of measures.
A.M.S. Classification : 28A33 – 04A15.
A compact subset $K$ of $T$ is a set of type $D$ or a Dirichlet set if for all $\varepsilon > 0$ and $N \in \mathbb{N}$ there exists $n \geq N$ such that $|\sin 2\pi nx| < \varepsilon$ for all $x \in K$.

A compact subset $K$ of $T$ is a set of type $H$ if there exist a non empty interval $I$ of $T$ and a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ of integers such that $n_k K \cap I = \emptyset$ for each integer $k$.

A compact subset $K$ of $T$ is a set of type $L$ or a lacunary set if there exist a sequence $\varepsilon_n \to 0^+$, a sequence $\alpha_n \to +\infty$ and for each integer $n$ a finite sequence $(I_k)$ of intervals such that $|I_k| \leq \varepsilon_n$ for each $k$, $d(I_k, I_{k'}) \geq \alpha_n \varepsilon_n$ for each $k \neq k'$ and $K \subseteq \bigcup I_k$.

A compact subset $K$ of $T$ is a set of type $L_0$ if there exist a sequence $\varepsilon_n \to 0^+$, $\alpha > 0$ and for each integer $n$ a finite sequence $(I_k)$ of intervals such that $|I_k| \leq \varepsilon_n$ for each $k$, $d(I_k, I_{k'}) \geq \alpha \varepsilon_n$ for each $k \neq k'$ and $K \subseteq \bigcup I_k$.

Note that both $H$ and $L$ are supersets of $D$ and subsets of $L_0$. The classes $D$ and $L$ are $\mathcal{G}_6$ subsets of $\mathcal{K}(T)$ and $H$ and $L_0$ are $\mathcal{K}_{\sigma_6}$ subsets [1].

A measure concentrated on a $D^\dagger$-set is called a Dirichlet measure. For every $\mu \in \mathcal{M}(T)$ and $n \in \mathbb{N}$, we denote $\hat{\mu}(n) = \int e^{2\pi inx} \, d\mu(x)$ and $\check{\mu}(n) = \int |\sin 2\pi nx| \, d\mu(x)$. For every $\mu \in \mathcal{M}(T)$, the following conditions are equivalent:

\begin{align*}
(1) \quad & \mu \in \mathcal{M}(D^\dagger) \\
(2) \quad & \limsup_{n \to \infty} |\hat{\mu}(n)| = \int d\mu \\
(3) \quad & \liminf_{n \to \infty} \check{\mu}(n) = 0.
\end{align*}

Note that $\mathcal{M}(D^\dagger)$ is a norm-closed $\mathcal{G}_6$ subset of $\mathcal{M}_1(T)$.

**Theorem 1.1.** — There does not exist a Borel subset $B$ of $\mathcal{M}_1(T)$ such that $B \cap L_0^\perp = \emptyset$ and $\mathcal{M}(D^\dagger) + \mathcal{M}(D^\dagger) \subset B$.

For all $M \subset \mathcal{M}(T)$ and $n \in \mathbb{N}$, we denote $M^{(n)}$ the set of all sums of $n$ elements of $M$.

**Corollary 1.2.** — The sets $\mathcal{M}(C^\dagger)^{(n)}$, $\overline{\mathcal{M}(C^\dagger)^{(n)}}$ and $\mathcal{M}(C^\sigma)$ are analytic non Borel for all $n \geq 2$ and $C = D$, $H$, $L$ or $L_0$.

We obtain also the following property which has been studied successively by Host, Louveau and Parreau [3], Kechris and Lyons [3] and Kaufman [2].
Corollary 1.3. — $C^\perp$ is a coanalytic non Borel set for $C = D$, $H$, $L$ or $L_0$.

Corollary 1.4. — None of the sets in the two previous corollaries can be pairwise separated by a Borel set.

We prove also that the sets $\mathcal{M}(C^\perp)^{(n)}$, for $n \geq 2$ and $C = D$, $H$, $L$ or $L_0$, are not norm-closed.

Theorem 1.5. — There exists a measure in $\mathcal{M}(D^\perp) + \mathcal{M}(D^\perp)$ which is not a finite sum of measures in $\mathcal{M}(L_0^\perp)$.

Theorem 1.6. — For every $n \geq 3$, there exists a measure in $\mathcal{M}(D^\perp) + \mathcal{M}(D^\perp)$ which is the sum of $n$ measures in $\mathcal{M}(D^\perp)$ and is not the sum of $n-1$ measures in $\mathcal{M}(L_0^\perp)$.

2. Kaufman’s reduction.

We follow Kaufman’s construction used to prove that $H^\perp$ is not a Borel set [2]. Let $\mathbb{N}$ be the set of positive integers, $[\mathbb{N}]$ be the set of all infinite subsets of $\mathbb{N}$, $\mathbb{N}^< \mathbb{N}$ be the set of all finite sequences of positive integers and $T$ be the set of trees on $\mathbb{N}$, i.e., $T \subset \mathcal{P}(\mathbb{N}^< \mathbb{N})$ and $T \in T$ if and only if all initial segments of $s \in T$ are also in $T$. We say that $T \in T$ is a well founded tree if $T$ has no infinite branch, i.e., there does not exist $\sigma \in \mathbb{N}^\mathbb{N}$ all whose initial segments belong to $T$. The set of all well founded trees is denoted by $WF$. Recall that $T$ is a Polish space in the product topology on $\mathcal{P}(\mathbb{N}^< \mathbb{N})$ and $WF$ is the classical example of a coanalytic non Borel set.

We denote $2^\mathbb{N}$ the compact, metrizable space $\{0,1\}^\mathbb{N}$. If $x \in 2^\mathbb{N}$, $x = (x(n))_{n \in \mathbb{N}}$ with $x(n) = 0$ or 1. Let $\lambda$ be the Lebesgue measure on $2^\mathbb{N}$. Let $\Sigma^*$ be the Polish space of all Borel sets on $2^\mathbb{N}$ with metric $d(A,B) = \lambda(A \Delta B)$, quotiented by the relation $d(A,B) = 0$; $\Sigma^*$ can be viewed as a closed subspace of $L^1(2^\mathbb{N})$. Consider the sets

$\mathcal{X} = \{ (A_n)_{n \in \mathbb{N}} \in \Sigma^*; \lambda(\bigcap_{R} A_n) = 0 \text{ for all } R \in [\mathbb{N}] \}$

and

$\mathcal{Y} = \{ (A_n)_{n \in \mathbb{N}} \in \Sigma^*; \lambda(\liminf_{R} A_n) = 1 \text{ for some } (R,S) \in [\mathbb{N}]^2 \},$
where \( \liminf_R A_n = \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m, n \in R} A_n \). Note that \( \mathcal{X} \) is a coanalytic subset of \( \Sigma^N \) [2] and that \( \mathcal{Y} \) is an analytic subset of \( \Sigma^N \).

**Lemma 2.1.** There is a continuous mapping \( \Phi \) from \( T \) to \( \Sigma^N \) such that \( \Phi(WF) \subset \mathcal{X} \) and \( \Phi(WF^c) \subset \mathcal{Y} \). Therefore, there is no Borel subset \( B \) of \( \Sigma^N \) such that \( \mathcal{Y} \subset B \) and \( \mathcal{X} \cap B = \emptyset \).

**Proof.** **Construction of \( \Phi \).** To each \( s \in \mathbb{N}^\mathbb{N} \), we attach subsets \( E(s) \) and \( F(s) \). Let \( <, > \) be a one-to-one mapping from \( \mathbb{N}^2 \) to \( \mathbb{N} \). We define \( E(s) \) and \( F(s) \) by induction on the length \( |s| \) of \( s \). Let \( E(\emptyset) = 2^\mathbb{N} \) and \( F(\emptyset) = \emptyset \). If \( s \in \mathbb{N}^\mathbb{N} \) has length \( |s| = k - 1 \) and \( n_k \in \mathbb{N} \), put

\[
E(s \upharpoonright n_k) = \{ x \in 2^\mathbb{N}; (x \in E(s) \text{ and } \exists i \in [kn_k, k(n_k + 1)], x(<k, i>) = 0) \\
\quad \text{or } (x \in F(s) \text{ and } \forall i \in [kn_k, k(n_k + 1)], x(<k, i>) = 1) \}
\]

and

\[
F(s \upharpoonright n_k) = \{ x \in 2^\mathbb{N}; (x \in F(s) \text{ and } \exists i \in [kn_k, k(n_k + 1)], x(<k, i>) = 0) \\
\quad \text{or } (x \in E(s) \text{ and } \forall i \in [kn_k, k(n_k + 1)], x(<k, i>) = 1) \}.
\]

We have \( E((n_1)) = \{ x \in 2^\mathbb{N}; x(<1, n_1>) = 0 \} \) and \( F((n_1)) = \{ x \in 2^\mathbb{N}; x(<1, n_1>) = 1 \} \) if \( n_1 \in \mathbb{N} \). Note that \( E(s) = F(s)^c \) and \( \lambda(E(s)) = \lambda(F(s)) = \frac{1}{2} \) for all \( s \in \mathbb{N}^{\mathbb{N}} \setminus \{ \emptyset \} \). Let \( \sigma \in \mathbb{N}^\mathbb{N} \). The length \( k \) initial segment of \( \sigma \) is denoted by \( \sigma_{\upharpoonright k} \). We have

\[
\lambda \left( \bigcap_{k \geq n} E(\sigma_{\upharpoonright k}) \right) \geq \lambda(E(\sigma_{\upharpoonright n})) \times \prod_{k > n} (1 - 2^{-k})
\]

for each \( n \in \mathbb{N} \). But \( \lim_{n \to +\infty} \prod_{k > n} (1 - 2^{-k}) = 1 \), whence

\[
\lambda \left( \liminf_{n \to +\infty} E(\sigma_{\upharpoonright k}) \cup \liminf F(\sigma_{\upharpoonright k}) \right) = 1.
\]

Let us enumerate \( \mathbb{N}^{\mathbb{N}} = \{ s_n; n \in \mathbb{N} \} \) and consider the mapping \( \Phi: T \to \Sigma^N, T \mapsto (\Phi_n(T))_{n \in \mathbb{N}} \) defined by

\[
\Phi_n(T) = \begin{cases} 
E(s_p) & \text{if } n = 2p \text{ and } s_p \in T, \\
F(s_p) & \text{if } n = 2p + 1 \text{ and } s_p \in T, \\
\emptyset & \text{otherwise}.
\end{cases}
\]

Clearly, \( \Phi \) is continuous and \( \Phi(WF^c) \subset \mathcal{Y} \).

To complete the proof of Lemma 2.1, it remains only to show that \( \Phi(WF) \subset \mathcal{X} \). Let \( T \in T \) such that there exists \( R \in [\mathbb{N}] \) with \( \lambda \left( \bigcap_R \Phi_n(T) \right) > 0 \). Let us suppose that \( R \cap 2\mathbb{N} \) is infinite (the case
ON THE COMPLEXITY OF SUMS OF DIRICHLET MEASURES

Let $P \in [\mathbb{N}]$ such that $2P \subset R$. We have $\lambda \left( \bigcap_{p} E(s_p) \right) > 0$. Let $s_p = (n_{p_1}^p, n_{p_2}^p, \ldots, n_{|s_p|}^p)$ for each $p \in P$. Let us prove that $\{ n_k^p; p \in P \}$ is finite for all $k \in \mathbb{N}$. Otherwise, there exist $k \in \mathbb{N}$, $s \in \mathbb{N}^{<\mathbb{N}}$ and an infinite subset $P'$ of $P$ such that $s_p = s^p n_k^p t_p$ with $t_p \in \mathbb{N}^{<\mathbb{N}}$ for all $p \in P'$ and $n_k^p \neq n_k^{p'}$ for distinct $p, p' \in P'$. Let $p \in P'$. For all $x \in 2^\mathbb{N}$, we have

$$x \in E(s_p) \iff \begin{cases} \exists i \in [kn_k^p, k(n_k^p + 1)] \text{ such that } x(<k,i>) = 0, \\
\forall i \in [kn_k^p, k(n_k^p + 1)] \text{ such that } x(<k,i>) = 1, \end{cases}$$

where $E_k(t)$ and $F_k(t)$ can be defined by induction as follows: $E_k(\emptyset) = 2^\mathbb{N}$ and $F_k(\emptyset) = \emptyset$; if $t \in \mathbb{N}^{<\mathbb{N}}$ has length $|t| = j - 1$ and $m_j \in \mathbb{N}$, put

$$E_k(t \cup m_j) = \{ x \in 2^\mathbb{N}; (x \in E_k(t) \text{ and } \exists i \in [(k + j)m_j, (k + j)(m_j + 1)], x(<k + j,i>) = 0) \}
\text{ or } (x \in F_k(t) \text{ and } \forall i \in [(k + j)m_j, (k + j)(m_j + 1)], x(<k + j,i>) = 1) \}$$

and

$$F_k(t \cup m_j) = \{ x \in 2^\mathbb{N}; (x \in E_k(t) \text{ and } \exists i \in [(k + j)m_j, (k + j)(m_j + 1)], x(<k + j,i>) = 0) \}
\text{ or } (x \in F_k(t) \text{ and } \forall i \in [(k + j)m_j, (k + j)(m_j + 1)], x(<k + j,i>) = 1) \}.$$

Note that $E_k(t) = F_k(t)^c$ for all $t \in \mathbb{N}^{<\mathbb{N}}$. Moreover in the probability space $(2^\mathbb{N}, \lambda)$, the conditions $\{ x \in E(s) \}$, $\{ \exists i \in [(k + j)m_j, (k + j)(m_j + 1)], x(<k + j,i>) = 0 \}$ and $\{ x \in E_k(t) \}$ are independent, because the mappings $x \mapsto x(j)$, $j \in \mathbb{N}$, are independent. The conditions $\{ \exists i \in [kn_k^p, k(n_k^p + 1)], x(<k,i>) = 0 \}$ and $\{ \exists i \in [kn_k^{p'}, k(n_k^{p'} + 1)], x(<k,i>) = 0 \}$ are also independent for distinct $p, p' \in P'$. So we can explicitly calculate

$$\lambda \left( \bigcap_{p \in I} E(s_p) \right) = \sum_{i=0}^{|I|} \alpha_i (2^{-k})^i (1 - 2^{-k})^{|I|-i}$$
where $\alpha_0 = \lambda\left([E(s) \cap \bigcap_{p \in I} E_k(t_p)] \cup [F(s) \cap \bigcap_{p \in I} F_k(t_p)]\right)$ and $\alpha_i \geq 0$, 
\[ \sum_{i=0}^{|I|} \alpha_i = 1. \] So $\alpha_0 \leq \frac{1}{2}$, whence

\[ \lambda\left(\bigcap_{p \in I} E(s_p)\right) \leq \frac{1}{2}(1 - 2^{-k})^{|I|} + \frac{1}{2}2^{-k}(1 - 2^{-k})^{|I|-1} = \frac{1}{2}(1 - 2^{-k})^{|I|-1}. \]

Thus $\lambda\left(\bigcap_{p \in P'} E(s_p)\right) = 0$ which is a contradiction, and proves that 
\[ \{ n_k^p, p \in P \} \] is finite for all $k \in \mathbb{N}$. So the tree $T' = \{ s \in \mathbb{N}^\infty; \exists p \in P, s is an initial segment of s_p \}$ is an infinite tree ($P$ is infinite) with finite branching, so $T' \not\subseteq WF$, whence $T \not\subseteq WF$. \hfill \Box 

### 3. The abstract case.

We introduce a subset $I$ of $\mathcal{K}(2^\mathbb{N})$ which plays the role of $D$ in this simpler case.

A compact subset $K$ of $2^\mathbb{N}$ is a set of type $I$ if for all $N \in \mathbb{N}$ there exists $n \geq N$ such that $x(n) = 0$ for all $x \in K$. Note that $I$ is a $G_\delta$ subset of $\mathcal{K}(2^\mathbb{N})$.

For each $A \in [\mathbb{N}]$, put 
\[ K_A = \{ x \in 2^\mathbb{N}; \forall n \in A, x(n) = 0 \}, \]
\[ K_A^\uparrow = \{ x \in 2^\mathbb{N}; \exists m \in \mathbb{N}, \forall n \in A \cap [m, +\infty[, x(n) = 0 \} \]

and let $\mu_A$ be the Haar measure on the subgroup $K_A$ of $2^\mathbb{N} \cong (\mathbb{Z}/2\mathbb{Z})^\mathbb{N}$. More precisely, $\mu_A$ is the product measure $\bigotimes_{n \in \mathbb{N}} \nu_n$ with $\nu_n = \delta_0$ if $n \in A$ and $\nu_n = \frac{1}{2}(\delta_0 + \delta_1)$ otherwise.

We will use the following elementary, but fundamental fact.

**Lemma 3.1.** Let $A$ and $B \in [\mathbb{N}]$. If $B \setminus A$ is finite, then $\mu_A(K_B^\uparrow) = 1$. If $B \setminus A$ is infinite, then $\mu_A(K_B^\uparrow) = 0$.

Note that
\[ I = \{ K \in \mathcal{K}(2^\mathbb{N}); \exists A \in [\mathbb{N}], K \subset K_A \} \]

and
\[ I^\uparrow = \{ K \in \mathcal{K}(2^\mathbb{N}); \exists A \in [\mathbb{N}], K \subset K_A^\uparrow \}. \]
Let \( \hat{\mu}(n) = \int x(n) \, d\mu(x) \). We have
\[
\mathcal{M}(I^\dagger) = \{ \mu \in \mathcal{M}(2^\mathbb{N}) ; \liminf \hat{\mu}(n) = 0 \}.
\]
Note that \( \mathcal{M}(I^\dagger) \) is a \( \mathcal{G}_\delta \) subset of \( \mathcal{M}_1(2^\mathbb{N}) \).

Following Kaufman's ideas [2], we assign to each sequence \( \bar{A} = (A_n)_{n \in \mathbb{N}} \in \Sigma^\mathbb{N} \) a mapping \( A \) from \( 2^\mathbb{N} \) to \( \mathcal{P}(\mathbb{N}) \), defined by \( A(x) = \{ n \in \mathbb{N} ; x \in A_n \} \), and a measure \( \nu_{\bar{A}} \) defined by \( \nu_{\bar{A}} = \int \mu_{A(x)} \, d\lambda(x) \). Let \( \Theta \) be the mapping from \( \Sigma^\mathbb{N} \) to \( \mathcal{M}_1(2^\mathbb{N}) \) defined by \( \Theta(\bar{A}) = \nu_{\bar{A}} \). Note that \( \Theta \) is continuous.

**Lemma 3.2.** — \( \Theta(\mathcal{X}) \subset I^\perp \) and \( \Theta(\mathcal{Y}) \subset \mathcal{M}(I^\dagger) + \mathcal{M}(I^\dagger) \).

**Proof.** — Using Lemma 3.1 we have
\[
\lambda(\liminf_{R} A_n) = \lambda(\{ x \in 2^\mathbb{N} ; R \setminus A(x) \text{ finite} \}) = \nu_{\bar{A}}(K^1_R),
\]
for all \( \bar{A} = (A_n)_{n \in \mathbb{N}} \in \Sigma^\mathbb{N} \) and \( R \in [\mathbb{N}] \). This remark allows us to finish easily the proof.

We have an abstract version of Theorem 1.1.

**Theorem 3.3.** — There does not exist a Borel subset \( B \) of \( \mathcal{M}_1(2^\mathbb{N}) \) such that \( \mathcal{M}(I^\dagger) + \mathcal{M}(I^\dagger) \subset B \) and \( B \cap I^\perp = \emptyset \).

**Proof.** — Such \( B \) insure \( (\Phi \circ \Theta)^{-1}(B) = WF^c \) and cannot be a Borel set, because \( \Phi \circ \Theta \) is continuous.

---

**4. How to go from the abstract case to \( T \).**

Every element \( x \) of \( T \) can be expressed in the form \( x = \sum_{n \in \mathbb{N}} x(n) 2^{-n} \) with \( x(n) \) either 0 or 1, and \( x(n) = 0 \) for large enough \( n \) if \( x \) is rational.

For each \( A \in [\mathbb{N}] \), let
\[
K_A = \{ x \in T ; \forall n \in A, x(n) = 0 \},
\]
\[
K^1_A = \{ x \in T ; \exists m \in \mathbb{N}, \forall n \in A \cap [m, +\infty[, x(n) = 0 \}
\]
and
\[
\mu_A = \mathcal{X}_{n \in \mathbb{N}} \left( \frac{1}{2}\delta_0 + \frac{1}{2}\delta_{2^{-n}} \right)
\]
be the canonical Bernoulli product measure concentrated on $K_A$. A set $A$ is called colacunary if for each $n \in \mathbb{N}$, there exists $a \in \mathbb{N}$ such that $[a, a + n] \subset A$. Note that $K_A \in D$ if $A$ is colacunary.

Lemma 3.1 still holds with these new notations. Our next goal is to extend this property to the $L_0$-sets.

**Lemma 4.1.** — Let $K \in L_0$ and $\alpha > 0$ and $(\varepsilon_n)_{n \in \mathbb{N}}$ witnessing this. Let $A \in [\mathbb{N}]$ and $c = \sup(-[\log_2 \alpha], 0) + 2$. If $\limsup d\left( \frac{1}{\varepsilon_n}, A \right) \geq c$, then $\mu_A(K) = 0$, where $d(x, A) = \inf \{ |x - n| ; n \in A \} \ (x \in \mathbb{R})$.

This property is derived from a result of Lyons [4] whose conclusion is much more precise, but which concerns only the case $K \in H$ and $A$ lacunary. The proof of Lemma 4.1 uses the following simple result ([1] Lemma 2.9).

**Lemma 4.2.** — Let $K \in L_0$ and $\alpha > 0$ and $\varepsilon_n \in ]0, \frac{1}{8}[$ witnessing this. Let $m = -\lfloor \log_2 \varepsilon_n \rfloor$ and $p = \sup(-\lfloor \log_2 \alpha \rfloor, 0)$. For each $(x_i)_{i \in [1, m - 2]} \in \{0, 1\}^{m-2}$, there exists $(x_i)_{i \in [m-1, m+p+1]} \in \{0, 1\}^{p+3}$ such that for each $x \in T$, 

$$(\forall i \in [1, m+p+1], x(i) = x_i) \implies x \notin K.$$

**Proof of Lemma 4.1.** — Let $K \in L_0$ and let $\alpha > 0$ and $(\varepsilon_n)_{n \in \mathbb{N}}$ witness this. Let $p = \sup(-\lfloor \log_2 \alpha \rfloor, 0)$ and $m_n = -\lfloor \log_2 \varepsilon_n \rfloor$ for each $n \in \mathbb{N}$. Without loss of generality, we can suppose that the intervals $[m_n - 1, m_n + p + 1]$, $n \in \mathbb{N}$, are pairwise disjoint and disjoint from $A$. Let $n \in \mathbb{N}$. There exists, by Lemma 4.2, a mapping $\varphi_n$ from $\{0, 1\}^{1, m_n - 2}$ to $\{0, 1\}^{[m_n - 1, m_n + p + 1]}$ such that the set $B_n$ of all $x \in T$ such that

$$\forall s \in \{0, 1\}^{1, m_n - 2} \left( s = (x(i))_{i \in [1, m_n - 2]} \implies \varphi(s) = (x(i))_{i \in [m_n - 1, m_n + p + 1]} \right)$$

is disjoint from $K$. But $\mu_A(B_n) = 2^{-p-3}$ and the $B_n$'s, $n \in \mathbb{N}$, are independent events in the probability space $(T, \mu_A)$, so $\mu_A(K) \leq \mu_A(\bigcap B_n^c) = \prod \mu_A(B_n^c) = 0$.

5. Proof of Theorem 1.1.

Let $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ be two sequences of positive integers such that $\lim (b_k - a_k) = +\infty$ and $\lim (a_{k+1} - b_k) = +\infty$. Put $I_k = [a_k, b_k] \subset \mathbb{N}$. 


For $A \subseteq \mathbb{N}$, put $\tilde{A} = \bigcup_{k \in A} I_k$. Note that $\tilde{A}$ is colacunary if and only if $A$ is infinite.

To each sequence $\tilde{A} = (A_n)_{n \in \mathbb{N}} \in \Sigma^\mathbb{N}$, we assign a mapping $A$ from $2^\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$ defined by $A(x) = \{ n \in \mathbb{N}; x \in A_n \}$, and next a measure $\tilde{\nu}_A = \int \mu_{\tilde{A}(x)} \, d\lambda(x)$. Let $\tilde{\Theta}$ be the mapping from $\Sigma^\mathbb{N}$ to $\mathcal{M}_1(T)$ defined by $\tilde{\Theta}(\tilde{A}) = \tilde{\nu}_A$. Note that $\tilde{\Theta}$ is continuous.

**Lemma 5.1.** — $\tilde{\Theta}(\mathcal{X}) \subseteq L_0$ and $\tilde{\Theta}(\mathcal{Y}) \subseteq \mathcal{M}(D^\updownarrow) + \mathcal{M}(D^\updownarrow)$.

**Proof.** — Using Lemma 3.1, we have for each $\tilde{A} = (A_n)_{n \in \mathbb{N}} \in \Sigma^\mathbb{N}$ and each $R \in [\mathbb{N}]$,
\[
\lambda(\liminf_{R_n} A_n) = \lambda(\{ x \in 2^\mathbb{N}; R \setminus A(x) \text{ finite} \})
\]
\[
= \lambda(\{ x \in 2^\mathbb{N}; \tilde{R} \setminus \tilde{A}(\tilde{x}) \text{ finite} \})
\]
\[
= \tilde{\nu}_A(K^\updownarrow_{\tilde{R}}).
\]

But $K^\updownarrow_{\tilde{R}} \subseteq D^\updownarrow$, because $\tilde{R}$ is colacunary, whence $\tilde{\Theta}(\mathcal{Y}) \subseteq \mathcal{M}(D^\updownarrow) + \mathcal{M}(D^\updownarrow)$.

The previous remark does not allow us to prove that $\tilde{\Theta}(\mathcal{X}) \subseteq L_0$. Let $\tilde{A} = (A_n)_{n \in \mathbb{N}} \in \Sigma^\mathbb{N}$ such that $\tilde{\Theta}(\tilde{A}) \notin L_0$, i.e., there exists $K \in L_0$ such that $\tilde{\nu}_A(K) > 0$. Let $\alpha > 0$ and $(\varepsilon_n)_{n \in \mathbb{N}}$ witness that $K \in L_0$. We have $\lambda(H) > 0$ with $H = \{ x \in 2^\mathbb{N}; \mu_{\tilde{A}(x)}(K) > 0 \}$.

Now $H \subseteq \{ x \in 2^\mathbb{N}; \limsup d\left(\log_2 \frac{1}{\varepsilon_n}, \tilde{A}(x)\right) < c \}$ by Lemma 4.1. Thus $\limsup d\left(\log_2 \frac{1}{\varepsilon_n}, \tilde{N} \right) < c$, because $\lambda(H) > 0$, so $d\left(\log_2 \frac{1}{\varepsilon_n}, \tilde{N} \right) \leq c$ for large enough $n$. Moreover $a_{k+1} - b_k > 2c$ for large enough $k$, so there exists a unique $k_n$ such that $d\left(\log_2 \frac{1}{\varepsilon_n}, I_k \right) < c$ for large enough $n$ ($n \geq n_0$). Let $R = \{ k_n; n \geq n_0 \}$. We have $H \subseteq \{ x \in 2^\mathbb{N}; R \setminus A(x) \text{ finite} \} = \liminf_{R_n} A_n$, so there exists $a \in \mathbb{N}$ such that $\lambda(\bigcap_{R \cap [a, +\infty[} A_n) > 0$, whence $\tilde{A} \notin \mathcal{X}$. □

Clearly, we can deduce Theorem 1.1 from this.

**6. Theorems 1.5 and 1.6 in the abstract case.**

We denote $\mathcal{C} = \{ X \cup Y; (X, Y) \in \mathcal{C}^2 \}$ for $\mathcal{C} \subseteq \mathcal{P}(E)$ where $E$ is a metrizable, compact set. It is easy to verify that
\[
\mathcal{M}(\mathcal{C}^\updownarrow) + \mathcal{M}(\mathcal{C}^\downarrow) = \mathcal{M}(\mathcal{C}^{\updownarrow}\downarrow)
\]
and
\[ M(C^\top) + M(C^\top) = M((C^\top)^\top). \]

We use again the notations of Part 3. Let \((A_n)_{n \in \mathbb{N}}\) be a sequence of infinite, pairwise disjoint subsets of \(\mathbb{N}\). Consider the set
\[ X_0 = \bigcup_{n \in \mathbb{N}} \bigcap_{n \geq m} (K_{A_{2n}} \cup K_{A_{2n+1}}) \]
which belongs to \((I^U)^\top\). Note that \(X_0 \not\subseteq (I^U)^\top\). To all \(x \in 2^\mathbb{N}\) and \(m \in \mathbb{N}\), we attach \(C_m(x) = \bigcup_{n \geq m} A_{2n+x(n)}\); note that \(K_{C_m(x)} \subseteq X_0\). Consider the weak*-integral
\[ \mu_\infty = \sum_{m \in \mathbb{N}} 2^{-m} \int \mu_{C_m(x)} \, d\lambda(x). \]
Clearly \(\mu_\infty \in M_1(X_0)\) and \(M_1(X_0) \subseteq \overline{M(I^\top) + M(I^\top)}\).

**Lemma 6.1.** — \(\mu_\infty\) is not a finite sum of measures in \(M(I^\top)\).

We can immediately deduce an abstract version of Theorem 1.5.

**Theorem 6.2.** — There exists a measure in \(\overline{M(I^\top) + M(I^\top)}\) which is not a finite sum of measures in \(M(I^\top)\).

We can generalize the previous construction. Let \((F_m)_{m \in \mathbb{N}}\) be a sequence of finite subsets of \(\mathbb{N}\). We define
\[ \mu(F_m) = \sum_{m \in \mathbb{N}} 2^{-m} \int \mu_{C(x,F_m)} \, d\lambda(x), \]
where \(C(x,F_m) = \bigcup_{n \not\in F_m} A_{2n+x(n)}\). Note that \(\mu(F_m) \in M_1(X_0)\). In particular, \(\mu_\infty = \mu([1,m])\).

Let \(k \in \mathbb{N}\) and \((F_m^k)_{m \in \mathbb{N}}\) be an enumeration of all subsets of \(\mathbb{N}\) containing \(k\) elements and
\[ \mu_k = \mu(F_m^k). \]
In particular \(\mu_1 = \mu_{\{m\}}\). Note that \(\mu_k\) is concentrated on \(\bigcup_{n \in F} (K_{A_{2n}} \cup K_{A_{2n+1}})\) for each subset \(F\) of \(\mathbb{N}\) containing \(k + 1\) elements, whence \(\mu_k \in M(I^\top)^{(2k+2)}\).

**Lemma 6.3.** — \(\mu_k \not\in M(I^\top)^{(2k+1)}\) for each \(k \geq 0\).

We can immediately deduce an abstract version of Theorem 1.6.
THEOREM 6.4. — For every $n \geq 3$, there exists a measure in $\mathcal{M}(I^\uparrow) + \mathcal{M}(I^\uparrow)$ which is the sum of $n$ measures in $\mathcal{M}(I^\uparrow)$ and is not the sum of $n - 1$ measures in $\mathcal{M}(I^\uparrow)$.

PROPOSITION 6.5. — For every $n \geq 2$, there exists a measure in $\mathcal{M}(I^\uparrow)^{(n)}$ which is not in $\mathcal{M}(I^\uparrow)^{(n-1)}$.

Proof. — Consider $\nu_n = \frac{1}{n} \sum_{k=1}^{n} \mu_{A_k}$. If $B \in [N]$, there exists at most one $k$ such that $B \setminus A_k$ finite, so, by Lemma 3.1, $\nu_n(K_B^\uparrow) \leq \frac{1}{n}$. If $X$ is a union of $n - 1$ $I^\uparrow$-sets, then $\nu_n(X) \leq \frac{n-1}{n}$, whence $\nu_n \notin \mathcal{M}(I^\uparrow)^{(n-1)}$. □

We deduce from Theorem 6.4 and Proposition 6.5 the following fact.

COROLLARY 6.6. — The sets $\mathcal{M}(I^\uparrow)^{(n)}$ and $\mathcal{M}(I^\uparrow)^{(n)}$, $n \geq 2$, are all distinct.

We will now prove Lemmas 6.1 and 6.3.

LEMMA 6.7. — Let $\mu = \mu(F_m)$ and $X \in I^\uparrow$. Then $\mu|_X$ is concentrated on $K_{A_p}$ for some $p$.

Proof. — Let $X \in I^\uparrow$ such that $\mu(X) > 0$. There exists $B \in [N]$ such that $X \subset K_B^\uparrow$. If $B \setminus \bigcup_{p \in \mathbb{N}} A_p$ is infinite, then $B \setminus C(x, F_m)$ is infinite for all $m \in \mathbb{N}$ and $x \in 2^\mathbb{N}$. Using Lemma 3.1, we have $\mu_C(x, F_m)(K_B^\uparrow) = 0$, so $\mu(K_B^\uparrow) = 0$ which contradicts our hypothesis, whence $B \setminus \bigcup_{p \in \mathbb{N}} A_p$ is finite.

Consider $C = \{ p \in \mathbb{N}; B \cap A_p \neq \emptyset \}$. If $C$ is infinite, $C = \{ 2n_k + \zeta_k; k \in \mathbb{N}, \zeta_k = 0, 1 \}$. If $m \in \mathbb{N}$ and $x \in 2^\mathbb{N}$ are such that $\mu_C(x, F_m)(K_B^\uparrow) > 0$, then $B \setminus C(x, F_m)$ is finite by Lemma 3.1, so $x(n_k) = \zeta_k$ for large enough $k$, because $F_m$ is finite. But $\lambda(\{ x \in 2^\mathbb{N}; x(n_k) = \zeta_k \text{ for large enough } k \}) = 0$, so $\mu(K_B^\uparrow) = 0$. This contradiction prove that $C$ is finite and $B \cap A_p$ is infinite for some $p$.

If $m \in \mathbb{N}$ and $x \in 2^\mathbb{N}$ are such that $\mu_C(x, F_m)(K_B^\uparrow) > 0$, then $A_p \subset C(x, F_m)$, so $\mu_C(x, F_m)(K_{A_p}) = 1$, whence $\mu|_{K_B^\uparrow}$ is concentrated on $K_{A_p}$. □

Proof of Lemma 6.1. — Let $X$ be a finite union of $I^\uparrow$-sets. Using Lemma 6.7, we can suppose that $X = \bigcup_{n \in F} K_{A_p}$ for some finite subset $F$. 

of \( \mathbb{N} \). Let \( m_0 \) with \( p < 2m_0 \) for all \( p \in F \). For all \( m \geq m_0 \) and \( x \in 2^\mathbb{N} \), 
\[ \mu_{C_m(x)}(X) = 0 \] by Lemma 3.1. So \( \mu(\infty)(X^c) > 0 \). \( \square \)

**Proof of Lemma 6.3.** — Using Lemma 6.7, we have just to prove that \( \mu_k \) cannot be concentrated on \( X = \bigcup_{n \in F} K_{A_p} \) for every \( F \) with cardinality \( \leq 2k + 1 \). Let \( F \) be a set having this property. Thus \( F = \{2n; n \in G\} \cup \{2n + 1; n \in G\} \cup \{2n + \zeta_n; n \in H\} \) with \( \zeta_n \) either 0 or 1. Now \( G \) has cardinality \( \leq k \), so \( G \subset F^{k}_m \) for some \( m_0 \). Using Lemma 3.1, we have \( \mu_{C(x,F^{k}_m)}(X^c) > 0 \) for every \( x \in 2^\mathbb{N} \) such that \( x(n) = 1 - \zeta_n \) for each \( n \in H \), whence \( \mu_k(\infty)(X^c) > 0 \). \( \square \)

**7. Proof of theorems 1.5 and 1.6.**

To prove Theorems 1.5 and 1.6, we follow the ideas and techniques of Part 6. We introduce the same notations and the same lemmas, expect that, in this case, \((A_n)_{n \in \mathbb{N}}\) is a sequence of colacunary subsets of \( \mathbb{N} \) such that for \( k \) going to \( +\infty \), \( d(A_n \cap [k, +\infty[, A_m \cap [k, +\infty[) \to +\infty \) uniformly for all \( n \neq m \). Moreover, \( K_A \) and \( \mu_A \), \( A \in [\mathbb{N}] \), are the same as in Part 4. Finally, Lemma 6.7 is replaced by the following result.

**LEMMA 7.1.** — Let \( \mu = \mu(F_m) \) and \( X \in L^1_0 \). Then \( \mu\lambda X \) is concentrated on \( K_{A_p} \) for some \( p \).

**Proof.** — We start by proving the result for \( K \in L_0 \). Let \( \alpha > 0 \) and \((\varepsilon_k)_{k \in \mathbb{N}}\) witness that \( K \in L_0 \). Let \( p = \sup(-[\log_2 \varepsilon_k], 0) \), \( m_k = -[\log_2 \varepsilon_k] \) and \( J_k = [m_k - 1, m_k + p + 1], k \in \mathbb{N} \). If \( \mu(K) > 0 \), then \( \mu_{C(x,F_m)}(K) > 0 \) for some \( x \in 2^\mathbb{N} \) and \( m \in \mathbb{N} \). But \( C(x,F_m) \subset \bigcup A_p \), so, using Lemma 4.1, we deduce that \( J_k \) meets at least one \( A_p \) for large enough \( k \). Now \( |J_k| \) is constant and as \( k \to +\infty \), \( d(A_n \cap [k, +\infty[, A_m \cap [k, +\infty[) \to +\infty \) uniformly for all \( n \neq m \), so \( J_k \) meets exactly one \( A_{p_k} \) for large enough \( k \). If \((p_k)_{k \in \mathbb{N}}\) is unbounded, then \((p_k)_{k \in D}\) is injective for some \( D \in [\mathbb{N}] \). Put \( p_k = 2n_k + \zeta_k \) with \( \zeta_k = 0 \) or 1. By Lemma 4.1, if \( \mu_{C(x,F_m)}(K) > 0 \) for some \( x \in 2^\mathbb{N} \), then \( x(n_k) = \zeta_k \) for large enough \( k \). But \( \lambda\{x \in 2^\mathbb{N}; x(n_k) = \zeta_k \text{ for large enough } k\} \) = 0, so \( \mu(K) = 0 \). If \((p_k)_{k \in \mathbb{N}}\) is bounded, there exists \( p \) such that \( p = p_k \) for infinitely many \( k \). If \( m \in \mathbb{N} \) and \( x \in 2^\mathbb{N} \) are such that \( \mu_{C(x,F_m)}(K) > 0 \), then \( A_p \subset C(x,F_m) \), so \( \mu_{C(x,F_m)}(K_{A_p}) = 1 \), whence \( \mu(K) \) is concentrated on \( K_{A_p} \).
Let $X \in L_0^\dagger$. There exists a sequence $(K_j)_{j \in \mathbb{N}}$ of $L_0$-sets such that $X \subseteq \liminf K_j$. Now, for each $j$, there exists $p_j$ that $\mu_{|K_j}$ is concentrated on $K_{A_{p_j}}$, so $\mu_{|X}$ is concentrated on $\liminf K_{A_{p_j}}$. As before, $\mu(\liminf K_{A_{p_j}}) = 0$ if $(p_j)_{j \in \mathbb{N}}$ is unbounded. So $\mu_{|X}$ is concentrated on $K_{A_p}$ for some $p$. \hfill \Box

BIBLIOGRAPHY


Sylvain KAHANE,
Université de Paris VI
Equipe d'Analyse
4, place Jussieu
75252 Paris Cedex 05 (France).