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MAXIMUM MODULUS SETS AND REFLECTION SETS

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INTRODUCTION

The study of « maximum modulus sets » (i.e. subsets of the boundary of a domain, in $\mathbf{C}^n$, on which a holomorphic function takes its maximum modulus) was initiated by Duchamp and Stout [9], to whom most known results are due. Their work dealt mainly with manifolds. Some preliminary work on general maximum modulus sets had been done by Sibony [17], and some subsequent work on manifolds has been done by A. Iordan [12], [13].

Duchamp and Stout mainly studied the case of *real analytic* maximum modulus manifolds. It is precisely our goal to show how big the gap is between the smooth (*i.e.* $\mathscr{C}^2$ or $\mathscr{C}^\infty$) case, and the real analytic case. One of our results is the following : Let $\mathscr{M}$ be a smooth manifold of real dimension n in the boundary of a strictly pseudoconvex domain Ω in $\mathbf{C}^n$. If the boundary of Ω is real analytic and if $\mathscr{M}$ is a maximum modulus set then $\mathscr{M}$ is real analytic (see Corollary 1 and Proposition 6 for precise statements). We wish to emphasize how this result contrasts with the theory of peak sets. For peak sets, local constructions are often immediate in case of real analytic date, and then can easily be carried over to the smooth case by using standard tools such as first finding almost analytic extensions and then using solutions to the $\bar{\partial}$ problem to correct approximate solutions (see e.g. [7], [10], or [18] for a different approach). In particular, peak sets which are manifolds, even of the maximal possible dimension (i.e. dimension $(n-2)$), need not be real analytic.

We also study the case of maximum modulus manifolds of lower dimension, in real analytic pseudoconvex boundaries. It turns out that for lower dimensions, maximum modulus manifolds need not be real analytic, and the situation is quite complicated. Consider a curve in the boundary of a strictly pseudoconvex domain with real analytic boundary. Assume that this curve bounds an analytic disk « from inside ». (Precise definitions are given in section § I.) Then this curve is a (local) maximum modulus set if and only if it is real analytic (Proposition 5).

But the situation is strikingly different for curves which bound a disk « from outside ». Every such curve is a maximum modulus set

(Proposition 7). For the proof, we have to employ « one sided straightening » of a Levi flat hypersurface.

Maximum modulus sets in real analytic strictly pseudoconvex boundaries have a remarkable property : along a maximum modulus set one can match $\bar{z}_1, \ldots, \bar{z}_n$ with holomorphic functions. This property is of interest in its own right, and leads to the notion of what we call a *reflection set*, which, roughly, is a set along which appropriate collections of holomorphic and antiholomorphic functions agree. Another object of this paper is then to obtain regularity results for reflection sets.

The paper is organized as follows.

In § 0 we set up notations and give precise definitions. For the convenience of the reader we also summarize some results from [10]. In § I we study reflection sets. The reflection sets which are manifolds of maximum possible dimension are easily shown to be real analytic. (However a question is raised that we cannot even answer for curves in the plane.) We then observe that, under some minimal smoothness assumptions (on the function having maximum modulus), maximum modulus sets in real analytic strictly pseudoconvex boundaries are reflection sets. § II is more technical, and deals mainly with the problem of relaxing smoothness assumptions made in I. In § III, we study curves which bound an analytic disk « from outside ». This provides non trivial examples of maximum modulus sets. Then in § IV, we try to exploit the immediate relation between maximum modulus sets and some type of holomorphic foliations. Having failed to formulate satisfactory general results, we have preferred to work out completely two examples to illustrate this relation. These examples indicate the difficulty in finding a simple characterization of maximum modulus curves, even in the case of the sphere.

When discussing maximum modulus sets, we have always supposed these sets to be manifolds (which is of course a very unnatural hypothesis to make on a critical set), and we have considered various smoothness hypothesis on the « set ». But we have much more worried about not imposing unnecessary smoothness assumptions on the function which attains the maximum modulus. The reason for it can be understood by considering the following situation. Let γ be a « complex tangential » curve in the boundary of a strictly pseudo convex domain Ω. Then γ is known to be a peak interpolation set. This means that there exists a function f continuous on $\bar{\Omega}$, holomorphic on Ω, so that $|f| = 1$ on γ and $|f| < 1$ on Ω. Thus γ is a maximum modulus curve. But unless f is constant on γ, f cannot be $\mathscr{C}^1$ on $\bar{\Omega}$ (this question is discussed in § III, in relation with pluriharmonic interpolation).

In the title of this paper, we have used the notion of « maximum modulus sets » mainly for historical reasons. At least for necessary conditions it is better to consider sets on which a pluriharmonic function, continuous up to the boundary, reaches its maximum, without insisting that the conjugate function be continuous. (To see the relation with maximum modulus, consider $\log |f|$.)

All our studies will be local.

0. NOTATIONS AND DEFINITIONS

Let Ω be a strictly pseudoconvex (bounded) domain, in $\mathbf{C}^n$, with $\mathscr{C}^2$ boundary. The boundary of Ω is denoted by $b\Omega$. Let $E \subset b\Omega$. E is said to be a *maximum modulus set* if for every $p \in E$, there exists U a neighborhood of p in $\mathbf{C}^n$, and a function f defined and continuous on $\bar{\Omega} \cap U$, holomorphic on $\Omega \cap U$ so that $|f| < 1$ on $\Omega \cap U$ and $|f| \equiv 1$ on $E \cap U$.

In case the function f in the definition of maximum modulus set can be chosen of class $\mathscr{C}^k$, on $\bar{\Omega} \cap U$, we will say that E is a $\mathscr{C}^k$-*maximum modulus set.*

The set E will be said to be a *pluriharmonic peak set* if for every $p \in E$, there exists U as above and λ a function pluriharmonic on $\Omega \cap U$, so that $\lambda < 0$ and $\lambda(z)$ tends to 0 as z approaches E. Of course every maximum modulus set is a pluriharmonic peak set.

Duchamp and Stout [10] have shown that if E is a (germ of a) manifold of class $\mathscr{C}^2$, if $E \subset b\Omega$ (strictly pseudoconvex), and if E is a $\mathscr{C}^2$ maximum modulus set, then E must be totally real. In $\mathbf{C}^2$, they show that if $b\Omega$ is real analytic, every totally real, real analytic submanifold of $b\Omega$ is a maximum modulus set. In $\mathbf{C}^n$ $(n > 2)$, they find a necessary and sufficient condition for real analytic, totally real submanifolds to be maximum modulus sets. This condition is related to the integrability of the distribution of subspaces obtained by intersecting the tangent to the submanifold with the complex tangent space to $b\Omega$. For additional discussions of this, see [13].

I. REFLECTION SETS, AND $\mathscr{C}^k$ MAXIMUM MODULUS SETS

As indicated in the introduction, we shall study sets along which appropriate collections of holomorphic and antiholomorphic functions match. We were led to the study of such sets by our studies in $\mathbf{C}^n$, but we begin with the special case of curves in $\mathbf{C}$, since no special definitions are needed, and even here there are interesting questions which we are unable to answer.

I.1. Reflection curves in $\mathbf{C}$.

PROPOSITION 1. – *Let Γ be a (germ of a) $\mathscr{C}^1$ curve through 0 in $\mathbf{C}$. Assume that there are holomorphic functions f and g, each defined on one side of Γ and not on the same side, which are continuous up to Γ, so that $f = \bar{g}$ on Γ. If f and g are of class $\mathscr{C}^k$, $(k \geqslant 1)$ up to Γ and $f^{(k)}(0) \neq 0$, then Γ is real analytic in a neighborhood of 0.*

Thus, the real analyticity of γ shows that g is obtained from f by Schwarz's reflection. Notice therefore that no extra smoothness of f and g is to be expected.

COUTEREXAMPLE. – *There exists a $\mathscr{C}^1$ curve Γ, f and g, as above, $\mathscr{C}^\infty$ up to Γ (realizing the matching $f=\bar{g}$ along Γ) so that Γ is not real analytic.*

According to Proposition 1, in the couterexample, f and g need to have all their derivatives vanishing at some point. Also note that the functions z^2 and $\bar{z}^2$ match along the positive real axis and the positive imaginary axis, and so it is easy to produce counterexamples where the curve γ is Lipschitz. Thus in our counterexample, the fact that γ is of class $\mathscr{C}^1$ is significant.

QUESTION. – *Are there any counterexample as above with a curve Γ of class $\mathscr{C}^\infty$ or oven $\mathscr{C}^2$?*

Proof of Proposition 1. — 1) Case $k = 1$.

For $r > 0$, r small enough, let Ω^+, respectively Ω^-, be the side of Γ in the set $\{z \in \mathbf{C}, |z|<r\}$ on which f, respectively g, is defined. The hypothesis $f'(0) \neq 0$ implies $g'(0) \neq 0$, since $f = g$ along Γ. Thus if r is small enough, f and g are diffeomorphisms from $\bar{\Omega}^{\pm}$ onto respectively $f(\bar{\Omega}^+)$ and $g(\bar{\Omega}^-)$. We have $f(\Gamma) = \bar{g}(\Gamma)$. The crucial but trivial observation is that Ω^+ and Ω^- are mapped under f and $\bar{g}$ to the same side of the curve $f(\Gamma)$, as a consequence of the Cauchy-Riemann equations.

Now we claim that the restriction of the function $\bar{z}$ to the curve Γ has a holomorphic extension to a neighborhood of 0. Indeed this extension is given $\overline{\bar{g}^{-1}(f(z))}$ for $z \in \Omega^+$, and $\overline{f^{-1}(\bar{g}(z))}$ for $z \in \Omega^-$. Let Ψ be this holomorphic extension. Of the two equations

$$\operatorname{Re} \Psi = \operatorname{Re} z, \qquad \operatorname{Im} \Psi = -\operatorname{Im} z,$$

one at least must be non-degenerate (rank 1), since due the Cauchy Riemann equations it is not possible that $\nabla \operatorname{Re} \Psi = \nabla \operatorname{Re} z$ and $\nabla \operatorname{Im} \Psi = -\nabla \operatorname{Im} z$ at 0. The implicit function theorem then shows that this equation defines Γ as a real analytic curve.

2) The case $k \in \mathbf{N}$, $k > 1$, follows easily from the case $k = 1$. Indeed if $f(0) = f^1(0) = \cdots = f^{(k-1)}(0) = 0$ and $f^{(k)}(0) \neq 0$, one considers the functions $f^{1/k}$ defined on one side of Γ, and similarly $g^{1/k}$. One needs to check carefully that $f^{1/k}$ and $f^{1/k}$ are $\mathscr{C}^1$, and, with a correct choice of determinations of k-th roots, that $f^{1/k}$ and $g^{1/k}$ match along Γ. The details are left to the reader.

Counterexample. — Off the negative real axis we define $z^{1/4}$ by $(\rho e^{i\theta})^{1/4} = \rho^{1/4} e^{i\frac{\theta}{4}}$ where $-\pi < \theta < +\pi$. Let Γ be the curve made of the negative real axis and the set of $z = x + iy$ so that $\operatorname{Re} z > 0$, $\operatorname{Im}(1/z^{1/4}) = \pi$. Then Γ is a $\mathscr{C}^1$ curve and we set $f = e^{-(1/z^{1/4})}$ above Γ and $g = e^{-(1/z^{1/4})}$ below Γ. Along Γ, $f = \bar{g}$. At 0, f and g are $\mathscr{C}^\infty$ but all their derivative vanish. The curve Γ is not real analytic, but is of class $\mathscr{C}^1$.

Remark. — It can be shown that it is not possible to replace $e^{-(1/z^{1/4})}$ in the couterexample by another function defined also off the negative real axis in order to get a curve Γ of class $\mathscr{C}^2$.

I.2. Reflection sets in $\mathbf{C}^n$.

We begin with two definitions.

DEFINITION 1. – *If $E \subset \mathbf{C}^n$, a wedge W^+ with edge E is a set obtained in the following manner. Let Γ be nonvoid open cone in $\mathbf{C}^n$, and $r > 0$. Then set*

$$W^+ = \{x+\gamma \in \mathbf{C}^n, x \in E, \gamma \in \Gamma |\gamma| < r\}.$$

The opposite wedge W^- is given by

$$W^- = \{x-\gamma \in \mathbf{C}^n, x \in E, \gamma \in \Gamma |\gamma| < r\}.$$

Note. – The reader may prefer to restrict attention to the case where there is some transversality of the edge and the cone. In our definition, however, we accept the possibility that $E = R^n$ and that Γ contains a non zero vector in R^n. In this case, W^+ and W^- are neighborhoods of R^n. This is certainly not the case of interest, but there has been no need to rule out this case explicitly.

DEFINITION 2. – *A set E in $\mathbf{C}^n$ is a* reflection set *if there exist two opposite wedges W^+ and W^- with edge E, and two n tuples of holomorphic functions $f_1, \ldots, f_n$ defined on W^+, and $g_1, \ldots, g_n$ defined on W^- of class $\mathscr{C}^1$ on the closure of respectively W^+ and W^- so that $df_1 \wedge \cdots df_n \neq 0$ and $f_j = \bar{g}_j$ on E.*

Every totally real, real analytic manifold is a reflection set (Schwarz reflection). It is immediate that a manifold which is a reflection set must be totally real. In § II, we will see that, in case E is a totally real manifold of dimension n, of class $\mathscr{C}^2$, the $\mathscr{C}^1$ smoothness requirement can be dropped for either the $f_j's$ or the $g_j's$. This is used in applications, where $g_j = z_j$.

First, we make the following easy observation.

LEMMA 1. – *Let $\mathscr{M}$ be a (germ of) $\mathscr{C}^1$ manifold of real dimension n through 0 in $\mathbf{C}^n$. Assume that there exist U a neighborhood of 0 in $\mathbf{C}^n$, $\Psi_1, \ldots, \Psi_n$ holomorphic functions in U so that $\Psi_j = \bar{z}_j$ on $\mathscr{M} \cap U$. Then $\mathscr{M}$ is real analytic (in a neighborhood of 0), and totally real.*

Proof. — From the $2n$ equations

$$\operatorname{Re} \Psi_j = \operatorname{Re} z_j , \qquad \operatorname{Im} \psi_j = - \operatorname{Im} z_j ,$$

one can extract n independent equations, which shows that $\mathscr{M}$ (defined by these n equations) is real analytic. And $\mathscr{M}$ is totally real due to the Cauchy-Riemann equations.

A generalization of Proposition 1 is the following.

PROPOSITION 2. — *If E is a $\mathscr{C}^1$ manifold of dimension n in $\mathbf{C}^n$, and is a reflection set, then E is real analytic.*

Proof. — Set $f = (f_1, \ldots, f_n)$, $g = (g_1, \ldots, g_n)$. Then f and g are local diffeomorphisms. In a neighborhood of some point $p \in E$, $z \mapsto \overline{f^{-1}(\bar{g}(z))}$ and $z \mapsto (\bar{g}^{-1}(f(z))$ define local extension of the restriction of $\bar{z}$ to opposite wedges (after shrinking W^+ and W^-) by Lemma 2 below. Due to the edge of the wedge theorem (which applies even if $\mathscr{M}$ is only of class $\mathscr{C}^1$, see [16]), this shows that the restriction of $\bar{z}_j$ to E has an holomorphic extension to some neighborhood of p. Therefore, according to Lemma 1, E is analytic.

The proof is complete, except for the fact which has been used above : that $\overline{f}^{-1} \circ g$ and $\bar{g}^{-1} \circ f$ are defined on opposite wedges.

LEMMA 2. — *Let E be a $\mathscr{C}^1$ manifold of dimension n in $\mathbf{C}^n$, $W^{\pm}$ be opposite wedges with edge E and $f = (f_1, \ldots, f_n)$, $g = (g_1, \ldots, g_n)$ as in Definition 2. Let $p \in E$. Then there exist U a neighborhood of p in $\mathbf{C}^n$ and opposite wedges $\mathscr{W}^{\pm}$ with edge $E \cap U$, $\mathscr{W}^{\pm} \subset W^{\pm}$ so that $f(\mathscr{W}^+) \subset \bar{g}(W^-)$ and $\bar{g}(\mathscr{W}^-) \subset f(W^+)$.*

Proof. — We first have to do some preliminary work, due to the fact that our definition of wedge is not the most convenient one. We can assume that $p = 0$ and that $\mathbf{R}^n$ is the tangent space to E at 0. Fix $\gamma \in \Gamma$, $\gamma \notin R^n$. We can find $\rho_1, \ldots, \rho_\ell$, $\mathscr{C}^1$ real valued functions so that, for some neighborhood U_0 of 0 :

$$W'^+ = \{z \in U_0, \rho_j(z) > 0 \; j = 1, \ldots, p\} \subset W^+$$

$$W'^- = \{z \in U_0, \rho_j(z) < 0 \; j = 1, \ldots, p\} \subset W^-$$

and

$$\frac{d}{dt}[\rho_j(0 + t\gamma)]\bigg|_{t=0} > 0 .$$

Of course $\rho_j = 0$ along $E \cap \mathrm{U}_0$. Write $\gamma = \gamma_0 + i\gamma_1$, γ_0 and $\gamma_1 \in \mathbf{R}^n$. Since γ_0 is tangent to E at 0 one gets

$$\frac{d}{dt}[\rho_j(0+ti\gamma_1)]\bigg|_{t=0} > 0\,.$$

Therefore we can choose Γ' and $r' > 0$ a small open conic neighborhood of $i\gamma_1$ so that after shrinking U_0 if necessary:

$$W^{\pm} \supset W'^{\pm} \supset \{x \pm \gamma', x \in E \cap U_0, \gamma' \in \Gamma', |\gamma'| < r'\}\,.$$

Extend f and $\bar{g}$ to local diffeomorphisms $\tilde{f}$ and $\tilde{\bar{g}}$ on a neighborhood of 0. Then $f(W'^{+})$ (resp $\bar{g}(W'^{-})$) is defined by

$$\{\rho_j \circ \tilde{f}^{-1} > 0\} \text{ (resp. } \{\rho_j \circ \tilde{\bar{g}}^{-1} < 0\})\,.$$

Since γ_1 is tangent to E at 0 one has

$$df\Big|_0 (\gamma_1) = d\bar{g}\Big|_0 (\gamma_1)\,.$$

And since df and dg are $\mathbf{C}$ linear by the Cauchy-Riemann equations,

$$df\Big|_0 (i\gamma_1) = -d\bar{g}\Big|_0 (i\gamma_1)\,.$$

By the chain rule

$$\frac{d}{dt}[\rho_j \circ \tilde{f}^{-1} \circ \bar{g}(0+ti\gamma_1)]\bigg|_{t=0} < -0$$

and

$$\frac{d}{dt}[\rho_j \circ \tilde{\bar{g}}^{-1} \circ f(0+ti\gamma_1)]\bigg|_{t=0} < 0\,.$$

By continuity there exist $A > 0$, U a neighborhood of 0 in $\mathbf{C}^n$, $U \subset U_0$, Γ'' a conic neighborhood of $i\gamma$, $\Gamma'' \subset \Gamma'$, so that for every $q \in E \cap U$ and $\gamma'' \in \Gamma'$:

$$\frac{d}{dt}[\rho_j \circ \tilde{f}^{-1} \circ \bar{g}(q+t\gamma'')]\bigg|_{t=0} < -A|\gamma''|$$

and

$$\frac{d}{dt}[\rho_j \circ \tilde{\bar{g}}^{-1} \circ f(q+t\gamma'')]\bigg|_{t=0} > -A|\gamma''|\,.$$

If $q \in E \cap U$ and $\gamma'' \in \Gamma''$, $|\gamma''| = 1$, then for $t < 0$, $|t|$ small, one has

$$\rho_j \circ \tilde{f}^{-1} \circ \bar{g}(q + t\gamma'') > 0, \qquad \text{so} \qquad \bar{g}(q + t\gamma'') \in f(W'^{+}).$$

Similarly, for $t > 0$ $\rho_j \circ \tilde{\bar{g}}^{-1} \circ f(q + t\gamma'') < 0$, so $f(q + t\gamma'') \in \bar{g}(W'^{-})$. Therefore we can take

$$\mathscr{W}^{\pm} = \{q \pm \gamma'' \in \mathbf{C}^n \,;\, q \in E \cap U, \gamma'' \in \Gamma, |\gamma''| < r''\},$$

where $r'' > 0$, is taken small enough. Q.E.D.

I.3. $\mathscr{C}^k$ maximum modulus sets in real analytic strictly pseudoconvex boundaries.

PROPOSITION 3. — *Let Ω be a domain in $\mathbf{C}^n$, let $E \subset b\Omega$, and assume that $b\Omega$ is strictly pseudoconvex and real analytic in a neighborhood of E. Suppose there exists a neighborhood U of E and a function f of class $\mathscr{C}^k (k \geqslant 1)$ on $\bar{\Omega} \cap U$, holomorphic on $\Omega \cap U$, so that $|f| = 1$ on E and $|f| < 1$ on $\Omega \cap U$. (This means that E is a $\mathscr{C}^k$ maximum modulus set.) Then there exist V a neighborhood of E and functions $g_1, \ldots, g_n$, of class $\mathscr{C}^{k-1}$ on $\bar{\Omega} \cap V$, which are holomorphic on $\Omega \cap V$ so that $g_j = \bar{z}_j$ on E. If $k \geqslant 2$, it follows that E is a reflection set.*

Proof. — By shrinking U if necessary, we can assume that $U \cap \Omega = \{z \in U | \rho(z) < 0\}$, where ρ is a real analytic strictly plurisubharmonic function on U. Let

$$M = \{(z,\xi) \in \mathbf{C}^n \times P_{n-1}(\mathbf{C}) | z \in b\Omega \cap U, \xi = \widetilde{\partial\rho}(z)\},$$

where $P_{n-1}(\mathbf{C})$ is the $(n-1)$-dimensional complex projective space, and $\xi = \widetilde{\partial\rho}(z)$ means that ξ is the point given in homogeneous coordinates by $\left(\dfrac{\partial\rho}{\partial z_1}(z), \ldots, \dfrac{\partial\rho}{\partial z_n}(z)\right)\cdot$ Since $b\Omega$ is strictly pseudoconvex in U, it is well known that M is totally real. (This is the essence of the reflection principle in several variables as developed by Lewy, Webster, Pinčuk, *et al.*, see [20].)

We will denote by $\chi = (\chi_1, \chi_2)$ the antiholomorphic reflection accross M. That is,

$$\chi : (z,\xi) \to (z' = \chi_1(z,\xi), \xi' = \chi_2(z,\xi))$$

is defined in a neighborhood of M, is antiholomorphic, and satisfies

$$\chi(z,\xi) = (z,\xi) \quad \text{if } (z,\xi) \in M .$$

At every point of E, $\partial_b(f) = 0$. This follows immediately from the implicit function theorem, or if one prefers, by differentiating $f\bar{f}$. Moreover, at any point of E, we have $df \neq 0$ by Hopf's Lemma. So along E, $\partial\rho$ and ∂f are proportional. Consequently, they define the same element

$$\widetilde{\partial\rho}(z) = \widetilde{\partial f}(z)$$

in $P_{n-1}(\mathbf{C})$ when $z \in b\Omega$. (Again, we denote by $\widetilde{\partial f}(z)$ the element in P_{n-1} given in homogeneous coordinates by $\left(\frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z)\right)\cdot\Big)$ Then to obtain the first part of the Proposition, set

$$(g_1(z), \ldots, g_n(x)) = \overline{\chi_1(z,\widetilde{\partial f}(z))} .$$

On E,

$$(g_1(z), \ldots, g_n(x)) = \overline{\chi_1(z,\widetilde{\partial\rho}(z))} = (\bar{z}_1, \ldots, \bar{z}_n) .$$

And for some neighborhood V of E, $g_1, \ldots, g_n$ are defined and of class $\mathscr{C}^{k-1}$ on $\bar{\Omega} \cap V$, and are holomorphic on $\Omega \cap V$.

If f is $\mathscr{C}^2$, $g_1, \ldots, g_n$ are $\mathscr{C}^1$ on $\bar{\Omega} \cap V$. In Definition 2, on then has to take wedges $W^{\pm}$ with edge E. Choose $W^- \subset \Omega \cap V$, and take $(f_1, \ldots, f_n) = (z_1, \ldots, z_n)$ on W^+. This shows that E is a reflection set.

From Propositions 2 and 3 we deduce immediately :

Corollary 1. — *Let $\Omega \subset \mathbf{C}^n$ be domain and let E be a submanifold of $b\Omega$ of class $\mathscr{C}^1$ and of real dimension n. If $b\Omega$ is strictly pseudoconvex and real analytic in a neighborhood of E, and if E is a $\mathscr{C}^2$ maximum modulus set, then E is real analytic.*

Notice that this also has consequences for functions f which have maximum modulus along E. Let f be any holomorphic function on Ω (or on $\Omega \cap U$) which extends continuously to E, (E is as in Corollary 1), so that $|f|$ is constant along E. Then by the edge of the wedge theorem f has an analytic extensions to some neighborhood of E.

Let us mention another Corollary, which provides examples of smooth curves which are not maximum modulus sets. We say that a (piece of simple) curve Γ in $b\Omega$ *bounds a disk from inside* Ω if, first, there exists a continuous parameterization $\gamma : (-1,+1) \to \Gamma$ which, for some $\varepsilon > 0$, extends holomorphically to the rectangle R_ε where

$$R_\varepsilon = \{(x+iy) \in \mathbf{C} \mid -1<x<+1, 0<y<\varepsilon\},$$

and second, $\gamma(R_\varepsilon) \subset \Omega$. (In this case, the extension is still denoted by γ).

COROLLARY 2. — *Let Ω be domain in $\mathbf{C}^n$. Let Γ be a curve in $b\Omega$ which is a $\mathscr{C}^1$ maximum modulus set. If Γ bounds a disk from inside Ω and if $b\Omega$ is strictly pseudoconvex and real analytic in a neighborhood of Γ, then Γ is real analytic.*

Proof. — The problem being local, we may assume that there exists a neighborhood U of Γ and a $\mathscr{C}^1$ function f on $\bar{\Omega} \cap U$, holomorphic on $\Omega \cap U$, so that $|f| = 1$ on Γ, and $|f| < 1$ on $\Omega \cap U$. Let γ be a parameterization as above. The map γ extends analytically across $(-1,+1)$ by setting:

$$\gamma(\zeta) = (\overline{g_1(\gamma(\bar{\zeta}))}, \ldots, \overline{g_n(\gamma(\bar{\zeta}))})$$

for ζ in some neighborhood of $(-1,+1)$ in $\mathbf{C}$ and $\operatorname{Im}(\zeta) < 0$. Here the g_j's are given in Proposition 3. Along $(-1,+1)$, $\gamma' \neq 0$; this follows from Hopf's Lemma applied either to $f \circ \gamma$ or $\rho \circ \gamma$ (ρ a plurisubharmonic defining function). Hence Γ is real analytic.

II. PLURIHARMONIC PEAK SETS

II.1. Regularity of functions.

For simplicity we wil first consider the case of curves. Let $\Omega \subset \mathbf{C}^n$ be a domain with $\mathscr{C}^2$ boundary, and let Γ be an open arc of a simple $\mathscr{C}^1$ curve in $b\Omega$ which is transverse to the complex tanget space to $b\Omega$ at each point. We suppose that $b\Omega$ is strictly pseudoconvex in a neighborhood U of the curve Γ, and that $\Omega \cap U = \{z \in U \mid \rho(z)<0\}$, where ρ is a plurisubharmonic function on U.

DEFINITION. – *The curve* Γ *satisfies the condition* (AH), (*standing for* almost holomorphic), *if and only if there is a* $\mathscr{C}^1$ *parameterization*

$$\gamma : (-1,+1) \to \Gamma$$

with $\gamma \neq 0$ *such that* γ *extends to* $\bar{R}_1$, *the closure of the rectangle*

$$R_1 = \{\zeta = x + \imath y \mid -1 < x < +1, 0 < y < 1\},$$

in such a way that γ *is of class* $\mathscr{C}^1$ *on* $\bar{R}_1$, $\gamma(R_1) \subset \Omega$, $\dfrac{\partial\gamma}{\partial\bar{\zeta}}$ *is of class* $\mathscr{C}^2$ *on* $\bar{R}_1$, *and* $\dfrac{\partial\gamma}{\partial\bar{\zeta}}$ *vanishes to second order along* $[-1,+1]$. *Such a parametrization will be called a special parameterization.*

Examples of curves satisfying (AH) *are the following.*

Any $\mathscr{C}^1$ curve which bounds a $\mathscr{C}^1$ disk from inside Ω; (*i.e.* as in I.3, Corollary 2, but with γ of class $\mathscr{C}^1$ on $\bar{R}_\varepsilon$). Here the transversality condition comes from Hopf's Lemma.

Any $\mathscr{C}^3$ curve with is transverse to the complex tangent space to $b\Omega$.

PROPOSITION 4. – *Assume that* Γ *is a* $\mathscr{C}^1$ *curve in* $b\Omega$ *which satisfies* (AH). *Let* γ *be a special parameterization of* Γ. *Let* λ *be a negative pluriharmonic function defined on* Ω *such that* $\lambda(z)$ *tends to* 0 *as* z *approaches* Γ *with* $z \in \Omega$. *Then the functions* $\dfrac{\partial\lambda}{\partial z_j} \circ \gamma$, *defined on* R_1, *extend continuously to* $(-1,+1)$. *Moreover, if* $x \in (-1,+1)$, *if* $p = \gamma(x)$, *and if we set*

$$\lambda_j(p) = \lim_{\zeta \to x} \frac{\partial\lambda}{\partial z_j} \circ \gamma(\zeta),$$

then

$$\sum_{j=1}^{n} |\lambda_j(p)| \neq 0,$$

and for every j, $k \in \{1,\ldots,n\}$,

$$\left[\frac{\partial\rho}{\partial z_j}\lambda_k - \frac{\partial\rho}{\partial z_k}\lambda_j\right](p) = 0.$$

If λ is assumed to be of class $\mathscr{C}^1$ on $\bar{\Omega}$ the Proposition is straightforward. The meaning of the last assertion is that $\partial_b \lambda = 0$ along Γ (and since λ is real valued, $\bar{\partial}_b \lambda = 0$ as well). The necessity of this is clear since one must have $d_{b\Omega}\lambda = 0$ along Γ. The condition that $\sum |\lambda_p(p)| \neq 0$ comes from applying Hopf's lemma. Thus the significance of the Proposition is the absence of *a priori* smoothness assumptions on λ.

Proof of Proposition 4. — The function $\lambda \circ \gamma$ is defined on R_1 and has limit 0 along the edge $[-1,+1]$. For any $a \in (0,1)$ the Laplacian of $\lambda \circ \gamma$ is bounded in a neighborhood of $[-a,+a]$. Indeed, since λ is pluriharmonic, one has

$$\frac{\partial^2(\lambda \circ \gamma)}{\partial \zeta\, \partial \bar{\zeta}}(\zeta) = 2\,\mathrm{Re} \sum_{j=1}^{n} \frac{\partial \lambda}{\partial z_j}(\gamma(\zeta)) \frac{\partial^2 \gamma_j}{\partial \zeta\, \partial \bar{\zeta}}(\zeta) + 2\,\mathrm{Re} \sum_{j,k=1}^{n} \frac{\partial^2 \lambda}{\partial z_j\, \partial z_k}(\gamma(\zeta)) \frac{\partial \gamma_j}{\partial \bar{\zeta}}(\zeta) \frac{\partial \gamma_k}{\partial \zeta}(\zeta),$$

where $\gamma = (\gamma_1, \ldots, \gamma_n)$. But as ζ approaches $[-a,+a]$ the following estimates hold :

$$\begin{aligned}
|\nabla \lambda(\zeta)| &= O\left(\frac{1}{\mathrm{Im}\,\zeta}\right); \\
|\nabla^2 \lambda(\zeta)| &= O\left(\frac{1}{(\mathrm{Im}\,\zeta)^2}\right); \\
\left|\frac{\partial^2 \gamma_j}{\partial \zeta\, \partial \bar{\zeta}}(\zeta)\right| &= O(\mathrm{Im}\,\zeta); \\
\left|\frac{\partial \gamma_j}{\partial \bar{\zeta}}(\zeta)\right| &= O((\mathrm{Im}\,\zeta)^2); \\
\left|\frac{\partial \gamma_k}{\partial \zeta}(\zeta)\right| &= O(1).
\end{aligned}$$

The following standard Lemma applied to smaller rectangles now shows that $\lambda \circ \gamma$ has a $\mathscr{C}^1$ extension to $\mathbf{R}_1 \cup (-1,+1)$.

LEMMA 3. — *Let $u(x,y)$ be a function defined on R_1. If Δu is bounded and $u(x,y)$ tends to 0 as y tends to 0, then u has a $\mathscr{C}^1$ extension to $R_1 \cup (-1,+1)$.*

Proof of Lemma 3. — Define $\tilde{u}$ (resp $\widetilde{\Delta u}$) an extension of u (resp Δu) to $(-1,+1) \times (-1,+1)$ by

$$\tilde{u}(x,y) = -u(x,-y), \quad \text{if} \quad y < 0$$

and

$$\widetilde{\Delta u}(x,y) = -\Delta u(x,-y), \quad \text{if} \quad y < 0 .$$

In the sense of distributions on $(-1,+1) \times (-1,+1)$, $\Delta \tilde{u} = \widetilde{\Delta u}$. There is no « jump term » of the function or the derivative along $(-1,+1) \times \{0\}$. (Approximate u by $u(x,y+\varepsilon)$ and let ε tend to 0.) Thus $\Delta\tilde{u}$ is bounded on R_1 and hence u is of class $\mathscr{C}^1$ since the gradient of the logarithmic (Newtonian) potential is locally integrable.

This ends the proof of Lemma 3, and we continue with the proof of Proposition 4. We have obtained that $\lambda \circ \gamma$ has a $\mathscr{C}^1$ extension to $R_1 \cup (-1,+1)$. Let $x \in (-1,+1)$ and set $p = \gamma(x)$. Due to Hopf's Lemma, $\lambda \circ \gamma(\zeta) \leqslant -C \operatorname{dist}(\gamma(\zeta), b\Omega)$ for some constant $C > 0$. Using the transversality hypothesis on Γ, one gets that $\dfrac{\partial}{\partial y}(\lambda \circ \gamma)(x) \neq 0$. Therefore the function

$$\sum_{j=1}^{n} \frac{\partial \lambda}{\partial z_j}(\gamma(\zeta)) \frac{d\gamma_j}{d\zeta}(\zeta)$$

has a nonzero limit as ζ tends to x $(\zeta \in R_1)$. Next we claim that for any $j, k \in \{1, \ldots, n\}$,

$$\left(\frac{\partial \rho}{\partial z_j}\frac{\partial \lambda}{z_k} - \frac{\partial \rho}{\partial z_k}\frac{\partial \lambda}{\partial z_j}\right)(\gamma(\zeta)) \to 0$$

as $\zeta \to x$. For ζ close to x, and for some constant $C > 0$ we have:

$$\operatorname{dist}(\gamma(\zeta), b\Omega) \leqslant C \operatorname{Im} \zeta, \qquad \text{and} \qquad -\frac{1}{C}\operatorname{Im}\zeta < \lambda(\gamma(\zeta)) < 0$$

since $\lambda \circ \gamma$ is $\mathscr{C}^1$ up to the real axis. Since Γ is transverse to the complex tangents, we also have for some constant $C > 0$,

$$\operatorname{dist}(\gamma(\zeta), b\Omega) \geqslant \frac{1}{C}\operatorname{Im}\zeta, \qquad \text{and} \qquad \rho(\gamma(\zeta)) \leqslant -C\operatorname{Im}\zeta .$$

Let $1 \leqslant j < k \leqslant n$. For $r > 0$ set

$$\Delta_\zeta(r) = \left\{\gamma(\zeta) + t\left(0, \ldots, -\frac{\partial \rho}{\partial z_k}(\gamma(\zeta)), \ldots, \frac{\partial \rho}{\partial z_j}(\gamma(\zeta)), \ldots, 0\right) \; t \in \mathbf{C}, |t| < r\right\}$$

(where $\dfrac{\partial\rho}{\partial z_k}$ and $-\dfrac{\partial\rho}{\partial z_j}$ appear respectively in the j-th and the k-th entry and there are zeros elsewhere). This is a complex disk centered at $\gamma(\zeta)$, tangent at $\gamma(\zeta)$ to the level set of ρ through $\gamma(\zeta)$. For C large enough, independent of ζ close enough to 0, $\Delta_\zeta\left(\dfrac{1}{C}\sqrt{\operatorname{Im}\zeta}\right) \subset \Omega$. Then one gets

$$\left(\frac{\partial\rho}{\partial z_j}\frac{\partial\lambda}{\partial z_k} - \frac{\partial\rho}{\partial z_k}\frac{\partial\lambda}{\partial z_j}\right)(\gamma(\zeta)) = \mathcal{O}(\sqrt{\operatorname{Im}\zeta}).$$

This is consequence of the following simple fact, applied after rescaling to the function

$$t \mapsto \lambda(\gamma(\zeta)) + t\left(0, \dots, \frac{\partial f}{\partial z_k}(\gamma(\zeta)), \dots, \frac{-\partial\rho}{\partial z_j}(\gamma(\zeta)), \dots, 0\right):$$

if μ *is a harmonic function defined on the unit disk which satisfies* $\mu < 0$, $\mu(0) \geqslant -1$, *then* $|\nabla\mu(0)| \leqslant 2$.

After a linear change of coordinates, we can assume that

$$\frac{\partial\rho}{\partial z_1}(p) \neq 0, \quad \frac{\partial\rho}{\partial z_2}(p) = \cdots = \frac{\partial\rho}{\partial z_n}(p) = 0.$$

The quantities $\dfrac{\partial\lambda}{\partial z_j}(\gamma(\zeta))$ solve the following linear system (*) of n equations with n unknown X_j:

$$\sum_{j=1}^{n}\frac{d\gamma_j}{d\zeta}(\zeta)X_j = \sum_{j=1}^{n}\frac{\partial\lambda}{\partial z_j}(\gamma(\zeta))\frac{d\gamma_j}{d\zeta}(\zeta)$$

$$\frac{\partial\rho}{\partial z_1}(\gamma(\zeta))X_2 - \frac{\partial\rho}{\partial z_2}(\gamma(\zeta))X_1 = \left(\frac{\partial\rho}{\partial z_1}\frac{\partial\lambda}{\partial z_2} - \frac{\partial\rho}{\partial z_2}\frac{\partial\lambda}{\partial z_1}\right)(\gamma(\zeta))$$

$$\vdots$$

$$\frac{\partial\rho}{\partial z_1}(\gamma(\zeta))X_n - \frac{\partial\rho}{\partial z_n}(\gamma(\zeta))X_1 = \left(\frac{\partial\rho}{\partial z_1}\frac{\partial\lambda}{\partial z_n} - \frac{\partial\rho}{\partial z_n}\frac{\partial\lambda}{\partial z_1}\right)(\gamma(\zeta)).$$

We have just proved that the right hand sides have limits as ζ approaches x.

By the transversality hypothesis on Γ, $\dfrac{\partial\gamma_1}{d\zeta}(x) \neq 0$. The coefficients of the equations have limits as ζ approaches x. For $\zeta = x$, the system has rank n and reduces to

$$\frac{d\gamma_1}{d\zeta}(x)X_1 + \cdots + \frac{d\gamma_n}{d\zeta}(x)X_n = \lim_{\zeta\to x}\sum\frac{\partial\lambda}{\partial z_j}(\gamma(\zeta))\frac{d\gamma_j}{d\zeta}(\zeta)$$

$$X_2 = 0$$

$$\vdots$$

$$X_n = 0.$$

By solving the system of equations (*), one sees that as ζ tends to $x(\zeta\in R_1)$, $\frac{\partial\lambda}{\partial z_j}(\gamma(\zeta))$ has limit $\lambda_j(p)$. In the above coordinates $\lambda_2(p) = \cdots = \lambda_n(p) = 0$, and $\sum_{j=1}^{n}|\lambda_j(p)| \neq 0$. The last statement of the Proposition is just the formulation of this fact in arbitrary coordinates.

II.2. Curves which bound disks from inside Ω.

The following generalizes Corollary 2 for $\mathscr{C}^1$ curves:

PROPOSITION 5. – *Let $\Omega \subset \mathbf{C}^n$ be a domain with $\mathscr{C}^2$ boundary. Let Γ be a curve in $b\Omega$ with a $\mathscr{C}^1$ parameterization $\gamma : (-1,+1) \to \Gamma$ which extends holomorphically to a rectangle*

$$R_\varepsilon = \{(x+\iota y)\in\mathbf{C}\,|-1 < x < +1, 0 < y < \varepsilon\}$$

so that $\gamma(R_\varepsilon)\subset\Omega$. Suppose $b\Omega$ is real analytic and strictly pseudoconvex in a neighborhood of Γ and suppose that Γ is a pluriharmonic peak set. Then Γ is real analytic.

Proof. – By applying Hopf's Lemma to $\rho\circ\gamma$ (with ρ a local defining function), one sees that $\dot\gamma \neq 0$ and that Γ is transverse to the complex tangents. We can apply the results from Proposition 4. The problem being local, we can assume that there exists λ a pluriharmonic function defined on all of Ω so that $\lambda < 0$, and $\lambda(z)$ tends to 0 as z approaches Γ. Notice that the functions $\frac{\partial\lambda}{\partial z_1}, \ldots, \frac{\partial\lambda}{\partial z_n}$ are holomorphic on Ω. As $\zeta\in R_\varepsilon$ approaches $x\in(-1,+1)$, Proposition 4 shows that the point $\xi(\zeta)$ in $P_{n-1}(\mathbf{C})$ given in homogeneous coordinates by $\left(\frac{\partial\lambda}{\partial z_1}(\gamma(\zeta)),\ldots,\frac{\partial\lambda}{\partial z_n}(\gamma(\zeta))\right)$ tends to the point in $P_{n-1}(\mathbf{C})$ given in homogeneous coordinates by $\left(\frac{\partial\rho}{\partial z_1}(\gamma(x)),\ldots,\frac{\partial\rho}{\partial z_n}(\gamma(x))\right)\cdot$ And, as in Proposition 3 and Corollary 2, the holomorphic extension of γ accross $(-1,+1)$ is obtained by setting (for $\operatorname{Im}\zeta<0$):

$$\gamma(\zeta) = \overline{\chi_1(\gamma(\bar\zeta),\xi(\bar\zeta))}.$$

II.3. Maximum modulus sets of maximal dimension.

We now turn to the generalization of Corollary 1.

PROPOSITION 6. – *Let $\Omega \subset \mathbf{C}^n$ be a domain. Let E be a submanifold of $b\Omega$ of real dimension n and of class $\mathscr{C}^3$. If E is a pluriharmonic peak set, and if $b\Omega$ is real analytic and strictly pseudoconvex in a neighborhood of E, then E is real analytic.*

Proof. – Let $p\in E$ and parameterize a neighborhood of p in E by a mapping $\gamma:(-1,+1)^n\to E$, where γ is of class $\mathscr{C}^3$, so that $\frac{\partial\gamma}{\partial x_1}(x)$ is transverse at each point to the complex tangent space at the point $\gamma(x)$, and $i\frac{\partial\gamma}{\partial x_1}$ points towards Ω. We can extend γ to a $\mathscr{C}^3$ function defined and one-to-one on $R_1\times(-1,+1)^{n-1}$ where $R_1 = \{\zeta=(x_1+iy_1)\in\mathbf{C}\,|\,-1<x_1<+1, 0<y_1<1\}$, so that $\gamma(R_2\times(-1,+1)^{n-1})\subset\Omega$ and $\frac{\partial\gamma}{\partial\bar\zeta_1}$ vanishes to second order along $y_2=0$.

Thus E is locally the boundary of a $\mathscr{C}^2$ manifold $\tilde E$ in Ω, of real dimension $(n+1)$, transverse to $b\Omega$ along E, and foliated by two dimensional leaves $E_{x_2,\ldots,x_n}$ parameterized by $\zeta\to\gamma(\zeta,x_2,\ldots,x_n)$, $(\zeta\in R_1)$.

Let λ be a pluriharmonic function that again we can assume to be defined on Ω, so that $\lambda<0$ and $\lambda(z)$ tends to 0 as z approaches E. Going through the proof of Proposition 4, carrying $(x_2,\ldots,x_n)$ as a parameter, it follows that as $z\in\tilde E$ approaches some point $p\in E$, then $\left(\frac{\partial\lambda}{\partial z_1}(z),\ldots,\frac{\partial\lambda}{\partial z_n}(z)\right)$ has a limit $(\lambda_1(p),\ldots,\lambda_n(p))$. Also,

$\sum_{j=1}^{n} |\lambda_j(p)| \neq 0$, and for every $j, k \in \{1, \dots, n\}$,

$$\left[\frac{\partial \rho}{\partial z_j}\lambda_k - \frac{\partial \rho}{\partial z_k}\lambda_j\right](p) = 0 .$$

But more is true. Let W_0 be a wedge with edge E so that $W_0 \subset \Omega$, and let W^+ be a strictly smaller wedge with edge E, in the sense that $\bar{W}^+ \subset W_0 \cup E$. Let $p \in E$. Then $\left(\frac{\partial \lambda}{\partial z_1}(z), \dots, \frac{\partial \lambda}{\partial z_n}(z)\right)$ tends to $(\lambda_1(p), \dots, \lambda_n(p))$ as z tends to p, $z \in W^+$ and not only $z \in \tilde{E}$. (See [16], Proposition 2, with $k = 1$: $\frac{\partial \lambda}{\partial z_j}$ is a function whose growth in W_0 is controlled by $\frac{\partial \lambda}{\partial z_j} = O(\operatorname{dist}(z,E)^{-1})$, and which has continuous boundary values along $E(\lambda_j)$. So the restriction of $\frac{\partial \lambda}{\partial z_j}$ to W^+ is continuous up to the edge E).

The equations satisfied by $(\lambda_1(p), \dots, \lambda_n(p))$ show that $(\lambda_1(p), \dots, \lambda_n(p))$ and $\left(\frac{\partial \rho}{\partial z_1}(p), \dots, \frac{\partial \rho}{\partial z_n}(p)\right)$ define the same element in $P_{n-1}(\mathbf{C})$. The proof of Proposition 3 can then immediately be adapted to provide us with functions $g_1, \dots, g_n$ continuous on $\bar{W}^+$, holomorphic on W^+, so that on E, $g_j = \bar{z}_j$. But the smoothness of g_j restricted to E implies the smoothness of g_j on smaller wedges (see *e.g.* [8]). And since we could as well start with a wedge strictly larger that W^+, we get that g_j is of class $\mathscr{C}^1$ on $\bar{W}^+$. So E is a reflection set, and therefore is real analytic. This completes the proof.

Remark 1. – After proving that $\frac{\partial \lambda}{\partial z}$ is continuous on $\bar{W}^+$, we know that λ is the real part of a holomorphic function which is continuous on $\bar{W}^+$. By [8], or by theorem 3 in [15], we can then get directly the $\mathscr{C}^1$ smoothness of the g_j's on $\bar{W}^+$.

Remark 2. – Let Γ be a curve as in Proposition 5. Any holomorphic function which has maximum (therefore constant) modulus along Γ extends locally, accross $b\Omega$ to the complexification of Γ, by Schwarz reflection. From this, one can sketch an alternative approach to the case of maximum modulus sets of maximal dimension. Consider M as in Proposition 6. Pinčuk has constructed disks with (part of the) boundary in M [14]. A precise version of this construction, as in [9], shows that one can foliate M by curves (Γ_t) which bound analytic disks « from inside ». If f is a holomorphic function on Ω, with maximum modulus along M, each Γ_t is real analytic, and f extends locally accross $b\Omega$, to the complexification of Γ_t. This gives a CR extension of f accross $b\Omega$ to $\tilde{M}$ a $(n+1)$ dimensional manifold (foliated by holomorphic disks) with M as boundary. If one has obtained enough smoothness of $\tilde{M}$, and by applying results of [1] or [2], this is enough to guarantee that f has a holomorphic extension to a neighborhood of M. The real analyticity of M is then easy to get by studying the equation $\rho = 0$ in the real analytic hypersurface : $|f| = 1$.

III. CURVES WHICH BOUND DISKS FROM OUTSIDE Ω

III.1.

In this section we study curves Γ in the boundary of a real analytic strictly pseudoconvex domain which bound an analytic disc from outside the domain. Our main objective is to show that such curves are maximum modulus sets for the domain.

The main idea behind the construction of the appropriate holomorphic function is the following : At each point w of the boundary of a domain $\Omega \subset \mathbf{C}^n$ we wish to find a nonsingular complex hypersurface Σ_w passing through w, depending real analytically on w, which stays outside the domain Ω. Assume that Γ is transverse to the complex tangents, then the union

$$\Sigma = \bigcup_{w \in \Gamma} \Sigma_w$$

of the hypersurfaces Σ_w for $w \in \Gamma$ form a real hypersurface in $\mathbf{C}^n$, which is foliated by complex hypersurfaces, and hence is Levi flat. If the curve Γ is real analytic, then this hypersurface is real analytic, and hence it is locally biholomorphically equivalent to

$$\{(z_1, \ldots, z_n) \in \mathbf{C}^n \,|\, \mathrm{Im}\,(z_1) = 0\} = \{\mathbf{R} \times \mathbf{C}^{n-1} \subset \mathbf{C}^n\}.$$

The first component of this change of variables is then a holomorphic function on a neighborhood of a fixed boundary point of the domain whose imaginary part vanishes on Γ and is non-vanishing off Γ (since the hypersurfaces Ω_w stay outside Ω). Hence this first component would give the desired « maximum modulus function ».

In case the curve Γ, and hence the real hypersurface Σ is not real analytic, one can still look a « one-sided » holomorphic change of variables, (defined and holomorphic only on the side of Σ containing Ω) but still mapping Σ to $\mathbf{R} \times \mathbf{C}^{n-1}$. It is rather easy to see that such one-sided change of variables need not exist in general (see [4] for a detailed study, and see also [3]). However, in our case, we shall see that if Γ bounds an analytic disc from outside, then an appropriate one-sided change of variables does exist.

For a strictly pseudoconvex domain, one has considerable latitude in choosing a complex hypersurface Σ_w which contains $w \in \partial\Omega$ but which stays outside Ω. For our purposes, the correct choice of such hypersurfaces is given by the vanishing of the polarization of the real analytic defining function. This gives an antiholomorphic dependance of Σ_w, on w. We begin by recalling these concepts.

Let $\Omega \subset \mathbf{C}^n$ be a domain with real analytic boundary. Then there is a neighborhood U of $\partial\Omega$ and a real analytic defining function $\rho : U \to \mathbf{R}$, so that for all $z \in U$,

$$\nabla\rho(z) \neq 0$$

and

$$\Omega \cap U = \{z \in U \mid \rho(z) < 0\}.$$

DEFINITION. — A polarization *of the defining function* ρ *is a holomorphic function* $R : W \times \bar{W} \to \mathbf{C}$, *where* W *is an open neighborhood of* $\partial\Omega$ *and* $W \times \bar{W} \subset U \times \bar{U} \subset \mathbf{C}^n \times \mathbf{C}^n = \mathbf{C}^{2n}$ *such that for* $z \in \partial\Omega$,

$$R(z,\bar{z}) = \rho(z).$$

(At the risk of confusion, we use the notation $\bar{W}$, and $\bar{U}$ to denote complex conjugation. In the rest of the paper, unless otherwise noted, overlining continues to mean closure).

The polarization is uniquely determined by the defining function, since at any point $z \in \partial\Omega$,

$$\frac{\partial^{\alpha+\beta} R}{\partial z^{\alpha}\, \partial w^{\beta}}(z,\bar{z}) = \frac{\partial^{\alpha+\beta}\rho}{\partial z^{\alpha}\, \partial \bar{z}^{\beta}}(z).$$

For every $w \in W$ set

$$\Sigma_w = \{z \in W \mid R(z,\bar{w})=0\}.$$

Then Σ_w is an analytic variety in W. Moreover, since $\nabla\rho(w) \neq 0$, it follows that the variety Σ_w is nonsingular at the point w. Hence, after shrinking W if necessary, we may assume that for each $w \in W$, Σ_w is a nonsingular holomorphic hypersurface in W. The following proposition is well known:

LEMMA 4. – *If the domain Ω is strictly pseudoconvex at a point $w_0 \in \partial\Omega$, then there is a neighborhood U of w_0 so that for every $w \in \partial\Omega \cap U$,*

$$\Sigma_w \cap \bar{\Omega} \cap U = \{w\}.$$

Remark. – The family of hypersurfaces $\{\Sigma_w\}$ defined above seems to depend on the choice of the defining function for the real analytic domain Ω. However, it is not hard to see that the family of germs of the hypersurfaces $\{\Sigma_w\}$ is in fact independent of the choice of defining function, and hence is an invariant of the domain.

First, note that the real analytic defining function for a domain with real analytic boundary is uniquely determined up to multiplication by a non-vanishing real analytic function. Precisely, if ρ_1 and ρ_2 are two real analytic defining functions, defined on neighborhoods U_1 and U_2 of $\partial\Omega$, then there is an open neighborhood U_3 of $\partial\Omega$ in $U_1 \cap U_2$, and a non-vanishing real analytic function h defined on U_3 so that for all $z \in U_3$,

$$\rho_1(z) = h(z)\rho_2(z).$$

Next, two different defining functions ρ_1 and ρ_2 for the same domain with real analytic boundary lead to two different polarizations $R_1(z,w)$ and $R_2(z,w)$. However, it follows that on some neighborhood of $\partial\Omega \times \overline{\partial\Omega}$ there is a nonvanishing holomorphic function h so that

$$R_1(z,w) = h(z,w)R_2(z,w).$$

From this it follows that the family of germs of hypersurfaces $\{\Sigma_w\}$ defined by the polarization R_1 is the same as the family defined by R_2.

We now turn to certain preliminaries required in the construction of « maximum modulus functions ». Assume that $0 \in \partial\Omega$, and that we have fixed a defining function ρ for Ω with associated polarization R such that

$$\frac{\partial \rho}{\partial z_1}(0) = \frac{1}{2i},$$

and

$$\frac{\partial \rho}{\partial z_j}(0) = 0, \qquad \text{for} \qquad j = 2, \ldots, n.$$

This condition can always be achieved by a translation and a unitary change of variables. They amount to requiring that for z near the origin,

$$\rho(z) = \operatorname{Im}(z_1) + O(|z|^2).$$

LEMMA 5. – *There are neighborhoods W_1 of the origin 0_{n-1} in C^{n-1} and W_2 of the origin 0_n in $\mathbf{C}^n$ and a unique holomorphic function F: $W_1 \times W_2 \to \mathbf{C}$ so that*

$$F(0_{n-1}, 0_n) = 0;$$

$$R(F(z_2, \ldots, z_n; w_1, \ldots, w_n), z_2, \ldots, z_n; w_1, \ldots, w_n) = 0;$$

$$\frac{\partial F}{\partial w_1}(0_{n-1}, 0_n) = 1; \qquad \frac{\partial F}{\partial w_j}(0_{n-1}, 0_n) = 0 \quad \textit{if} \quad j > 1.$$

Moreover, for $w \in W_2$, if the holomorphic mapping $G_w : W_1 \to \mathbf{C}^n$ is given by

$$G_w(z_2, \ldots, z_n) = (F(z_2, \ldots, z_n; \bar{w}), z_2, \ldots, z_n),$$

then $G_w : W_1 \to \Sigma_w$ is a nonsingular parameterization of the holomorphic hypersurface Σ_w.

Proof. – The proposition is an immediate consequence of the chain rule and the holomorphic implicit function theorem applied to the equation

$$R(z_1, z_2, \ldots, z_n; w_1, \ldots, w_n) = 0$$

since we have

$$\frac{\partial R}{\partial z_1}(0,0) = \frac{\partial \rho}{\partial z_1}(0) = \frac{1}{2i} \neq 0.$$

Notation. — Let $\Gamma \subset \partial\Omega$ be a curve of class $\mathscr{C}^k$, $k \geqslant 1$, passing through 0, which is transverse to the complex tangent space at 0.

Let

$$\gamma = (\gamma_1, \dots, \gamma_n) : (-1, +1) \to W_2$$

be a $\mathscr{C}^k$ parameterization of Γ near 0. The transversality of the curve Γ means that

$$\gamma_1'(0) \neq 0 .$$

For each $x \in (-1, +1)$, we let $\Sigma_x = \Sigma_{\overline{\gamma(x)}}$ denote the germ of the complex hypersurface defined by $R(z, \gamma(x)) = 0$. We also let

$$\Sigma = \bigcup_{-1<x<+1} \Sigma_x .$$

As we now see, Σ is a smooth real hypersurface in $\mathbf{C}^n$. Indeed, define a mapping

$$\Psi : (-1, +1) \times W_1 \to \mathbf{C}^n$$

by setting

$$\Psi(x; z_2, \dots, z_n) = (F(z_2, \dots, z_n; \overline{\gamma(x)}), z_2, \dots, z_n) .$$

Clearly Ψ is again of class $\mathscr{C}^k$, and one has :

LEMMA 6. — *There is an open neighborhood W of 0 in $\mathbf{R} \times \mathbf{C}^{n-1}$ so that Ψ is a $\mathscr{C}^k$ diffeomorphism of W onto $\Psi(W)$. Also, there is an open neighborhood V of the origin in $\mathbf{C}^n$ so that $\Psi(W) = \Sigma \cap V$, and hence $\Sigma \cap V$ is a real hypersurface in $V \subset \mathbf{C}^n$ of class $\mathscr{C}^k$.*

Proof. — If we write

$$\Psi_1(x, z_2, \dots, z_n) = F(z_2, \dots, z_n; \overline{\gamma_1(x)}, \dots, \overline{\gamma_n(x)})$$

then this proposition is an immediate consequence of the fact that

$$\frac{\partial \Psi_1}{\partial s}(0; 0, \dots, 0) = \sum_{j=1}^{n} \frac{\partial F}{\partial w_j}(0;0)\, \overline{\gamma_j'(0)} = \frac{\partial F}{\partial w_1}(0;0)\, \overline{\gamma_j'(0)} \neq 0 .$$

Remark. — Since Σ is a smooth real hypersurface through the origin in $\mathbf{C}^n$, it divides sufficiently small neighborhoods V of the origin into two parts, say $V^{\pm}$. We can describe these two sides as follows. The curve $\gamma(s)$ is contained in Σ and hence the vector $\gamma'(0)$ is a vector at the origin in $\mathbf{C}^n$ which is tangent to Σ. Since the curve Γ was assumed to be transverse to the complex directions, the vectors $\pm\, \imath\gamma'(0)$ are not tangent to $b\Omega$, and hence are not tangent to Σ. Thus they point to opposite sides of the hypersurface Σ. If V is a small neighborhood of 0, we call $V^{\pm}$ the parts of V towards which the vectors $\pm\, \imath\gamma'(0)$ point.

The parameterization γ of the curve Γ is initially defined on an interval about the origin in $\mathbf{R}$, but we can always extend the mapping γ to a $\mathscr{C}^k$ function in a neighborhood of the origin in $\mathbf{C}$ so that

$$\text{(A)} \qquad \frac{\partial\gamma}{\partial\bar{z}}(0) = 0 .$$

Any such extension then gives an extension of the mapping Ψ to a neighborhood of the origin in $\mathbf{C} \times \mathbf{C}^{n-1}$ by setting

$$\Psi(x+\imath y, z_2, \ldots, z_n) = (F(z_2, \ldots, z_n, \overline{\gamma(z-\imath y)}, z_2, \ldots, z_n) .$$

We let

$$H^- = \{(z_1, z_2, \ldots, z_n) \in \mathbf{C}^n \mid \operatorname{Im}(z_1) < 0\} .$$

LEMMA 7. — *Under the hypothesis that equation* (A) *is satisfied, there is a neighborhood $\tilde{W}$ of the origin in $\mathbf{C} \times \mathbf{C}^{n-1} = \mathbf{C}^n$ so that the mapping Ψ maps $\tilde{W}$ diffeomorphically onto a neighborhood $\tilde{V}$ in $\mathbf{C}^n$, and Ψ maps $H^- \cap \tilde{W}$ diffeomorphically onto $\tilde{V}^-$, i.e., onto the side of the hypersurface Σ in $\tilde{V}$ towards which the vector $-\, \imath\gamma'(0)$ points.*

Proof. — Write $z_1 = x + \imath y$. Equation (A) says that the differential of the mapping Ψ from $\mathbf{C}^n$ to $\mathbf{C}^n$ at 0 is a complex linear map, and its Jacobian determinant is thus given by

$$\left|\frac{\partial\Psi_1}{\partial z_1}(0;0,\ldots,0)\right| = \left|\sum_{j=1}^{n}\frac{\partial F}{\partial w_j}(0;0)\,\overline{\gamma_j'(0)}\right| = \left|\frac{\partial F}{\partial w_1}(0;0)\,\overline{\gamma_1'(0)}\right| \neq 0 .$$

H^- is the side of $\mathbf{R} \times \mathbf{C}^{n-1}$ given by the direction of the vector $\dfrac{\partial}{\partial y}$, and the differential of Ψ at the origin carries this vector to $-\, \imath y'(0)$.

We are now in a position to state and prove the main result of this section. If $\Omega \subset \mathbf{C}^n$ is a domain, and $\Gamma \subset b\Omega$ is a curve, we say that Γ bounds an analytic disc from outside Ω if the parameterization γ of the curve Γ can be choosen so that

(1) $\gamma(s)$ extends smoothly to a holomorphic function in a rectangle in the upper half plane near the origin $R_\varepsilon = \{(s+\imath t) \,|\, |s| < 1, 0 < t < \varepsilon\}$.

(2) $\gamma(R_\varepsilon) \cap \Omega = \emptyset$.

So far, in this section, in order to simplify the statements, we have considered domains with global real analytic boundary. But it is clear that this is local theory, and that it is enough to have the real analyticity of the boundary in the neighborhood of some point. For the final statement, we wish to be precise:

PROPOSITION 7. — *Let $\Omega \subset \mathbf{C}^n$ be a domain and let $w_0 \in b\Omega$. Suppose Ω has real analytic boundary near w_0. Also suppose $b\Omega$ is strictly pseudoconvex at w_0, or more generally, suppose there is a neighborhood U of w_0 so that for $w \in \partial\Omega \cap U$*

$$\Sigma_w \cap \bar{\Omega} \cap U = \{w\}.$$

Suppose Γ is a curve in $b\Omega$ through w_0 class $\mathscr{C}^k$ which is transverse to the complex directions, and suppose that Γ bounds an analytic disc from outside the domain Ω. Then there is a neighborhood $\tilde{U} \subset U$ of w_0 and a function g of class $\mathscr{C}^k$ on $\bar{\Omega} \cap \tilde{U}$ and holomorphic on $\Omega \cap \tilde{U}$ so that

(1) $\mathrm{Im}\,(g(z)) = 0$ *if* $z \in \Gamma \cap \tilde{U}$;

(2) $\mathrm{Im}\,(g(z)) < 0$ *if* $z \in (\bar{\Omega} \cap \tilde{U})\backslash\Gamma$.

Proof. — Since γ extends to a holomorphic function on R_ε, it follows that

$$\frac{\partial\gamma}{\partial\bar{z}}(0) = 0,$$

i.e. equation (A) is satisfied. But then the equation

$$\Omega(z_1, z_2, \ldots, z_n) = (F(z_2, \ldots, z_n; \overline{\gamma(\bar{z})}), z_2, \ldots, z_n)$$

shows that Ψ is holomorphic on $\bar{R}_\varepsilon \times W_2$. (Here the bar over R_ε denotes complex conjugation.) Hence Ψ is a diffeomorphism in a neighborhood of the origin and which gives a biholomorphic mapping of $\bar{R}_\varepsilon \times W_2$ to $\tilde{V}^-$.

On the other hand, the condition that the curve Γ bounds a disc from outside the domain means (using transversality) that

$$\rho(\gamma_1(x+\imath y), \ldots, \gamma_n(x+\imath y)) \geqslant Cy$$

for $y > 0$. Taking the derivative with respect to t then shows that

$$Re\left(\imath \sum_{j=1}^{n} \frac{\partial\rho}{\partial z_1}(0)\, \gamma_j'(0)\right) > 0.$$

This means the vector $i\gamma'(0)$ points outside the domain, and hence the domain is locally contained in the image of $\bar{R}_\varepsilon \times \mathbf{C}^{n-1}$ under the biholomorphic mapping γ. The first component of the inverse of this biholomorphic mapping is then the required function g. This completes the proof.

III.2. Almost analytic version – a question.

If Γ is a smooth curve, say in the unit sphere, which is transverse to the complex tangent space, Γ may fail to be a maximum modulus set for holomorphic functions as shown in I and II. However, it is immediate to adapt the proof given above to show that such a curve is the maximum modulus set of an « almost analytic » function f ; that is to say, if Γ is $\mathscr{C}^\infty$, then $\bar{\partial} f$ vanishes to infinite order along Γ.

This is not without application. It provides a « geometric explanation » for pluriharmonic interpolation [6]. Thus suppose $|F| = 1$ along Γ. To interpolate $u \in \mathscr{C}(\Gamma)$ by a pluriharmonic function U, one interpolates $u \circ (F|_\Gamma)^{-1}$ on $F(\Gamma)$ (an arc in the unit circle) by a harmonic function on the disk. If F is holomorphic, $U = u \circ F$ is a solution to the problem. If F is only « almost analytic » one has to solve a $\partial\bar{\partial}$ problem, which leads to a compact operator (as in [6]). If one wants to get in the same way the generalization by Berndtson and Bruna [5] of the result of Bruna and Ortega, one is lead to the following :

QUESTION. – *Given Γ a « smooth » arc in the sphere (with NO hypothesis of transversality to the complex tangent) is it possible to find a function F defined on the closed ball so that : $|F| = 1$ along Γ, F is $1 - 1$ on Γ, $\bar{\partial}F$ vanishes to (high or) infinite order along Γ (i.e. $\bar{\partial}F = \mathscr{O}(\mathrm{dist}(\cdot, \Gamma)^K)$?*

If the curve Γ is complex tangential, which is the case « opposite » to the one considered so far, the answer is positive but the function F *cannot* be $\mathscr{C}^1$ along Γ. The same question could of course be asked for general strictly pseudoconvex hypersurfaces. Recall that pluriharmonic interpolation is a phenomenon which basically happens only on interpolation manifolds or along arcs ([17]).

IV. HOLOMORPHIC FOLIATIONS - EXAMPLES

Any holomorphic function with nonvanishing gradient on a domain gives raise to a foliation of the domain by the level sets of the function. In particular, a $\mathscr{C}^1$ holomorphic function has nonvanishing gradient near its maximum modulus set, and hence gives rise to a (local) foliation of the domain. The object of this section is to construct a function with maximum modulus along some curve or set, by first obtaining the associated foliation. Having failed to reach a satisfactory general result, we just illustrate this approach with two examples. As indicated in the Introduction, this sheds some light on the difficulty of characterizing those transverse curves which are maximum modulus sets.

IV.1.

Let us consider a curve Γ in S_2 the unit sphere of $\mathbf{C}^2$. (The dimension $n=2$ is used here for simplicity of notations only.) If Γ is real analytic then Γ is a maximum modulus set. Indeed this follows from Corollary 2.5 in [10] by writing Γ as the intersection of two real analytic, totally real manifolds of dimension 2. In § II we saw that Γ is also a maximum modulus set if Γ bounds a disk from outside. We wish here to give an **example** *of a curve* Γ, *transverse to the complex tangent, which does not bound a disk from outside but is a maximum modulus set.* Despite the fact that the curve is easy to define, the construction of a function which takes maximum modulus along Γ is far from explicit!

Construction of the curve.

Let h be a holomorphic function defined on the unit disk in $\mathbf{C}$, which extends smoothly to the closed unit disk, and so that $h(z) - (1-z)$ vanishes to infinite order at the point 1, but $h \not\equiv 1 - z$. For example, take $h(z) = (1-z) + \exp\left(\dfrac{-1}{(1-z)^{\frac{1}{2}}}\right)$, where $(1-z)^{\frac{1}{2}}$ denotes the square root of $(1-z)$ with positive real part.

For $\varepsilon > 0$ let Γ_ε be the set in S_2 defined by the equation $\bar{z}_2 = \varepsilon h(z_1)$.

The equation $\bar{z}_2 = 0$, considered as a system of two real equations on S_2, has rank 2 along the circle $\Gamma_0 = \{e^{i\theta},0)\}$. By the implicit function theorem, for ε small enough, Γ_ε is a smooth curve, close to Γ_0, that one can parametize by its projection on the z_1 plane which is a smooth simple closed curve, that we denote by γ_ε. In all the statements below, ε will have to be taken small enough, even when not explicitly mentioned. Notice that $(1,0) \in \Gamma_\varepsilon$.

Along Γ_ε, $\bar{z}_1$ and $\bar{z}_2$ match with holomorphic functions :

$$\bar{z}_2 = \varepsilon h(z_1)$$
$$\bar{z}_1 = \frac{1 - z_\varepsilon \bar{z}_2}{z_1} = \frac{1 - z_2 \varepsilon h(z_1)}{z_1}.$$

We have already seen in § I that this matching is crucial in the theory of maximum modulus sets on the sphere.

Claim 1. – In any neighborhood of (1,0) the curve Γ_ε does not bound an analytic disk from outside (or from inside) the unit ball.

Indeed such a disk (from outside) could be parametrized by z_1, i.e. by a map $z_1 \to (z_1, z_2(z_1))$ defined, in a neighborhood of 1, on the « right » side of γ_ε. The antiholomorphic function $\bar{z}_2(z_1)$ defined on one side of γ_ε would then coincide along γ_ε with the holomorphic function $\varepsilon h(z_1)$ defined on the other side. According to Proposition 1, γ_ε would therefore be real analytic at the point 1 (and coincide with the circle $\varepsilon^2|1-z_1|^2+|z_1|^2=1)$. Along γ_ε $\varepsilon^2|h(z_1)|^2 = 1 - |z_1|^2$. This would force $\varepsilon^2|h(z_1)|^2\Big|_{\gamma_\varepsilon}$ to be real analytic, and since $|h(z_1)|^2 - |1-z_1|^2$ vanishes to infinite order at the point 1, one would have, in a neighborhood of 1 : $|h(z_1)| = |1-z_1|$ if $z_1 \in \gamma_\varepsilon$. By Schwarz's reflection principle, one sees that this would imply $h(z_1) = (1-z_1)H(z_1)$ where H is holomorphic in some neighborhood of 1 and $H(1) = 1$. Therefore $h(z_1) \equiv (1-z_1)$, since the difference vanishes to infinite order at 1, a contradiction as desired.

It has been seen in the proof of Corollary 2 that since Γ_ε is not real analytic at (1,0), and since along $\Gamma_\varepsilon \bar{z}_1$ and $\bar{z}_2$ match with holomorphic functions defined in this ball, Γ_ε cannot bound a disk from inside.

Claim 2. – For ε small enough Γ_ε is a maximum modulus set. More precisely *there exists F a smooth function on $\bar{B}_\varepsilon$, which is holomorphic on B_2 and satisfies $|F| = 1$ on Γ_ε, $|F| < 1$ on $\bar{B}_\varepsilon - \Gamma_\varepsilon$.*

Proof of Claim 2. — The holomorphic vector field

$$Z = \frac{-\varepsilon z_1 h(z_1)}{1 - \varepsilon z_2 h(z_1)} \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}$$

is tangent to the sphere precisely along Γ_ε. For ε small enough Z defines a holomorphic foliation of the ball whose leaves will be the level sets of F. We now proceed to the construction of F.

The curve Γ_ε being parametrized by γ_ε, by $z_1 \mapsto (z_1, \theta(z_1))$, we can attach a differentiable disk to Γ_ε by taking a smooth extension of θ to Ω_ε the bounded component of $\mathbf{C} - \gamma_\varepsilon$, say the harmonic extension, that we still denote by θ. Then let $\bar{D}$ be the image of the map $z_1 \mapsto (z_1, \theta(z_1))$ defined on $\bar{\Omega}_\varepsilon$, this is a differentiable disk attached to Ω_ε.

LEMMA 8. — *There is a smooth (almost) complex structure J on $\bar{D}$ and a smooth holomorphic retract Π of $\bar{B}_2$ onto $\bar{D}$, endowed with the complex structure J, so that $\Pi(\bar{B}_2 - \Gamma_\varepsilon) \subset D = \bar{D} \cap B_2$ (provided that ε is small enough).*

a) Let us first not worry about holomorphicity and state a purely « differentiable fact ». Set $Z = X + iY$, so

$$X = \frac{\partial}{\partial x_2} + u\frac{\partial}{\partial x_1} + v\frac{\partial}{\partial y_1}, \qquad Y = \frac{\partial}{\partial y_2} - v\frac{\partial}{\partial x_1} + u\frac{\partial}{\partial y_1},$$

where

$$\frac{-\varepsilon z_1 h(z_1)}{1 - \varepsilon z_2 h(z_1)} = u + iv.$$

We claim that there is a smooth foliation of a neighborhood of $\bar{B}_2$, by two dimensional surfaces, so that if p is any point in $\bar{B}_2$ then at p, X and Y are tangent to the leaf of the foliation through p, and this (connected) leaf has a unique point of intersection with $\bar{D}$. We denote this point of intersection by $\pi(p)$. If $p \in \bar{B}_2 \backslash \Gamma_\varepsilon$, $\pi(p) \in D$, and $p \mapsto \pi(p)$ is a smooth map.

To see this, extend X smoothly to some neighborhood of $\bar{B}_2$ and denote the extension by $\tilde{X}$. We can change variables $(x_1, y_1, x_2, y_2) \xrightarrow{\Psi_0} (x'_1, y'_1, x'_2, y'_2)$ so that in the new variables $\tilde{X} = \dfrac{\partial}{\partial x'_2}\cdot$ If ε is small enough the change of variable is close to the identity, and $\Psi(\bar{B}_2)$ the image of this ball is still strictly convex. From the fact that $[X,Y]=0$ we get that in the new coordinates,

$$Y=\alpha_1\frac{\partial}{\partial x_1'}+\alpha_2\frac{\partial}{\partial y_1'}+\alpha_3\frac{\partial}{\partial x_2'}+\alpha_4\frac{\partial}{\partial y_2'}$$

where

$$\frac{\partial\alpha_j}{\partial x_2'}=0,\qquad j=1,\ldots,4.$$

The α_j are therefore functions of x_1', y_1' and y_2', and do not depend on x_2' (since the insertion of $\Psi_0(\bar{B}_2)$ with any line is connected). By subtracting $\alpha_2\frac{\partial}{\partial x_2'}$ from Y and changing variables in the (x_1',y_1',y_2') space, one finally reaches the following situation:

There is a smooth vector field $\tilde{Y}$ defined on some neighborhood of $\bar{B}_2$ $\left(\text{with } \tilde{Y}=Y-\alpha_2\frac{\partial}{\partial x_2'} \text{ on } \bar{B}_2\right)$ so that the span of $(X,\tilde{Y})$ is the same as the span of (X,Y) at every point of $\bar{B}_2$, and there is a change of variable Ψ defined on some neighborhood U of $\bar{B}_2$

$$\Psi:\ (x_1,y_2,x_2,y_2)\to(x_1'',y_1'',x_2'',y_2'')$$

so that: $\Psi(\bar{B}_2)$ is strictly convex, and in the variables $(x_1'',y_1'',x_2'',y_2'')$,

$$\tilde{X}=\frac{\partial}{\partial x_2''},\qquad \tilde{Y}=\frac{\partial}{\partial y_2''}.$$

As ε tends to 0 the neighborhood U does not shrink and Ψ is as close as needed to the identity map. Then $\Psi(\bar{D})$ is bounded by the curve $\Psi(\Gamma_\varepsilon)$, defined in $\Psi(S_2)$ by the fact that along this curve $\frac{\partial}{\partial x_2''}$ and $\frac{\partial}{\partial y_2''}$ are tangent to $\Psi(S_2)$. The projection of $\Psi(\Gamma_\varepsilon)$ in the (x_1'',y_1'') plane bounds a domain ω, which is the projection of $\Psi(B_2)$. And $\Psi(\bar{D})$ can be parametrized by

$$\omega\ni(x_1'',y_1'')\mapsto(x_1'',y_1'',X_1''(x_2'',y_1''),X_2''(x_1'',y_1'')),$$

where X_1'', X_2'' are smooth maps, close to 0. In the variables $(x_1'',y_1'',x_2'',y_2'')$ the map π is the map

$$(x_1'',y_1'',x_2'',y_2'')\mapsto(x_1'',y_1'',X_1''(x_1'',y_1''),X_2''(x_1'',y_1'')),$$

and the leaves of the foliation are given by fixing (x_1'',y_1'').

b) Now we are going to use the fact that Z is a holomorphic vector field. It therefore defines holomorphic local foliation of the ball, and in fact a global foliation as seen in *a*). The set of leaves of a holomorphic foliation has a complex structure (which locally can be identified with the complex structure on any holomorphic manifold transverse to the leaves), which makes the mapping, which associates to a point the leaf which contains it, holomorphic. By identifying a leaf with its intersection with $\bar{D}$, we have a complex structure J defined on $\bar{D}$. The complex structure J (« multiplication by i » in the tangent space) can be described in the following way. If V is a vector tangent to $\bar{D}$ at some point p. Then $J(V) = \Pi_*(iV)$ (the image under the differential of Π at p of iV considered as a tangent vector to $\mathbf{C}^2$ at p). This ends the proof of the Lemma.

Now we finish the proof of Claim 2.

Consider a differentiable « open » disk $\tilde{D}$ containing $\bar{D}$, with an (almost) complex structure $\tilde{J}$ extending J (in one complex variable there is no integrability condition). According to the uniformization theorem $(\tilde{D},\tilde{J})$ is biholomorphically equivalent to $\mathbf{C}$ or to the unit disk. At any rate (D,J) is biholomorphically equivalent to a smooth bounded domain in $\mathbf{C}$, and the correspondence is smooth up to the boundary (i.e. on $\bar{D}$). Composing with a conformal map onto the unit disk, one gets a smooth function f on $\bar{D}$ which is holomorphic on D with respect to the complex structure J and satisfies $|f| < 1$ on D, $|f| = 1$ on $\bar{D} - D = \Gamma_\varepsilon$. Finally $F = f \circ \Pi$ is the desired function.

IV.2.

Recall that if a smooth curve in S_2 is a maximum modulus curve, then the functions $\bar{z}_1$ and $\bar{z}_2$ must match with holomorphic functions. We now construct an **example** of *a smooth curve in* S_2 *along which* $\bar{z}_1$ *and* $\bar{z}_2$ *do match with holomorphic functions, but which is not a* $\mathscr{C}^2$ *maximum modulus set*. The explanation for this is that the local foliation obtained as in IV.1 fails to have the required properties (the leaves have too much « curvature »).

Let h be a non-zero holomorphic function defined on the unit disk in $\mathbf{C}$, so that h extends smoothly to the closed disk and vanishes to infinite order at the point 1. Let $\Gamma \subset S_2$ be the curve defined near $(1,0)$ by the equation $\bar{z}_2 = 2z_2 + h(z_1)$. Along Γ, $\bar{z}_1$ and $\bar{z}_2$ do match with holomorphic functions defined in the ball near $(1,0)$. For $\bar{z}_2$ this is clear, and one has

$$\bar{z}_1 = \frac{1 - z_2\bar{z}_2}{z_1} = \frac{1 - 2z_2^2 - z_2 h(z_1)}{z_1}.$$

However, we have

CLAIM. — *If W is any neighborhood of* $(1,0)$, *there is no function F, $\mathscr{C}^2$ on $\bar{B}_2 \cap W$, holomorphic on $B_2 \cap W$ so that $|F| = 1$ along $\Gamma \cap W$ and $|F| < 1$ on $B_2 \cap W$.*

Proof. — Assume, to get a contradiction, that there exists W and F as above. On Γ we would have

$$\frac{\dfrac{\partial F}{\partial z_2}}{\dfrac{\partial F}{\partial z_1}} = \frac{\bar{z}_2}{\bar{z}_1}.$$

Now we claim that if H is a $\mathscr{C}^1$ function on $\bar{B}_2 \cap W$, holomorphic on $B_2 \cap W$ and H is zero along Γ then $dH(1,0) = 0$. It is clear that $\dfrac{\partial H}{\partial z_1}(1,0) = 0$ since Γ is tangent to the circle $\{(e^{i\theta},0)\}$. If one had $\dfrac{\partial H}{\partial z_2}(1,0) \neq 0$, Γ would bound an analytic disk from inside, near $(1,0)$ by the implicit function theorem. Since along Γ, $\bar{z}_1$ and $\bar{z}_2$ match with holomorphic functions, this would force Γ to be real analytic (as seen in the proof of Corollary 2), which is not the case since Γ has contact to infinite order with the circle $\{(e^{i\theta},0)\}$, but is not the circle.

The conclusion is that since

$$\frac{\dfrac{\partial F}{\partial z_2}}{\dfrac{\partial F}{\partial z_1}} = \frac{z_1(2z_2 + h(z_1))}{1 - 2z_2^2 - z_2 h(z_1)} \qquad \text{on } \Gamma,$$

we have

$$\frac{\dfrac{\partial F}{\partial z_2}}{\dfrac{\partial F}{\partial z_1}} = \frac{z_1(2z_2 + h(z_1))}{1 - 2z_2^2 - z_2 h(z_1)} + o(|z_2| + |1 - z_1|).$$

For simplicity of the exposition, let us extend F to be $\mathscr{C}^2$ on some neighborhood of $(1,0)$. Consider the germ of curve $\lambda = (\lambda_1, \lambda_2)$ obtained by integrating the differential equation :

$$\frac{d\lambda}{dt} = \left(- \frac{\dfrac{\partial F}{\partial z_2}}{\dfrac{\partial F}{\partial z_1}} (\lambda(t)), 1 \right), \qquad \lambda(0) = (1,0).$$

Then λ is of class $\mathscr{C}^2$, $\lambda_2(t) = t$, $\lambda_1(t) = 1 + at^2 + o(t^2)$. From the differential equation one gets

$$2at + o(t) = -2t + o(t).$$

Hence $a = -1$. So $\lambda(t) = (1-t^2+o(t^2),t)$ which shows that for t small, $t \neq 0$, we have $|\lambda(t)| < 1$. The curve λ (restricted to small t's) is in the ball, where F is holomorphic. Then chain rule gives $\frac{d}{dt}(F \circ \lambda) = 0$, and so the function F is therefore constant along λ, which shows that there exists point p in $B_2 \cap w$ at which $|F(p)| = |F(1,0)| = 1$. This contradiction establishes the claim.

Note. — Since we quote the paper [16], the second author would like to take this opportunity to correct a statement in [16].

Page 64, in Proposition 1, the hypothesis in 2° should be, as in 3°, that $\frac{\partial g}{\partial \bar{w}_j}$ be bounded for $j = 1, \ldots, l$, and not only for $j = 1$ as stated.

The formula giving $\frac{\partial G_q}{\partial t_1}$, page 66 line 16, should be replaced by :

$$\frac{\partial \tilde{G}_q}{\partial s_j} = 2\frac{\partial G_q}{\partial \bar{w}_j} = \sum_k a_{jk} \frac{\partial G_q}{\partial t_k},$$

with $a_{jk} = \frac{\partial \varphi_k}{\partial s_j}$ if $j \neq k$, and $a_{jj} = -i + \frac{\partial \varphi_j}{\partial s_j}$.

By solving this system of l equations in the unknown $\left(\frac{\partial G_q}{\partial t_1}, \ldots, \frac{\partial G_q}{\partial t_t}\right)$, one gets :

$$\frac{\partial G_q}{\partial t_1} = \sum_j \alpha_j \cdot \left(\frac{\partial \tilde{G}_q}{\partial s_j} - 2\frac{\partial G_q}{\partial \bar{w}_j}\right),$$

and the $\alpha_j's$ depend (smoothly) on the functions $\frac{\partial \varphi_k}{\partial s_j}$.

The terms $\frac{\partial \tilde{G}_q}{\partial s_j}$ have to be integrated by parts. The terms $\frac{\partial G_q}{\partial \bar{w}_j}$ are continuous up to the edge, since one has $\frac{\partial G_q}{\partial \bar{w}_j} = -\int_{t_1}^{+\infty} \frac{\partial G_{q-1}}{\partial \bar{w}_j}$ (for $j > 1$, this is straightforward, by differentiation of the integral defining G_q).

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