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CLASSIFYING TOPOSES AND FOLIATIONS

by Ieke MOERDIJK

This paper makes no special claim to originality. Its sole purpose is to point out that in some circumstances, classifying toposes are more convenient to work with than classifying spaces.

Let G be a topological groupoid. I will focus on the case where G is etale, in the sense that the domain and codomain maps d_0 and $d_1 : G_1 \rightrightarrows G_0$ are etale (that is, are local homeomorphisms). A prime example is Haefliger's [14] groupoid Γ^q of germs of local diffeomorphisms of $\mathbf{R}^q$, which enters into the construction of the classifying space $B\Gamma^q$ for foliations of codimension q. But the holonomy groupoid [15] of any foliation is an example of an etale topological groupoid, as is any S-atlas [9]. For such a groupoid G, one can construct its classifying space BG by taking the geometric realization of the nerve $G_\bullet$ of G (Segal [29], [30], Haefliger [14]). Alternatively, one may consider the category $\Gamma(G)$ of sheaves on the simplicial space $G_\bullet$ (Deligne [6]). In topos theory, one studies a third object associated to G, namely its classifying topos $\mathcal{B}G$ which is simply the category of equivariant G-sheaves (see 2.1 below). I will show that these three approaches are compatible, for etale topological groupoids:

COMPARISON THEOREM. — *There are canonical maps $BG \to \mathcal{B}G \leftarrow \Gamma G$ which are weak equivalences (hence induce isomorphisms in homotopy and cohomology).*

The classifying topos $\mathcal{B}G$ is often easier to work with. For example, the maps $\mathcal{B}G \to \mathcal{B}H$ between classifying toposes can be described directly

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in terms of the groupoids G and H. And it is immediate that $\mathcal{B}G$ classifies principal G-bundles, in the sense that isomorphism classes of principal G-bundles on a space X correspond to isomorphism classes of maps of toposes $X \to \mathcal{B}G$. (The corresponding up-to-homotopy classification theorem relating $[X, BG]$ to concordance classes of principal G-bundles, proved in Haefliger [13], [14], Bott [4], follows from this.)

As in the statement of the comparison theorem, one can regard toposes as 'generalized spaces', so that spaces are a special kind of toposes. The classifying topos $\mathcal{B}G$ of an etale topological groupoid is precisely the *quotient* of the groupoid, by which I mean the quotient as a topos, not as a space. In this sense, the consideration of the classifying topos $\mathcal{B}G$ is related to the work on "quotients" of manifolds, such as that of Satake [33], Barre [3], Molino [25], Pradines-Wouafa-Kamga [27], Van Est [9], Tapia [34], and others.

In particular, van Est [9] proposes as such a "quotient-space" the notion of an S-atlas. An S-atlas is a special kind of etale topological groupoid. Van Est associates with any foliated manifold an S-atlas (essentially the holonomy groupoid of the foliation), and as a definition of the fundamental group of the "quotient space" of a foliated manifold he then proposes to take the fundamental groupoid of the associated S-atlas, as constructed in loc.cit.

In this paper, I will describe the fundamental groupoid of the topos $\mathcal{B}G$ in terms of the groupoid G, and show that in the special case where G is an S-atlas, this fundamental group coincides with the fundamental group of Van Est. It thus follows by the comparison theorem that Van Est's fundamental group also coincided with the fundamental group of the classifying space BG. As a particular case, this means that Van Est's fundamental group of a foliation agrees with Haefliger's, since the latter is defined as the (ordinary) fundamental group of the classifying space of the holomony groupoid of the foliation.

As a further illustration of the use of the classifying topos, I will give a proof of Segal's theorem [31] that Haefliger's classifying space $B\Gamma^q$ mentioned above is homotopy equivalent to the classifying space BM of the discrete monoid $M = M(\mathbb{R}^q)$ of smooth embeddings of $\mathbb{R}^q$ into itself, using a little topos theory. The method will of course be to apply the comparison theorem and show that $\mathcal{B}\Gamma^q$ is homotopy equivalent to the classifying topos $\mathcal{B}M$ of the monoid M. We shall see that much of Segal's work can be replaced by the construction of an explicit deformation retract (rather than a weak equivalence) at the level of toposes.

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1. Preliminaries.

1.1. *Topological groupoids.* — Let me fix some notation concerning topological groupoids. A topological groupoid G is a topological category in which each morphism is invertible. I write G_0 for the space of objects, G_1 for the space of morphisms, and $d_0, d_1 : G_1 \rightrightarrows G_0$, $s : G_0 \to G_1$, $m : G_1 \times_{G_0} G_1 \to G_1$ for the structure maps (d_0 is the domain, d_1 the codomain, $s(x)$ is the identity on $x \in G_0$, and m is the composition; as usual $m(f,g) = f \circ g$). G is called *etale* if d_0 and d_1 are etale maps (i.e. local homeomorphisms). All topological groupoids considered in this paper are etale. A *homomorphism* of topological groupoids $\varphi : G \to H$ is a continuous functor; i.e φ consists of two maps $\varphi_0 : G_0 \to H_0$ and $\varphi_1 : G_1 \to H_1$ commuting with the structure maps d_0, d_1, s, m for G and H. A homomorphism φ is called an *essential equivalence* if (i) the map $d_0\pi_2 : G_0 \times_{H_0} H_1 \to H_0$ is an open surjection (where the fibered product $G_0 \times_{H_0} H_1$ is along $d_1 : H_1 \to H_0$), and (ii) the diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi_1} & H_1 \\ {\scriptstyle (d_0,d_1)}\downarrow & & \downarrow{\scriptstyle (d_0,d_1)} \\ G_0 \times G_0 & \xrightarrow[\varphi_0 \times \varphi_0]{} & H_0 \times H_0 \end{array}$$

is cartesian. If G is a topological groupoid and $f : X \to G_0$ is a map, there is an induced topological groupoid f^*G with $f^*(G)_0 = X$ as space of objects, and $f^*(G)_1 = X \times_{G_0} G_1 \times_{G_0} X$ as space of morphisms; f extends to a homomorphism $f^*G \to G$, which is an essential equivalence if $X \to G_0$ is an open surjection.

1.2. *Principal G-bundles* (cf. Haefliger [13], [14], Bott [4]). — Let G be a topological groupoid. A G-bundle over a space B is a map $Y \xrightarrow{p} B$ equipped with a fiberwise (left) G-space structure; i.e. there is a projection $\pi : Y \to G_0$ and an action $\mu : G_1 \times_{G_0} Y \to Y$, $\mu(g,y) = g \cdot y$, such that for $g, h \in G_1$ and $y \in Y$, (i) $p(g \cdot y) = p(y)$, (ii) $\pi(g \cdot y) = d_1 g$, (iii) $s(\pi y) \cdot y = y$, (iv) $(g \circ h) \cdot y = g \cdot (h \cdot y)$ (whenever these make sense). Such a bundle is called *principal* if (a) $Y \to B$ is an open surjection, and (b) the map $G_1 \times_{G_0} Y \to Y \times_B Y$, $(g, y) \mapsto (g \cdot y, y)$ is an isomorphism. A map of principal G-bundles $\alpha : Y \to Y'$ over B is a map preserving all the structure involved (i.e. $p'\alpha y = py$, $\pi'\alpha y = \pi y$, $\alpha(g \cdot y) = g \cdot \alpha(y)$). Any map of principal G-bundles is an isomorphism. If $Y \xrightarrow{p} B$ is a principal G-bundle and G is etale, then p is also etale; i.e. Y corresponds to a sheaf on B.

Let $p : X \to B$ be an open surjection. A *cocycle* on X with values in G consists of maps $k : X \to G_0$ and $c : X \times_B X \to G_1$ such that $d_0 c(x, x') = kx'$, $d_1 c(x, x') = kx$, $c(x, x) = sk(x)$, $c(x, x') \circ c(x', x'') = c(x, x'')$. If (c, k) is a cocyle on $p : X \to B$, the quotient $G \otimes_c X$ obtained from $G_1 \times_{G_0} X$ by identifying (g, x) and $(g \cdot c(x, x'), x')$ is a principal G-bundle over B. Conversely, if $(Y \xrightarrow{p} B, \pi, \mu)$ is a principal G-bundle over B, any open surjection $q : X \twoheadrightarrow Y$ defines a cocycle $(\pi q, c(q \times q))$ on X with values in G, where $c : X \times_B X \to G_1$ is the map uniquely determined by the equation $c(x, x') \cdot q(x') = q(x)$. If G is etale, it suffices to consider cocycles on open surjections of the form $\coprod_\alpha U_\alpha \to B$, where $\{U_\alpha\}$ is a cover of B by open sets U_α. (In [14] and [4] only open surjections of this form $\coprod U_\alpha \to B$ are considered.)

Two principal G-bundles Y and Y' over B are said to be *concordant* if there exists a principal G-bundle Z over $B \times [0,1]$ such that $Y \cong i_0^* Z$, $Y' \cong i_1^* Z$ (where $i_0, i_1 : B \hookrightarrow B \times [0,1]$ are the inclusions). This divides the principal G-bundles over B into so-called concordance classes.

1.3. *The classifying space.* — Let G be an etale topological groupoid, and let $G_\bullet$ be its nerve; $G_\bullet$ is the simplicial space whose space of n-simplices G_n consists of composable n-tuples $(x_0 \xleftarrow{g_1} x_1 \leftarrow \dots \xleftarrow{g_n} x_n)$ of morphisms in the groupoid G. The classifying space BG of G is the geometric realization of $G_\bullet$. (This realization is of course defined in a way which takes the topology of the spaces $G_n (n \geq 0)$ into account, see Segal [29], [30], appendix.) It can be shown that BG classifies principal G-bundles, in the sense that for good spaces X (e.g. CW-complexes), the set of concordance classes of principal G-bundles is isomorphic to $[X, BG]$; cf. [13], [14], and also remark 2.5 below.

1.4. *Sheaves on simplicial spaces* (Deligne [6]). — Let $X_\bullet$ be a simplicial space. A sheaf E on $X_\bullet$ consists of a sequence $\{E^n\}_{n \geq 0}$ where E^n

is a sheaf on X_n, together with structure maps: for each map $\alpha : [n] \to [m]$ of finite ordered sets, with corresponding map $X(\alpha) : X_m \to X_n$, there is a morphism of sheaves $E(\alpha) : X(\alpha)^*(E^n) \to E^m$ over X_m, such that $E(id) = id$, $E(\alpha\beta) = E(\alpha) \circ E(\beta)$ hold. With the obvious notion of morphism, one defines a category $\Gamma(X_\bullet)$ of sheaves on $X_\bullet$. $\Gamma(X_\bullet)$ is a topos (cf 1.5), closely related to the geometric realization $| X_\bullet |$. In particular, $\Gamma G_\bullet$ is sometimes used in place of BG; cf. Deligne, loc.cit.

1.5. *Toposes.* — For a topological space X, the category $Sh(X)$ of sheaves on X is an example of a topos. In general, a topos is defined as a category which has exactness properties similar to such a category $Sh(X)$ (these properties are listed in SGA IV, p. 303, or Johnstone [19], p. 15). The other examples of toposes occurring in this paper are the category of sheaves on a simplicial space (as defined in 1.4), the category of equivariant sheaves (see 2.1) and the category $\mathcal{B}\mathbb{C} = Sets^{\mathbb{C}^{op}}$ of contravariant functors from a fixed small category $\mathbb{C}$ into the category *Sets* of all sets and functions (see 5.2).

If $f : X \to Y$ is a continuous map of spaces, then f induces two functors at the level of sheaves: an inverse image functor $f^* : Sh(Y) \to Sh(X)$ and a direct image functor $f_* : Sh(X) \to Sh(Y)$; here f^* is left adjoint to f_*, while moreover f^* is left exact. In general, a map of toposes $f : \mathcal{T} \to \mathcal{T}'$ is by definition a pair of functors $f^* : \mathcal{T}' \to \mathcal{T}$ and $f_* : \mathcal{T} \to \mathcal{T}'$ such that f^* is left adjoint to f_* and f^* is left exact. A map of toposes $Sh(X) \to Sh(Y)$ is necessarily induced by a uniquely determined continuous map for X into Y (if Y is Hausdorff). Thus, continuous maps of spaces $X \to Y$ are really the same thing as maps of toposes $Sh(X) \to Sh(Y)$. Therefore, we will generally *identify* a space X with the topos $Sh(X)$ of sheaves on X, and denote the latter topos simply by X.

For every space X, there is a unique map γ from X into the one-point space 1. If we identify the space 1 with the topos of sheaves on the 1-point space, i.e with the category of *Sets*, the same is true for toposes: any topos $\mathcal{T}$ admits an essentially unique map $\gamma : \mathcal{T} \to 1$. (Its inverse image part γ^* is the constant sheaf functor, while its direct image part γ_* is the global sections functor.)

For an abelian group A in a topos $\mathcal{T}$, one defines the cohomology groups $H^n(\mathcal{T}, A)$ (see SGA IV, part II, exp. V [1]). Moreover, if $\mathcal{T}$ is locally connected and p is a point of $\mathcal{T}$, one defines the homotopy groups $\pi_n(\mathcal{T}, p)$, for $n \geq 0$, see Artin-Mazur [2]. In particular, the fundamental group $\pi_1(\mathcal{T},p)$ can be described as follows([2], [12]). A *covering space* of $\mathcal{T}$ is an object E of $\mathcal{T}$ such that there exists a set S (the fiber of the covering) and a map $U \twoheadrightarrow 1$ in $\mathcal{T}$ for which $E \times U \cong \gamma^*(S) \times U$ by an isomorphism over U. One also says that E is a *locally constant object* of $\mathcal{T}$. The full subcategory of $\mathcal{T}$ consisting of sums of covering spaces is equivalent to the category of sets equipped with an action by a uniquely determined group; this group is $\pi_1(\mathcal{T},p)$. (It is really a pro-group; $\pi_1(\mathcal{T},p)$ is an ordinary group if $\mathcal{T}$ is locally simply connected, as all toposes considered here are.) The fundamental group $\pi_1(\mathcal{T},p)$ can be characterized by a universal property, [24] p.297. If X is a good space, the homotopy groups of X viewed as a topos coincide with the usual homotopy groups, [2] p. 129. There is a Whitehead theorem for connected locally connected pointed toposes: a morphism $f : (\mathcal{T},p) \to (\mathcal{T}',p')$ induces isomorphisms in all homotopy groups iff f induces an isomorphism $\pi_1(\mathcal{T},p) \xrightarrow{\sim} \pi_1(\mathcal{T},p')$ as well as isomorphisms $H^n(\mathcal{T}',A) \xrightarrow{\sim} H^n(\mathcal{T},f^*A)$, $n \geq 0$, for all locally constant coefficients A over $\mathcal{T}'$; see [2], p. 36. A map with these properties is called a *weak equivalence* of toposes.

2. The classifying topos.

2.1. *Definition of $\mathcal{B}G$.* — Let G be an etale topological groupoid. A *G-sheaf* (or etale G-space) is an etale map $p : E \to G_0$ equipped with a right action $\mu : E \times_{G_0} G_1 \to E$ (i.e. writing $e \cdot g = \mu(e,g)$, we have $p(e \cdot g) = d_0 g$, $e \cdot (spe) = e$, $(e \cdot g) \cdot h = e \cdot (g \circ h)$). A map of G-sheaves $(E,p,\mu) \to (E',p',\mu')$ is a map $u : E \to E'$ such that $p'u(e) = p(e)$ and $u(e \cdot g) = u(e) \cdot g$. The category of G-sheaves is denoted by $\mathcal{B}G$. It follows from Giraud's theorem (1.5) that $\mathcal{B}G$ is a topos, called the classifying topos of G. (It is extensively discussed in [23], [24].)

2.2. *Functoriality.* — If $\varphi : G \to H$ is a homomorphism of groupoids (see 1.1), an H-sheaf $E = (E,p,\mu)$ induces a G-sheaf $\varphi^*(E)$ by pullback along $\varphi_0 : G_0 \to H_0$ in the obvious way. The functor $\varphi^* : \mathcal{B}H \to \mathcal{B}G$ is part of a morphism of toposes $\mathcal{B}\varphi : \mathcal{B}G \to \mathcal{B}H$, making $\mathcal{B}$ into a functor. If $\varphi : G \to H$ is an essential equivalence, then $\varphi^* : \mathcal{B}H \to \mathcal{B}G$ is an equivalence of categories, i.e. $\mathcal{B}\varphi$ is an equivalence of toposes $\mathcal{B}G \xrightarrow{\sim} \mathcal{B}H$ ([23],p. 656).

2.3. *Maps between classifying toposes.* — In [23] it is shown that the category of toposes can be obtained from the category of (essentially) topological groupoids, by a calculus of fractions in the sense of [10]. For the special case of etale topological groupoids, this has the following meaning: Given etale topological groupoids G and H and a morphism of toposes $f : \mathcal{B}G \to \mathcal{B}H$, there exists a diagram $G \xleftarrow{\varepsilon} K \xrightarrow{\varphi} H$ of homomorphisms of etale topological groupoids such that ε is an essential equivalence (as in 1.1) and $f \circ \mathcal{B}\varepsilon \cong \mathcal{B}\varphi$. Conversely, given such a diagram $G \xleftarrow{\varepsilon} K \xrightarrow{\varphi} H$, it follows that the map $\mathcal{B}\varepsilon : \mathcal{B}K \to \mathcal{B}G$ is an equivalence of toposes (2.2), so one obtains a geometric morphism $\mathcal{B}\varphi \circ (\mathcal{B}\varepsilon)^{-1} : \mathcal{B}G \to \mathcal{B}H$. Two such diagrams $G \xleftarrow{\varepsilon} K \xrightarrow{\varphi} H$ and $G \xleftarrow{\varepsilon'} K' \xrightarrow{\varphi'} H$ induce the same (up to isomorphism) morphism of toposes $\mathcal{B}G \to \mathcal{B}H$ (and are then called *equivalent diagrams*) iff there exists a diagram

$$\begin{tikzcd} & K \arrow[dr, "\varepsilon"] \arrow[drr, "\varphi"] & & \\ K'' \arrow[ur, "\delta"] \arrow[dr, "\delta'"'] & & G & H \\ & K \arrow[ur, "\varepsilon'"'] \arrow[urr, "\varphi'"'] & & \end{tikzcd}$$

where δ, δ' are essential equivalences and $\varphi\delta \cong \varphi'\delta'$, $\varepsilon\delta \cong \varepsilon'\delta'$. (In the context of etale topological groupoids, it actually suffices to consider essential equivalences $\varepsilon(\delta$, etc.) for which $\varepsilon_0 : K_0 \to G_0$ is an etale surjection.)

So for example, if G and H are S-atlases [9], a morphism $f : G \to H$ of atlases corresponds exactly to a morphism of toposes $\mathcal{B}G \to \mathcal{B}H$, provided f has what Van Est [9] calls trivial isotropy.

2.4. COROLLARY. — *$\mathcal{B}G$ classifies principal G-bundles; (i.e.) for any space (or topos) X, there is an equivalence between principal G-bundles over X and morphisms of toposes $X \to \mathcal{B}G$.*

Proof. — Recall that we identify the space X with the topos of sheaves on X (so it makes sense to speak of maps from the space X into some topos, in casu $\mathcal{B}G$). Consider the groupoid $\tilde{X}$ given by $\tilde{X}_0 = X = \tilde{X}_1$, and the domain and codomain maps $d_0, d_1 : X \rightrightarrows X$ both the identity. Then an etale $\tilde{X}$-space is obviously the same thing as a sheaf on X, so (following our convention of writing X for the topos of sheaves on X), we have an equality of toposes $X = \mathcal{B}\tilde{X}$. By 2.3, morphism of toposes $X = \mathcal{B}\tilde{X} \to \mathcal{B}G$ correspond to equivalence classes of diagrams $\tilde{X} \xleftarrow{\varepsilon} K \xrightarrow{\varphi} G$. But ε being an essential equivalence, the map of topological spaces $K_0 \to \tilde{X}_0 = X$ must be an open surjection (in fact it can be taken to be an etale surjection, but this is irrelevant here), and $K_1 \cong K_0 \times_X K_0$. Thus φ corresponds to a cocycle on $K_0 \twoheadrightarrow X$, and hence to a principal G-bundle over the space X. It is now a routine matter to conclude from 2.3 and 1.2 that this sets up an equivalence. Essentially the same argument applies to the case where X is a topos. (For a different argument using Giraud-stacks, see Bunge [5].)

2.5. Remark. — By obstruction theory, it follows from 2.4 and the comparison theorem 3.1 below that for a CW-complex X, homotopy classes of maps $X \to BG$ (cf. 1.3) correspond to concordance classes of principal G-bundles over X. (This is the classification result referred to in 1.3.)

3. The comparison theorem.

Let G be an etale topological groupoid, with associated classifying space BG (1.3), classifying topos $\mathcal{B}G$ (2.1), and category of sheaves on the nerve $\Gamma G_\bullet$ (1.4). The purpose of this section is to prove the theorem stated in the introduction :

3.1. THEOREM. — *There are canonical maps $p : BG \to \mathcal{B}G$ and $q : \Gamma G_\bullet \to \mathcal{B}G$ which are weak equivalences.*

I shall begin with the second morphism of toposes $q : \Gamma G_\bullet \to \mathcal{B}G$. Its inverse image p^* is essentially the inclusion-functor, since $\mathcal{B}G$ can be identified with the category of sheaves E on $G_\bullet$ as in 1.4, for which all structure maps $E(\alpha) : G(\alpha)^*(E^n) \to E^m$ (for $\alpha : [n] \to [m]$, $G(\alpha) : G_m \to G_n$) are isomorphisms.

3.2. *Categories and simplicial sets.* — Let X be a simpicial set. Applying the Grothendieck construction to X yields a category $E(X)$ with as objects pairs (n, x), where $x \in X_n$, and as morphisms $(n, x) \to (m, y)$ those morphisms $\alpha : [n] \to [m]$ of Δ for which $\alpha^* y = x$. The well-known homotopy theoretic equivalence between categories and simplicial sets comes about by weak equivalences $NE(X) \to X$ and $EN(\mathbb{C}) \to \mathbb{C}$, for any simplicial set X and any small category $\mathbb{C}$, where N denotes the nerve (see Illusie [16], p. 20-22).

These results have a straightforward extension to toposes (as is apparent from the discussion of the category $\mathcal{T}^{\Delta^{\mathrm{op}}}$ of simplicial objects in a topos $\mathcal{T}$ given by Joyal [20]; see also Jardine [17], [18]). If X is a simplicial object in a topos $\mathcal{T}$ and $\mathbb{C}$ is a category object in $\mathcal{T}$, there are weak equivalences of simplicial objects of $\mathcal{T}$, $NEX \to X$ and $EN\mathbb{C} \to \mathbb{C}$. In particular, there are weak equivalences of toposes $\mathcal{T}^{\Delta^{\mathrm{op}}}/NEX \xrightarrow{\sim} \mathcal{T}^{\Delta^{\mathrm{op}}}/X$

and $\mathcal{T}^{EN(\mathbb{C})^{op}} \xrightarrow{\sim} \mathcal{T}^{\mathbb{C}^{op}}$. Since $\mathcal{T}^{EN(\mathbb{C})^{op}} \cong \mathcal{T}^{\Delta^{op}}/N\mathbb{C}$, this gives a weak equivalence $\mathcal{T}^{\Delta^{op}}/N\mathbb{C} \to \mathcal{T}^{\mathbb{C}^{op}}$.

I also recall that for a small category $\mathbb{C}, N(\mathbb{C})$ and $N(\mathbb{C}^{op})$ are weakly equivalent simplicial sets (by weak equivalences $N(\mathbb{C}) \leftarrow X \to N(\mathbb{C}^{op})$ for a suitable simplicial set X, see Quillen [28], p.94). The same construction yields that for a category object $\mathbb{C}$ in a topos $\mathcal{T}$, there are weak equivalences of toposes over $\mathcal{T}$, namely $\mathcal{T}^{\Delta^{op}}/N(\mathbb{C}) \xleftarrow{\sim} \mathcal{T}^{\Delta^{op}}/X \xrightarrow{\sim} \mathcal{T}^{\Delta^{op}}/N(\mathbb{C}^{op})$.

3.3. *Simplicial toposes.* — Let $\mathcal{T}_\bullet$ be a simplicial topos; so for each $n \geq 0$ there is a topos $\mathcal{T}_n$, for $\alpha : [n] \to [m]$ in Δ there is a map of toposes $\mathcal{T}(\alpha) : \mathcal{T}_m \to \mathcal{T}_n$, and the simplicial identities hold up to coherent isomorphism. Recall from Saint-Donat [32] that $\Gamma(\mathcal{T}_\bullet)$ is the following category: an object X of $\Gamma(\mathcal{T}_\bullet)$ consists of a sequence $X^n, n \geq 0$, where X^n is an object of $\mathcal{T}_n$, together with maps $X(\alpha) : \mathcal{T}(\alpha)^* X^n \to X^m$ in $\mathcal{T}_m$ for $\alpha : [n] \to [m]$, such that the identities $X(\alpha)X(\beta) = X(\alpha\beta)$ and $X(id) = id$ hold (modulo the coherent isomorphisms just mentioned). $\Gamma(\mathcal{T}_\bullet)$ is a topos.

If $X_\bullet$ is a simplicial object of a topos $\mathcal{T}$, one can form the simplicial topos $\mathcal{T}/X_\bullet = (\mathcal{T}/X_0 \leftrightarrows \mathcal{T}/X_1 \overleftarrow{\leftarrows} \ldots)$. Then $\Gamma(\mathcal{T}/X_\bullet)$ is precisely the same as the category $\mathcal{T}^{E(X)}$ of internal covariant functors from $E(X)$ into $\mathcal{T}$.

In particular, if T is an object of $\mathcal{T}$, one can form the trivial category $C(T)$ with $C_0(T) = T$ as object of objects, $C_1(T) = T \times T$ as object of morphisms, and the two projections as domain and codomain maps. Then $NC(T) = T_\bullet$ is the trivial simplicial set with $T_n = T \times \ldots \times T$ ($n+1$-times). So one obtains an equivalence of categories $\Gamma(\mathcal{T}/T_\bullet) \simeq \mathcal{T}^{E(T_\bullet)}$, and weak equivalences $\mathcal{T}^{E(T_\bullet)} \xleftarrow{\sim} \cdot \xrightarrow{\sim} \mathcal{T}^{E(T_\bullet)^{op}}$ and $\mathcal{T}^{E(T_\bullet)^{op}} = \mathcal{T}^{ENC(T)^{op}} \xrightarrow{\sim} \mathcal{T}^{C(T)^{op}}$, by 3.2. If $T \to 1$ is epi, then $\mathcal{T}^{C(T)^{op}}$ is equivalent to $\mathcal{T}$ (by an equivalence of toposes, see [22], or [23], p.285). Thus we obtain :

3.4. PROPOSITION. — *For any topos $\mathcal{T}$ and any object T of $\mathcal{T}$ with $T \to 1$ epi, there is a canonical weak equivalence $\Gamma(\mathcal{T}/T_\bullet) \xrightarrow{\sim} \mathcal{T}$.*

3.5. *The map $p_G : \Gamma G_\bullet \to \mathcal{B}G$ is a weak equivalence, for an etale topological groupoid G.* — This is just a special case of 3.4. Indeed, consider the object

$$T = (G_1, d_0 : G_1 \to G_0, m)$$

of $\mathcal{B}G$ (which is, incidentally, the 'underlying space' of the universal principal G-bundle). Now consider the category $\mathcal{B}G/T$, whose objects are pairs (E, f), where E is an object of $\mathcal{B}G$, and f is a map $E \to T$ of equivariant sheaves. We claim that there is an equivalence of categories

$$\mathcal{B}G/T \overset{\sim}{\leftrightarrows} Sh(G_0).$$

Indeed, for a sheaf S on G_0, i.e. an etale space $S \to G_0$, define an object $(\varphi(S), f_S : \varphi(S) \to T)$ of $\mathcal{B}G/T$ as follows: $\varphi(S)$ is the G-sheaf $d_0\pi_2 : S \times_{G_0} G_1 \to G_0$ on which G acts from the right by composition, while $f_S : \varphi(S) \to T$ is the second projection. This defines a functor $\varphi : Sh(G_0) \to \mathcal{B}G/T$. Conversely, define $\psi : \mathcal{B}G/T \to Sh(G_0)$ by the pullback

$$\begin{tikzcd} \psi(E) \arrow[r] \arrow[d] & E \arrow[d, "f"] \\ G_0 \arrow[r, "s"] & G_1 \end{tikzcd}$$

One readily verifies that φ and ψ are mutually inverse (up to natural isomorphism). Similarly,

$$\mathcal{B}G/T_{n-1} = \mathcal{B}G/T \underbrace{\times \ldots \times}_{-n-} T \cong G_{n-1} = G_1 \underbrace{\times_{G_0} \ldots \times_{G_0}}_{-n-1-} G_1.$$

So $\Gamma(\mathcal{B}G/T_\bullet) \cong \Gamma(G_\bullet)$, and as a special case of 3.4 one obtains a weak equivalence $\Gamma(G_\bullet) \to \mathcal{B}G$. This shows one half of theorem 3.1.

It remains to prove the existence of a weak equivalence $BG \to \mathcal{B}G$. Rather than giving an ad hoc proof, I will derive this from some general properties of 'shift' or decalage.

3.6. *Decalage* (Illusie [16], Duskin [8]). — Let $\mathbb{C}$ be a topological category. Define $\mathrm{Dec}(\mathbb{C})$ to be the topological category whose objects are arrows of $\mathbb{C}$, and whose morphisms from α to β are commuting triangles

$$\begin{tikzcd} \cdot \arrow[rr] \arrow[dr, "\alpha"'] & & \cdot \arrow[dl, "\beta"] \\ & \cdot & \end{tikzcd}$$

(so there are morphisms from α to β only in case α and β have the same codomain). The domain map defines a functor Dec $(\mathbb{C}) \to \mathbb{C}$. The codomain map defines a functor Dec $(\mathbb{C}) \to \mathbb{C}_0$ (where the space $\mathbb{C}_0$ of objects is viewed as a topological category whose only morphisms are identities). d_1 : $\mathrm{Dec}(\mathbb{C}) \to \mathbb{C}_0$ has a splitting $s : \mathbb{C}_0 \to \mathrm{Dec}(\mathbb{C})$ for which there is a natural transformation $id_{\mathrm{Dec}(\mathbb{C})} \to sd_1$, so (cf. [29]) d_1 induces a homotopy equivalence between the nerve of $\mathrm{Dec}(\mathbb{C})$ and the constant simplicial space $\mathbb{C}_0$, and hence by realizing ([30], appendix) a homotopy equivalence $B\mathrm{Dec}(\mathbb{C}) \to \mathbb{C}_0$. Notice that $N\mathrm{Dec}(\mathbb{C})$ is obtained from $N(\mathbb{C})$ by shifting the index one place and omitting the last face. Shifting n times gives the nerve of $\mathrm{Dec}^n(\mathbb{C})$, which is the category whose objects are sequences $(\cdot \xleftarrow{\alpha_0} \cdot \leftarrow \ldots \xleftarrow{\alpha_n} \cdot)$; there are maps from one such sequence $(\alpha_0, \ldots, \alpha_n)$ to another $(\beta_0, \ldots, \beta_n)$ only if $\alpha_0 = \beta_0, \ldots, \alpha_{n-1} = \beta_{n-1}$, and they are given by maps $\alpha_n \to \beta_n$ in $\mathrm{Dec}(\mathbb{C})$. Chopping off the last element of a sequence defines a similar homotopy equivalence $B\ \mathrm{Dec}^n(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}_{n-1} = \mathbb{C}_1 \times_{\mathbb{C}_0} \ldots \times_{\mathbb{C}_0} \mathbb{C}_1$.

3.7. *The weak equivalence $BG \to \mathcal{B}G$.* — We now prove the remaining half of theorem 3.1. Let G be an etale topological groupoid. Let $T \in \mathcal{B}G$ be the universal principal G-bundle as in 3.4, so that $\mathcal{B}G/T \cong$ (the category of sheaves on) G_0. The functor $d_0 : \mathrm{Dec}(G) \to G$ defines an etale map $Bd_0 : B\mathrm{Dec}(G) \to BG$, and $d_1 : \mathrm{Dec}(G) \to G_0$, $m : G_1 \times_{G_0} \mathrm{Dec}(G) \to G_0$ give $B\mathrm{Dec}(G)$ the structure of a principal G-bundle over BG. This bundle is classified by a map of toposes $f : BG \to \mathcal{B}G$, so that one gets a fibered product diagram

$$\begin{tikzcd}
B\,\mathrm{Dec}\,G \arrow[r, "d_1"] \arrow[d] & G_0 = \mathcal{B}G/T \arrow[d] \\
BG \arrow[r, "f"] & \mathcal{B}G
\end{tikzcd}$$

But for a groupoid G, $N\mathrm{Dec}(G) \times_{NG} N\mathrm{Dec}(G) \times \ldots \times_{NG} N\mathrm{Dec}(G) \cong N\mathrm{Dec}^n(G)$, so the simplicial space $\mathrm{Dec}_\bullet(G)$ augmented over G gives a hypercover $B\ \mathrm{Dec}_\bullet(G)$ of BG, and one obtains a diagram

$$\begin{tikzcd}
BG & B\,\mathrm{Dec}\,G \arrow[l] \arrow[d] & B\,\mathrm{Dec}^2 G \arrow[l, shift left] \arrow[l, shift right] \arrow[d] & \ldots \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \\
\mathcal{B}G \arrow[u, leftarrow] \arrow[d, equal] & G_0 \arrow[l] \arrow[d, equal] & G_1 \arrow[l, shift left] \arrow[l, shift right] \arrow[d, equal] & \ldots \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \\
\mathcal{B}G & \mathcal{B}G/T \arrow[l] & \mathcal{B}G/T \times T \arrow[l, shift left] \arrow[l, shift right] & \ldots \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}$$

Since $B\mathrm{Dec}^n G \to G_{n-1}$ is a weak equivalence by 3.6, the result follows from the following lemma (which we only use, in fact, for the case $X' = f^*X$).

3.8. LEMMA. — *Let $f : \mathcal{T}' \to \mathcal{T}$ be a morphism of toposes, and let $X_\bullet$ and $X'_\bullet$ be hypercovers of $\mathcal{T}$ and $\mathcal{T}'$. Let $X'_\bullet \to f^*(X_\bullet)$ be a morphism of hypercovers, and suppose for each $n \geq 0$ the composition $f_n : \mathcal{T}'/X_n \to \mathcal{T}'/f^*X_n \to \mathcal{T}/X_n$ is a weak equivalence. Then $f : \mathcal{T} \to \mathcal{T}'$ is a weak equivalence.*

Proof. — For a hypercover $X_\bullet$ of $\mathcal{T}$ and an abelian group A in $\mathcal{T}$, there is a spectral sequence $H^p(\mathcal{T}/X_q, A) \Rightarrow H^{p+q}(\mathcal{T}, A)$. So the hypotheses clearly imply that $f : \mathcal{T}' \to \mathcal{T}$ induces isomorphisms in cohomology. Moreover, a covering space of $\mathcal{T}^{\Delta^{\mathrm{op}}}/X_\bullet$ with a set S as fiber consists of a system of covering spaces E_n of $\mathcal{T}/X_n$ with S as fiber (plus some structural maps between these). So if f_n is a weak equivalence for each n, it follows that the corresponding map of toposes $f_\bullet : \mathcal{T}'^{\Delta^{\mathrm{op}}}/X'_\bullet \to \mathcal{T}^{\Delta^{\mathrm{op}}}/X_\bullet$ induces an equivalence at the level of covering spaces. Since $X_\bullet$ and $X'_\bullet$ are hypercovers, so do the vertical maps in the diagram below.

$$\begin{array}{ccc}
\mathcal{T}'^{\Delta^{\mathrm{op}}}/X'_\bullet & \longrightarrow & \mathcal{T}^{\Delta^{\mathrm{op}}}/X_\bullet \\
\downarrow & & \downarrow \\
\mathcal{T}' & \longrightarrow & \mathcal{T}
\end{array}$$

Consequently, $f : \mathcal{T}' \to \mathcal{T}$ induces an equivalence at the level of covering spaces, and hence an isomorphism in π_1. By the Whitehead theorem (1.5), $f : \mathcal{T}' \to \mathcal{T}$ is a weak equivalence.

This completes the proof of theorem 3.1.

4. The fundamental group.

Let G be a fixed etale topological groupoid, with classifying topos $\mathcal{B}G$.

4.1 *Connectedness assumptions.* — One can define an equivalence relation on the space G_0 of objects of G, by stating that two points $x, y \in G_0$ are equivalent iff there exists a point $g \in G_1$ with $d_0 g = x$ and $d_1 g = y$.

If the quotient space obtained from G_0 by this equivalence relation is a connected space, we call G a *connected* topological groupoid. Then G is connected iff $\mathcal{B}G$ is a connected topos. To describe the fundamental group of $\mathcal{B}G$, it suffices to consider the connected components of $\mathcal{B}G$ separately, and therefore we may as well assume that G is connected. We will also assume that G_0 is locally connected (this implies that $\mathcal{B}G$ is a locally connected topos so that the homotopy groups [2] are defined), and that G_0 has a cover $\{U_\alpha\}$ by connected simply connected open sets $U_\alpha \subseteq G_0$. In the applications [9], [15], G_0 is generally a manifold, so these conditions are certainly satisfied.

To describe the fundamental group of $\mathcal{B}G$, we shall need to identify the covering spaces of $\mathcal{B}G$ (cf. 1.5).

4.2. LEMMA. — *Let G be an etale topological groupoid, let E be a G-sheaf, and let S be a set. Then E is a covering space of $\mathcal{B}G$ with fiber S iff E is a covering space of G_0 (i.e. a locally constant sheaf) with fiber S.*

Proof. — "Only if" is clear, by simply forgetting the action of G. Conversely, suppose E is locally constant as a sheaf on G_0, with fiber S so that there is an etale surjection $\tau : V \to G_0$ and an isomorphism $\alpha : V \times_{G_0} E \xrightarrow{\sim} V \times S$ over V, say $\alpha(v,e) = (v, \bar{\alpha}(v,e))$. The problem is that this isomorphism doesn't take the action of G on E into account. But consider the space $W = V \times_{G_0} G_1 = \{(v,g) | v \in V, g \in G_1, d_1 g = \tau v\}$. Then W has the structure of a G-sheaf, with projection $p : W \to G_0$ defined by $p(v,g) = d_0 g$, and action $\mu : W \times_{G_0} G_1 \to W$ defined by $\mu(v,g,h) = (v, gh)$. The map $\beta : W \times_{G_0} E \to W \times S$ defined by $\beta(v,g,e) = (v, g, \bar{\alpha}(v, e \cdot g^{-1}))$ is an isomorphism of G-sheaves from $W \times E$ (product in $\mathcal{B}G$) into $W \times S$. This proves the lemma.

Since G_0 is assumed to have a cover by simply connected open sets, there exists an etale surjection $p_0 : U_0 \to G_0$ from a simply connected (but not necessarily connected) space U_0 onto G_0. Consider the induced etale topological groupoid $U = p_0^* G$ as defined in 1.1; so $U_1 = U_0 \times_{G_0} G_1 \times_{G_0} U_0$. There is an essential equivalence $p : U \to G$, and hence $\mathcal{B}U \cong \mathcal{B}G$ (equivalence of toposes). So $\pi_1(\mathcal{B}U) = \pi_1(\mathcal{B}G)$, relative to a given basepoint $x_0 \in U_0$.

Since U_0 is simply connected, lemma 3.2 applied to the groupoid U yields that a covering space of $\mathcal{B}U$ with fiber S can be identified with the constant sheaf $U_0 \times S$ on U_0 equipped with some action by U. But such an action $\mu : (U_0 \times S) \times_{U_0} U_1 \to (U_0 \times S)$ is the same things as a map

$\hat{\mu} : U_1 \to \mathrm{Aut}(S)$ such that for $x \in U_0$ and $x \xrightarrow{g} y, y \xrightarrow{h} z$ in U_1, it holds for all $\sigma \in S$ that $\hat{\mu}(s(x))(\sigma) = \sigma$ and $\hat{\mu}(hg)(\sigma) = \hat{\mu}(g)(\hat{\mu}(h)(\sigma))$. Since U_1 is locally connected, $\hat{\mu}$ factors through $U_1 \twoheadrightarrow \pi_0(U_1)$.

Let $\pi_0(U)$ be the discrete groupoid generated by $\pi_0(U_1 \times_{U_0} U_1) \xrightarrow{\pi_0(m)} \pi_0(U_1) \rightrightarrows \pi_0(U_0)$; that is, the objects of $\pi_0(U)$ are the connected components $[x]$ of U_0 $(x \in U_0)$, and the morphisms of $\pi_0(U)$ are generated by the components $[g]$ of morphisms $g \in U_1$, modulo the relations

$$[d_0 g] = d_0[g], [d_1 g] = d_1[g]$$

$$[g \circ h[= [g] \circ [h] \text{ (whenever } d_0 g = d_1 h)$$

$$s[x] = [sx].$$

The groupoid $\pi_0(U)$ is connected since G is assumed to be connected.

It is clear from the preceding discussion that the full subcategory of $\mathcal{B}G$ consisting of sums of covering spaces of $\mathcal{B}G$ can be identified with the functor category $\mathrm{Sets}^{\pi_0(U)^{\mathrm{op}}} \cong \mathcal{B}(\pi_0(U))$ of sets equipped with an action by $\pi_0(U)$. So $\pi_0(U)$ is the fundamental groupoid of $\mathcal{B}G$. (If one uses a different covering $U_0 \to G_0$, one ends up with an equivalent groupoid.) Now take a point $x_0 \in G_0$, with lifting $u_0 \in U_0$ so that $p_0(u_0) = x_0$. Since $\pi_0(U)$ is a connected groupoid, it follows that the fundamental group $\pi_1(\mathcal{B}G, x_0)$ of the topos $\mathcal{B}G$ can thus be identified with the vertex group at $[u_0]$ of the groupoid $\pi_0(U)$.

But this is precisely the description given by Van Est of the fundamental group of an S-atlas (an S-atlas is a special kind of etale topological groupoid). So we have proved :

4.3. Proposition. — *The fundamental group of an S-atlas G [9] coincides with the fundamental group of the classifying topos $\mathcal{B}G$ (and hence by 3.1 with the fundamental group of the classifying space BG).*

4.4. *Remark.* — As said in 1.5, the fundamental group of any topos, and in particular that of $\mathcal{B}G$, can be characterized by a universal property. There is a map of toposes $\mathcal{B}G \cong \mathcal{B}U \to \mathcal{B}(\pi_0 U) \cong \mathcal{B}(\pi_1(\mathcal{B}G, x_0))$ which is universal for maps of toposes from $\mathcal{B}G$ into toposes of the form $\mathcal{B}D = \mathrm{Sets}^{D^{\mathrm{op}}}$ where D is a discrete group. In particular, using the localization theorem (2.3) plus the well-known fact that for discrete groups D and D', morphisms of toposes $\mathcal{B}D \to \mathcal{B}D'$ correspond to group homomorphisms $D \to D'$, one derives a universal property purely in terms of topological groupoids. (A different approach to defining the fundamental group of an S-atlas by a universal property is given in [26].)

5. Haefliger's classifying space and Segal's theorem.

5.1. *Segal's theorem.* — Let M^q be the monoid of smooth embeddings of $\mathbb{R}^q$ into itself (viewed as a discrete category). Let Γ^q be Haefliger's topological groupoid of germs of local diffeomormophisms of $\mathbb{R}^q$. Explicitly, the space $(\Gamma^q)_0$ of objects is $\mathbb{R}^q$, and the space $(\Gamma^q)_1$ of morphisms has as its points equivalence classes of maps $f : (U, x) \to (V, y)$ where U and V are open neighbourhoods of points x and y of $\mathbb{R}^q$, and f is a diffeomorphism $U \xrightarrow{\sim} V$ with $f(x) = y$. Two such maps $f : (U, x) \to (V, y)$ and $f' : (U', x') \to (V', y')$ are equivalent if $x = x'$, $y = y'$ and $f = f'$ on a neighbourhood of x. The domain and codomain maps $d_0, d_1 : \Gamma_1^q \rightrightarrows \Gamma_0^q$ send the equivalence class of such an f to x and y respectively. The topology on Γ_1^q is the unique one for which d_0 and d_1 are etale maps. Segal's theorem referred to above asserts that there is a weak homotopy equivalence between BM^q and $B\Gamma^q$, see [31]. I shall prove this result by explicitly constructing a weak homotopy equivalence at the level of toposes, and then applying the comparison theorem 3.1 below. First, we need to describe some more classifying toposes.

5.2. *The classifying topos of a small category.* — Let $\mathbb{C}$ be a small category (i.e. $\mathbb{C}_0$, $\mathbb{C}_1$ are sets with the discrete topology). The functor category $\mathcal{B}\mathbb{C} = \mathrm{Sets}^{\mathbb{C}^{\mathrm{op}}}$ is a topos, and it is well-known that the homotopy and cohomology groups of $\mathcal{B}\mathbb{C}$ are the same as those of $\mathbb{C}$, or equivalently, as those of the classifying space $B\mathbb{C}$ (i.e. the realization of the nerve of $\mathbb{C}$). It is equally well-known, and easy to prove that the topos $\mathcal{B}\mathbb{C}$ classifies flat $\mathbb{C}$-diagrams (this is a very special case of 'Diaconescu's theorem', see [19], p. 113, or [7], [11]).

To be more concrete, let me spell out a special case of this. Let X be a topological space. A $\mathbb{C}$-bundle over X is an etale map $p : E \to X$ equipped with a fiberwise action by $\mathbb{C}$ on the left; i.e. there are maps $\pi : E \to \mathbb{C}_0$ and $\nu : \mathbb{C}_1 \times_{\mathbb{C}_0} E \to E$ such that for any $C \xrightarrow{\alpha} C' \xrightarrow{\beta} C''$ in $\mathbb{C}$ and $e \in \pi^{-1}(C) \subseteq E$: $p(\nu(\alpha, e)) = p(e)$, $\pi(\nu(\alpha, e)) = C'$, $\nu(1_C, e) = e$, and $\nu(\beta, (\nu(\alpha, e)) = \nu(\beta\alpha, e)$. A $\mathbb{C}$-bundle $E = (E \xrightarrow{p} X, \pi, \nu)$ is called *flat* if (i) the stalk E_x is non-empty for each $x \in X$; (ii) for each $x \in X$ and for each $e, e' \in E_x$ there are arrows $\pi(e) \xleftarrow{\alpha} C \xrightarrow{\beta} \pi(e')$ in $\mathbb{C}$ and $d \in E_x$

such that $\nu(\alpha, d) = e$ and $\nu(\beta, d) = e'$; (iii) for each $x \in X$ and $e \in E_x$, and arrows $\alpha, \beta : \pi(e) \rightrightarrows C$ in $\mathbb{C}$, if $\nu(\alpha, x) = \nu(\beta, x)$ then there exists an arrow $\gamma : C' \to \pi(e)$ in $\mathbb{C}$ and an $e' \in E_x$ with $\pi(e') = C'$, $\nu(\gamma, e') = e$, and $\alpha\gamma = \beta\gamma$. Then as a special case of Diaconescu's theorem just mentioned, we have the following proposition.

5.3. PROPOSITION (DIACONESCU). — *There is a natural equivalence of categories between morphisms of toposes $X \to \mathcal{B}\mathbb{C}$ and flat $\mathbb{C}$-bundles over X.*

Define two flat $\mathbb{C}$-bundles E and E' over X to be concordant if, up to isomorphism, they lie at the two ends of some flat $\mathbb{C}$-bundle over $X \times [0, 1]$. Using the standard weak equivalence between the space $B\mathbb{C}$ and the topos $\mathcal{B}\mathbb{C}$ it follows from 5.3 that for a CW-complex X, homotopy classes of maps from X into the classifying space (not topos) $B\mathbb{C}$ correspond to concordance classes of flat $\mathbb{C}$-bundles over X. (Without going through topos theory, this is worked out for a discrete monoid M in Segal [31], §5.)

5.4. *Etale topological categories.* — There is a common generalization of 5.3 and 2.4. Let $\mathbb{C}$ be a topological category such that d_0 and $d_1 : \mathbb{C}_1 \rightrightarrows \mathbb{C}_0$ are etale maps. Let $\mathcal{B}\mathbb{C}$ be the category of $\mathbb{C}$-sheaves (defined exactly as $\mathcal{B}G$ in 2.1). The constructions referred to in 5.2 extend in a straightforward way to show that $\mathcal{B}\mathbb{C}$ classifies flat $\mathbb{C}$-bundles. Thus, morphisms of topos $\mathcal{T} \to \mathcal{B}\mathbb{C}$ correspond to flat $\mathbb{C}$-bundles over $\mathcal{T}$. I will now spell this out more explicitly for the case that $\mathcal{T}$ is a topos of the form $\mathcal{B}G$, where G is an etale topological groupoid (or category). A flat $\mathbb{C}$-bundle over $\mathcal{B}G$ consists of an object $E = (E, p, \mu)$ of $\mathcal{B}G$ equipped with an action by $\mathbb{C}$ on the left given by a projection $\pi : E \to \mathbb{C}_0$ and an action $\nu : \mathbb{C}_1 \times_{\mathbb{C}_0} E \to E$, such that (a) the actions by G and by $\mathbb{C}$ on E are mutually compatible, in the sense that the identities $\pi(e \cdot g) = \pi(e)$, $p(\alpha \cdot e) = p(e)$ and $(\alpha \cdot e) \cdot g = \alpha(e \cdot g)$ hold (where $\alpha \cdot e = \nu(\alpha, e)$ and $e \cdot g = \mu(e, g)$), and moreover (b) E is a flat $\mathbb{C}$-bundle over the space G_0, as explained in 5.2 (cf. the conditions (i)-(iii) there).

The universal flat $\mathbb{C}$-bundle U lies over $\mathcal{B}\mathbb{C}$: it is the object $(\mathbb{C}_1, d_0, m)$ of $\mathcal{B}\mathbb{C}$, where d_0 and m are domain and composition of $\mathbb{C}$; the bundle structure $(\pi : U \to \mathbb{C}_0, \nu : \mathbb{C}_1 \times_{\mathbb{C}_0} U \to U)$ of U is given by $\pi = d_1$, and $\nu = m$ again.

5.5. PROPOSITION. — *There is a natural equivalence between maps $f : \mathcal{B}G \to \mathcal{B}\mathbb{C}$ of toposes and flat $\mathbb{C}$-bundles E over $\mathcal{B}G$.*

The proof of 5.5 goes by well-known constructions: given f, define $E = f^*(U)$; given E, define the inverse image part f^* of f by

$$f^*(X) = X \otimes_{\mathbb{C}} E.$$

More precisely, for an object $X = (X, p, \mu)$ of $\mathcal{B}G$, $f^*(X)$ is the quotient of $X \times_{\mathbb{C}_0} E$ obtained by identifying $(x \cdot \alpha, e)$ with $(x, \alpha \cdot e)$. The G-sheaf structure of $X \otimes_{\mathbb{C}} E$ is inherited from that of E.

5.6. *Diagrams of spaces.* — Let $\mathbb{C}$ be a small (discrete) category, and let X be a $\mathbb{C}$-diagram of spaces, i.e. a functor $C \mapsto X_C$ from $\mathbb{C}$ into the category of topological spaces. In analogy with 1.4, one defines a sheaf on X as a system $\{E_C : C \in \mathbb{C}_0\}$ where E_C is a sheaf on X_C, together with maps $E(\alpha) : X(\alpha)^*(E_D) \to E_C$ of sheaves on E_C for any morphism $\alpha : C \to D$ in $\mathbb{C}$, such that $E(\alpha)E(\beta) = E(\alpha\beta), E(id) = id$. A morphism $\varphi : E \to E'$ between two such sheaves on X consists of a system of morphisms $\varphi_C : E_C \to E'_C$ of sheaves on X_C, such that $\varphi_C \circ E(\alpha) = E(\alpha) \circ X(\alpha)^*(\varphi_D)$, for any $\alpha : C \to D$ in $\mathbb{C}$. This defines a category $\Gamma(X)$ of sheaves on X; $\Gamma(X)$ is a topos.

To obtain an alternative description of the topos $\Gamma(X)$, one can apply the Grothendieck construction to the diagram X and obtain an etale topological category $X_{\mathbb{C}}$, whose objects are pairs (x, C) with $x \in X_C$, and whose morphisms $(x, C) \to (y, D)$ are morphisms $\alpha : C \to D$ with $X(\alpha)(x) = y$. The classifying topos $\mathcal{B}X_{\mathbb{C}}$ is (essentially literally) the same as the topos $\Gamma(X)$. There is a projection functor $p : X_{\mathbb{C}} \to \mathbb{C}$, which induces a morphism

$$\mathcal{B}p : \mathcal{B}X_{\mathbb{C}} \to \mathcal{B}\mathbb{C}$$

of toposes. The inverse image functor $(\mathcal{B}p)^* : \mathcal{B}\mathbb{C} \to \mathcal{B}X_{\mathbb{C}} = \Gamma X$ sends a functor $A : \mathbb{C}^{\mathrm{op}} \to \mathit{Sets}$ into the system of constant sheaves $X_C \times A_C$ on X_C.

5.7. LEMMA. — *If all the spacesù X_C are contractible, then $\mathcal{B}p : \Gamma(X) = \mathcal{B}X_{\mathbb{C}} \to \mathcal{B}\mathbb{C}$ is a weak equivalence.*

Proof. — This follows immediately frôm the Whitehead theorem. Indeed, it is obvious that $(\mathcal{B}p)^*$ induces an equivalence between covering spaces of $\mathcal{B}\mathbb{C}$ and of $\mathcal{B}X_{\mathbb{C}}$. Moreover, if A is an abelian group in $\mathcal{B}\mathbb{C}$ (i.e. A is a functor $C \mapsto A_C$ from $\mathbb{C}^{\mathrm{op}}$ into abelian groups) there is a standard spectral sequence

$$E_2^{p,q} = H^p(\mathbb{C}, H^q(X_C, A_C)) \Rightarrow H^{p+q}(\mathcal{B}X_{\mathbb{C}}, (\mathcal{B}p)^*A).$$

If the X_C are contractible, the spectral sequence degenerates, and $(\mathcal{B}p)^*$ induces an isomorphism in cohomology.

5.8. *Notation* (from Segal [31]). — Let $\tilde{\mathcal{D}}^q$ be the discrete category whose objects are open disks in $\mathbb{R}^q$, and whose morphisms are smooth embeddings. Since every open disk is diffeomorphic to $\mathbb{R}^q$, there is an equivalence of categories $\varepsilon : M^q \to \tilde{\mathcal{D}}^q$, which induces an equivalence of toposes

$$\mathcal{B}\tilde{\mathcal{D}}^q \cong \mathcal{B}M^q.$$

Let $P : \tilde{\mathcal{D}}^q \hookrightarrow$ (topological spaces) be the obvious inclusion functor, and let $\mathcal{D}^q = P_{(\tilde{\mathcal{D}}^q)}$ be the associated etale topological category as in 5.6, with projection functor $p : \mathcal{D}^q \to \tilde{\mathcal{D}}^q$. So the objects of $\mathcal{D}^q$ are pairs (W, x) with $x \in W$ and W an open disk in $\mathbb{R}^q$, and the morphisms $(W, x) \to (V, y)$ of $\mathcal{D}^q$ are smooth embeddings $f : W \hookrightarrow V$ with $f(x) = y$. By lemma 5.7, p induces a weak equivalence of classifying toposes

$$\mathcal{B}p : \mathcal{B}\mathcal{D}^q \to \mathcal{B}\tilde{\mathcal{D}}^q.$$

Let $r : \mathcal{D}^q \to \Gamma^q$ be the functor which sends an embedding $(W, x) \hookrightarrow (V, y)$ to its germ at x.

5.9. PROPOSITION. — *$\mathcal{B}r : \mathcal{B}\mathcal{D}^q \to \mathcal{B}\Gamma^q$ is a natural deformation retraction of toposes; i.e. there is a map $j : \mathcal{B}\Gamma^q \to \mathcal{B}\mathcal{D}^q$ such that $(\mathcal{B}r) \circ j \cong \mathrm{id}_{\mathcal{B}\Gamma^q}$ and such that there is a natural transformation $\mathrm{id}_{\mathcal{B}\mathcal{D}^q} \Rightarrow j \circ \mathcal{B}r$. In particular, $\mathcal{B}r$ is a weak equivalence of toposes.*

Proof. — There is not much more involved than spelling out the definitions. The morphism $\mathcal{B}r$ has as inverse image part $(\mathcal{B}r)^*$ the functor induced by r: so $(\mathcal{B}r)^*(E)_{(W,x)} = E_x$. Thus $\mathcal{B}r$ corresponds (cf. 2.4, 5.4) to the principal Γ^q-bundle T over $\mathcal{B}\mathcal{D}^q$ defined (as an object of $\mathcal{B}\mathcal{D}^q$) by giving the fiber $T_{(W,x)}$ of $T \to \mathcal{D}_0^q$ over (W, x) as

$$T_{(W,x)} = \Gamma^q(x, -) = \{g \in \Gamma^q \mid d_0 g = x\},$$

with action by $\mathcal{D}^q$ from the right induced by composition. The bundle structure comes from the map $T \xrightarrow{\pi} \Gamma_0^q = \mathbb{R}^q$, $\pi(g) = d_1 g$, and the action by Γ^q from the left also comes from composition in Γ^q.

To define a morphism $j : \mathcal{B}\Gamma^q \to \mathcal{B}\mathcal{D}^q$, we need to describe a flat $\mathcal{D}^q$-bundle S in $B\Gamma^q$ (cf 5.4, 5.5). S is really the same as T, except that the actions by $\mathcal{D}^q$ and Γ^q are reversed:

$$S = \coprod_W d_1^{-1}(W)$$

where the coproduct ranges over open disks W in $\mathbb{R}^q$, and $d_1^{-1}(W) = \{g \in \Gamma_1^q : d_1(g) \in W\}$. S is given the structure of an object of $\mathcal{B}\Gamma^q$ by the domain map $d_0 : S \to \Gamma_0^q$, and by the obvious action of Γ^q on S from the right by composition. S is given the structure of a $\mathcal{D}^q$-bundle by the projection $\pi : S \to \mathcal{D}_0^q$, $\pi(g) = (W, d_1 g)$ for $g \in d_1^{-1}(W) \subseteq S$, and the action $\nu : \mathcal{D}_1^q \times_{\mathcal{D}_0^q} S \to S$ comes from composition in Γ again; i.e. for $f : (W, x) \to (V, y)$ in $\mathcal{D}^q$ and $g \in d_1^{-1}(W)$ with $d_1(g) = x$, $\nu(f, g) = r(f) \circ g$. It is easy to see that S is a flat $\mathcal{D}^q$-bundle.

For an object E of $\mathcal{B}\Gamma^q$, $j^* \circ (\mathcal{B}p)^*(E)$ is the Γ^q-sheaf whose fiber over $x \in \Gamma^q$ is $(\mathcal{B}p)^*(E) \otimes_{\mathcal{D}^q} S_x$, where $S_x = \{g \in S \mid d_0 g = x\}$. But this is the quotient of the set $\coprod_{(W,z)}\{(e, g) \mid e \in E_z, g : x \to z \text{ in } \Gamma^q\}$ (the sum being taken over objects (W, z) of $\mathcal{D}^q$), obtained by identifying $(e \cdot r(h), g)$ and $(e, r(h) \circ g)$ for any map $(W, z) \xrightarrow{h} (W', z')$ of $\mathcal{D}^q$. Clearly then, $((\mathcal{B}r)^*(E) \otimes_{\mathcal{D}^q} S_x) \cong E_x$ via the map sending the equivalence class of (e, g) to $e \cdot g$. Thus $(\mathcal{B}r) \circ j \cong \mathrm{id}$.

The other way round (writing U for the universal flat $\mathcal{D}^q$-bundle as in 5.4), $j \circ \mathcal{B}r$ corresponds to the flat $\mathcal{D}^q$-bundle $(\mathcal{B}r)^* j^*(U) = (\mathcal{B}r)^*(S)$ in $\mathcal{B}\mathcal{D}^q$, whose fiber (as an object of $\mathcal{B}\mathcal{D}^q$) over an object (V, y) of $\mathcal{D}^q$ is

$$S_{r(V,y)} = S_y = \coprod_W \{g \in d_1^{-1}(W) \mid d_0(g) = y\}.$$

This is not the same as the universal bundle U, whose fiber over (V, y) is $\mathcal{D}^q((V, y), -) = \coprod_W \{g : V \rightarrowtail W \mid g \text{ is a smooth embedding}\}$, but there is an obvious map of $\mathcal{D}^q$-bundles $U \to (\mathcal{B}r)^* S$ sending an embedding to its germ. But U is classified by $id : \mathcal{B}\mathcal{D}^q \to \mathcal{B}\mathcal{D}^q$ and $(\mathcal{B}r)^*(S)$ is classified by $j \circ \mathcal{B}r : \mathcal{B}\mathcal{D}^q \to \mathcal{B}\mathcal{D}^q$, so there is a natural map $id \to j \circ \mathcal{B}r$. It immediately follows that $\mathcal{B}r$ is a weak equivalence (with "homotopy inverse" j, see Joyal-Wraith [21]). This proves the proposition.

5.10. COROLLARY. — *There is a weak equivalence of classifying toposes between $\mathcal{B}M^q$ and $\mathcal{B}\Gamma^q$.*

Proof. — This follows by the equivalence $\mathcal{B}M^q \simeq \mathcal{B}\tilde{\mathcal{D}}^q$ of 5.8 and the weak equivalences $\mathcal{B}\tilde{\mathcal{D}}^q \leftarrow \mathcal{B}\mathcal{D}^q$ of 5.8 and $\mathcal{B}\mathcal{D}^q \to \mathcal{B}\Gamma^q$ of 5.9.

Segal's theorem, which asserts a similar weak equivalence of classifying spaces, now follows by the Comparison Theorem 3.1 and the standard weak equivalence $B\mathbb{C} \to \mathcal{B}\mathbb{C}$ for any discrete category $\mathbb{C}$.

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