

GEORGI VODEV

**On the distribution of scattering poles for
perturbations of the Laplacian**

Annales de l'institut Fourier, tome 42, n° 3 (1992), p. 625-635

http://www.numdam.org/item?id=AIF_1992__42_3_625_0

© Annales de l'institut Fourier, 1992, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

ON THE DISTRIBUTION OF SCATTERING POLES FOR PERTURBATIONS OF THE LAPLACIAN

by Georgi VODEV (*)

1. Introduction.

In this note we study the distribution of the scattering poles associated to second order differential operators of the form

$$G = c(x)^{-1} \left(- \sum_{i,j=1}^n \partial_{x_i} (g_{ij}(x) \partial_{x_j}) + \sum_{j=1}^n b_j(x) \partial_{x_j} + a(x) \right)$$

in $\mathbb{R}^n$, $n \geq 3$, odd, where the coefficients are such that the following conditions are fulfilled :

- (i) The operator G admits a selfadjoint realization, which will be again denoted by G , in the Hilbert space $H = L^2(\mathbb{R}^n; c(x)dx)$ with domain $D(G)$;
- (ii) There exists a constant $\rho_0 > 0$ so that for any $u \in D(G)$ such that $u = 0$ for $|x| \leq \rho_0$ we have $u \in H^2(\mathbb{R}^n)$ and $Gu = -\Delta u$, Δ being the Laplacian in $\mathbb{R}^n$;
- (iii) G is positively definite, i.e. $(Gu, u)_H \geq 0$, $\forall u \in D(G)$.

In what follows $\|\cdot\|$ will denote the norm in $\mathfrak{L}(H, H)$, the space of all linear bounded operators acting from H into H . It is easy to see by (i) and (iii) that the resolvent $R(z) = (G - z^2)^{-1} \in \mathfrak{L}(H, H)$ is well defined and holomorphic in $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$, and

$$(1.1) \quad \|R(z)\| \leq C(\text{Im } z)^{-2} \quad \text{for} \quad \text{Im } z > 0.$$

Choose a function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi = 1$ for $|x| \leq \rho_0 + 1$ and set $R_\chi(z) = \chi R(z) \chi$ for $z \in \mathbb{C}_+$. When

- (iv) $R_\chi(z_0)$ is a compact operator in $\mathfrak{L}(H, H)$ for some $z_0 \in \mathbb{C}_+$,

(*) Partially supported by Bulgarian Scientific Foundation under grant MM8/91.

it is well known that the cutoff resolvent $R_\chi(z)$ admits a meromorphic continuation from $\mathbb{C}_+$ to the entire complex plane $\mathbb{C}$ (see the analysis in the next section). The poles of this continuation are known as scattering poles or resonances and in our case they all are in $\bar{\mathbb{C}}_-$, where $\mathbb{C}_- = \{z \in \mathbb{C} : \text{Im } z < 0\}$. Note that if (iv) holds for at least one z_0 , it holds for all z_0 . Let $\{\lambda_j\}$ be the poles of $R_\chi(z)$, repeated according to multiplicity, and set

$$N(r) = \# \{\lambda_j : |\lambda_j| \leq r\}.$$

When the operator G is elliptic, in [8] and [14] (see also [13]) it is proved (without assuming (iii)) that

$$(1.2) \quad N(r) \leq Cr^n + C.$$

It also follows from the analysis in [8] and [14] that for hypoelliptic operators, i.e. when we have the estimates

$$(1.3) \quad \|f\|_{s+2\delta} \leq C_s(\|Gf\|_s + \|f\|_s), \quad \forall s \geq 0, \forall f \in D(G), Gf \in H^s,$$

where $0 < \delta < 1$ and $\|\cdot\|_s$ denotes the norm in the usual Sobolev space H^s , (again without assuming (iii)) the number of the poles satisfies the bound

$$(1.4) \quad N(r) \leq Cr^{n/\delta} + C.$$

Note that (1.3) implies (iv) at once. By (1.4) one actually concludes that the less regular the operator G is, the worse bound for $N(r)$ one has. In this work we show that outside a conic neighbourhood of the real axis the number of the scattering poles satisfies a much better estimate than (1.4) no matter how regular the operator G is. It actually has a bound of the type (1.2). To be more precise, given any ε , $0 < \varepsilon \ll 1$, set $\Lambda_\varepsilon = \{z \in \mathbb{C} : \varepsilon \leq \arg z \leq \pi - \varepsilon\}$ and

$$N(\varepsilon, r) = \# \{\lambda_j : |\lambda_j| \leq r, -\lambda_j \in \Lambda_\varepsilon\}.$$

Our main result is the following :

THEOREM 1. — *Assume (i)-(iv) fulfilled. Then for any ε , $0 < \varepsilon \ll 1$, there exists a constant $C_\varepsilon > 0$ so that*

$$(1.5) \quad N(\varepsilon, r) \leq C_\varepsilon r^n + C_\varepsilon.$$

The estimate (1.5) shows that to study the counting function $N(r)$ modulo terms $O(r^n)$ for positively definite selfadjoint hypoelliptic operators it suffices to study the number of the scattering poles in a conic ε -neighbourhood of the real axis for any small $\varepsilon > 0$.

The idea for the proofs of polynomial bounds of the scattering poles originates from Melrose [4] (see also [2], [5], [11], [12], [13], [14], [17]). One first needs to find an entire family of compact operators, $K(z)$, so that $(1 - K(z))R_\chi(z)$ is an entire operator-valued function and $1 - K(z)$ is invertible for at least one $z \in \mathbb{C}$. Thus one concludes that the poles of $R_\chi(z)$, with multiplicity, are among the poles of $(1 - K(z))^{-1}$ and hence among the zeros of an entire function $h(z) = \det(1 - K(z)^p)$, where $p \geq 1$ is an integer taken so that $K(z)^p$ is trace class. Thus the problem is reduced to obtaining suitable estimates for $|h(z)|$.

To prove (1.5) we need to find a family $K(z)$ as above so that $(1 - K(z))^{-1}$ can be expressed in terms of $R(z)$ for $z \in \mathbb{C}_+$ (see (2.5)), and $K(z) - K(-z)$ is trace class for any $z \in \mathbb{C}$. This enables us to characterize the poles of $R_\chi(z)$ in $\mathbb{C}_-$, with multiplicity, as zeros of a function $h(z)$, defined and holomorphic in $\mathbb{C}_-$, such that for any $\gamma > 0$ there exists a constant $C_\gamma > 0$ so that

$$(1.6) \quad |h(-z)| \leq C_\gamma \exp(C_\gamma |z|^\gamma) \quad \text{for} \quad \text{Im } z \geq \gamma.$$

Then, we derive (1.5) from (1.6) and a classical result due to Carleman (see Lemma 2).

Acknowledgments. The author would like to thank Vesselin Petkov for his support and encouragement.

2. Representation of the cutoff resolvent.

Denote by G_0 the selfadjoint realization of $-\Delta$ in the Hilbert space $H_0 = L^2(\mathbb{R}^n)$ and let $R_0(z)$ denote the outgoing resolvent of $-\Delta - z^2$, $z \in \mathbb{C}$. Then $R_0(z) = (G_0 - z^2)^{-1} \in \mathfrak{L}(H_0, H_0)$ for $z \in \mathbb{C}_+$ and as is well-known the kernel of $R_0(z)$ is given in terms of Hankel's functions by

$$(2.1) \quad R_0(z)(x, y) = (i/4)(z/2\pi|x-y|)^{(n-2)/2} H_{(n-2)/2}^{(1)}(z|x-y|).$$

It is easy to see that $\chi R_0(z)\chi \in \mathfrak{L}(H_0, H_0)$ for all $z \in \mathbb{C}$ and it forms an entire family of compact pseudodifferential operators of order -2 . Using this together with the assumption (iv) we shall build the meromorphic continuation of the cutoff resolvent of G . Set $Q = G - G_0$ and fix a $z_0 \in \mathbb{C}_+$. Clearly, for all $z \in \mathbb{C}_+$ we have

$$(2.2) \quad R(z) = R_0(z) + R(z)QR_0(z)$$

and

$$(2.3) \quad R(z) = R(z_0) + (z^2 - z_0^2)R(z)R(z_0).$$

Combining (2.2) and (2.3) yields

$$R(z)(1 - (z^2 - z_0^2)QR_0(z)R(z_0)) = R(z_0) + (z^2 - z_0^2)R_0(z)R(z_0)$$

for $z \in \mathbb{C}_+$. Multiplying the both sides of this identity by χ , since $Q = \chi Q$, we get

$$(2.4) \quad R_\chi(z)(1 - K(z)) = R_\chi(z_0) + K_1(z) \quad \text{for } z \in \mathbb{C}_+,$$

where

$$K(z) = (z^2 - z_0^2)QR_0(z)R(z_0)\chi$$

$$K_1(z) = (z^2 - z_0^2)\chi R_0(z)R(z_0)\chi.$$

Moreover, since $R(z)$ is well defined in $\mathbb{C}_+$, it is easy to see by (2.4) that $1 - K(z)$ is invertible in $\mathfrak{L}(H, H)$ for all $z \in \mathbb{C}_+$ and

$$(2.5) \quad (1 - K(z))^{-1} = 1 + (z^2 - z_0^2)QR_0(z)(R_0(z_0) + R_0(z_0)QR(z))\chi$$

for $z \in \mathbb{C}_+$. Now, since $R_0(z)$ and $R(z)$ are holomorphic in $\mathbb{C}_+$ with values in $\mathfrak{L}(H, H)$ and since $QR_0(z) = QR_0(z_0)(1 + (z^2 - z_0^2)R_0(z))$ for $z \in \mathbb{C}_+$, we deduce from (2.5) that $(1 - K(z))^{-1}$ is holomorphic in $\mathbb{C}_+$ with values in $\mathfrak{L}(H, H)$. Moreover, by (1.1), which clearly holds with $R(z)$ replaced by $R_0(z)$ as well, for any $\gamma > 0$ there exists a constant $C_\gamma > 0$ so that

$$(2.6) \quad \|(1 - K(z))^{-1}\| \leq C_\gamma(1 + |z|)^4 \quad \text{for } \operatorname{Im} z \geq \gamma.$$

Now let us see that the operator-valued functions $K(z)$ and $K_1(z)$, defined in $\mathbb{C}_+$, extend analytically to the entire $\mathbb{C}$ with values in the compact operators in $\mathfrak{L}(H, H)$. We shall consider $K(z)$ only, since $K_1(z)$ is treated similarly. Using that $R(z_0) = R_0(z_0) + R_0(z_0)QR(z_0)$ it is easy to see that

$$(2.7) \quad K(z) = (z^2 - z_0^2)QR_0(z)R_0(z_0)\chi(1 + QR_\chi(z_0))$$

for $z \in \mathbb{C}_+$. Choose functions $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^n)$ such that $\chi_1 = 1$ on $\operatorname{supp} Q$, $\chi_2 = 1$ on $\operatorname{supp} \chi_1$ and $\chi = 1$ on $\operatorname{supp} \chi_2$. After a standard computation (2.7) takes the form

$$(2.8) \quad K(z) = (z^2 - z_0^2)QR_0(z)R_0(z_0)\chi K_2 + (z^2 - z_0^2)QR_0(z)\chi K_3 R_\chi(z_0)$$

for $z \in \mathbb{C}_+$, where

$$K_2 = 1 + [\chi_2, G_0] R_0(z_0) [\chi_1, G_0] R_0(z_0) Q R_\chi(z_0),$$

$$K_3 = \chi_1 R_0(z_0) Q + \chi_2 R_0(z_0) [\chi_1, G_0] R_0(z_0) Q.$$

Here $[\cdot, \cdot]$ denotes the comutator. Clearly, we have $K_2, K_3 \in \mathfrak{L}(H, H)$. Further on, by a similar computation, for $z \in \mathbb{C}_+$, one obtains

$$(2.9) \quad \begin{aligned} (z^2 - z_0^2) Q R_0(z) R_0(z_0) \chi \\ = (K_4 + (z^2 - z_0^2) K_5) \chi R_0(z) \chi - K_4 \chi R_0(z_0) \chi \end{aligned}$$

and

$$(2.10) \quad Q R_0(z) \chi = (K_4 + (z^2 - z_0^2) K_5) \chi R_0(z) \chi + K_5,$$

where

$$K_4 = Q R_0(z_0) [G_0, \chi_1] R_0(z_0) [G_0, \chi_2],$$

$$K_5 = Q R_0(z_0) \chi_1 + Q R_0(z_0) [G_0, \chi_1] R_0(z_0) \chi_2.$$

Clearly, $K_4, K_5 \in \mathfrak{L}(H, H)$. Thus, by (2.8)-(2.10) we deduce

$$(2.11) \quad K(z) = K_6(z) \chi R_0(z) \chi K_2 + K_7(z) \chi R_0(z) \chi K_8 + K_9(z)$$

for $z \in \mathbb{C}_+$, where

$$K_6(z) = K_4 + (z^2 - z_0^2) K_5,$$

$$K_7(z) = (z^2 - z_0^2) K_6(z),$$

$$K_8 = K_3 R_\chi(z_0),$$

$$K_9(z) = -K_4 \chi R_0(z_0) \chi K_2 + (z^2 - z_0^2) K_5 K_3 R_\chi(z_0).$$

Clearly, these four operators are analytic $\mathfrak{L}(H, H)$ -valued functions. Now, since $\chi R_0(z) \chi$ forms an entire family of compact operators and by (iv) so does $K_9(z)$, by (2.11) we can extend $K(z)$ analytically to the entire $\mathbb{C}$. Then, since $K(z_0) = 0$, by Fredholm theorem, $(1 - K(z))^{-1}$ is a meromorphic $\mathfrak{L}(H, H)$ -valued function on $\mathbb{C}$. Thus, by (2.4) we obtain the desired meromorphic continuation of $R_\chi(z)$. Moreover, clearly the poles of this continuation coincide, with multiplicity, with the poles of $(1 - K(z))^{-1}$. Thus, since $1 - K(z)$ is invertible for $z \in \mathbb{C}_+$, we have that all the poles are in $\bar{\mathbb{C}}_-$. Now, for $z \in \mathbb{C}_+$, we have

$$(2.12) \quad 1 - K(-z) = (1 - K(z))(1 - T(z)),$$

where

$$T(z) = (1 - K(z))^{-1} (K(-z) - K(z)).$$

By (2.11) we have

$$(2.13) \quad T(z) = T_1(z)\chi S(z)\chi K_2 + T_2(z)\chi S(z)\chi K_8$$

where

$$\begin{aligned} S(z) &= R_0(-z) - R_0(z) \\ T_1(z) &= (1 - K(z))^{-1} K_6(z) \\ T_2(z) &= (1 - K(z))^{-1} K_7(z). \end{aligned}$$

By (2.6), for any $\gamma > 0$, we get

$$(2.14) \quad \|T_j(z)\| \leq C_\gamma(1 + |z|)^8 \quad \text{for} \quad \operatorname{Im} z \geq \gamma, \quad j = 1, 2.$$

On the other hand, by (2.1) and the well known properties of the Hankel functions, we have the following formula for the kernel of $S(z)$:

$$\begin{aligned} (2.15) \quad S(z)(x, y) &= (i/2)(z/2\pi|x-y|)^{(n-2)/2} J_{(n-2)/2}(z|x-y|) \\ &= (i/2)(2\pi)^{-n+1} z^{n-2} \int_{\mathbb{S}^{n-1}} \exp(iz\langle x-y, w \rangle) dw, \quad x, y \in \mathbb{R}^n, \end{aligned}$$

where $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$. Denote by $\tilde{S}(z)$ the operator with kernel given by the integral above. Now it is easy to see by (2.15) that $\chi S(z)\chi$ forms an entire family of trace class operators in $\mathfrak{L}(H, H)$. Hence, by (2.13), $T(z)$ is holomorphic in $\mathbb{C}_+$ with values in the trace class operators in $\mathfrak{L}(H, H)$. Now, by (2.12) it is easy to see that $1 - T(z)$ is invertible in $\mathfrak{L}(H, H)$ for those $z \in \mathbb{C}_+$ for which so is $1 - K(-z)$, and then we have

$$(2.16) \quad (1 - K(-z))^{-1} = (1 - T(z))^{-1}(1 - K(z))^{-1}.$$

Since $(1 - K(z))^{-1}$ is holomorphic in $\mathbb{C}_+$, by (2.16) we conclude that the poles of $(1 - K(-z))^{-1}$ lying in $\mathbb{C}_+$, with multiplicity, coincide with the poles of $(1 - T(z))^{-1}$. Introduce the function

$$h(z) = \det(1 - T(z)),$$

which is well defined and holomorphic in $\mathbb{C}_+$. Now, by the above analysis we conclude that if λ_j , $\lambda_j \in \mathbb{C}_-$, is a scattering pole, then $-\lambda_j$ is a zero of $h(z)$ with the corresponding multiplicity. Thus we can characterize the scattering poles as zeros of $h(-z)$, $z \in \mathbb{C}_-$. Notice that the fact that $T(z)$ is trace class does not depend on whether (iv) is

fulfilled or not. Hence the function $h(z)$ is always defined, under the conditions (i)-(iii), and holomorphic in $\mathbb{C}_+$. Now we are going to study the distribution of the zeros of $h(z)$ without assuming (iv). Note that in general the zeros of $h(z)$ may accumulate at points on the real axis. Let $\{z_j\} \subset \mathbb{C}_+$ be the zeros of $h(z)$, repeated according to multiplicity, and given $0 < \varepsilon$, $\delta \ll 1$, $r \gg 1$, set

$$N(\varepsilon, \delta, r) = \#\{z_j : \delta \leq |z_j| \leq r, z_j \in \Lambda_\varepsilon\}.$$

We have the following :

THEOREM 2. — *Assume (i)-(iii) fulfilled. Then, for any ε , δ , r as above there exists a constant $C_{\varepsilon, \delta} > 0$, independent of r , so that*

$$(2.17) \quad N(\varepsilon, \delta, r) \leq C_{\varepsilon, \delta} r^n \quad \text{for} \quad r \geq 1.$$

When (iv) is fulfilled the number of the scattering poles in $\{z \in \mathbb{C} : |z| \leq \delta\}$ is finite for any $\delta > 0$, and hence (1.5) is obtained as an immediate consequence of (2.17).

3. Proof of Theorem 2.

We start with the following :

LEMMA 1. — *Under the assumptions (i)-(iii), for any $\gamma > 0$ there exists a constant $C_\gamma > 0$ so that*

$$(3.1) \quad |h(z)| \leq C_\gamma \exp(C_\gamma |z|^n) \quad \text{for} \quad \text{Im } z \geq \gamma.$$

Proof. — The estimate (3.1) is established in the same way as in [13] (see also [17]). Here we shall sketch the proof. Given a compact operator A , $\mu_j(A)$ will denote the characteristic values of A , i.e. the eigenvalues of $(A^*A)^{1/2}$, repeated according to multiplicity and ordered to form a nonincreasing sequence. First, recall some well known properties of $\mu_j(A)$:

$$(3.2) \quad \mu_j(A) \leq \|A\|, \quad \forall j,$$

$$(3.3) \quad \{\mu_j(AB), \mu_j(BA)\} \leq \mu_j(A)\|B\|, \quad \forall j,$$

$$(3.4) \quad \mu_j\left(\sum_{i=1}^k A_i\right) \leq \sum_{i=1}^k \mu_{j_k}(A_i), \quad \forall j,$$

where $j_k \sim [j/k]$, $[a]$ denotes the integer part of a . By (2.13)-(2.15) and (3.2)-(3.4) it is easy to see that

$$(3.5) \quad \mu_j(T(z)) \leq C_\gamma(1+|z|)^{n+6} \mu_{j_2}(\chi \tilde{S}(z) \chi) \quad \text{for} \quad \text{Im } z \geq \gamma.$$

On the other hand, clearly we have

$$(3.6) \quad \|\chi \tilde{S}(z) \chi\| \leq C \exp(C|z|), \quad \forall z \in \mathbb{C}.$$

Combining (3.5) and (3.6) yields

$$(3.7) \quad \mu_j(T(z)) \leq C_\gamma \exp(C|z|), \quad \forall j, \quad \text{for} \quad \text{Im } z \geq \gamma.$$

Further on, we shall show that there exists a constant $C > 0$ so that

$$(3.8) \quad \mu_j(\chi \tilde{S}(z) \chi) \leq C e^{-|z|j^{-n/(n-1)}} \quad \text{if} \quad j \geq C|z|^{n-1}, \quad \forall z \in \mathbb{C}.$$

This is actually proved in [13], but for the sake of completeness we shall repeat the key points. The key observation is the representation

$$(3.9) \quad \tilde{S}(z) = S_1(z) S_2(z),$$

where $S_1(z)$ is the operator with kernel $S_1(z)(x, w) = \exp(iz \langle x, w \rangle)$, $S_2(z)$ is the operator with kernel $S_2(z)(w, x) = \exp(-iz \langle x, w \rangle)$, $x \in \mathbb{R}^n$, $w \in \mathbb{S}^{n-1}$. Then, using (3.3) and (3.9) we have

$$(3.10) \quad \mu_j(\chi \tilde{S}(z) \chi) \leq \|\chi S_1(z)\|_1 \|(1 - \Delta_w)^m S_2(z) \chi\|_2 \mu_j((1 - \Delta_w)^{-m}), \quad \forall j,$$

for any integer $m \geq 1$, where Δ_w denotes the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$, $\|\cdot\|_1$ and $\|\cdot\|_2$ denote the norms in $\mathcal{L}(L^2(\mathbb{S}^{n-1}), L^2(\mathbb{R}^n))$ and $\mathcal{L}(L^2(\mathbb{R}^n), L^2(\mathbb{S}^{n-1}))$, respectively. On the other hand, we have with a constant $C > 0$,

$$(3.11) \quad \mu_j((1 - \Delta_w)^{-m}) \leq C^m j^{-2m/l},$$

where $l = \dim \mathbb{S}^{n-1} = n - 1$, and

$$(3.12) \quad \|\chi S_1(z)\|_1 \leq C \exp(C|z|),$$

$$(3.13) \quad \begin{aligned} \|(1 - \Delta_w)^m S_2(z) \chi\|_2 &\leq C \sup_{x, w} |\chi(x) (1 - \Delta_w)^m (e^{-z \langle x, w \rangle})| \\ &\leq C^{2m+1} (|z|^{2m} + (2m)^{2m}) e^{C|z|}. \end{aligned}$$

Thus, by (3.10)-(3.13),

$$(3.14) \quad \mu_j(e^{|z|} \chi \tilde{S}(z) \chi) \leq C^{2m+1} (|z|^{2m} + (2m)^{2m}) e^{C|z|} j^{-2m/l},$$

with a new constant $C > 0$. Now, (3.8) is an easy consequence of (3.14) (see [13], [17]).

Thus, by (3.5) and (3.8), we have

$$(3.15) \quad \mu_j(T(z)) \leq C_\gamma j^{-n/(n-1)} \quad \text{if } j \geq C|z|^{n-1}, \quad \text{for } \operatorname{Im} z \geq \gamma,$$

with new constants C_γ , $C > 0$. Now, it is a straightforward calculation that (3.7) and (3.15) together with Weyl's convexity estimate imply (3.1) (see [13], [17]. The proof of Lemma 1 is completed.

To derive (2.17) from (3.1), instead of Jensen's inequality, we shall use the following classical result (see [9], Section 3, Carleman's theorem).

LEMMA 2. — Given $r > r_0 > 0$, set $\Omega = \{z \in \mathbb{C} : r_0 \leq |z| \leq r, \operatorname{Im} z \geq 0\}$. Let $f(z)$ be a function holomorphic in Ω and let $r_1 \exp(i\varphi_1), r_2 \exp(i\varphi_2), \dots, r_k \exp(i\varphi_k)$ be the zeros of $f(z)$ in Ω repeated according to multiplicity. Then,

$$\begin{aligned} \sum_{j=1}^k (r_j^{-1} - r_j r^{-2}) \sin \varphi_j &= (\pi r)^{-1} \int_0^\pi \log |f(re^{i\varphi})| \sin \varphi \, d\varphi \\ &\quad + (2\pi)^{-1} \int_{r_0}^r (t^{-2} - r^{-2}) \log |f(t)f(-t)| \, dt \\ &\quad - (\pi r_0)^{-1} \int_0^\pi \log |f(r_0 e^{i\varphi})| \sin \varphi \, d\varphi. \end{aligned}$$

Note that each term in the sum above is ≥ 0 . Fix $\varepsilon, \delta, 0 < \varepsilon, \delta \ll 1$, and let $r > 6$. Let

$$z_1 = r_1 \exp(i\varphi_1), \quad z_2 = r_2 \exp(i\varphi_2), \dots, z_k = r_k \exp(i\varphi_k)$$

be the zeros of $h(z)$, repeated according to multiplicity, satisfying the conditions: $3 \leq r_j \leq r/2$; $\varepsilon \leq \varphi_j \leq \pi - \varepsilon$. Clearly,

$$(3.16) \quad N(\varepsilon, \delta, r/2) \leq k + N(\varepsilon, \delta, 3).$$

Set $f(z) = h(z + i\gamma)$ where $\gamma = \sin \varepsilon$. Clearly, $f(z)$ is holomorphic in $\bar{\mathbb{C}}_+$ and by (3.1) we have

$$(3.17) \quad |f(z)| \leq C_\varepsilon \exp(C_\varepsilon |z|^n), \quad \forall z \in \bar{\mathbb{C}}_+.$$

Moreover, $z'_j = z_j - i\gamma$, $j = 1, \dots, k$, are zeros of $f(z)$. Set $r'_j = |z'_j|$ and $\varphi'_j = \arg z'_j$. It is easy to check that $2 \leq r'_j \leq 2r/3$ and $\sin \varphi'_j \geq$

$2^{-1} \sin \varepsilon$, $j = 1, \dots, k$. Hence

$$(3.18) \quad (r_j'^{-1} - r_j' r^{-2}) \sin \varphi_j' \geq (5\gamma/12) r^{-1}, \quad j = 1, \dots, k.$$

Now, applying Lemma 2 to $f(z)$ with $r_0 = 2$ and using (3.17) and (3.18), we get

$$\begin{aligned} (5\gamma/12) r^{-1} k &\leq \sum_{j=1}^k (r_j'^{-1} - r_j' r^{-2}) \sin \varphi_j' \\ &\leq (\pi r)^{-1} \int_0^\pi \log |f(re^{i\varphi})| \sin \varphi \, d\varphi \\ &\quad + (2\pi)^{-1} \int_2^r (t^{-2} - r^{-2}) \log |f(t)f(-t)| \, dt + C_\varepsilon \\ &\leq C'_\varepsilon r^{-1}(r^n + 1) + C'_\varepsilon \int_2^r t^{-2}(t^n + 1) \, dt + C_\varepsilon \\ &\leq C''_\varepsilon r^{-1}(r^n + 1). \end{aligned}$$

Hence

$$(3.19) \quad k \leq (12/5\gamma) C''_\varepsilon (r^n + 1).$$

Now (2.17) follows from (3.16) and (3.19) at once.

BIBLIOGRAPHY

- [1] C. BARDOS, J. GUILLOT, J. RALSTON, La relation de Poisson pour l'équation des ondes dans un ouvert non borné. Application à la théorie de la diffusion, Commun. Partial Differ. Equations, 7 (1982), 905-958.
- [2] A. INTISSAR, A polynomial bound on the number of scattering poles for a potential in even dimensional space $\mathbb{R}^n$, Commun. Partial Differ. Equations, 11 (1986), 367-396.
- [3] P. D. LAX, R. S. PHILLIPS, Scattering Theory, Academic Press, 1967.
- [4] R. B. MELROSE, Polynomial bounds on the number of scattering poles, J. Func. Anal., 53 (1983), 287-303.
- [5] R. B. MELROSE, Polynomial bounds on the distribution of poles in scattering by an obstacle, Journées « Équations aux dérivées partielles », Saint-Jean-de-Monts (1984).
- [6] R. B. MELROSE, Weyl asymptotics for the phase in obstacle scattering, Commun. Partial Differ. Equations, 13 (1988), 1431-1439.
- [7] J. SJÖSTRAND, Geometric bounds on the number of resonances for semiclassical problems, Duke Math. J., 60 (1990), 1-57.

- [8] J. SJÖSTRAND, M. ZWORSKI, Complex scaling and the distribution of scattering poles, *J. Amer. Math. Soc.*, (1991), 729-769.
- [9] E. C. TITCHMARSH, *The Theory of Functions*, Oxford University Press, 1968.
- [10] B. VAINBERG, *Asymptotic Methods in Equations of Mathematical Physics*, Gordon and Breach Sci. Publ., 1988.
- [11] G. VODEV, Polynomial bounds on the number of scattering poles for symmetric systems, *Ann. Inst. Henri Poincaré (Physique théorique)*, 54 (1991), 199-208.
- [12] G. VODEV, Polynomial bounds on the number of scattering poles for metric perturbations of the Laplacian in $\mathbb{R}^n$, $n \geq 3$, odd, *Osaka J. Math.*, 28 (1991), 441-449.
- [13] G. VODEV, Sharp polynomial bounds on the number of scattering poles for metric perturbation of the Laplacian in $\mathbb{R}^n$, *Math. Ann.*, 291 (1991), 39-49.
- [14] G. VODEV, Sharp bounds on the number of scattering poles for perturbations of the Laplacian, *Commun. Math. Phys.*, 145 (1992), to appear.
- [15] M. ZWORSKI, Distribution of poles for scattering in the real line, *J. Func. Anal.*, 73 (1987), 227-296.
- [16] M. ZWORSKI, Sharp polynomial bounds on the number of scattering poles of radial potentials, *J. Func. Anal.*, 82 (1989), 370-403.
- [17] M. ZWORSKI, Sharp polynomial bounds on the number of scattering poles, *Duke Math. J.*, 59 (1989), 311-323.

Manuscrit reçu le 24 juin 1991,
révisé le 8 octobre 1991.

Georgi VODEV,
Institute of Mathematics
Bulgarian Academy of Sciences
Acad. G. Bonchev str. bl. 8
1113 Sofia (Bulgarie).