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FRANK MANTLIK

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PARTIAL DIFFERENTIAL OPERATORS DEPENDING ANALYTICALLY ON A PARAMETER

by Frank MANTLIK

0. Introduction.

Consider a linear differential operator in $\mathbf{R}^n$,

$$P(\lambda, D) = \sum_{|\alpha| \leq m} a_\alpha(\lambda) D^\alpha : D = -i\partial \ , \ \partial = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) \ ,$$

where the coefficients $a_\alpha(\lambda)$ – constant with respect to the variable of differentiation x – may depend analytically on a parameter λ in a complex manifold Λ. We assume that $P(\lambda, D)$ is equally strong for each $\lambda \in \Lambda$.

In [H2], p. 59 L. Hörmander posed the question whether under these conditions there exists a fundamental solution $\mathfrak{f}_\lambda$ of $P(\lambda, D)$ which depends analytically on λ. In 1962 F. Treves [T2] had shown that this is true locally in Λ and that the assumption of constant strength is necessary for this to hold [T1]. Recently the author could construct a global solution in the hypoelliptic case [M]. The proof of this result based on the fact that for each compact subset Λ' of Λ there exists an integration contour in $\mathbf{C}^n$ which yields fundamental solutions of $P(\lambda, D)$ simultaneously for all $\lambda \in \Lambda'$. In a second step we could apply a theorem of J. Leiterer [L] to obtain a global solution $\mathfrak{f}_\lambda$ by means of a Mittag-Leffler procedure.

The aim of the present paper is to eliminate the assumption of hypoellipticity. In section 1 we show that also in the general case one can

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always find a uniform integration contour $H_{\Lambda'}$ for all λ in a compact subset Λ' of Λ. As a consequence we obtain an explicit formula for $\mathfrak{f}_\lambda : \lambda \in \Lambda'$. Our proof uses some ideas of Hörmander [H2] concerning asymptotic properties of multivariate polynomials. The rest of this article is essentially an adaptation of the methods of [M] : in section 2 certain distribution spaces are introduced by means of the contours $H_{\Lambda'}$. These spaces constitute the setting for our application of the Leiterer theorem [L]. Section 3 contains the statements and proofs of our main results. We consider the equation $P(\lambda, D)\mathfrak{f}_\lambda = \mathfrak{g}_\lambda$ where $\mathfrak{g}_\lambda$ is a given analytic function of λ with values in some distribution space and prove the existence of a solution $\mathfrak{f}_\lambda$ which also depends analytically on λ. In the special case $\mathfrak{g}_\lambda \equiv \delta$ (the Dirac distribution) we obtain a solution to the problem described above.

1. Construction of a uniform integration contour.

We begin by fixing some notations : for any $n, m \in \mathbb{N}$ let

$$\mathrm{Pol}(n,m) := \{P \in \mathbb{C}[x_1, \dots, x_n] \mid \deg P \leq m\};$$

$$\mathrm{Pol}'(n,m) := \{P \in \mathrm{Pol}(n,m) \mid \deg P = m\}\ .$$

If $P, Q \in \mathbb{C}[x_1, \dots, x_n]$ then we write

$$\delta_P(\xi) := \mathrm{dist}(\xi, \{\zeta \in \mathbb{C}^n \mid P(\zeta) = 0\}) : \quad \xi \in \mathbb{C}^n;$$

$$\widetilde{P}(\xi, t) := \sum_\alpha t^{|\alpha|} |P^{(\alpha)}(\xi)| : \quad \xi \in \mathbb{C}^n,\ t > 0\ ,$$

where $|\alpha| := \sum_{j=1}^n \alpha_j$ and $P^{(\alpha)} := \partial^\alpha P$;

$$\widetilde{P}(\xi) := \widetilde{P}(\xi, 1);$$

$$P < Q :\iff \sup\{\widetilde{P}(\xi)/\widetilde{Q}(\xi) \mid \xi \in \mathbb{R}^n\} < \infty;$$

$$P \sim W :\iff P < Q \wedge Q < P;$$

$$\mathsf{W}(Q) := \{P \in \mathbb{C}[x_1, \dots, x_n] \mid P < Q\};$$

$$\mathsf{E}(Q) := \{P \in \mathbb{C}[x_1, \dots, x_n] \mid P \sim Q\}\ .$$

1.1. *Remarks.*

(i) Note that our definition of $\widetilde{P}(\xi, t)$ differs from that of Hörmander [H2], §10.4, who used the notation $\widetilde{P}(\xi, t) := \left(\sum_\alpha t^{2|\alpha|} |P^{(\alpha)}(\xi)|^2\right)^{1/2}$.

According to [H2], 10.4.3 we have

$$P < Q \iff \sup\{\widetilde{P}(\xi,t)/\widetilde{Q}(\xi,t) \mid \xi \in \mathbf{R}^n,\ t \geq 1\} < \infty\ .$$

In this case we say that P *is weaker than* Q. If $P \sim Q$ then we say that P *and* Q *are equally strong.*

(ii) $P < Q \Longrightarrow \deg P \leq \deg Q$. This is clear by definition of $\widetilde{P}$. In particular, $\mathbf{W}(Q)$ is a finite-dimensional complex vector space (consequence of [H2], 10.4.1).

(iii) $\mathsf{E}(Q)$ is a linearly convex, open subset of $\mathbf{W}(Q)$ ([H2], 10.4.7). For our purposes it suffices to know that $\mathsf{E}(Q)$ is holomorphically convex (*cf.* [M]).

We assume the integers n, m to be fixed throughout this paper. The letters c, C denote positive constants which only depend on n and m. We use the notations

$$|\xi| := \sum |\xi_j|\ ,\ |\xi|_\infty := \max |\xi_j| : \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbf{C}^n\ .$$

For $\mathbf{K} = \mathbf{R}, \mathbf{C}$ and $\rho \geq 0$ let

$$\mathbf{B}_{\mathbf{K}^n}(\rho) := \{\xi \in \mathbf{K}^n \mid |\xi|_\infty \leq \rho\}\ .$$

In the case $\rho = 1$ we simply write $\mathbf{B}_{\mathbf{K}^n}$. Further let

$$\mathbf{T}^r := \{z \in \mathbf{C}^r \mid |z_1| = \cdots = |z_r| = 1\} \ \text{ if } \ r \in \mathbf{N}\ .$$

1.2. THEOREM. — *Let* $Q \in \mathrm{Pol}'(n,m)$, $\Pi \subseteq \mathsf{E}(Q)$ *be a compact set and* $\rho \geq 0$. *Then there exists* $A \geq 1$ *and a bounded measurable function* $\eta : \mathbf{R}^n \to \mathbf{R}^n$ *such that*

$$\widetilde{P}(\xi) \leq A|P(\xi + \zeta + z\eta(\xi))| : \quad P \in \Pi, \xi \in \mathbf{R}^n, \zeta \in \mathbf{B}_{\mathbf{C}^n}(\rho),\ z \in \mathbf{T}^1\ . \tag{1.1}$$

Our proof of this theorem is long and will occupy the rest of this section. First it requires a detailed study of the function $\widetilde{P}(\xi,t)$:

1.3. LEMMA. — *Let* $Q \in \mathrm{Pol}'(n,m)$ *and* $\Pi \subseteq \mathsf{E}(Q)$ *be compact. Then there exists* $B \geq 1$ *such that*

$$B^{-1} \leq \widetilde{P}(\xi,t)/\widetilde{Q}(\xi,t) \leq B : \quad P \in \Pi,\ \xi \in \mathbf{R}^n,\ t \geq 1\ . \tag{1.2}$$

Proof. — By 1.1 (i) the expression $N_Q(P) := \sup\{\widetilde{P}(\xi,t)/\widetilde{Q}(\xi,t) \mid \xi \in \mathbf{R}^n,\ t \geq 1\}$ defines a norm on $\mathbf{W}(Q)$. Now let $R \in \Pi$ be fixed. Since $Q < R$ we have

$$b_R := \inf\{\widetilde{R}(\xi,t)/\widetilde{Q}(\xi,t) \mid \xi \in \mathbf{R}^n,\ t \geq 1\} > 0\ .$$

For any $P \in \omega_R := \{P \in \mathbf{W}(Q) \mid N_Q(R-P) < b_R/2\}$ we get

$$\frac{\widetilde{P}(\xi,t)}{\widetilde{Q}(\xi,t)} \geq \frac{\widetilde{R}(\xi,t) - (R-P)^{\sim}(\xi,t)}{\widetilde{Q}(\xi,\tau)} > b_R/2 : \quad \xi \in \mathbf{R}^n, t \geq 1.$$

Since ω_R is an open neighborhood of R it follows from the compactness of Π that there exists $b_0 > 0$ with

$$\widetilde{P}(\xi,t) \geq b_0 \widetilde{Q}(\xi,t) : \quad P \in \Pi,\ \xi \in \mathbf{R}^n,\ t \geq 1 .$$

On the other hand the boundedness of Π implies that

$$B_0 := \sup\{N_Q(P) \mid P \in \Pi\} < \infty ,$$

hence

$$\widetilde{P}(\xi,t) \leq B_0 \widetilde{Q}(\xi,t) : \quad P \in \Pi,\ \xi \in \mathbf{R}^n,\ t \geq 1 .$$

With $B := \max\{1/b_0, B_0\}$ the assertion follows . □

1.4. Lemma (*cf.* [H2], 11.1.4). — *There exists $C \geq 1$ such that for any $P \in \mathrm{Pol}'(n,m)$ the following holds :*

$$|P^{(\alpha)}(\xi)|\delta_P(\xi)^{|\alpha|} \leq C|P(\xi)| : \quad \xi \in \mathbf{C}^n,\ |\alpha| \leq m . \tag{1.3}$$

$$C^{-1} \leq \delta_P(\xi) \sum_{\alpha \neq 0} |P^{(\alpha)}(\xi)/P(\xi)|^{1/|\alpha|} \leq C : \quad \xi \in \mathbf{C}^n,\ P(\xi) \neq 0 . \tag{1.4}$$

$$|P(\xi)| \leq \widetilde{P}(\xi, \delta_P(\xi)) \leq C|P(\xi)| : \quad \xi \in \mathbf{C}^n . \tag{1.5}$$

Proof. — (1.4) is due to Hörmander [H2], 11.1.4. (1.5) is a consequence of (1.3) which follows from (1.4). □

1.5. Lemma (*cf.* [H2], 11.1.9). — *There exists $c > 0$ such that for any $P, Q \in \mathrm{Pol}'(n,m)$ and $\xi \in \mathbf{C}^n$ we have : if*

$$B^{-1} \leq \widetilde{P}(\xi,t)/\widetilde{Q}(\xi,t) \leq B : \quad t \geq 1 \tag{1.6}$$

holds with some $B \geq 1$ then

$$\frac{c}{1+B^2} \leq \frac{1+\delta_P(\xi)}{1+\delta_Q(\xi)} \leq \frac{1+B^2}{c} . \tag{1.7}$$

Proof. — If $\delta_Q(\xi) \geq 1$ then

$$\begin{aligned}\sum_\alpha |P^{(\alpha)}(\xi)|\delta_Q(\xi)^{|\alpha|} &\overset{(1.6)}{\leq} B \sum_\alpha |Q^{(\alpha)}(\xi)|\delta_Q(\xi)^{|\alpha|} \\ &\overset{(1.5)}{\leq} C_1 B|Q(\xi)| \overset{(1.6)}{\leq} C_1 B^2 \sum_\alpha |P^{(\alpha)}(\xi)| .\end{aligned}$$

When $\delta_Q(\xi) \geq 2C_1B^2 =: D$ (hence $\frac{1}{2}\delta_Q(\xi)^{|\alpha|} \leq \delta_Q(\xi)^{|\alpha|} - \frac{D}{2}$, $\alpha \neq 0$) this yields

$$\sum_{\alpha} |P^{(\alpha)}(\xi)|\delta_Q(\xi)^{|\alpha|} \leq D|P(\xi)| \,.$$

In particular then $P(\xi) \neq 0$ and

$$|P^{(\alpha)}(\xi)/P(\xi)|^{1/|\alpha|}\delta_P(\xi) \leq D\delta_P(\xi)/\delta_Q(\xi) : \quad \alpha \neq 0 \,.$$

Summing up we get

$$C_2B^2\delta_P(\xi)/\delta_Q(\xi) \geq \delta_P(\xi)\sum_{\alpha\neq 0}|P^{(\alpha)}(\xi)/P(\xi)|^{1/|\alpha|} \overset{(1.4)}{\geq} C_3^{-1} \,,$$

hence

$$\frac{1+\delta_P(\xi)}{1+\delta_Q(\xi)} \geq \frac{1}{2}\frac{\delta_P(\xi)}{\delta_Q(\xi)} \geq (2C_2C_3B^2)^{-1} \quad \text{if} \quad \delta_Q(\xi) \geq D \,.$$

In the case $\delta_Q(\xi) \leq D$ we have

$$\frac{1+\delta_P(\xi)}{1+\delta_Q(\xi)} \geq \frac{1}{1+2C_1B^2} \,.$$

With suitable $c > 0$ we obtain the lefthand side of (1.7). The second inequality follows from this one by interchanging the roles of P and Q. □

1.6. LEMMA (*cf.* [H2], 10.4.2). — *There exists $C \geq 1$ such that for any $P \in \mathrm{Pol}(n,m)$, $\xi \in \mathbb{C}^n$ and $\tau > 0$:*

(1.8) $$C^{-1}\tilde{P}(\xi,\tau) \leq \max\{|P(\xi+\eta)| \mid \eta \in \mathbf{B}_{\mathbb{K}^n}(\tau)\} \leq C\tilde{P}(\xi,\tau) \,;$$

(1.9) $$C^{-1}\tau \leq \max\{\delta_P(\xi+\eta) \mid \eta \in \mathbf{B}_{\mathbb{K}^n}(\tau)\} \quad \textit{if } P \textit{ is nonconstant} \,.$$

This holds for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

Proof. — Assertion (1.8) corresponds to [H2], 10.4.2. (Our use of the ℓ_1-norm in the definition of $\tilde{P}(\xi,t)$ only results in a change of the constants.)

Ad (1.9) : first we note that for $\tau > 0$ and $\eta \in \mathbf{B}_{\mathbb{K}^n}(\tau)$,

$$|P^{(\alpha)}(\xi+\eta)| \leq \sum_{\beta}|P^{(\alpha+\beta)}(\xi)|\tau^{|\beta|} \leq \tau^{-|\alpha|}\tilde{P}(\xi,\tau)$$

by Taylor's formula. As a consequence we have the estimate

(1.10) $$\tilde{P}(\xi+\eta,\tau) \leq C_1\tilde{P}(\xi,\tau) : \quad P \in \mathrm{Pol}(n,m), \xi \in \mathbb{C}^n, \eta \in \mathbf{B}_{\mathbb{C}^n}(\tau),$$

which will be used later. By (1.8) there exists for fixed $\xi \in \mathbb{C}^n$ and $\tau > 0$ an $\eta \in \mathbf{B}_{\mathbb{K}^n}(\tau)$ such that

$$\tilde{P}(\xi,\tau) \leq C_2|P(\xi+\eta)| \,.$$

In particular then $P(\xi+\eta) \neq 0$ and

$$\sum_{\alpha\neq 0} |P^{(\alpha)}(\xi+\eta)/P(\xi+\eta)|^{1/|\alpha|} \leq \sum_{1\leq|\alpha|\leq m} (C_2\tau^{-|\alpha|})^{1/|\alpha|} \leq C_3\tau^{-1} .$$

From (1.4) it follows that $\delta_P(\xi+\eta) \geq C_4^{-1}\tau$, hence the assertion. □

Now we can already prove a preliminary version of Theorem 1.2 :

1.7. COROLLARY. — *Let $Q \in \mathrm{Pol}'(n,m)$ and $\Pi \subseteq \mathsf{E}(Q)$ compact. Then there exist A, $\mu \geq 1$ such that*

(1.11) $\quad \forall\tau \geq \mu,\ \xi \in \mathbf{R}^n\ \exists\eta \in \mathbf{B}_{\mathbf{R}^n}(\tau)\ \forall P \in \Pi\ : \widetilde{P}(\xi,\tau) \leq A|P(\xi+\eta)| .$

Proof. — By Lemma 1.3 there exists $B \geq 1$ such that

$$B^{-1} \leq \widetilde{P}(\xi,t)/\widetilde{Q}(\xi,t) \leq B : \quad P \in \Pi,\ \xi \in \mathbf{R}^n,\ t \geq 1 .$$

With $A_1 := (1+B^2)/c \geq 1$ we get from (1.7),

$$A_1^{-1}(1+\delta_Q(\xi)) \leq 1+\delta_P(\xi) : \quad P \in \Pi,\ \xi \in \mathbf{R}^n .$$

By (1.9) we have

(1.12) $\quad \max\{\delta_Q C\xi+\eta) \mid \eta \in \mathbf{B}_{\mathbf{R}^n}(\tau)\} \geq C_0^{-1}\tau : \quad \xi \in \mathbf{R}^n,\ \tau > 0 .$

Choose $A_2 \geq 1$ with $C_0^{-1} - A_1/A_2 > 0$ and put

$$\mu := \max\{1, (A_1-1)/(C_0^{-1} - A_1/A_2)\} .$$

If $\tau \geq \mu$ then $(1+C_0^{-1}\tau)/A_1 \geq 1+\tau/A_2$. For such a τ and arbitrary $\xi \in \mathbf{R}^n$ we may now choose $\eta \in \mathbf{B}_{\mathbf{R}^n}(\tau)$ with $\delta_Q(\xi+\eta) \geq C_0^{-1}\tau$ according to (1.12). For any $P \in \Pi$ we then obtain

$$1+\delta_P(\xi+\eta) \geq A_1^{-1}(1+\delta_Q(\xi+\eta)) \geq A_1^{-1}(1+C_0^{-1}\tau) \geq 1+\tau/A_2 ,$$

i.e. $\tau \leq A_2\delta_P(\xi+\eta)$. Because of (1.5) this yields

$$\begin{aligned}\widetilde{P}(\xi+\eta,\tau) \leq \widetilde{P}(\xi+\eta, A_2\delta_P(\xi+\eta)) &\leq A_2^m\widetilde{P}(\xi+\eta,\delta_P(\xi+\eta)) \\ &\leq A_3|P(\xi+\eta)|.\end{aligned}$$

Finally, replacing in (1.10) η by $-\eta$ and ξ by $\xi+\eta$, we obtain

$$\widetilde{P}(\xi,\tau) \leq C_1\widetilde{P}(\xi+\eta,\tau) \leq C_1A_3|P(\xi+\eta)| : \quad P \in \Pi . \qquad \square$$

For any $R \in \mathbb{C}[x_1,\ldots,x_n]$ and $k \in \mathbb{N}_0$ we put

$$(\Phi_kR)(\xi) := \sum_{|\alpha|=k} R^{(\alpha)}(\xi)\overline{R}^{(\alpha)}(\xi) ,$$

where $\overline{R}$ is obtained from R by taking complex conjugates of the coefficients. Note that $\Phi_k R \in \mathbf{R}[x_1, \dots, x_n]$ and $(\Phi_k R)(\xi) \geq 0$ for $\xi \in \mathbf{R}^n$. With the notation

$$(\Psi_k R)(\xi) := \sum_{|\alpha|=k} |R^{(\alpha)}(\xi)|$$

we have

$$\widetilde{R}(\xi, t) = \sum_{k=0}^{m} t^k (\Psi_k R)(\xi) : \quad R \in \mathrm{Pol}(n, m) .$$

1.8. LEMMA. — *There exists $C \geq 1$ such that for any $P \in \mathrm{Pol}(n,m)$, $k \in \mathbb{N}_0$, $\xi \in \mathbf{R}^n$ and $t > 0$:*

$$C^{-1}(\Phi_k P)^{\sim}(\xi, t) \leq \Big(\sum_{j=k}^{m} t^{j-k}(\Psi_j P)(\xi)\Big)^2 \leq C(\Phi_k P)^{\sim}(\xi, t) . \tag{1.13}$$

Proof. — First we have by (1.8) (note that $\Phi_k P \in \mathrm{Pol}(n, 2m)$),

$$C_1^{-1}(\Phi_k P)^{\sim}(\xi, t) \leq \max_{\eta \in \mathbf{B}_{\mathbf{R}^n}} (\Phi_k P)(\xi + t\eta) \leq C_1(\Phi_k P)^{\sim}(\xi, t) \tag{1.14}$$

and

$$C_1^{-1} \sum_{|\alpha|=k} (P^{(\alpha)})^{\sim}(\xi, t) \leq \sum_{|\alpha|=k} \max_{\eta \in \mathbf{B}_{\mathbf{R}^n}} |P^{(\alpha)}(\xi + t\eta)| \leq C_1 \sum_{|\alpha|=k} (P^{(\alpha)})^{\sim}(\xi, t) .$$

Furthermore an easy calculation shows that

$$C_2^{-1} \sum_{|\alpha|=k} (P^{(\alpha)})^{\sim}(\xi, t) \leq \sum_{j=k}^{m} t^{j-k}(\Psi_j P)(\xi) \leq C_2 \sum_{|\alpha|=k} (P^{(\alpha)})^{\sim}(\xi, t) ,$$

hence

$$\begin{aligned} C_3^{-1} \sum_{j=k}^{m} t^{j-k}(\Psi_j P)(\xi) &\leq \sum_{|\alpha|=k} \max_{\eta \in \mathbf{B}_{\mathbf{R}^n}} |P^{(\alpha)}(\xi + t\eta)| \\ &\leq C_3 \sum_{j=k}^{m} t^{j-k}(\Psi_j P)(\xi) . \end{aligned} \tag{1.15}$$

Now let $\mathbf{M}(n, k) = \{\alpha \in \mathbb{N}_0^n \mid |\alpha| = k\}$. Obviously the expressions

$$N_1((R_\alpha)_{\alpha \in \mathbf{M}(n,k)}) := \Big(\max_{\eta \in \mathbf{B}_{\mathbf{R}^n}} \sum_{|\alpha|=k} R_\alpha(\eta)\overline{R}_\alpha(\eta)\Big)^{1/2} ,$$

$$N_2((R_\alpha)_{\alpha \in \mathbf{M}(n,k)}) := \sum_{|\alpha|=k} \max_{\eta \in \mathbf{B}_{\mathbf{R}^n}} |R_\alpha(\eta)|$$

define norms on the finite-dimensional vector space $\mathrm{Pol}(n,m)^{\mathbf{M}(n,k)}$, hence they are equivalent. On replacing $R_\alpha(\eta)$ by $P^{(\alpha)}(\xi+t\eta)$ we get

$$C_4^{-1}\sum_{|\alpha|=k}\max_{\eta\in\mathbf{B}_{\mathbf{R}^n}}|P^{(\alpha)}(\xi+t\eta)|\le\Big(\max_{\eta\in\mathbf{B}_{\mathbf{R}^n}}(\Phi_kP)(\xi+t\eta)\Big)^{1/2}$$
$$\le C_4\sum_{|\alpha|=k}\max_{\eta\in\mathbf{B}_{\mathbf{R}^n}}|P^{(\alpha)}(\xi+t\eta)|\ .$$

With (1.14) and (1.15) we obtain the assertion. □

1.9. LEMMA. — *There exist $0<c\le 1\le C$ such that for any $P,Q\in\mathrm{Pol}'(n,m)$ and $\xi\in\mathbf{R}^n$ the following holds : let $0\le k\le m-1$ and $B\ge 1$ with*

$$B^{-1}\le\Big(\sum_{j=k}^{m}t^{j-k}(\Psi_jP)(\xi)\Big)/\Big(\sum_{j=k}^{m}t^{j-k}(\Psi_jQ)(\xi)\Big)\le B:\quad t\ge 1. \tag{1.16}$$

Further let $\nu\ge 1$ such that $\hat\nu:=\big(\frac{c\nu}{1+B^4}-1\big)/C\ge 1$. Then we have with $\check\nu:=C(1+\nu)(1+B^4)$:

(i) $(\Psi_kQ)(\xi)\ge\sum_{j=k+1}^{m}\nu^{\,j-k}(\Psi_jQ)(\xi)\Longrightarrow(\Psi_kP)(\xi)\ge\sum_{j=k+1}^{m}\hat\nu^{\,j-k}(\Psi_jP)(\xi)$,

(ii) $(\Psi_kQ)(\xi)\le\sum_{j=k+1}^{m}\nu^{j-k}(\Psi_jQ)(\xi)\Longrightarrow(\Psi_kP)(\xi)\le\sum_{j=k+1}^{m}\check\nu^{j-k}(\Psi_jP)(\xi)$.

Proof.

(i) Let $\nu\ge 1$ with $(\Psi_kQ)(\xi)\ge\sum_{j=k+1}^{m}\nu^{j-k}(\Psi_jQ)(\xi)$. Then we have

$$|Q^{(\alpha)}(\xi)|\le\nu^{-(|\alpha|-k)}(\Psi_kQ)(\xi)\le C_1\nu^{-(|\alpha|-k)}\sqrt{(\Phi_kQ)(\xi)}\ :\ |\alpha|\ge k\ .$$

This implies by Leibniz' rule,

$$|(\Phi_kQ)^{(\beta)}(\xi)|=\Big|\sum_{|\alpha|=k}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}Q^{(\alpha+\gamma)}(\xi)\overline{Q}^{(\alpha+\beta-\gamma)}(\xi)\Big|$$
$$\le C_2\nu^{-|\beta|}(\Phi_kQ)(\xi)$$

for any multiindex β ($C_2\ge 1$). In particular then $(\Phi_kQ)(\xi)\ne 0$ and

$$|(\Phi_kQ)^{(\beta)}(\xi)/(\Phi_kQ)(\xi)|^{1/|\beta|}\le C_2\nu^{-1}:\quad\beta\ne 0\ .$$

An application of (1.4) yields

$$C_3^{-1}\le\delta_{\Phi_kQ}(\xi)\sum_{\beta\ne 0}|(\Phi_kQ)^{(\beta)}(\xi)/(\Phi_kQ)(\xi)|^{1/|\beta|}\le C_4\nu^{-1}\delta_{\Phi_kQ}(\xi)\ .$$

By (1.13) and (1.16) we also have

$$(C_5B^2)^{-1} \leq (\Phi_k P)^{\sim}(\xi,t)/(\Phi_k Q)^{\sim}(\xi,t) \leq C_5B^2 : \quad t \geq 1 .$$

Using (1.7) we obtain

$$\frac{1+\delta_{\Phi_k P}(\xi)}{1+C_3^{-1}C_4^{-1}\nu} \geq \frac{1+\delta_{\Phi_k P}(\xi)}{1+\delta_{\Phi_k Q}(\xi)} \geq \frac{c_1}{1+C_5^2B^4} ,$$

$$\delta_{\Phi_k P}(\xi) \geq \frac{c_1(1+C_3^{-1}C_4^{-1}\nu)}{1+C_5^2B^4} - 1 \geq \frac{c_2\nu}{1+B^4} - 1 =: \tilde{\nu}$$

with $0 < c_2 \leq 1$. Let ν be so large that $\tilde{\nu} \geq 1$. Then

$$\begin{aligned}(\Phi_k P)(\xi) &\overset{(1.5)}{\geq} C_6^{-1}(\Phi_k P)^{\sim}(\xi,\delta_{\Phi_k P}(\xi)) \geq C_6^{-1}(\Phi_k P)^{\sim}(\xi,\tilde{\nu}) \\ &\overset{(1.13)}{\geq} C_7^{-1}\Big(\sum_{j=k}^{m} \tilde{\nu}^{\,j-k}(\Psi_j P)(\xi)\Big)^2\end{aligned}$$

with $C_7 \geq 1$, hence

$$\begin{aligned}(\Psi_k P)(\xi) \geq \sqrt{(\Phi_k P)(\xi)} &\geq C_7^{-1/2}\sum_{j=k}^{m} \tilde{\nu}^{\,j-k}(\Psi_j P)(\xi) \\ &\geq \sum_{j=k+1}^{m} (\tilde{\nu}/C_7)^{j-k}(\Psi_j P)(\xi) .\end{aligned}$$

With $c := c_2$, $C \geq C_7$ we obtain the first assertion.

(ii) Now assume that $(\Psi_k Q)(\xi) \leq \sum_{j=k+1}^{m} \nu^{\,j-k}(\Psi_j Q)(\xi)$. If then

$$(\Psi_k P)(\xi) \geq \sum_{j=k+1}^{m} \mu^{j-k}(\Psi_j P)(\xi) \text{ and } \tilde{\mu} := \frac{c_2\mu}{1+B^4} - 1 \geq 1$$

with some $\mu \geq 1$ we obtain as above (on interchanging the roles of P and Q) : $(\Psi_k Q)(\xi) \geq \sum_{j=k+1}^{m} (\tilde{\mu}/C_7)^{j-k}(\Psi_j Q)(\xi)$, hence

$$\sum_{j=k+1}^{m} (\tilde{\mu}/C_7)^{j-k}(\Psi_j Q)(\xi) \leq \sum_{j=k+1}^{m} \nu^{\,j-k}(\Psi_j Q)(\xi) .$$

This implies $\tilde{\mu}/C_7 \leq \nu$, i.e.

$$\mu \leq (1+C_7\nu)(1+B^4)/c_2 \leq C_7(1+\nu)(1+B^4)/c_2 .$$

Thus, with $C := C_7/c_2$ the second assertion also holds. □

Proof of Theorem 1.2. — The subsequent procedure will yield a decomposition of $\Omega_0 := \mathbf{R}^n$ into $m+1$ disjoint subsets, $\Omega_0 = \Omega_0'\dot\cup\Omega_1'\dot\cup\cdots\dot\cup\Omega_m'$, such that the following holds :

$$\exists A \geq 1\ \forall k = 0,\ldots,m\ \exists \tau_k \geq 1\ \forall \xi \in \Omega_k'\ \exists \eta_\xi \in \mathbf{B}_{\mathbf{R}^n}(\tau_k)\ :$$

$$(1^k) \qquad |P(\xi + z\eta_\xi)| \geq \frac{1}{2A}\widetilde{P}(\xi,\tau_k) : \quad P \in \Pi,\ z \in \mathbf{T}^1 .$$

Now note that the set

$$\Pi_\rho := \{P(\cdot + \zeta) \mid P \in \Pi,\ \zeta \in \mathbf{B}_{\mathbf{C}^n}(\rho+1)\}$$

is a compact subset of $\mathbf{E}(Q)$ since for fixed ζ the polynomial $P(\cdot+\zeta)$ is equally strong as P. So we may assume that $(1^0),\ldots,(1^m)$ is already proved for Π_ρ instead of Π. It follows that for any $\vartheta \in \mathbb{Z}^n$ there exists $\eta_\vartheta \in \mathbf{B}_{\mathbf{R}^n}(\tau)$, where $\tau := \max\{\tau_0,\ldots,\tau_m\}$, such that if $|\xi - \vartheta|_\infty \leq 1$ we have for each $P \in \Pi$, $\zeta \in \mathbf{B}_{\mathbf{C}^n}(\rho)$ and $z \in \mathbf{T}^1$:

$$|P(\xi+\zeta+z\eta_\vartheta)| = |P(\vartheta + z\eta_\vartheta + (\xi - \theta + \zeta))| \geq \frac{1}{2A}\widetilde{P}(\vartheta) \overset{(1.10)}{\geq} \frac{1}{2CA}\widetilde{P}(\xi) .$$

In particular we may choose $\eta(\xi) \equiv \eta_\vartheta$ in any cube $\{\xi \mid \vartheta_j \leq \xi_j < \vartheta_j + 1\}$, where $\vartheta_1,\ldots,\vartheta_n$ are integers, such that (1.1) holds and $\sup_\xi |\eta(\xi)|_\infty \leq \tau$. This completes the proof. The sets Ω_k' will be defined inductively as follows :

$$\Omega_k' := \{\xi \in \Omega_k \mid (\Psi_k Q)(\xi) \geq \sum_{j=k+1}^{m} \nu_k^{j-k}(\Psi_j Q)(\xi)\}\ (0 \leq k \leq m-1)$$

with suitable constants $\nu_k \geq 1$, and

$$\Omega_{k+1} := \Omega_k \setminus \Omega_k' \ ;\ \Omega_m' := \Omega_m .$$

In what follows the statements (2^k) $(0 \leq k \leq m)$ will be needed :

$$\exists B_k \geq 1\ \forall P \in \Pi,\ \xi \in \Omega_k,\ t \geq 1\ :$$

(2^k)

$$B_k^{-1} \leq \Big(\sum_{j=k}^{m} t^{j-k}(\Psi_j P)(\xi)\Big)\Big/\Big(\sum_{j=k}^{m} t^{j-k}(\Psi_j Q)(\xi)\Big) \leq B_k .$$

With the constants c, C in Lemma 1.9 we set

$$\hat\nu_k := \Big(\frac{c\nu_k}{1+B_k^4} - 1\Big)\Big/C \quad\text{and}\quad \check\nu_k := C(1+\nu_k)(1+B_k^4) .$$

Then for each $0 \leq k \leq m-1$ we have by (2^k) and Lemma 1.9, if $\hat\nu_k \geq 1$,

$$(3^k) \qquad (\Psi_k P)(\xi) \geq \sum_{j=k+1}^{m} \hat\nu_k^{j-k}(\Psi_j P)(\xi) : \quad P \in \Pi,\ \xi \in \Omega_k' ,$$

$$(4^k) \qquad (\Psi_k P)(\xi) \le \sum_{j=k+1}^{m} \check{\nu}_k^{j-k}(\Psi_j P)(\xi): \quad P \in \Pi,\ \xi \in \Omega_{k+1}\,.$$

Now the proof of (1^k), (2^k) proceeds by induction on k. Recall that by Corollary 1.7 there exist A, $\mu \ge 1$ such that

$$(5) \qquad \forall \tau \ge \mu,\ \xi \in \mathbb{R}^n\ \exists \eta \in \mathbf{B}_{\mathbf{R}_n}(\tau)\ \forall P \in \Pi\ : \widetilde{P}(\xi,\tau) \le A|P(\xi+\eta)|\,.$$

Without loss of generality we may assume that $Q \in \Pi$.

Case $k = 0$. — Lemma 1.3 yields the existence of B_0 satisfying (2^0). Choose $\nu_0 \ge 1$ such that $\hat{\nu}_0 \ge 1$ and define Ω_0', Ω_1 as above. Let $\tau_0 := \hat{\nu}_0$ and for any $\xi \in \Omega_0'$ choose $\eta_\xi := 0 \in \mathbf{B}_{\mathbf{R}_n}(\tau_0)$. We obtain

$$2|P(\xi + z\eta_\xi)| = 2(\Psi_0 P)(\xi) \overset{(3^0)}{\ge} \sum_{j=0}^{m} \hat{\nu}_0^j(\Psi_j P)(\xi) = \widetilde{P}(\xi,\tau_0)$$

for $P \in \Pi$, $z \in \mathbf{T}^1$, i.e. (1^0) is satisfied.

Case $1 \le k \le m$. — The inductive assumption yields (2^{k-1}) and $(4^0), \dots, (4^{k-1})$. Since $\Omega_k \subseteq \Omega_{k-1}$ this implies for $\xi \in \Omega_k$, $t \ge \check{\nu}_{k-1}$:

$$\begin{aligned}(2B_{k-1})^{-1}\sum_{j=k}^{m} t^{j-k}(\Psi_j Q)(\xi) &\le (2B_{k-1})^{-1}\frac{1}{t}\sum_{j=k-1}^{m} t^{j-(k-1)}(\Psi_j Q)(\xi)\\ &\overset{(2^{k-1})}{\le} \frac{1}{2t}\sum_{j=k-1}^{m} t^{j-(k-1)}(\Psi_j P)(\xi)\\ &\overset{(4^{k-1})}{\le} \sum_{j=k}^{m} t^{j-k}(\Psi_j P)(\xi)\,.\end{aligned}$$

For $1 \le t \le \check{\nu}_{k-1}$ this yields

$$\begin{aligned}(2B_{k-1})^{-1}\sum_{j=k}^{m} t^{j-k}(\Psi_j Q)(\xi) &\le \sum_{j=k}^{m} \check{\nu}_{k-1}^{j-k}(\Psi_j P)(\xi)\\ &\le \check{\nu}_{k-1}^{m-k}\sum_{j=k}^{m} t^{j-k}(\Psi_j P)(\xi)\,.\end{aligned}$$

Analogous estimates hold with P and Q interchanged. Setting $B_k := 2B_{k-1}\check{\nu}_{k-1}^{m-k}$ we obtain (2^k). Now let

$$\mu_k := \max\{\mu, \check{\nu}_0, \dots, \check{\nu}_{k-1}\} \ \ (\ge 1)\,.$$

For $P \in \Pi$, $\xi \in \Omega_{j+1}$ $(j = 0, \dots, k-1)$, $\tau \ge \mu_k$ it follows from (4^j) :

$$(\Psi_j P)(\xi) \le \sum_{i=j+1}^{m} \left(\frac{\mu_k}{\tau}\right)^{i-j} \tau^{i-j}(\Psi_i P)(\xi) \le \frac{\mu_k}{\tau}\sum_{i=j+1}^{m} \tau^{i-j}(\Psi_i P)(\xi)\,.$$

Multiplying by τ^j and summing up this yields (note that $\Omega_k \subseteq \Omega_{j+1}$) :

$$\text{(6)} \qquad \sum_{j=0}^{k-1} \tau^j (\Psi_j P)(\xi) \leq \frac{k\mu_k}{\tau} \widetilde{P}(\xi,\tau) : \quad P \in \Pi,\ \xi \in \Omega_k,\ \tau \geq \mu_k \ .$$

In the case $k \leq m-1$ we choose $\tau_k, \nu_k \geq 1$ such that

$$\text{(7)} \qquad \mu_k \leq \tau_k \leq \hat{\nu}_k \ , \ A^{-1} - \frac{2k\mu_k}{\tau_k} - \frac{2\tau_k}{\hat{\nu}_k} \geq \frac{1}{2A}$$

and define Ω'_k, Ω_{k+1} as above. By (3^k) (consequence of (2^k)) we have

$$\text{(8)} \quad \sum_{j=k+1}^{m} \tau_k^j (\Psi_j P)(\xi) \leq \frac{\tau_k}{\hat{\nu}_k} \tau_k^k (\Psi_k P)(\xi) \leq \frac{\tau_k}{\hat{\nu}_k} \widetilde{P}(\xi,\tau_k) : \quad P \in \Pi,\ \xi \in \Omega'_k \ .$$

Now let $\xi \in \Omega'_k$ be fixed and choose $\eta_\xi \in \mathbf{B}_{\mathbf{R}^n}(\tau_k)$ such that

$$\text{(9)} \qquad \widetilde{P}(\xi,\tau_k) \leq A|P(\xi+\eta_\xi)| : \quad P \in \Pi \ \text{(cf. (5))} \ .$$

An application of Taylor's formula gives for $P \in \Pi$, $z \in \mathbf{T}^1$:

$$\begin{aligned}
|P(\xi + z\eta_\xi)| &\geq \Big| \sum_{|\alpha|=k} \frac{P^{(\alpha)}(\xi)}{\alpha!} \eta_\xi^\alpha \Big| - \sum_{j\neq k} \tau_k^j (\Psi_j P)(\xi) \\
&\geq \sum_{j=0}^{m} \Big| \sum_{|\alpha|=j} \frac{P^{(\alpha)}(\xi)}{\alpha!} \eta_\xi^\alpha \Big| - 2\sum_{j\neq k} \tau_k^j (\Psi_j P)(\xi) \\
&\overset{(6),(8)}{\geq} |P(\xi+\eta_\xi)| - 2\Big\{ \frac{k\mu_k}{\tau_k} + \frac{\tau_k}{\hat{\nu}_k} \Big\} \widetilde{P}(\xi,\tau_k) \\
&\overset{(9)}{\geq} \Big\{ A^{-1} - \frac{2k\mu_k}{\tau_k} - \frac{2\tau_k}{\hat{\nu}_k} \Big\} \widetilde{P}(\xi,\tau_k) \\
&\overset{(7)}{\geq} \frac{1}{2A} \widetilde{P}(\xi,\tau_k) \ .
\end{aligned}$$

This yields (1^k).

In the case $k = m$ we choose $\tau_m \geq 1$ such that

$$\text{(10)} \qquad \mu_m \leq \tau_m,\ A^{-1} - \frac{2m\mu_m}{\tau_m} \geq \frac{1}{2A} \ .$$

Let $\xi \in \Omega'_m := \Omega_m$ be fixed and choose $\eta_\xi \in \mathbf{B}_{\mathbf{R}^n}(\tau_m)$ such that

$$\text{(11)} \qquad \widetilde{P}(\xi,\tau_m) \leq A|P(\xi+\eta_\xi)| : \quad P \in \Pi \ \text{(cf. (5))} \ .$$

Using (6), (10) and (11) an analogous computation as above yields (1^m) :

$$|P(\xi+z\eta_\xi)| \geq \Big\{ A^{-1} - \frac{2m\mu_m}{\tau_m} \Big\} \widetilde{P}(\xi,\tau_m) \geq \frac{1}{2A} \widetilde{P}(\xi,\tau_m) : P \in \Pi, z \in \mathbf{T}^1. \ \square$$

2. Some distribution spaces.

We adopt the standard notations for spaces of test functions and distributions (*cf.* [H1], [H2]) :

$\mathcal{D} = C_c^\infty(\mathbf{R}^n)$ — C^∞-functions with compact support;

$\mathcal{D}' = \mathcal{D}'(\mathbf{R}^n)$ — space of all distributions;

$\mathcal{S} = \mathcal{S}(\mathbf{R}^n)$ — space of rapidly decreasing C^∞-functions;

$\mathcal{S}' = \mathcal{S}'(\mathbf{R}^n)$ — space of tempered distributions.

Recall that each of these spaces carries a natural locally convex vector space topology. The scalar product of two vectors $\xi, \zeta \in \mathbf{C}^n$ will be denoted by $[\xi,\zeta] := \sum_{\nu=1}^{n} \xi_\nu \bar{\zeta}_\nu$. If $\varphi \in \mathcal{S}$ then the Fourier transform $\hat{\varphi}$ of φ is the function

$$\hat{\varphi}(\zeta) := \int_{\mathbf{R}^n} \exp(-i[\zeta, x])\varphi(x)dx \ : \ \zeta \in \mathbf{R}^n \ .$$

The Fourier transform $\hat{u}$ of $u \in \mathcal{S}'$ is defined by the formula

$$\langle \hat{u}, \varphi \rangle := \langle u, \hat{\varphi} \rangle : \quad \varphi \in \mathcal{S} \ ,$$

where $\langle \cdot, \cdot \rangle$ denotes the disbribution pairing. The following definitions and results are taken from Hörmander [H2], §10.1.

2.1. Definition.

(a) *A function $k : \mathbf{R}^n \to (0,\infty)$ will be called a* temperate weight function *if there exist constants $a, b > 0$ such that*

$$k(\xi + \zeta) \leq (1 + a|\xi|)^b k(\zeta) : \quad \xi, \zeta \in \mathbf{R}^n \ .$$

The set of all such functions will be denoted by $\mathcal{K}$.

(b) *If $k \in \mathcal{K}$ and $1 \leq p \leq \infty$ we denote by $\mathbf{B}_{p,k}$ the set of all distributions $u \in \mathcal{S}'$ such that $\hat{u}$ is a function and*

$$\|u\|_{p,k} := \left((2\pi)^{-n} \int_{\mathbf{R}^n} |k(\xi)\hat{u}(\xi)|^p d\xi\right)^{1/p} < \infty \ .$$

In the case $p = \infty$ this expression has to be interpreted as $\operatorname{ess.sup}_{\xi \in \mathbf{R}^n} |k(\xi)\hat{u}(\xi)|$.

By [H2], 10.1.7 we have

$$\mathcal{S} \hookrightarrow \mathbf{B}_{p,k} \hookrightarrow \mathcal{S}' \ ,$$

where $\mathfrak{F} \hookrightarrow \mathfrak{G}$ means a continuous embedding of topological vector spaces $\mathfrak{F}, \mathfrak{G}$. The spaces $\mathbf{B}_{p,k}$ are Banach spaces which for $1 \leq p < \infty$ contain $\mathcal{D}$

as a dense subset. In this case the dual $(\mathbf{B}_{p,k})'$ of $\mathbf{B}_{p,k}$ is (isometrically) isomorphic to $\mathbf{B}_{p',k'}$, where

$$1/p + 1/p' = 1\ ,\quad k'(\xi) := 1/k(-\xi)\ .$$

Any continuous linear form on $\mathbf{B}_{p,k}$ is given by continuous extension of a form $\varphi \mapsto \langle v, \varphi\rangle$, defined for $\varphi \in \mathcal{D}$ with $v \in \mathbf{B}_{p',k'}$. The norm of this functional equals $\|v\|_{p',k'}$ ([H2], 10.1.14). Let

$$\mathbf{B}^{\mathrm{loc}}_{p,k} := \{u \in \mathcal{D}' \mid \psi \cdot u \in \mathbf{B}_{p,k},\ \psi \in \mathcal{D}\}$$

denote the local space associated with $\mathbf{B}_{p,k}$. This is a Fréchet space with the system of seminorms $u \mapsto \|\psi \cdot u\|_{p,k}$, $\psi \in \mathcal{D}$.

In the following we shall consider certain subspaces of $\mathbf{B}^{\mathrm{loc}}_{p,k}$:

2.2. DEFINITION. — *Let $\sigma : [0,\infty) \to \mathbb{R}$ be a $\mathcal{C}^\infty$-function satisfying $\lim_{\rho\to+\infty} \sigma(\rho) = +\infty$ and $\sigma^{(j)}$ is bounded for all $j \geq 1$.*

Further let $\tilde\sigma(x) := \exp(\sigma([x,x])\cdot\sqrt{1+[x,x]})$, $x \in \mathbb{R}^n$. For $1 \leq p \leq \infty$ and $k \in \mathcal{K}$ we consider the distribution spaces

$$\mathbf{B}^{+\sigma}_{p,k} := \{u/\tilde\sigma \mid u \in \mathbf{B}_{p,k}\}\ ;\quad \mathbf{B}^{-\sigma}_{p,k} := \{\tilde\sigma \cdot v \mid v \in \mathbf{B}_{p,k}\}\ .$$

Obviously these are Banach spaces with the norms

1) $\|u/\tilde\sigma\|^{+\sigma}_{p,k} := \|u\|_{p,k}$

2) $\|\tilde\sigma \cdot v\|^{-\sigma}_{p,k} := \|v\|_{p,k}$.

Remarks.

(i) Since $\tilde\sigma$, $1/\tilde\sigma \in \mathcal{C}^\infty(\mathbb{R}^n)$ we have $\mathbf{B}^{\pm\sigma}_{p,k} \subseteq \mathbf{B}^{\mathrm{loc}}_{p,k}$ by [H2], 10.1.23.

(ii) It is our intention to keep the spaces $\mathbf{B}^{-\sigma}_{p,k}$ as small as possible. This can be achieved by letting the function σ tend to $+\infty$ very slowly. For example, choose $\sigma_0 \in \mathcal{C}^\infty(\mathbb{R})$ with $\sigma_0(\rho) = \begin{cases} 0, & \rho \leq 0 \\ 1, & \rho \geq 1 \end{cases}$ and put $\sigma(\rho) := \sum_{j=1}^{\infty} \sigma_0(\rho/a_j - a_j)$, where the sequence (a_j) tends to $+\infty$ very fast (e.g. $a_1 := 2$, $a_{j+1} := a_j^{a_j}$).

2.3. LEMMA. — *Let $1 \leq p \leq \infty$, $k \in \mathcal{K}$ and σ as in Definition 2.2. Then we have*

$$\mathbf{B}^{-\sigma}_{p,k} \hookrightarrow \mathbf{B}^{\mathrm{loc}}_{p,k}\ . \tag{2.1}$$

Proof. — Let $\psi \in \mathcal{D}$ and $v \in \mathbf{B}^{-\sigma}_{p,k}$ arbitrary. Since $\psi \cdot \tilde{\sigma} \in \mathcal{D} \subseteq \mathcal{S}$ it follows from [H2], 10.1.15 that

$$\|\psi \cdot v\|_{p,k} = \|\psi \cdot \tilde{\sigma} \cdot v/\tilde{\sigma}\|_{p,k} \le K\|v/\tilde{\sigma}\|_{p,k} = K\|v\|^{-\sigma}_{p,k} ,$$

with $K < \infty$ depending only on $\tilde{\sigma}$, k and ψ. Since the topology of $\mathbf{B}^{\mathrm{loc}}_{p,k}$ is given by the seminorms $v \mapsto \|\psi \cdot v\|_{p,k}$ the proof is complete. □

The same proof shows that if σ_1, σ_2 are such that $\tilde{\sigma}_1/\tilde{\sigma}_2 \in \mathcal{S}$ (e.g. if $\limsup_{\rho\to\infty} \sigma_1(\rho) - \sigma_2(\rho) < 0$) then $\mathbf{B}^{-\sigma_1}_{p,k} \hookrightarrow \mathbf{B}^{-\sigma_2}_{p,k}$.

2.4. *Remark.* — Let $Q \in \mathrm{Pol}'(n,m)$ be fixed and $\Pi \subseteq \mathsf{E}(Q)$ a compact set. By Theorem 1.2 there is a bounded measurable function $\eta : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$\widetilde{P}(-\xi) \le A|P(-\xi - z\eta(\xi))| : \quad P \in \Pi,\ \xi \in \mathbf{R}^n,\ z \in \mathbf{T}^1 .$$

Using this we can for every $P \in \Pi$ define a distribution $\mathfrak{f}_P \in \mathcal{D}'$ through

$$(2.2)\quad \langle \mathfrak{f}_P, \varphi\rangle := (2\pi)^{-n} \int_{\mathbf{R}^n} \int_{z\in\mathbf{T}^1} \frac{\widehat{\varphi}(\xi + z\eta(\xi))}{P(-\xi - z\eta(\xi))} \frac{dz}{2\pi i z}\, d\xi : \quad \varphi \in \mathcal{D} .$$

This type of formula has been introduced by L. Hörmander. Similarly as in [T2] we could now show that $\mathfrak{f}_P$ is an analytic function of $P \in \Pi$ with values in $\mathbf{B}^{-\sigma}_{\infty,\widetilde{Q}}$ and $\mathfrak{f}_P$ is a fundamental solution of $P(D)$ for each P. (In fact, $\mathfrak{f}_P$ takes its values in the smaller space $\mathbf{B}^{*H^1}_{\infty,\widetilde{Q}}$ defined below, where $H^1 = (\eta)$.) We shall not do so since it is our aim to prove a more general result (Theorem 3.1 below). However, formula (2.2) serves as a motivation for the following

2.5. Definition. — *In order to simplify notations we introduce the measure $|dz| := |dz_1|\cdots|dz_r|$ on the torus $\mathbf{T}^r$ ($r \in \mathbb{N}$). Let $1 \le p \le \infty$, $k \in \mathcal{K}$ and $H^r = (\eta_s)_{s=1}^r : \mathbf{R}^n \longrightarrow (\mathbf{R}^n)^r$ a bounded measurable function. For any $\varphi \in \mathcal{D}$ we set*

$$\|\varphi\|^{H^r}_{p,k} := \Big((2\pi)^{-n-r} \int_{\mathbf{R}^n} \int_{\mathbf{T}^r} |k(\xi)\widehat{\varphi}(\xi + \widetilde{H}^r(\xi,z))|^p\, |dz|\, d\xi\Big)^{1/p} (p < \infty) ,$$

where $\widetilde{H}^r(\xi,z) := \sum_{s=1}^{r} z_s \cdot \eta_s(\xi)$,

$$\|\varphi\|^{H^r}_{\infty,k} := \sup\{|k(\xi)\widehat{\varphi}(\xi + \widetilde{H}^r(\xi,z))| \mid \xi \in \mathbf{R}^n,\ z \in \mathbf{T}^r\} .$$

The theorem of Paley-Wiener-Schwartz ([H1], §7.3) ensures that $\|\varphi\|^{H^r}_{p,k}$ is finite for each $\varphi \in \mathcal{D}$. Obviously, $(\mathcal{D}, \|\cdot\|^{H^r}_{p,k})$ is a normed space. Its "dual space",

$$\mathbf{B}^{*H^r}_{p',k'} := \{v \in \mathbf{B}^{\mathrm{loc}}_{p',k'} \mid \|v\|^{*H^r}_{p',k'} := \sup\{|\langle v,\varphi\rangle|/\|\varphi\|^{H^r}_{p,k} \mid 0 \ne \varphi \in \mathcal{D}\} < \infty\}$$

will be endowed with the norm $\|\cdot\|^{*H^r}_{p',k'}$. Here $p' := 1$ if $p = \infty$.

The reason why we have introduced the space $\mathbf{B}^{-\sigma}_{q,k}$ is that it contains each $\mathbf{B}^{*H^r}_{q,k}$, yet it is small enough to give quite precise information on the growth at infinity of solutions of the equation $P(D)\mathfrak{f}_P = \delta$ when P runs through $\mathsf{E}(Q)$ and $\mathfrak{f}_P$ depends analytically on P (*cf.* the remark at the end of [M]).

2.6. LEMMA. — *Let $H^{r+1} = (\eta_s)^{r+1}_{s=1}$ as in Definition 2.5. With $H^r := (\eta_s)^r_{s=1}$ we then have*

$$\|\varphi\|_{p,k} \leq \|\varphi\|^{H^r}_{p,k} \leq \|\varphi\|^{H^{r+1}}_{p,k} : \quad \varphi \in \mathcal{D}\,, \tag{2.3}$$

hence

$$\mathbf{B}_{p',k'} \hookrightarrow \mathbf{B}^{*H^r}_{p',k'} \hookrightarrow \mathbf{B}^{*H^{r+1}}_{p',k'}\,. \tag{2.4}$$

Proof. — By Cauchy's formula and the Hölder inequality we have, if $p < \infty$,

$$|\widehat{\varphi}(\xi + \widetilde{H}^r(\xi, z'))|^p \leq \int_{z_{r+1}\in\mathbf{T}^1} |\widehat{\varphi}(\xi + \widetilde{H}^{r+1}(\xi, z))|^p \frac{|dz_{r+1}|}{2\pi}\,,$$

where $z = (z', z_{r+1})$. Inserting this in the definition of $\|\varphi\|^{H^{r+1}}_{p,k}$ yields the second inequality in (2.3). In the case $p = \infty$ we can argue similarly using the maximum principle. Choosing $H^0 \equiv 0$ we also get $\|\varphi\|_{p,k} = \|\varphi\|^{H^0}_{p,k} \leq \|\varphi\|^{H^r}_{p,k}$. The embedding (2.4) is a direct consequence of these estimates. □

2.7. LEMMA. — *Let σ as in Definition 2.2 and H^r as in Definition 2.5. Then there exists a constant $K < \infty$ such that*

$$\|\varphi\|^{H^r}_{p,k} \leq K\|\varphi\|^{+\sigma}_{p,k} : \quad \varphi \in \mathcal{D}\,. \tag{2.5}$$

Proof. — Let $\rho := 1 + \sup\{|\widetilde{H}^r(\xi, z)|_\infty \mid \xi \in \mathbf{R}^n,\ z \in \mathbf{T}^r\}$. For any $\varphi \in \mathcal{D}$ and fixed $\xi \in \mathbf{R}^n$, $z \in \mathbf{T}^r$ we have

$$|\widehat{\varphi}(\xi + \widetilde{H}^r(\xi, z))|^p \leq \left(\frac{\rho^p}{2\pi}\right)^n \int_{\mathbf{T}^n} |\widehat{\varphi}(\xi + \rho\zeta)|^p |d\zeta| \text{ if } p < \infty\,.$$

This implies

$$\begin{aligned}(\|\varphi\|^{H^r}_{p,k})^p &\leq \frac{\rho^{np}}{(2\pi)^{2n}} \int_{\mathbf{R}^n}\int_{\mathbf{T}^n} |k(\xi)\cdot\widehat{\varphi}(\xi + \rho\zeta)|^p |d\zeta|\, d\xi \\ &= \left(\frac{\rho^p}{2\pi}\right)^n \int_{\mathbf{T}^n} (2\pi)^{-n} \int_{\mathbf{R}^n} |k(\xi)\cdot \exp(-i[\rho\zeta, \cdot])\varphi)^\wedge(\xi)|^p d\xi\, |d\zeta| \\ &= \left(\frac{\rho^p}{2\pi}\right)^n \int_{\mathbf{T}^n} (\|\exp(-i[\rho\zeta, \cdot])\varphi\|_{p,k})^p\, |d\zeta|\,.\end{aligned} \tag{2.6}$$

Now consider the functions

$$\Phi_\zeta(x) := \exp(-i[\rho\zeta, x])/\tilde{\sigma}(x) : \quad \zeta \in \mathbf{T}^n .$$

It is not hard to check that $\{\Phi_\zeta\}$ is a bounded subset of $\mathcal{S}$. With the weight function $M_k \in \mathcal{K}$ (*cf.* [H2], §10.1),

$$M_k(\xi) := \sup_{\xi' \in \mathbf{R}^n} k(\xi + \xi')/k(\xi') : \quad \xi \in \mathbf{R}^n ,$$

we have $\mathcal{S} \hookrightarrow \mathbf{B}_{1,M_k}$ ([H2], 10.1.7), hence

$$\sup\{\|\Phi_\zeta\|_{1,M_k} \mid \zeta \in \mathbf{T}^n\} =: K < \infty .$$

It follows from [H2], 10.1.15 that

$$\sup\{\|\Phi_\zeta \cdot \psi\|_{p,k} \mid \zeta \in \mathbf{T}^n\} \leq K\|\psi\|_{p,k} : \quad \psi \in \mathcal{D} .$$

From (2.6) we thus obtain with $\psi = \tilde{\sigma} \cdot \varphi$:

$$\|\varphi\|_{p,k}^{H^r} \leq \left(\left(\frac{\rho^p}{2\pi}\right)^n \int_{\mathbf{T}^n} (\|\Phi_\zeta \cdot \tilde{\sigma} \cdot \varphi\|_{p,k})^p |d\zeta|\right)^{1/p} \leq K\rho^n \|\tilde{\sigma} \cdot \varphi\|_{p,k} = K'\|\varphi\|_{p,k}^{+\sigma} .$$

The case $p = \infty$ can be treated analoguously. □

2.8. Corollary. — *Under the assumptions of Lemma 2.7 the mapping $v \mapsto \langle v, \cdot \rangle$ identifies $\mathbf{B}_{p',k'}^{*H^r}$ isometrically with the dual of the normed space $(\mathcal{D}, \|\cdot\|_{p,k}^{H^r})$. In particular, $\mathbf{B}_{p',k'}^{*H^r}$ is complete. Furthermore we have*

$$\mathbf{B}_{p',k'}^{*H^r} \hookrightarrow \mathbf{B}_{p',k'}^{-\sigma} . \tag{2.7}$$

Proof. — Clearly, $v \mapsto \langle v, \cdot \rangle$ defines an isometric embedding of $\mathbf{B}_{p',k'}^{*H^r}$ into $(\mathcal{D}, \|\cdot\|_{p,k}^{H^r})'$. We have to show that it is onto. So let ℓ be a continuous linear form on $(\mathcal{D}, \|\cdot\|_{p,k}^{H^r})$. By Lemma 2.7 we have

$$|\langle \ell/\tilde{\sigma}, \varphi \rangle| \leq \|\ell\| \, \|\varphi/\tilde{\sigma}\|_{p,k}^{H^r} \leq K\|\ell\| \, \|\varphi\|_{p,k} : \quad \varphi \in \mathcal{D} . \tag{2.8}$$

If $p < \infty$ then $\mathbf{B}_{p',k'}$ is the dual space of $\mathbf{B}_{p,k}$, so $\ell \in \mathbf{B}_{p',k'}^{-\sigma} \subseteq \mathbf{B}_{p',k'}^{\mathrm{loc}}$. Hence $\ell \in \mathbf{B}_{p',k'}^{*H^r}$ and $\|\ell\|_{p',k'}^{-\sigma} = \|\ell/\tilde{\sigma}\|_{p',k'} \leq K\|\ell\|_{p',k'}^{*H^r}$ by (2.8).

In the case $p = \infty$ we can analoguously derive (2.8) with σ replaced by $\sigma_1(\rho) := \sigma(\rho) - 1$. Since $\mathcal{S} \hookrightarrow \mathbf{B}_{\infty,k}$ the functional $\ell_1 := \ell/\tilde{\sigma}_1$ can be extended such that $|\langle \ell_1, \varphi \rangle| \leq K\|\ell\| \, \|\varphi\|_{\infty,k}$ holds for all $\varphi \in \mathcal{S}$. Hence $\ell_1 \in \mathcal{S}'$ and the Fourier transform of ℓ_1 is a continuous linear form on $\mathcal{S}$ equipped with the norm $\sup_\xi |k(-\xi)\varphi(\xi)|$. But then $\langle \hat{\ell}_1, \varphi \rangle = \int \varphi(\xi) d\mu(\xi)$ with a measure $d\mu$ in $\mathbf{R}^n$ of total mass $\int |d\mu(\xi)|/k(-\xi) < \infty$. Noting that $\tau := \tilde{\sigma}_1/\tilde{\sigma} \in \mathcal{S}$ we obtain $\ell/\tilde{\sigma} = \tau \cdot \ell_1 \in \mathcal{S}'$ and $(\ell/\tilde{\sigma})^\wedge = (2\pi)^{-n}\hat{\tau} * d\mu$ which is a C^∞-function satisfying $\int |(\ell/\tilde{\sigma})^\wedge(\xi)|/k(-\xi)\, d\xi < \infty$, i.e. $(\ell/\tilde{\sigma}) \in \mathbf{B}_{1,k'}$. As in the case $p < \infty$ we conclude that $\ell \in \mathbf{B}^{*H^r}_{1,k'}$ and $\|\ell\|^{-\sigma}_{1,k'} \le K'\|\ell\|^{*H^r}_{1,k'}$ by the closed graph theorem. □

Now we shall investigate how a differential operator with constant coefficients acts in the spaces $\mathbf{B}^{*H^r}_{q,k}$ $(1 \le q \le \infty, k \in \mathcal{K})$. If $P(x) = \sum_{|\alpha|\le m} a_\alpha x^\alpha$ is a polynomial in $x \in \mathbf{R}^n$ we consider the differential expression

$$P(D) := \sum_{|\alpha|\le m} a_\alpha D^\alpha \text{ where } D := -i\Big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\Big).$$

2.9. Proposition. — *Let $P, Q \in \mathrm{Pol}'(n,m)$ with $P < Q$ and $H^r = (\eta_s)_{s=1}^r$ as in Definition 2.5. Then the operator $P(D)$ maps $\mathbf{B}^{*H^r}_{q,k\widetilde{Q}}$ continuously into $\mathbf{B}^{*H^r}_{q,k}$.*

Proof. — Let $\rho := \sup\{|\widetilde{H}^r(\xi,z)|_\infty \mid \xi \in \mathbf{R}^n, z \in \mathbf{T}^r\}$ and $\xi \in \mathbf{R}^n$, $z \in \mathbf{T}^r$ fixed. With $\zeta := \widetilde{H}^r(\xi,z)$ we have for any $\varphi \in \mathcal{D}$:

$$\begin{aligned}|(k\widetilde{Q})'(\xi)\cdot(P(-D)\varphi)^\wedge(\xi+\zeta)| &= |(k\widetilde{Q})'(\xi)\cdot P(-\xi-\zeta)\cdot\widehat{\varphi}(\xi+\zeta)|\\ &\le |(k\widetilde{Q})'(\xi)\cdot\widetilde{P}(-\xi,\rho)\cdot\widehat{\varphi}(\xi+\zeta)|\\ &\le (1+\rho)^m\frac{\widetilde{P}(-\xi)}{\widetilde{Q}(-\xi)}|k'(\xi)\cdot\widehat{\varphi}(\xi+\zeta)|\,.\end{aligned}$$

Since $\sup_{\xi\in\mathbf{R}^n}\dfrac{\widetilde{P}(-\xi)}{\widetilde{Q}(-\xi)} < \infty$ we obtain

$$\|P(-D)\varphi\|^{H^r}_{q',(k\widetilde{Q})'} \le K\|\varphi\|^{H^r}_{q',k'}: \quad \varphi \in \mathcal{D}\,. \tag{2.9}$$

Now, if $v \in \mathbf{B}^{*H^r}_{q,k\widetilde{Q}} \subseteq \mathbf{B}^{\mathrm{loc}}_{q,k\widetilde{Q}}$ it follows from [H2], 10.1.22 that $P(D)v \in \mathbf{B}^{\mathrm{loc}}_{q,k}$. Furthermore, (2.9) implies that

$$\begin{aligned}|\langle P(D)v,\varphi\rangle| = |\langle v, P(-D)\varphi\rangle| &\le \|v\|^{*H^r}_{q,k\widetilde{Q}}\|P(-D)\varphi\|^{H^r}_{q',(k\widetilde{Q})'}\\ &\le K\|v\|^{*H^r}_{q,k\widetilde{Q}}\|\varphi\|^{H^r}_{q',k'}\end{aligned}$$

for any $\varphi \in \mathcal{D}$. In particular this means that $P(D)v \in \mathbf{B}^{*H^r}_{q,k}$ and

$$\|P(D)v\|^{*H^r}_{q,k} \le K\|v\|^{*H^r}_{q,k\widetilde{Q}}\,.$$ □

2.10. Proposition. — *Let P, $Q \in \mathrm{Pol}'(n,m)$ with $P \sim Q$, $H^r = (\eta_s)_{s=1}^r$ as in Definition 2.5 and $\rho := \sup\{|\widetilde{H}^{r-1}(\xi,z')|_\infty \mid \xi \in \mathbf{R}^n$,*

$z' \in \mathbf{T}^{r-1}\}$ *($\rho := 0$ if $r = 1$). Assume that with some constant $A > 0$ we have*

$$\widetilde{P}(-\xi) \le A|P(-\xi - \zeta - z_r\eta_r(\xi))| : \quad \xi \in \mathbf{R}^n, \zeta \in \mathbf{B}_{\mathbf{C}^n}(\rho),\ z_r \in \mathbf{T}^1 .$$

Then the operator $P(D) : \mathbf{B}^{*H^r}_{q,k\widetilde{Q}} \longrightarrow \mathbf{B}^{*H^r}_{q,k}$ is surjective.

Proof. — Since $\widetilde{Q}(-\xi) \le B\widetilde{P}(-\xi)$ the assumption implies that

$$(2.10) \qquad \|P(-D)\varphi\|^{H^r}_{q',(k\widetilde{Q})'} \ge (AB)^{-1}\|\varphi\|^{H^r}_{q',k'} : \quad \varphi \in \mathcal{D} .$$

Now let $w \in \mathbf{B}^{*H^r}_{q,k}$ be given. Then by (2.10) the mapping

$$P(-D)\varphi \longmapsto \langle w, \varphi \rangle$$

is a well-defined continuous linear form on the subspace $P(-D)\mathcal{D}$ of $E := (\mathcal{D}, \|\cdot\|^{H^r}_{q',(k\widetilde{Q})'})$. By the Hahn-Banach theorem there exists a continuous extension v of this form to the whole of E and Corollary 2.8 implies that $v \in \mathbf{B}^{*H^r}_{q,k\widetilde{Q}}$. Finally it is clear that

$$\langle P(D)v, \varphi \rangle = \langle v, P(-D)\varphi \rangle = \langle w, \varphi \rangle : \quad \varphi \in \mathcal{D} ,$$

i.e. $P(D)v = w$. □

3. Parameter depending differential operators.

We come back to the main topic of this article. Let $Q \in \mathrm{Pol}'(n, m)$ be fixed. Consider a family of differential operators

$$(3.1) \qquad P(\lambda, D) = \sum_{|\alpha| \le m} a_\alpha(\lambda) D^\alpha ,$$

where the coefficients a_α (constant with respect to x) are analytic functions of a parameter λ varying in a complex manifold Λ. The only assumption we make is that for each value of λ the polynomial $P(\lambda, \cdot)$ is equally strong as Q. Denoting by $\{R_1, \dots, R_\nu\}$ any fixed basis of the vector space $\mathbf{W}(Q)$ we can write

$$(3.2) \qquad P(\lambda, D) = \sum_{\mu=1}^{\nu} b_\mu(\lambda) R_\mu(D)$$

with analytic functions $b_\mu : \Lambda \to \mathbf{C}$. Recall (1.1 (iii)) that the set $\mathbf{E}(Q)$ is a holomorphically convex open submanifold of $\mathbf{W}(Q)$. Hence we may take in (3.2) $\Lambda = \mathbf{E}(Q)$ and $\{b_\mu\}$ as the coordinate functions of P with respect to the basis $\{R_\mu\}$.

It $\mathcal{E}$ is a locally convex vector space we denote by $\mathcal{H}(\Lambda, \mathcal{E})$ the set of all analytic functions $e : \Lambda \to \mathcal{E}$. Further let $\sigma \in \mathcal{C}^\infty[0,\infty)$ be any fixed weight function as in Definition 2.2. Recall that $\mathbf{B}^{-\sigma}_{q,k} \hookrightarrow \mathbf{B}^{\mathrm{loc}}_{q,k}$ for $1 \leq q \leq \infty$, $k \in \mathcal{K}$.

3.1. THEOREM. — *Let $1 \leq q \leq \infty$ and $k \in \mathcal{K}$. Assume that Λ is a Stein manifold. Then for any $\mathfrak{g} \in \mathcal{H}(\Lambda, \mathbf{B}_{q,k})$ there exists $\mathfrak{f} \in \mathcal{H}(\Lambda, \mathbf{B}^{-\sigma}_{q,k\tilde{Q}})$ such that*

(i) $P(\lambda, D)\mathfrak{f}(\lambda) = \mathfrak{g}(\lambda)$, $\lambda \in \Lambda$;

(ii) $R(D)\mathfrak{f} \in \mathcal{H}(\Lambda, \mathbf{B}^{-\sigma}_{q,k})$ *for any* $R \in \mathbf{W}(Q)$.

In the following corollaries we do not make any assumptions concerning Λ :

3.2 COROLLARY. — *Let $1 \leq q \leq \infty$ and $k \in \mathcal{K}$. Then for any $\mathfrak{g}_0 \in \mathbf{B}_{q,k}$ there exists $\mathfrak{f} \in \mathcal{H}(\Lambda, \mathbf{B}^{-\sigma}_{q,k\tilde{Q}})$ such that $P(\lambda, D)\mathfrak{f}(\lambda) \equiv \mathfrak{g}_0$, and 3.1* (ii) *holds.*

Proof. — By our above remark we may take P itself as a parameter varying in the Stein manifold $\mathbf{E}(Q)$. Theorem 3.1 yields a function $\tilde{\mathfrak{f}} \in \mathcal{H}(\mathbf{E}(Q), \mathbf{B}^{-\sigma}_{q,k\tilde{Q}})$ such that $P(D)\tilde{\mathfrak{f}}(P) = \mathfrak{g}_0$, $P \in \mathbf{E}(Q)$. Since the mapping $\lambda \mapsto \mathfrak{p}(\lambda) := P(\lambda, \cdot)$ is analytic with values in $\mathbf{E}(Q)$ we have $\mathfrak{f} := \tilde{\mathfrak{f}} \circ \mathfrak{p} \in \mathcal{H}(\Lambda, \mathbf{B}^{-\sigma}_{q,k\tilde{Q}})$ and $P(\lambda, D)\mathfrak{f}(\lambda) \equiv \mathfrak{g}_0$. □

By δ we denote the Dirac distribution at 0, $\langle \delta, \varphi \rangle := \varphi(0)$. The next corollary answers a question of L. Hörmander ([H2], p. 59) :

3.3. COROLLARY. — *There exists $\mathfrak{f} \in \mathcal{H}(\Lambda, \mathbf{B}^{-\sigma}_{\infty,\tilde{Q}})$ such that $P(\lambda, D)\mathfrak{f}(\lambda) \equiv \delta$, and 3.1* (ii) *holds with $q = \infty$, $k \equiv 1$.*

Proof. — This is a special case of Corollary 3.2 since with $k \equiv 1$ we have $\delta = \mathfrak{g}_0 \in \mathbf{B}_{\infty,k}$. □

3.4. *Remark.* — If Λ is an open subset of $\mathbf{R}^d$ (or a real analytic manifold) then the analogues of Theorem 3.1 and its corollaries hold with "analytic" replaced by "real analytic".

Proof. — By a result of Grauert [G] there exists a neighborhood basis of Λ in $\mathbf{C}^d$ consisting of holomorphically convex open sets. Using this the real analytic case can be reduced to the analytic one (*cf.* [M]). □

It remains to prove Theorem 3.1. If $\mathfrak{F}$, $\mathfrak{G}$ are Banach spaces we denote by $\mathcal{L}(\mathfrak{F}, \mathfrak{G})$ the space of all bounded linear operators from $\mathfrak{F}$ to $\mathfrak{G}$ equipped with the operator norm topology. In the proof of 3.1 we shall make use of the following result of J. Leiterer [L].

3.5. THEOREM. — *Let $\mathfrak{F}$, $\mathfrak{G}$ be Banach spaces and Λ a complex Stein manifold. Let $\mathfrak{T} \in \mathcal{H}(\Lambda, \mathcal{L}(\mathfrak{F}, \mathfrak{G}))$ such that $\mathfrak{T}(\lambda)\mathfrak{F} = \mathfrak{G}$ for each $\lambda \in \Lambda$. Then*

(a) *There exists for each function $\mathfrak{g} \in \mathcal{H}(\lambda, \mathfrak{G})$ a function $\mathfrak{f} \in \mathcal{H}(\Lambda, \mathfrak{F})$ such that $\mathfrak{T}(\lambda)\mathfrak{f}(\lambda) = \mathfrak{g}(\lambda)$, $\lambda \in \Lambda$.*

(b) *For any open subset Λ' of Λ let $\mathcal{N}(\Lambda') := \{\mathfrak{f} \in \mathcal{H}(\Lambda', \mathfrak{F}) \mid \mathfrak{I}(\lambda)\mathfrak{f}(\lambda) \equiv 0\}$. If Λ' is holomorphically convex then the set $\mathcal{N}(\Lambda)_{|\Lambda'}$ of restrictions to Λ' of functions in $\mathcal{N}(\Lambda)$ is dense in $\mathcal{N}(\Lambda')$.*

Proof of Theorem 3.1. — Let $\{\Lambda_r\}_{r\in\mathbf{N}}$ be an exhausting sequence of open submanifolds of Λ such that each Λ_r is holomorphically convex, $\overline{\Lambda}_r$ is compact and $\overline{\Lambda}_r \subseteq \Lambda_{r+1}$. For each $r \in \mathbf{N}$ we inductively choose a bounded measurable function $H^r = (\eta_s)_{s=1}^r : \mathbf{R}^n \longrightarrow (\mathbf{R}^n)^r$ in the following way : set $\rho_r := \sup\{|\widetilde{H}^{r-1}(\xi, z')|_\infty \mid \xi \in \mathbf{R}^n, z' \in \mathbf{T}^{r-1}\}$ ($\rho_1 := 0$). Then by Theorem 1.2 there exist $A_r \geq 1$ and a bounded measurable function $\eta_r : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ such that for all $\lambda \in \overline{\Lambda}_r$, $\xi \in \mathbf{R}^n$, $\zeta \in \mathbf{B}_{\mathbf{C}^n}(\rho_r)$, $z_r \in \mathbf{T}^1$ we have

$$\widetilde{P}(\lambda, -\xi) \leq A_r |P(\lambda, -\xi - \zeta - z_r\eta_r(\xi))| \,. \tag{3.3}$$

Thus, H^r is defined for each $r \in \mathbf{N}$. Now consider the spaces

$$\mathfrak{F}_r := \mathbf{B}^{*H^r}_{q,k\widetilde{Q}} \,, \quad \mathfrak{G}_r := \mathbf{B}^{*H^r}_{q,k} : \quad r \in \mathbf{N} \,.$$

By (2.1), (2.4) and (2.7) we have the embeddings

$$\mathfrak{F}_r \hookrightarrow \mathfrak{F}_{r+1} \hookrightarrow \mathfrak{F} := \mathbf{B}^{-\sigma}_{q,k\widetilde{Q}} \hookrightarrow \mathbf{B}^{\mathrm{loc}}_{q,k\widetilde{Q}} \,, \tag{3.4}$$

$$\mathbf{B}_{q,k} \hookrightarrow \mathfrak{G}_r \hookrightarrow \mathfrak{G}_{r+1} \hookrightarrow \mathfrak{G} := \mathbf{B}^{-\sigma}_{q,k} \hookrightarrow \mathbf{B}^{\mathrm{loc}}_{q,k} \,. \tag{3.5}$$

Consider the representation (3.2) of $P(\lambda, D)$. From Proposition 2.9 we know that each $R_\mu(D)$ induces a bounded linear operator from $\mathfrak{F}_r$ into $\mathfrak{G}_r$. Hence the mapping $\lambda \mapsto P(\lambda, D)$ is analytic with values in $\mathcal{L}(\mathfrak{F}_r, \mathfrak{G}_r)$. From (3.3) and Proposition 2.10 we conclude that $P(\lambda, D)\mathfrak{F}_r = \mathfrak{G}_r$ for each $\lambda \in \overline{\Lambda}_r$. Furthermore, $\mathfrak{g} \in \mathcal{H}(\Lambda, \mathfrak{G}_r)$ by (3.5). It follows from part (a) of Theorem 3.5 that there exists for each $r \in \mathbf{N}$ a function $\tilde{\mathfrak{f}}_r \in \mathcal{H}(\Lambda_r, \mathfrak{F}_r)$ such that

$$P(\lambda, D)\tilde{\mathfrak{f}}_r(\lambda) = \mathfrak{g}(\lambda) : \quad \lambda \in \Lambda_r \,.$$

We construct a sequence of functions $\mathfrak{f}_r \in \mathcal{H}(\Lambda_r, \mathfrak{F}_r)$ as follows. Put $\mathfrak{f}_1 := \tilde{\mathfrak{f}}_1$ and assume that $\mathfrak{f}_1, \ldots, \mathfrak{f}_r$ are already defined. Consider then

$$\delta_{r+1}(\lambda) := \tilde{\mathfrak{f}}_{r+1}(\lambda) - \mathfrak{f}_r(\lambda) : \quad \lambda \in \Lambda_r .$$

By (3.4) we have $\delta_{r+1} \in \mathcal{H}(\Lambda_r, \mathfrak{F}_{r+1})$ and we may assume inductively that

$$P(\lambda, D)\delta_{r+1}(\lambda) = 0 : \quad \lambda \in \Lambda_r .$$

By part (b) of Theorem 3.5 there exists for arbitrary $\varepsilon_{r+1} > 0$ a function $\mathfrak{c}_{r+1} \in \mathcal{H}(\Lambda_{r+1}, \mathfrak{F}_{r+1})$ with the properties

$$P(\lambda, D)\mathfrak{c}_{r+1}(\lambda) = 0 : \lambda \in \Lambda_{r+1} \ ; \ \sup_{\lambda \in \Lambda_{r-1}} \|\delta_{r+1}(\lambda) - \mathfrak{c}_{r+1}(\lambda)\|_{\mathfrak{F}_{r+1}} \leq \varepsilon_{r+1} ,$$

where for convenience we put $\Lambda_0 := \emptyset$. Since $\mathfrak{F}_{r+1} \hookrightarrow \mathfrak{F}$, $\mathfrak{G}_{r+1} \hookrightarrow \mathfrak{G}$ and the operators $R_\mu(D) : \mathfrak{F}_{r+1} \longrightarrow \mathfrak{G}_{r+1}$ ($\mu = 1, \ldots, \nu$) are continuous (Proposition 2.9) one can choose ε_{r+1} so small that

$$\sup_{\lambda \in \Lambda_{r-1}} \|\delta_{r+1}(\lambda) - \mathfrak{c}_{r+1}(\lambda)\|_{\mathfrak{F}} \leq 2^{-r} ,$$

$$\sup_{\lambda \in \Lambda_{r-1}} \|R_\mu(D)(\delta_{r+1}(\lambda) - \mathfrak{c}_{r+1}(\lambda))\|_{\mathfrak{G}} \leq 2^{-r} : \quad \mu = 1, \ldots, \nu .$$

With this choice of $\mathfrak{c}_{r+1}$ we set

$$\mathfrak{f}_{r+1}(\lambda) := \tilde{\mathfrak{f}}_{r+1}(\lambda) - \mathfrak{c}_{r+1}(\lambda) : \quad \lambda \in \Lambda_{r+1} .$$

We obtain a sequence of functions $\mathfrak{f}_r \in \mathcal{H}(\Lambda_r, \mathfrak{F}_r) \subseteq \mathcal{H}(\Lambda_r, \mathfrak{F})$ with the properties

$$P(\lambda, D)\mathfrak{f}_r(\lambda) = \mathfrak{g}(\lambda) : \quad \lambda \in \Lambda_r , \tag{3.6}$$

$$\sup_{\lambda \in \Lambda_{r-1}} \|\mathfrak{f}_{r+1}(\lambda) - \mathfrak{f}_r(\lambda)\|_{\mathfrak{F}} \leq 2^{-r} , \tag{3.7}$$

$$\sup_{\lambda \in \Lambda_{r-1}} \|R_\mu(D)(\mathfrak{f}_{r+1}(\lambda) - \mathfrak{f}_r(\lambda))\|_{\mathfrak{G}} \leq 2^{-r} : \quad \mu = 1, \ldots, \nu . \tag{3.8}$$

By (3.7) the limit

$$\mathfrak{f}(\lambda) := \lim_{r \to \infty} \mathfrak{f}_r(\lambda)$$

exists in $\mathfrak{F}$ for each $\lambda \in \Lambda$, and $\mathfrak{f} \in \mathcal{H}(\Lambda, \mathfrak{F})$. Since $\{R_\mu\}$ is a basis of $\boldsymbol{W}(Q)$ we conclude from (3.8) that $R(D)\mathfrak{f} \in \mathcal{H}(\Lambda, \mathfrak{G})$ for any $R \in \boldsymbol{W}(Q)$. Finally it is clear by (3.6) that $P(\lambda, D)\mathfrak{f}(\lambda) \equiv \mathfrak{g}(\lambda)$ since for fixed $\lambda \in \Lambda$ the sequence $\{\mathfrak{f}_r(\lambda)\}$ converges in $\mathbf{B}^{\mathrm{loc}}_{q,k\tilde{Q}}$ and the operator $P(\lambda, D) : \mathbf{B}^{\mathrm{loc}}_{q,k\tilde{Q}} \longrightarrow \mathbf{B}^{\mathrm{loc}}_{q,k}$ is continuous ([H2], 10.1.22). The proof is complete. □

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Frank MANTLIK,
Universität Dortmund
Fachbereich Mathematik
Postfach 50 05 00
4600 Dortmund 50 (Germany).