ANNALES DE L'INSTITUT FOURIER

PING XU

Poisson cohomology of regular Poisson manifolds

Annales de l'institut Fourier, tome 42, nº 4 (1992), p. 967-988

<http://www.numdam.org/item?id=AIF_1992__42_4_967_0>

Ann. Inst. Fourier, Grenoble
42, 4 (1992), 967–988

POISSON COHOMOLOGY OF REGULAR POISSON MANIFOLDS

by Ping XU

Introduction.

Poisson cohomology plays a very important role in the study of Poisson geometry. Finding Poisson cohomology of certain Poisson manifolds would allow us to solve some problems in deformation theory, as well as a number of other interesting problems [L] [VK1] [VK2]. For symplectic manifolds, Poisson cohomology is isomorphic to the usual de Rham cohomology [L]. In general, Poisson cohomology, roughly speaking, reflects two pieces of information of a Poisson manifold, the de Rham cohomology of symplectic leaves and the variation of symplectic structures along symplectic leaves. Despite of its importance, little work has been done in computing such cohomology because of the lack of a general powerful method.

The aim of the present paper is to suggest a way of computing Poisson cohomology by means of symplectic groupoids. Using this method, we carry out a computation for a special kind of regular Poisson manifolds where the symplectic foliations are trivial fibrations. Such an idea was already suggested by Vorob'ev and Karasev in [VK2], where they compute the Poisson cohomology for certain kinds of Poisson manifolds in dimensions 1, 2 and 3.

The key point of this paper can be stated as follows. For an integrable Poisson manifold (i.e., a Poisson manifold admitting a global symplectic groupoid), the Poisson cohomology is naturally isomorphic to the de Rham

Key words : Poisson cohomology – Symplectic groupoid – De Rham cohomology.
A.M.S. Classification : 58F05.

cohomology of left invariant forms on the symplectic groupoid. This fact allows one to convert the problem of computing Poisson cohomology to that of computing de Rham cohomology of certain manifolds, which is much easier to handle in general. In particular, in the case that the symplectic foliations are trivial fibrations, we show that the computation of Poisson cohomology is equivalent to the computation of de Rham cohomology of certain torus bundles.

The first section of this paper contains a general discussion on the preceding fact and its consequence for the regular Poisson manifolds in which the symplectic foliations are locally trivial fibrations.

The second section is devoted to an explicit construction of symplectic groupoids for integrable Poisson manifolds of the form $P = S \times Q$.

In Section 3, we compute directly the de Rham cohomology of torus bundles, a result needed in Section 4.

In Section 4, we obtain the main result of this paper, namely, a general formula describing Poisson cohomology of integrable Poisson manifolds of the form $P = S \times Q$.

As a by-product of this work, we can easily see that any volume form of an integrable Poisson manifold produces a Haar system for its symplectic groupoid. The relation between volume forms of the base Poisson manifold, Haar systems of the symplectic groupoid and the symplectic volume of the symplectic groupoid, as well as a number of possible applications to C^*-algebras of symplectic groupoids will be discussed elsewhere.

Finally, we would like to mention that a different approach to the computations of Poisson cohomology was recently carried out by I. Vaisman [V].

Acknowledgements : Most parts of this work were finished when the author was at the University of California, studying with Professor Alan Weinstein. The author would like to acknowledge his indebtedness to Professor Alan Weinstein for his invaluable guidance during its preparation. He also wishes to thank Professor Pierre Dazord, Professor Richard Lashof, Professor Kirill Mackenzie and Jianghua Lu for many stimulating discussions.

1. Poisson cohomology and de Rham cohomology of invariant forms on symplectic groupoids.

Let $(\Gamma \rightrightarrows P, \alpha, \beta)$ be a Lie groupoid. By $T_\alpha\Gamma$ we denote the tangent bundle of the foliation on Γ arising from the α-fibration and by $T^*_\alpha\Gamma$ its dual bundle. Write Ω^1_α for the $C^\infty(\Gamma)$-module of all smooth sections of the bundle $T^*_\alpha\Gamma$ and $\Omega^n_\alpha = \bigwedge^n \Omega^1_\alpha$; then $\Omega^n_\alpha \xrightarrow{d} \Omega^{n+1}_\alpha$ with d the usual exterior derivative along the fibres becomes a cochain complex. The subset $\Omega^n_{\alpha,L}$ of Ω^n_α consisting of all forms which are invariant under the groupoid left translations is a subcomplex of (Ω^n_α, d). The cohomology groups of this subcomplex are denoted by $H^*_{\alpha,L}(\Gamma, \mathbb{R})$. Let $A \longrightarrow \Gamma_0$ be the Lie algebroid of Γ with anchor map $\rho : A \longrightarrow TP$. It is quite obvious that any element of $C^n(A, \mathbb{R})$ can be naturally identified with an element of $\Omega^n_{\alpha,L}$ and vice versa. This identification commutes with the coboundary operators, and therefore establishes an isomorphism between $H^n(A, \mathbb{R})$ and $H^n_{\alpha,L}(\Gamma, \mathbb{R})$ (cf. [WX]).

THEOREM 1.1. — *If Γ is a Lie groupoid with Lie algebroid $A \longrightarrow \Gamma_0$, then*

$$H^n_{\alpha,L}(\Gamma, \mathbb{R}) \cong H^n(A, \mathbb{R}).$$

In particular, if $(\Gamma \rightrightarrows P, \alpha, \beta)$ is a symplectic groupoid, its Lie algebroid is the cotangent bundle $T^*P \longrightarrow P$ with anchor $\rho : T^*P \longrightarrow TP$ being the map naturally induced from the Poisson tensor. Therefore, $C^n(A, \mathbb{R})$ is naturally isomorphic to $\Gamma^\infty(\bigwedge^n TP)$, and the Lie algebroid differential d turns out [H] to be the Poisson differential [L] d_π for the multi-vector fields over P. Hence, the Lie algebroid cohomology $H^n(T^*P, \mathbb{R})$ is isomorphic to the Poisson cohomology $H^n_\pi(P)$. Therefore by Theorem 1.1, we have

THEOREM 1.2. — *If $\Gamma \rightrightarrows P$ is a symplectic groupoid, then $H^*_{\alpha,L}(\Gamma, \mathbb{R})$ is isomorphic to the Poisson cohomology $H^*_\pi(P)$.*

A Poisson manifold is called integrable, by Dazord [D1], if it is the unit space of a symplectic groupoid. For an integrable regular Poisson manifold P whose symplectic foliation is a locally trivial fibration $P \xrightarrow{\pi} Q$ as a differentiable fibration, we have a nice description for the cohomology $H^n_{\alpha,L}(\Gamma, \mathbb{R})$. To this end, we shall introduce a vector bundle $\Xi^n \longrightarrow Q$ by the following procedure. It is known that for each $u \in P$, $\alpha^{-1}(u)$ is a principal bundle over the symplectic leaf through u, with the structure group being the isotropy group I_u of Γ at u [CDW]. We denote by $H^n_{I_u}(\alpha^{-1}(u), \mathbb{R})$ the de Rham cohomology of I_u-invariant forms on $\alpha^{-1}(u)$. For any two points u

and v in the same symplectic leaf L, there always exists a point $z \in \Gamma$ such that $\alpha(z) = u$ and $\beta(z) = v$. The left translation $L_z : \alpha^{-1}(v) \longrightarrow \alpha^{-1}(u)$ induces an isomorphism :

$$L_z^* : H^n_{I_u}(\alpha^{-1}(u), \mathbb{R}) \longrightarrow H^n_{I_v}(\alpha^{-1}(v), \mathbb{R}).$$

Moreover, $L^*_{z_1} = L^*_{z_2}$ if z_1 and $z_2 \in \Gamma$ are two such points, since $H^n_{I_u}(\alpha^{-1}(u), \mathbb{R})$ is fixed by the isotropy group.

Since the symplectic foliation of P is a locally trivial fibration, Γ is α-locally trivial. Let $\{U_i\}$ be an open cover of Q, such that the symplectic fibration $\pi : P \longrightarrow Q$ has a local section ϵ_i over each U_i. For any U_i, the α-fibration $\alpha^{-1}(\epsilon_i(U_i)) \xrightarrow{\alpha} \epsilon_i(U_i)$ induces a canonical vector bundle $\Xi^n_i \longrightarrow \epsilon_i(U_i)$ $(\cong U_i)$ with $H^n_{I_{\epsilon_i(u)}}(\alpha^{-1}(\epsilon_i(u)), \mathbb{R})$ as the fibre over each point $u \in U_i$. If $U_i \cap U_j \neq \phi$, the transition : $H^n_{I_{\epsilon_i(u)}}(\alpha^{-1}(\epsilon_i(u)), \mathbb{R}) \longrightarrow H^n_{I_{\epsilon_j(u)}}(\alpha^{-1}(\epsilon_j(u)), \mathbb{R})$ is given by the canonical identification above. Thus in this way, we obtain a canonical vector bundle $\Xi^n \longrightarrow Q$, which is easily seen to be independent of the choice of open covers $\{U_i\}$. It is not difficult to see that

$$H^n_{\alpha,L}(\Gamma, \mathbb{R}) \cong C^\infty(Q, \Xi^n),$$

where $C^\infty(Q, \Xi^n)$ is the space of smooth sections of the bundle Ξ^n. Hence, we have the following :

COROLLARY 1.1. — *If P is an integrable regular Poisson manifold in which the symplectic fibration $P \longrightarrow Q$ is locally trivial as a differentiable fibration, then $H^n_\pi(P)$ is isomorphic to the space of smooth sections of the vector bundle Ξ^n.*

Example 1.1. — If S is a symplectic manifold, its symplectic groupoid $S \times S^- \rightrightarrows S$ is transitive and principal. Hence, $C^\infty(Q, \Xi^n) \cong \Xi^n \cong H^n_{dR}(S, \mathbb{R})$ and Corollary 1.1 implies that $H^n_\pi(S) \cong H^n_{dR}(S, \mathbb{R})$. This result was first proved by Lichnerowicz [L], by introducing a direct isomorphism between the two cochain complexes.

Remark. — In the preceding example, if one takes the fundamental groupoid $\Pi(S)$ instead of the coarse groupoid $S \times S^-$ as a symplectic groupoid over S, the α-fibre is the universal covering space of S and the isotropy group becomes the fundamental group of S. Hence, the corresponding de Rham cohomology of invariant forms coincides with the usual de Rham cohomology of S. So, one obtains the same result as before.

More generally, if $P = S \times M$, the direct product of a symplectic manifold S and a zero Poisson manifold M, then a symplectic groupoid of P is given by

$$S \times S^- \times T^*M \rightrightarrows S \times M.$$

Fixing any point $s_0 \in S$, we have a global section $\{s_0\} \times M$ for the symplectic fibration of P. Then the corresponding α-fibration over this section is isomorphic to $S \times T^*M \xrightarrow{\alpha} M$, where the isotropy group over each $m \in M$ is isomorphic to $T^*_m M$, and $T^*_m M$ acts on each $\alpha^{-1}(m) = S \times T^*_m M$ by translations on the second factor $T^*_m M$. Hence,

$$\Xi^n \cong \bigoplus_{i=1}^{n} H^i_{dR}(S, \mathbb{R}) \otimes \bigwedge^{n-i} TM;$$

therefore,

$$H^n_\pi(P) \cong \bigoplus_{i=1}^{n} H^i_{dR}(S, \mathbb{R}) \otimes \Gamma^\infty(\bigwedge^{n-i} TM).$$

Example 1.2. — Let $P = S^2 \times \mathbb{R}^+$, with $t\omega$ the symplectic structure on each leaf $S^2 \times \{t\}$, where ω is the standard symplectic structure on the unit sphere S^2. First, we note that P may be viewed as $\mathfrak{so}(3)^* - \{0\}$ with the linear Poisson structure. Hence, its symplectic groupoid is the transformation groupoid

$$S^2 \times \mathbb{R}^+ \times SO(3) \rightrightarrows S^2 \times \mathbb{R}^+.$$

It can be easily seen that the α-fibre of this groupoid is $SO(3)$, and the isotropy group is the circle group S^1. Since S^1 is compact and connected, the de Rham cohomology of invariant forms on any α-fibre coincides with the de Rham cohomology of $SO(3)$. Furthermore, since the symplectic fibration is trivial with the base space $\mathbb{R}^+$, the vector bundle Ξ^n is trivial over $\mathbb{R}^+$. Hence,

$$\Xi^n = \mathbb{R}^+ \times H^n_{dR}(SO(3), \mathbb{R});$$

therefore,

$$H^n_\pi(P) \cong C^\infty(\mathbb{R}^+, H^n_{dR}(SO(3), \mathbb{R})).$$

It follows immediately that

$$
\begin{aligned}
&H^0_\pi(P) \cong C^\infty(\mathbb{R}^+, \mathbb{R}) = C^\infty(\mathbb{R}^+) \quad \text{(Casimirs)},\\
&H^1_\pi(P) = 0 \quad \text{(Poisson vector fields/ Hamiltonian vector fields)},\\
&H^2_\pi(P) = 0 \quad \text{(infinitesimal deformations of the Poisson structure)},\\
&H^3_\pi(P) \cong C^\infty(\mathbb{R}^+, \mathbb{R}) = C^\infty(\mathbb{R}^+) \quad \text{(obstructions to deformation}\\
&\qquad\qquad \text{quantization), and}\\
&H^k_\pi(P) = 0 \qquad (k \geq 4).
\end{aligned}
$$

2. Symplectic groupoids of regular Poisson manifolds.

Given a Poisson manifold in which the characteristic foliation is a trivial fibration $\pi : P = S \times Q \longrightarrow Q$, the Poisson structure is described by a smooth map from Q to $Z^2(S)$, the space of all closed two forms on S, denoted by $\{\omega_y | y \in Q\}$.

According to Dazord [D1], P is integrable if and only if the map $y \mapsto [\omega_y]$ from Q to $H^2(S, \mathbb{R})$ is a submersion onto an affine subspace of $H^2(S, \mathbb{R})$ whose underlying vector space is generated by elements of $H^2(S, \mathbb{Z})$. In other words, there exist $[C_1], \cdots, [C_k] \in H^2(S, \mathbb{Z})$ and a submersion $(g_1(y), \cdots, g_k(y)) : Q \longrightarrow \mathbb{R}^k$ such that

$$[\omega_y] = [\omega_0] + \sum_{i=1}^{k} g_i(y)[C_i].$$

Symplectic groupoids over such Poisson manifolds have been investigated by Dazord [D1] [D2]. For completeness, in the following, we shall give an explicit construction of the symplectic groupoids directly. Let us first recall that a RIL ("réalisation isotrope de Libermann") is a complete symplectic realization which is a surjective submersion with connected and isotropic fibres [D2]. According to Dazord [D1], in order to construct a symplectic groupoid over the Poisson manifold $P = S \times Q$, it suffices to find a RIL for the Poisson manifold $P * P^- = S \times S^- \times Q$. To this end we need the following :

PROPOSITION 2.1. — *Let $P = S \times Q$ be a Poisson manifold with Poisson structure $\{\omega_y | y \in Q\}$ such that*

$$\omega_y = \omega_0 + \sum_{i=1}^{k} g_i(y) C_i + d_S \theta_y,$$

where $[C_1], \cdots, [C_k] \in H^2(S, \mathbb{Z})$ are linearly independent in $H^2(S, \mathbb{R})$, $(g_1(y), \cdots, g_2(y)) : Q \longrightarrow \mathbb{R}^k$ is a submersion, $\{\theta_y | y \in Q\}$ is a family of one forms on S and d_S is the exterior derivative with respect to S. Let $E \xrightarrow{p} S$ be a T^k-principal bundle with connection $(\theta_1, \cdots, \theta_k)$ and curvature $(C_1, \cdots, C_k)$. Let $\Gamma = E \times \left(T^*Q / \sum_{i=1}^{k} \mathbb{R} dg_i\right)$ and

$$\omega = p^* \left(\omega_0 + \sum_{i=1}^{k} g_i(y) C_i\right) + \phi^*(d\theta_y) + \sum_{i=1}^{k} \pi^*(dg_i(y)) \wedge \theta_i - \left(d\lambda + \sum_{i=1}^{k} dg_i \wedge \gamma_i\right),$$

where θ_y is considered as a one-form on P of type (1, 0), $\left(T^*Q / \sum_{i=1}^{k} \mathbb{R} dg_i\right)$ is the quotient space of T^*Q under the $\mathbb{R}^k$-action defined by :

$$(t_1, \cdots, t_k)(y, \xi) = \left(y, \xi + \sum_{i=1}^{k} t_i dg_i(y)\right), \quad \forall (y, \xi) \in T^*Q,$$

$\pi : \left(T^*Q / \sum_{i=1}^{k} \mathbb{R} dg_i\right) \longrightarrow Q$ is the natural projection, $(\gamma_1, \cdots, \gamma_k)$ is a connection of this principal bundle, $d\lambda$ is the standard symplectic 2-form on T^*Q and $\phi = p \times \pi$. Then $(\Gamma, \omega) \xrightarrow{\phi} P$ is a RIL.

Here, $\left(d\lambda + \sum_{i=1}^{k} dg_i \wedge \gamma_i\right)$ is considered as a two-form on the quotient space $\left(T^*Q / \sum_{i=1}^{k} \mathbb{R} dg_i\right)$. This is justified by the following lemma.

LEMMA 2.1. — *$d\lambda + \sum_{i=1}^{k} dg_i \wedge \gamma_i$ is the pull back of a two-form on the quotient space $\left(T^*Q / \sum_{i=1}^{k} \mathbb{R} dg_i\right)$, and therefore can be viewed as a two-form on $\left(T^*Q / \sum_{i=1}^{k} \mathbb{R} dg_i\right)$.*

Proof. — By ρ, we denote the natural projection $T^*Q \longrightarrow Q$. Let $X_1, \cdots, X_k$ be a family of vector fields on T^*Q generating the $\mathbb{R}^k$-action described above, and $\phi_1^t, \cdots, \phi_k^t$ their flows. Then,

$$\phi_i^t(y, \xi) = (y, \xi + tdg_i(y)), \quad \text{for any } (y, \xi) \in T^*Q.$$

For any given $v \in T_{(y,\xi)}(T^*Q)$, it follows from the definition of λ and the relation $\rho \circ \phi_i^t = \rho$ that

$$\begin{aligned} ((\phi_i^t)^*\lambda)(v) &= \lambda(T\phi_i^t v) \\ &= (\xi + tdg_i(y))(T\rho T\phi_i^t v) \\ &= (\xi + tdg_i(y))(T\rho v) \\ &= \lambda(v) + t(\rho^* dg_i)(v). \end{aligned}$$

Then, $(\phi_i^t)^*\lambda = \lambda + t\rho^* dg_i$. Therefore, $L_{X_i}\lambda = \rho^* dg_i$. However, it is easy to see that $(i_{X_i}\lambda)(y, \xi) = \xi(T\rho X_i) = 0$ for all $(y, \xi) \in T^*Q$, since $T\rho X_i = 0$. Thus $i_{X_i} d\lambda = L_{X_i}\lambda - di_{X_i}\lambda = \rho^* dg_i$. Hence,

$$\begin{aligned} & i_{X_i}\left(d\lambda + \sum_{j=1}^{k} \rho^* dg_j \wedge \gamma_j\right) \\ &= i_{X_i} d\lambda + \sum_{j=1}^{k} i_{X_i}(\rho^* dg_j) \wedge \gamma_j - \sum_{j=1}^{k} \rho^* dg_j \wedge i_{X_i}\gamma_j \\ &= \rho^* dg_i + \sum_{j=1}^{k} i_{(T\rho X_i)}(dg_j) \wedge \gamma_j - \sum_{j=1}^{k} \rho^* dg_j \wedge \delta_{ij} \\ &= 0. \end{aligned}$$

Therefore, $\left(d\lambda + \sum_{i=1}^{k} dg_i \wedge \gamma_i\right)$ is indeed the pull back of a certain two-form on $\left(T^*Q / \sum_{i=1}^{k} \mathbb{R} dg_i\right)$.

In fact, locally around any fixed point in Q, we may choose a coordinate system $\mathcal{U} \cong \mathbb{R}^k \times \mathbb{R}^{r-k}$ with coordinates $(y_1, \cdots, y_k, y_{k+1}, \cdots, y_r)$, such that $g_i(y) = y_i, (i = 1, \cdots, k)$. Then, $\left(T^*Q / \sum_{i=1}^{k} \mathbb{R} dy_i\right) \cong \mathbb{R}^r \times \mathbb{R}^{(r-k)}$ with coordinates $(y_1, \cdots, y_r, p_{k+1}, \cdots, p_r)$, and the connection γ_i can be taken to be dp_i. Hence, $d\lambda + \sum_{i=1}^{k} dg_i \wedge \gamma_i = \sum_{i=k+1}^{r} dp_i \wedge dy_i$, which is obviously a two-form on the quotient space $\left(T^*Q/\sum_{i=1}^{k} \mathbb{R} dy_i\right)$. Q.E.D.

Sketch of proof of Proposition 2.1. — It is obvious that ω is a closed 2-form. We shall use a local argument below to show that ω is non-degenerate. Take $\mathcal{U} \cong \mathbb{R}^k \times \mathbb{R}^{r-k}$ with coordinates $(y_1, \cdots, y_k, y_{k+1}, \cdots, y_r)$ to be a local system as in the proof of Lemma 2.1, and let $\mathcal{V}$ be any open submanifold of S, such that $E|_{\mathcal{V}}$ is trivial and $C_1, \cdots, C_k$ are exact on $\mathcal{V}$. We assume that $E|_{\mathcal{V}} \cong \mathcal{V} \times T^k$ with coordinates $(q, x_1, \cdots, x_k)$, such that $\theta_i = \alpha_i + dx_i$ and $C_i = d\alpha_i$ with $\alpha_i \in \Omega^1(\mathcal{V}), (i = 1, \cdots, k)$. Then, $\Gamma|_{\mathcal{V}\times\mathcal{U}}$ is isomorphic to $\mathcal{V} \times T^k \times \mathbb{R}^r \times \mathbb{R}^{r-k}$ with coordinates $(q, x_1, \cdots, x_k, y_1, \cdots, y_r, p_{k+1}, \cdots, p_r)$, and under these coordinates

$$\omega = \omega_0 + \sum_{i=1}^{k} y_i d\alpha_i + d(\theta_y) + \sum_{i=1}^{k} dy_i \wedge \alpha_i + \sum_{i=1}^{k} dy_i \wedge dx_i + \sum_{i=k+1}^{r} dy_i \wedge dp_i.$$

By using the fact that $\omega_0 + \sum_{i=1}^{k} y_i d\alpha_i + d(\theta_y)$ is non-degenerate when being restricted on each symplectic leaf y =constant, one can easily show that ω is non-degenerate.

The tangent space of each ϕ-fibre is spanned by $\frac{\partial}{\partial x_i}$, $(i = 1, \cdots, k)$ and $\frac{\partial}{\partial p_i}, (i = k+1, \cdots, r)$, and therefore is obviously isotropic. For any function $f(q, y) \in C^\infty(\mathcal{V} \times \mathcal{U})$, it is not difficult to see that

$$X_{\phi^* f} = X_f - \sum_{i=1}^{k} \frac{\partial f}{\partial y_i}\frac{\partial}{\partial x_i} - \sum_{i=k+1}^{r} \frac{\partial f}{\partial y_i}\frac{\partial}{\partial p_i}, \tag{1}$$

where X_f is the Hamiltonian vector field of f on the Poisson submanifold $\mathcal{V} \times \mathcal{U} \subseteq P$. It is trivial to see that for all $f, g \in C^\infty(\mathcal{V} \times \mathcal{U})$,

$$\begin{aligned}\{\phi^* f, \phi^* g\} &= \omega(X_{\phi^* f}, X_{\phi^* g}) \\ &= \omega(X_f, X_g) \\ &= \phi^*\{f, g\},\end{aligned}$$

since X_f and X_g are always tangent to $\mathcal{V} \times \mathcal{U}$. Therefore, ϕ is a Poisson morphism, or in other words, a symplectic realization.

The coefficients in the front of $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial p_i}$ in Equation (1) depend neither on the coordinates x_i nor on p_i, so $X_{\phi^* f}$ is complete provided X_f is complete. In other words, ϕ is a complete symplectic realization. Q.E.D.

An immediate consequence of Proposition 2.1 is the following theorem, in which an explicit construction of a symplectic groupoid over the Poisson manifold $P = S \times Q$ is described (see also [D1] [D2]). We will see in the next section that this construction plays a key role in our later computations.

THEOREM. 2.1. — *Suppose that the Poisson structure* $\{\omega_y\}$ *on* $P = S \times Q$ *is given by*

$$[\omega_y] = [\omega_0] + \sum_{i=1}^{k} g_i(y)[C_i]$$

as in Proposition 2.1. Let $E \xrightarrow{p} S \times S$ *be a* T^k*-principal bundle with curvature* $(C_1, -C_1), \cdots, (C_k, -C_k)$, $\Gamma = E \times \left(T^*Q/\sum_{i=1}^{k} \mathbb{R} dg_i\right)$ *and* $\Gamma \xrightarrow{\phi} S \times S^- \times Q$ *be the RIL of* $S \times S^- \times Q$ *as constructed in Proposition 2.1; then* $\Gamma \rightrightarrows P$ *with* $\alpha = \alpha_0 \circ \phi$ *and* $\beta = \beta_0 \circ \phi$ *is a symplectic groupoid over* P, *where* α_0 *and* $\beta_0 : S \times S^- \times Q \longrightarrow S \times Q \ (= P)$ *are given by* $\alpha_0 = (\mathrm{pr}_1, \mathrm{id})$ *and* $\beta_0 = (\mathrm{pr}_2, \mathrm{id})$, *respectively.*

3. De Rham cohomology of torus bundles.

In order to find the Poisson cohomology for Poisson manifolds discussed in the last section, we need to compute the de Rham cohomology of principal torus bundles. For circle bundles, the de Rham cohomology follows easily from the Gysin sequence [BT]. However, for higher k, the spectral sequence of a T^k-principal bundle becomes very complicated, which makes it difficult to find the de Rham cohomology directly by using the spectral sequence. Here, instead, we compute the de Rham cohomology directly by geometric methods.

Given a T^k-principal bundle $E \xrightarrow{\pi} M$, the relative cohomology [BT] is defined as the cohomology of the complex $d : \Omega^n(\pi) \longrightarrow \Omega^{n+1}(\pi)$, where $\Omega^n(\pi) = \Omega^n(M) \oplus \Omega^{n-1}(E)$ and $d(\omega, \theta) = (d\omega, \pi^*\omega - d\theta)$, for all

$\omega \in \Omega^n(M)$ and $\theta \in \Omega^{n-1}(E)$. The short exact sequence

$$0 \longrightarrow \Omega^{n-1}(E) \xrightarrow{j} \Omega^n(\pi) \xrightarrow{i} \Omega^n(M) \longrightarrow 0$$

yields a long exact sequence

$$\longrightarrow H^n(\pi) \xrightarrow{i^*} H^n(M) \xrightarrow{\pi^*} H^n(E) \xrightarrow{j^*} H^{n+1}(\pi) \xrightarrow{i^*} H^{n+1}(M) \longrightarrow .$$

Hence, we have

LEMMA 3.1. — *The following exact sequence holds for any $n \in N$:*

$$0 \longrightarrow H^n(M)/K^n \longrightarrow H^n(E) \longrightarrow L^n \longrightarrow 0,$$

where $K^n = i^*(H^n(\pi))$ and L^n is the kernel of $i^* : H^{n+1}(\pi) \longrightarrow H^{n+1}(M)$. On the cochain level, i^* is given by $i^*(\omega, \theta) = \omega$.

Remark. — Since T^k is a compact group, both de Rham cohomology and relative de Rham cohomology of the bundle E can be computed by using the de Rham subcomplex consisting of all invariant differential forms. In the rest of this section, we will work on this invariant subcomplex without any further mention.

In order to compute $H^n(E)$, it suffices to compute K^n and L^n according to Lemma 3.1. To this end, let us choose a connection on E given by k invariant one-forms $\alpha_1, \cdots, \alpha_k \in \Omega^1(E)$ and denote the corresponding curvature by $\omega_1, \cdots, \omega_k \in \Omega^2(M)$. Given any $(\omega, \theta) \in \Omega^n(\pi)$, we may always assume that

$$\theta = \pi^*\phi + \sum_{r=1}^{k} \sum_{1 \le i_1 < i_2 \cdots < i_r \le k} \alpha_{i_1} \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_r} \wedge \pi^*\phi_{i_1, \cdots, i_r},$$

with $\phi_{i_1,\cdots,i_r} \in \Omega^{n-r-1}(M)$ and $\phi \in \Omega^{n-1}(M)$. Since we are working with relative cohomology, we can assume that $\phi = 0$. By an elementary computation, we have

$$\begin{aligned} &\pi^*\omega - d\theta \\ &= \pi^*\left(\omega - \sum_i \omega_i \wedge \phi_i\right) + \sum_{r=1}^{k} \sum_{i_1 < \cdots < i_r} \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_r} \\ &\wedge \pi^*\left((-1)^{r-1} d\phi_{i_1,\cdots,i_r} + \sum_{t=0}^{r} (-1)^t \sum_{i_1 < \cdots < i_t < l < i_{t+1} < \cdots < i_r} \omega_l \wedge \phi_{i_1,\cdots,i_t,l,i_{t+1},\cdots,i_r}\right). \end{aligned}$$

Hence, $(\omega,\theta) \in Z^n(\pi)$ with

$$\theta = \sum_{r=1}^{k} \sum_{1\le i_1<i_2\cdots<i_r\le k} \alpha_{i_1}\wedge\alpha_{i_2}\wedge\cdots\wedge\alpha_{i_r}\wedge\pi^*\phi_{i_1,\cdots,i_r}$$

if and only if

$$\omega = \sum_i \omega_i \wedge \phi_i \tag{2}$$

and

(3)

$$(-1)^{r-1}d\phi_{i_1,\cdots,i_r} + \sum_{t=0}^{r}(-1)^t \sum_{i_1<\cdots<i_t<l<i_{t+1}<\cdots<i_r} \omega_l\wedge\phi_{i_1,\cdots,i_t,l,i_{t+1},\cdots,i_r} = 0,$$

for all r such that $1\le r\le k$. On the other hand, $(\omega,\theta)\in B^n(\pi)$ if and only if $\omega = d\gamma$ and

$$\theta = \pi^*\phi + \sum_{r=1}^{k} \sum_{1\le i_1<i_2\cdots<i_r\le k} \alpha_{i_1}\wedge\alpha_{i_2}\wedge\cdots\wedge\alpha_{i_r}\wedge\pi^*\phi_{i_1,\cdots,i_r},$$

where $\gamma,\phi\in\Omega^{n-1}(M)$ and

$$\phi_{i_1,\cdots,i_r} = (-1)^{r-1}d\psi_{i_1,\cdots,i_r} + \sum_{t=0}^{r}(-1)^t \sum_{i_1<\cdots<i_t<l<i_{t+1}<\cdots<i_r} \omega_l\wedge\psi_{i_1,\cdots,i_t,l,i_{t+1},\cdots,i_r}$$

for certain $\psi_{i_1,\cdots,i_r}\in\Omega^{n-r-2}(M)$.

$[(\omega,\theta)]\in L^n$ if and only if $(\omega,\theta)\in Z^{n+1}(\pi)$ and $[\omega]=0$. Let us denote by $L_l^{m,k}(M)$ the group consisting of all elements $\bigoplus[\phi_{i_1,\cdots,i_l}] \in \bigoplus_{1\le i_1<\cdots<i_l\le k} H^m(M)$, such that for all $1\le r\le l-1$, there exists $\phi_{i_1,\cdots,i_r}\in\Omega^{m+l-r}(M)$ satisfying Equation (3) and $\sum_i[\omega_i\wedge\phi_i]=0$. Then, it is not difficult to see that

$$L^n = L_k^{n-k,k}(M)\oplus\cdots\oplus L_{k-i}^{n-k+i,k}(M)\oplus\cdots\oplus L_1^{n-1,k}(M).$$

Similarly, we have

(4)

$$K^n = \left\{ [\sum_{i=1}^{k}\omega_i\wedge\phi_i]\in H^n(M) \,\middle|\, \begin{array}{l}\phi_i\in\Omega^{n-2}(M)\ s.t.\text{ there exists }\ \phi_{i_1,\cdots,i_r}\in\Omega^{n-r-1}(M)\\ \text{satisfying Equation (3) for all } r, 1\le r\le k.\end{array}\right\}.$$

For instance, we have by definition,

$$L_1^{m,k}(M) = \left\{ \bigoplus_i [\phi_i] \in \bigoplus_{i=1}^k H^m(M) \;\middle|\; \sum_{i=1}^k [\omega_i] \wedge [\phi_i] = 0 \right\},$$

which can be easily seen to depend only on the cohomology classes $[\omega_i]$, $(i = 1, \cdots, k)$. Indeed, this fact is true for all such groups $L_l^{m,k}(M)$.

LEMMA 3.2. — *$L_l^{m,k}(M)$ and K^n depend only on the Chern class $([\omega_1], \cdots, [\omega_k])$.*

Proof. — We will prove this lemma only in the case that $k = 2$. The general case follows from the same argument. By the remark above, the lemma already holds for $L_1^{m,2}(M)$. So, it suffices to show this for $L_2^{m,2}(M)$ and K^n. By definition,

$$L_2^{m,2}(M) = \left\{ [\phi_{1,2}] \in H^m(M) \;\middle|\; \begin{array}{l} \exists\ \phi_1, \phi_2 \in \Omega^{m+1}(M) \text{ such that} \\ d\phi_1 + \omega_2 \wedge \phi_{1,2} = 0,\ d\phi_2 - \omega_1 \wedge \phi_{1,2} = 0 \\ \text{and} \quad [\omega_1 \wedge \phi_1 + \omega_2 \wedge \phi_2] = 0 \end{array} \right\}$$

and

$$K^n = \left\{ [\omega_1 \wedge \phi_1 + \omega_2 \wedge \phi_2] \;\middle|\; \begin{array}{l} \phi_1, \phi_2 \in \Omega^{n-2}(M) \text{ s.t. } d\phi_1 + \omega_2 \wedge \phi_{1,2} = 0 \text{ and} \\ d\phi_2 - \omega_1 \wedge \phi_{1,2} = 0, \quad \text{for some } \phi_{1,2} \in Z^{n-3}(M). \end{array} \right\}$$

Assume that $\omega_1' = \omega_1 + d\theta_1$ and $\omega_2' = \omega_2 + d\theta_2$ with $\theta_1, \theta_2 \in \Omega^1(M)$, and suppose that $\phi_1, \phi_2 \in \Omega^1(M)$ satisfy $d\phi_1 + \omega_2 \wedge \phi_{1,2} = 0$ and $d\phi_2 - \omega_1 \wedge \phi_{1,2} = 0$. Set

$$\phi_1' = \phi_1 - \theta_2 \wedge \phi_{1,2}$$

and

$$\phi_2' = \phi_2 + \theta_1 \wedge \phi_{1,2};$$

then it can be checked directly that

$$d\phi_1' + \omega_2' \wedge \phi_{1,2} = d\phi_1 + \omega_2 \wedge \phi_{1,2} = 0.$$

Similarly, $d\phi_2' - \omega_1' \wedge \phi_{1,2} = 0$. However, by a straightforward computation, we have

$$\omega_1' \wedge \phi_1' + \omega_2' \wedge \phi_2' = \omega_1 \wedge \phi_1 + \omega_2 \wedge \phi_2 + d(\theta_1 \wedge \phi_1 + \theta_2 \wedge \phi_2 - \theta_1 \wedge \theta_2 \wedge \phi_{1,2}).$$

Hence,

$$[\omega_1' \wedge \phi_1' + \omega_2' \wedge \phi_2'] = [\omega_1 \wedge \phi_1 + \omega_2 \wedge \phi_2].$$

So, $L_2^{m,2}(M)$ and K^n depend only on the cohomology classes of ω_1 and ω_2.
Q.E.D.

We summarize the above results in the following :

THEOREM 3.1. — *Suppose that E is a T^k-principal bundle over M with the first Chern class $([\omega_1],\cdots,[\omega_k])$; then*

$$H^n(E)\cong L_k^{n-k,k}(M)\oplus\cdots\oplus L_{k-i}^{n-k+i,k}(M)\oplus\cdots\oplus L_1^{n-1,k}(M)\oplus\bar{H}^n(M),$$

where $\bar{H}^n(M)=H^n(M)/K^n$, $L_l^{m,k}(M)$ (K^n resp.) is the subgroup of $\bigoplus_{1\le i_1<\cdots<i_l\le k} H^m(M)$ ($H^n(M)$ resp.) defined preceding Lemma 3.2 and $L_l^{m,k}(M)$ is assumed to be 0 if $m<0$.

In particular, if E is a circle bundle with Chern class $[\omega]$, i.e., when $k=1$, Theorem 3.1 implies that

$$H^n(E)\cong L_1^{n-1,1}(M)\oplus H^n(M)/K^n,$$

where

$$L_1^{n-1,1}(M)=\{\ [\phi]\in H^{n-1}(M)\ \big|\ [\omega]\wedge[\phi]=0\}$$

and

$$K^n=\{\ [\omega]\wedge[\phi]\ \big|\ [\phi]\in H^{n-2}(M)\ \}.$$

This result coincides with the well-known Gysin sequence of circle bundles [BT]. Although Theorem 3.1 gives an explicit formula for $H^n(E)$, the subgroups $L_l^{m,k}(M)$ there are somehow elusive. Nevertheless for low dimensional cohomology, we can describe them more explicitly.

COROLLARY 3.1. — *Assume that $[\omega_1],\cdots,[\omega_k]$ are linearly independent in $H^2(M,\mathbb{R})$; then*

1. $L_r^{0,k}(M)=0,\quad$ *for all $k,r\in N$;*
2. $H^1(E)\cong H^1(M)$;
3. $H^2(E)\cong L_1^{1,k}(M)\oplus H^2(M)/(\mathrm{span}\{[\omega_i]\})$;
4. *moreover, if $H^1(M)=0$ (for example, when M is simply connected), $H^3(E)\cong L_1^{2,k}(M)\oplus H^3(M)$, and $H^4(E)\cong L_2^{2,k}(M)\oplus L_1^{3,k}(M)\oplus H^4(M)/K^4$, where $K^4=\left\{\sum_{i=1}^{k}[\phi_i]\wedge[\omega_i]\big|\ [\phi_i]\in H^2(M)\right\}$; and*

5. *if $H^5(M)$ is spanned by the wedge products of $H^3(M)$ with $[\omega_i]$, $(i = 1, \cdots, k)$, i.e., $H^5(M) = \bigoplus_{i=1}^{k}(H^3(M) \wedge \{[\omega_i]\})$; then*

$$L_2^{2,k}(M) = \{A \in \mathfrak{so}(k, H^2(M)) = \mathfrak{so}(k, \mathbb{R}) \otimes H^2(M) |\ A \wedge \omega = 0\},$$

where $\mathfrak{so}(k, H^2(M))$ is the space of all $k \times k$ skew-symmetric matrices with values in $H^2(M)$ and $\omega = ([\omega_1], \cdots, [\omega_k])^T$.

Proof. — (1) follows immediately from the definition of $L_r^{0,k}(M)$ and the fact that $[\omega_1], \cdots, [\omega_k]$ are linearly independent in $H^2(M, \mathbb{R})$.

(2) Obviously, $K^1 = 0$. Moreover $L_1^{0,k}(M) = 0$ by (1), so

$$H^1(E) \cong L_1^{0,k}(M) \oplus H^1(M)/K^1 \cong H^1(M).$$

(3) According to Equation (4), $[\theta] \in K^2 \subseteq H^2(M)$ if and only if $\theta = \sum_{i=1}^{k} \omega_i \wedge \phi_i$ for some $\phi_i, (i = 1 \cdots, k)$, such that $d\phi_i = 0$. Thus, $\phi_i = c_i, (i = 1, \cdots, k)$ are constants and $\theta = \sum_{i=1}^{k} c_i \omega_i$. Therefore, $K^2 = \text{span}\{[\omega_i]\}$ and

$$\begin{aligned} H^2(E) &\cong L_2^{0,k}(M) \oplus L_1^{1,k}(M) \oplus H^2(M)/K^2 \\ &= L_1^{1,k}(M) \oplus H^2(M)/(\text{span}\{[\omega_i]\}). \end{aligned}$$

(4) By Theorem 3.1,

$$H^3(E) \cong L_3^{0,k}(M) \oplus L_2^{1,k}(M) \oplus L_1^{2,k}(M) \oplus H^3(M)/K^3.$$

$H^1(M) = 0$ implies that $L_2^{1,k}(M) = 0$. Moreover by (1), we have $L_3^{0,k}(M) = 0$. We will show below that $K^3 = 0$.

By definition, $[\theta] \in K^3$ if and only if $\theta = \sum_{i=1}^{k} \omega_i \wedge \phi_i$ for some $\phi_1, \cdots, \phi_k \in \Omega^1(M)$, so that the following equations hold for some $\phi_{i,j} \in \Omega^0(M), (i, j = 1, \cdots, k)$:

$$d\phi_j + \sum_{i<j} \omega_i \wedge \phi_{i,j} - \sum_{j<i} \omega_i \wedge \phi_{j,i} = 0 \tag{5}$$

$$d\phi_{i,j} = 0. \tag{6}$$

Equation (6) implies that $\phi_{i,j} = c_{ij}$ are constants. Then, according to Equation (5), we have

$$\sum_{i<j} c_{ij}[\omega_i] - \sum_{j<i} c_{ji}[\omega_i] = 0$$

for all $j = 1, \cdots, k$. It follows from the linear independence of $[\omega_1], \cdots, [\omega_k]$ that $c_{ij} = 0$. Hence, Equation (5) reduces to $d\phi_j = 0$, $(j = 1, \cdots, k)$. Since $H^1(M) = 0$, ϕ_j must be exact; therefore θ is exact. Hence, $K^3 = 0$.

Similarly, we can show that $H^4(E) \cong L_2^{2,k}(M) \oplus L_1^{3,k}(M) \oplus H^4(M)/K^4$ and $K^4 = \left\{ \sum_{i=1}^{k} [\phi_i] \wedge [\omega_i] | \ [\phi_i] \in H^2(M) \right\}$.

(5) By definition, $\{[\phi_{i_1,i_2}]\} \in \bigoplus_{1 \le i_1 < i_2 \le k} H^2(M)$ belongs to $L_2^{2,k}(M)$ if and only if for all $1 \le i \le k$, there is $\phi_i \in H^2(M)$ such that

$$d\phi_i + \sum_{l<i} \omega_l \wedge \phi_{l,i} - \sum_{i<l} \omega_l \wedge \phi_{i,l} = 0 \quad \text{and}$$

$$\sum_{i=1}^{k} [\omega_i \wedge \phi_i] = 0.$$

In fact, the second condition can be implied by the first one, as in the following. Suppose that $\phi_i \in H^2(M)$, $(i = 1, \cdots k)$ satisfy the first condition; $\sum_{i=1}^{k} \omega_i \wedge \phi_i$ must be closed. Hence, one can find $[\psi_i] \in H^2(M)$, $(i = 1, \cdots k)$ such that $\sum_{i=1}^{k} [\omega_i \wedge \phi_i] = \sum_{i=1}^{k} [\omega_i] \wedge [\psi_i]$ by assumption. Therefore, the above two conditions are satisfied if ψ_i is replaced by $\phi_i - \psi_i$ for all $1 \le i \le k$. This shows that $\{[\phi_{i_1,i_2}]\} \in L_2^{2,k}(M)$ if and if it is a solution of the equations :

$$\sum_{l<i} [\omega_l] \wedge [\phi_{l,i}] - \sum_{i<l} [\omega_l] \wedge [\phi_{i,l}] = 0, \quad i = 1, \cdots k.$$

That is,

$$L_2^{2,k}(M) = \{A \in \mathfrak{so}(k, H^2(M)) = \mathfrak{so}(k, \mathbb{R}) \otimes H^2(M) | \ A \wedge \omega = 0\}.$$

Q.E.D.

4. Main theorem.

With the results in previous sections, we can give the main theorem of this paper. First of all, we need two lemmas, which can be easily proved by a direct verification.

To any vector bundle $V \xrightarrow{\pi} Q$, there corresponds a vector bundle, denoted by E^n, with $H^n_{inv}(\pi^{-1}(q), \mathbb{R})$ (the invariant de Rham cohomology of $\pi^{-1}(q)$ under the translations of $\pi^{-1}(q)$ itself) as the fibre at each point $q \in Q$. We have

LEMMA 4.1. — *E^n is isomorphic to $\bigwedge^n V^*$, where V^* is the dual bundle of V.*

By $\mathcal{F}_Q$, we denote the foliation on Q induced from the fibration $(g_1, \cdots, g_k) : Q \longrightarrow \mathbb{R}^k$.

LEMMA 4.2. — *The dual bundle of $\left(T^*Q/\sum_{i=1}^{k} \mathbb{R}dg_i\right) \xrightarrow{\pi} Q$ is isomorphic to the vector bundle $T\mathcal{F}_Q \longrightarrow Q$.*

THEOREM 4.1. — *Suppose that $P = S \times Q$ is a Poisson manifold with the Poisson structure $\{\omega_y | y \in Q\}$, such that*

$$[\omega_y] = [\omega_0] + \sum_{i=1}^{k} g_i(y)[C_i],$$

where $[C_1], \cdots, [C_k] \in H^2(S, \mathbb{Z})$ are linearly independent in $H^2(S, \mathbb{R})$ and $(g_1, \cdots, g_k) : Q \longrightarrow \mathbb{R}^k$ is a submersion. Then for the Poisson cohomology of P we have :

$$H^n_\pi(P) \cong \bigoplus_{i=0}^{n} H^i[\mathcal{E}] \otimes \Gamma^\infty(\bigwedge^{n-i}(T\mathcal{F}_Q)),$$

where $H^i[\mathcal{E}] = C^\infty(Q) \otimes H^i(\mathcal{E})$ and $\mathcal{E}$ is any principal bundle over S with the first Chern class $([C_1], \cdots, [C_k])$. In other words, $H^n_\pi(P)$ can be described by the following formula :

$$H^n_\pi(P) \cong \bigoplus_{i=0}^{n} (L_k^{i-k,k}[S] \oplus L_{k-1}^{i-k+1,k}[S] \oplus \cdots \oplus L_1^{i-1,k}[S] \oplus \bar{H}^i[S]) \otimes \Gamma^\infty(\bigwedge^{n-i}(T\mathcal{F}_Q)),$$

where

$$L_l^{m,k}[S] = \Gamma^\infty(Q, L_l^{m,k}(S)) \cong C^\infty(Q) \otimes L_l^{m,k}(S),$$

$$\bar{H}^i[S] = \Gamma^\infty(Q, \bar{H}^i(S)) \cong C^\infty(Q) \otimes \bar{H}^i(S),$$

and $L_l^{m,k}(S)$, $\bar{H}^i(S)$ *are as defined in Section 3, with* $([C_1], \cdots, [C_k])$ *as the first Chern class.*

Proof. — For any fixed point $s \in S$, $\{s\} \times Q$ is a global section of the symplectic fibration $P \longrightarrow Q$. According to Theorem 2.1,

$$\begin{aligned}\alpha^{-1}(\{s\} \times Q) &= \phi^{-1}(\alpha_0^{-1}(\{s\} \times Q)) \\ &= \phi^{-1}(\{s\} \times S \times Q) \\ &= p^{-1}(\{s\} \times S) \times \left(T^*Q/\sum_{i=1}^{k} \mathbb{R}dg_i\right).\end{aligned}$$

So the fibre bundle $\alpha^{-1}(\{s\} \times Q) \xrightarrow{\alpha} Q$ becomes $p^{-1}(\{s\} \times S) \times \left(T^*Q/\sum_{i=1}^{k} \mathbb{R}dg_i\right) \xrightarrow{\alpha'} Q$, where α' is the natural projection. The latter bundle is a direct product of the trivial bundle $p^{-1}(\{s\} \times S) \times Q \longrightarrow Q$ with the vector bundle $T^*Q/\sum_{i=1}^{k} \mathbb{R}dg_i \xrightarrow{\pi} Q$. Moreover, since $\beta = \beta_0 \circ \phi$ and $\phi = p \times \pi$, the principal bundle $\alpha^{-1}(s, y) \xrightarrow{\beta} S$ is isomorphic to

$$p^{-1}(\{s\} \times S) \times \left(T_y^*Q/\sum_{i=1}^{k} \mathbb{R}dg_i(y)\right) \xrightarrow{p\times\pi} \{s\} \times S \times \{y\},$$

where π maps all the points in $\left(T_y^*Q/\sum_{i=1}^{k} \mathbb{R}dg_i(y)\right)$ into one point y. The structure group $I_{(s,y)}$ is isomorphic to $T^k \times T_y^*Q/\sum_{i=1}^{k} \mathbb{R}dg_i(y)$ $(\cong T^k \times \mathbb{R}^{r-k})$, where the action of T^k on $p^{-1}(\{s\} \times S)$ is induced from that on E by viewing $p^{-1}(\{s\} \times S)$ as an invariant subspace of E, and $T_y^*Q/\sum_{i=1}^{k} \mathbb{R}dg_i(y)$ acts on itself by translations.

Hence, by Theorem 3.1, we have

$$H^n_{I_{(s,y)}}(\alpha^{-1}(s,y),\mathbb{R}) \cong \bigoplus_{i=0}^{n} H^i_{T^k}(p^{-1}(\{s\}\times S))\otimes H^{n-i}_{\mathbb{R}^{r-k}}\left(T^*_yQ/\sum_{i=1}^{k}\mathbb{R}dg_i(y)\right) \cong \bigoplus_{i=0}^{n} H^i(\mathcal{E})\otimes \bigwedge^{n-i} T_y\mathcal{F}_Q,$$

where $\mathcal{E}$ is the principal bundle $p^{-1}(\{s\}\times S)\xrightarrow{p}\{s\}\times S$, having curvature $(-C_1,\cdots,-C_k)$. Combining this fact with Lemma 4.1 and Lemma 4.2, we deduce that the vector bundle Ξ^n introduced preceding Corollary 1.1 is isomorphic to $\bigoplus_{i=0}^{n} H^i(\mathcal{E})\times(\bigwedge^{n-i} T\mathcal{F}_Q)\longrightarrow Q$, or isomorphic to

$$\bigoplus_{i=0}^{n}(L_k^{i-k,k}(S)\oplus L_{k-1}^{i-k+1,k}(S)\oplus\cdots\oplus L_1^{i-1,k}(S)\oplus\bar{H}^i(S))\times(\bigwedge^{n-i} T\mathcal{F}_Q)\longrightarrow Q.$$

Hence, the conclusion follows immediately from Corollary 1.1.Q.E.D.

In particular, combining Theorem 4.1 with Corollary 3.1, we obtain the following formulas expressing Poisson cohomology groups of low dimensions (n=1, 2, 3, and 4), the first three of which exactly coincide with the results obtained by Vorob'ev and Karasev in [VK2].

COROLLARY 4.1. — *Under the assumption of Theorem 4.1,*

1. $H^1_\pi(P)\cong\Gamma^\infty(\bigwedge^1(T\mathcal{F}_Q))\oplus H^1[S]$;
2. $H^2_\pi(P)\cong\Gamma^\infty(\bigwedge^2(T\mathcal{F}_Q))\oplus\Gamma^\infty(\bigwedge^1(T\mathcal{F}_Q))\otimes H^1[S]\oplus L_1^{1,k}[S]\oplus\bar{H}^2[S]$; *and*
3. *moreover, if we assume that* $H^1(S)=0$, *then*

   $H^3_\pi(P)\cong\Gamma^\infty(\bigwedge^3(T\mathcal{F}_Q))\oplus\Gamma^\infty(\bigwedge^1(T\mathcal{F}_Q))\otimes\bar{H}^2[S]\oplus L_1^{2,k}[S]\oplus H^3[S]$, *and*

4. $H^4_\pi(P)\cong\Gamma^\infty(\bigwedge^4(T\mathcal{F}_Q))\oplus\Gamma^\infty(\bigwedge^2(T\mathcal{F}_Q))\otimes\bar{H}^2[S]\oplus\Gamma^\infty(\bigwedge^1(T\mathcal{F}_Q))\otimes(L_1^{2,k}[S]\oplus H^3[S])\oplus L_2^{2,k}[S]\oplus L_1^{3,k}[S]\oplus\bar{H}^4[S]$.

On the other hand, as $k=1$, the formulas in Theorem 4.1 in fact become very simple.

COROLLARY 4.2. — *Suppose that* $\{\omega_y|y\in Q\}$ *is the Poisson structure on Poisson manifold* $P=S\times Q$ *and* $[\omega_y]=[\omega_0]+g(y)[C]$, *where*

$[C] \in H^2(S,\mathbb{Z})$ is not zero in $H^2(S,\mathbb{R})$ and $g : Q \longrightarrow \mathbb{R}$ is a submersion. Then

$$H^n_\pi(P) \cong \bigoplus_{i=0}^{n}(L_1^{i-1,1}[S] \oplus \bar{H}^i[S]) \otimes \Gamma^\infty(\bigwedge^{n-i}(T\mathcal{F}_Q)),$$

where $L_1^{m,1}[S] = C^\infty(Q) \otimes L_1^{m,1}(S)$, $\bar{H}^i[S] = C^\infty(Q) \otimes (H^i(S)/K^i)$, $L_1^{m,1}(S) = \{[\phi] \in H^m(S) | \ [C] \wedge [\phi] = 0\}$, $K^i = \{[C] \wedge [\phi] | \ [\phi] \in H^{i-2}(S)\}$ and $\mathcal{F}_Q$ is the foliation on Q induced from the fibration $g : Q \longrightarrow \mathbb{R}$.

Remark. — It is possible to generalize the formula in Theorem 4.1 to those Poisson manifolds in which the symplectic foliations are locally trivial fibrations. If P is such a Poisson manifold with symplectic fibration $P \xrightarrow{\pi} Q$, to this fibration, we can associate a canonical flat vector bundle for each $n \in N$, denoted by $\Xi^n \longrightarrow Q$, with $H^n(\pi^{-1}(q),\mathbb{R})$ being the fibre at each $q \in Q$. The flat connection there can be defined as follows. Given any two points $x, y \in Q$ and any path $\gamma(t)$ in Q connecting x and y, let $\Phi^t : \pi^{-1}(x) \longrightarrow \pi^{-1}(\gamma(t))$ be a family of diffeomorphisms such that $\Phi^0 = \mathrm{id}$. We define the parallel translation between $H^n(\pi^{-1}(x),\mathbb{R})$ and $H^n(\pi^{-1}(y),\mathbb{R})$ to be the isomorphism $(\Phi^1)^*$. It can be justified by the following argument that this connection is well-defined. Suppose that Ψ^t is another family of such diffeomorphisms; we let $\Theta^t = (\Phi^t)^{-1} \circ \Psi^t$, which are automorphisms of $\pi^{-1}(x)$. Thus, Θ^1 is homotopic to Θ^0, which is the identity. So $(\Theta^1)^* = \mathrm{id}$; therefore, $(\Phi^1)^* = (\Psi^1)^*$. It also can be easily checked that Ξ^n is a flat bundle. In particular, if the base space Q is simply connected, Ξ^n becomes a trivial bundle without holonomy.

The Poisson structure on P defines a global section ω of the vector bundle $\Xi^2 \longrightarrow Q$. We define a subbundle $\mathcal{D}$ of TQ by saying that $v \in T_qQ$ belongs to $\mathcal{D}$ if and only if $\nabla_v\omega = 0$. Since ∇ is a flat connection, i.e., $\nabla_{[X,Y]} = [\nabla_X, \nabla_Y]$ for all $X, Y \in \mathcal{X}(Q)$, $\mathcal{D}$ is a distribution. Given any fixed $q \in Q$, let Φ_q denote the morphism from T_qQ to $H^2(\pi^{-1}(q),\mathbb{R})$ defined by $\Phi_q(v) = \nabla_v\omega$ for all $v \in T_qQ$. It is obvious that $T_qQ/\mathcal{D}_q \cong \mathrm{Im}\Phi_q$ as a vector space. According to Dazord [D1], if P is integrable, Φ_q must be of constant rank while q varies in Q. In fact, the image of Φ_q in $H^2(\pi^{-1}(q),\mathbb{R})$ is spanned by the first Chern classes of the canonically associated principal bundle over the symplectic leaf $\pi^{-1}(q)$ of the symplectic groupoid. Therefore, $\mathcal{D}$ is of constant rank, and hence defines a foliation on Q, denoted by $\mathcal{F}_Q$. Obviously, this foliation on Q is the generalization of $\mathcal{F}_Q$ already defined when $P = S \times Q$.

Suppose that the rank $\Phi_q = k$ for all $q \in Q$, and there exist k parallel sections $[C_1], \cdots, [C_k]$ for the bundle $\Xi^2 \longrightarrow Q$, such that they span the space $\mathrm{Im}\Phi_q$ for all $q \in Q$. Then, the Poisson cohomology of P can still be described by the same formula in Theorem 4.1, where $L_l^{m,k}(S)$ should be considered as a subbundle of $\Xi^m \longrightarrow Q$ in an evident way, and $L_l^{m,k}[S]$ as the space of all the sections. $\bar{H}^i[S]$ should be understood similarly. This is true, because a symplectic groupoid of P can be obtained by patching together the symplectic groupoids over the Poisson submanifolds of P in which the symplectic foliations are trivial, as constructed in Theorem 2.1. However, we do not know a general formula.

BIBLIOGRAPHY

[BT] R. BOTT, L.W. TU, Differential forms in algebraic topology, Springer-Verlag, (1981).

[Br] J.-L. BRYLINSKI, A differential complex for Poisson Manifolds, J. Diff. Geom., 28 (1988), 93-114.

[CDW] A. COSTE, P. DAZORD, A. WEINSTEIN, Groupoïdes symplectiques, Publications du Départment de Mathématiques, Université Claude Bernard Lyon I, (1987).

[D1] P. DAZORD, Groupoïdes symplectiques et troisième théorème de Lie "non linéaire", Lecture Notes in Mathematics, vol. 1416 (1990), 39-74.

[D2] P. DAZORD, Réalisations isotropes de Libermann, Publ. Dept. Math. Lyon, (1989).

[DD] P. DAZORD, and T. DELZANT, le problème general des variables actions angles, J. Diff. Geom., 26 (1987), 223-251.

[F] D.B. FUKS, Cohomology of infinite-dimensional Lie algebras, Contemporary Soviet Mathematics, Consultants Bureau (1986).

[Ka] M.V. KARASEV, Analogues of objects of the theory of Lie groups for nonlinear Poisson brackets, Math. USSR Izvestiya, 28 (1987), 497-527.

[H] J. HUEBSCHMANN, Poisson cohomology and quantization, J. reine angew. Math., 408 (1990), 57-113.

[L] A. LICHNEROWICZ, Les variétés de Poisson et leurs algebres de Lie associées, J. Diff. Geom., 12 (1977), 253-300.

[M] K. MACKENZIE, Lie groupoids and Lie algebroids in differential geometry; LMS lecture Notes Series, 124 Cambridge Univ. Press, (1987).

[V] I. VAISMAN, Remarks on the Licherowicz-Poisson cohomology, Ann. Inst. Fourier, Grenoble, 40, 4 (1990), 951-963.

[VK1] Yu. M. VOROB'EV, M.V. KARASEV, Corrections to classical dynamics and quantization conditions which arise in the deformation of Poisson brackets, Dokl. Akad. Nauk USSR, 247, No. 6 (1987), 1294-1298.

[VK2] Yu. M. VOROB'EV, M.V. KARASEV, Poisson manifolds and the Schouten Bracket, Functional Analysis and its Applications, Vol. 22, No. 1 (1988), 1-9.

[W1] A. WEINSTEIN, The local structure of Poisson manifolds, J. Diff. Geom., 18 (1983), 523-557.

[W2] A. WEINSTEIN, Symplectic groupoids and Poisson manifolds, Bull. Amer. Math. Soc., 16 (1987), 101-104.

[WX] A. WEINSTEIN, P. XU, Extensions of symplectic groupoids and quantization, J. reine angew. Math., 417 (1991), 159-189.

Manuscrit reçu le 3 janvier 1991.

Ping XU,
Department of Mathematics
University of California
Berkeley, CA 94720 (USA).
Current address :
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104-6395 (USA).