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LOWER BOUNDS FOR PSEUDO-DIFFERENTIAL OPERATORS

by N. LERNER(*) and J. NOURRIGAT

Introduction.

In this paper, we wish to start an investigation on lower bounds for pseudo-differential operators. Our guideline will be one of C. Fefferman and D.H. Phong's conjecture (see §7 in [6] and [2]) : if $a(x, D_x)$ is a second order operator, its lower bound will be given by some average of its symbol on canonical images of the unit cube in the phase space. Namely, we wish to prove roughly (in some cases) :

$$a(x, D_x) \geq \inf_{\chi \in \Phi} \iint_{\chi(Q_0)} a(x, \xi) dx\, d\xi \ ,$$

as an operator, where Q_0 is the unit cube of $\mathbb{R}^{2n}$, Φ a family of canonical transformations to be specified. The inequality above gives a connection between the geometry of the symbol $a(x, \xi)$ and the spectral properties of its quantization $a(x, D_x)$. Many papers were devoted to these questions. The classical sharp Gårding inequality was first proved by Hörmander [9] :

$a(x, \xi)$ first order ≥ 0 implies $a(x, D_x)$ semi-bounded from below. We refer to ([11] section 18.1 or [1]) for a proof of this inequality, yielding also the case of systems previously studied by Lax and Nirenberg [12]. Later on, in his paper on the Weyl calculus, Hörmander [10] proved an inequality

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with a "gain" of 6/5 derivatives. Namely, if $a(x,\xi)$ is a symbol of order 6/5 such that

$$a(x,\xi) + \frac{1}{2}\,\mathrm{trace}_+\, a \geq 0 ,$$

then $a(x, D_x)$ is semi-bounded from below. Here $\mathrm{trace}_+\, a$ is a positive quantity related to the Hessian of the symbol. On the other hand, C. Fefferman and D.H. Phong proved a two derivative inequality [3] for non-negative symbols :

$a(x,\xi)$ second order ≥ 0 implies $a(x, D_x)$ semi-bounded from below (see also the proof in [11] section 18.6). On the other hand, C. Fefferman and D.H. Phong [6] discussed the conjecture above for non-negative symbols of order $2 - \varepsilon$, $\varepsilon > 0$.

The present paper is concerned with various cases involving symbols which can take large negative values. The first section is devoted to the Schrödinger equation with magnetic potential. In the second section, we discuss the one-dimensional Schrödinger equation, with very little assumptions on the potential. The third section, purely technical, is devoted to miscellaneous properties of the proper class of a symbol. The fourth section contains a proof of the conjecture for pseudo-differential operators in one dimension.

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1. THE SCHRÖDINGER EQUATION WITH MAGNETIC POTENTIAL

a. Statement of the result.

We are interested in the following operator

$$P = \sum_{j=1}^{n} \left(D_{x_j} - A_j(x)\right)^2 + V(x) \ , \tag{1.1}$$

where $D_{x_j} = \dfrac{1}{i}\dfrac{\partial}{\partial x_j}$, and $A_1, \ldots, A_n, V$ are real polynomials of degree $< m$. (Note that V is *not* assumed to be non-negative). We set-up

$$p(x,\xi) = \sum_{j=1}^{n} (\xi_j - A_j(x))^2 + V(x) \ , \tag{1.2}$$

the Weyl symbol of the operator P.

We denote by G_m the group of canonical transformations of $\mathbb{R}^{2n}$ of the following type :

$$(y,\eta) \longrightarrow (x_0 + \lambda y,\ \lambda^{-1}\eta + \nabla\varphi(y)) \tag{1.3}$$

where $x_0 \in \mathbb{R}^n$, $\lambda > 0$ and φ is a real polynomial of degree $\leq m$.

THEOREM 1.1. — *For each integer m, there exists $\delta_m > 0$ such that the following property holds. If $A_1, \ldots, A_n$, V are real polynomials of degree $< m$ and if the symbol $p(x,\xi)$ given by (1.2) satisfies*

$$\iint_{\max(|y|,|\eta|)\leq\delta_m} (p \circ \chi)(y,\eta)\, dy\, d\eta \geq 0 \ , \tag{1.4}$$

for any χ of G_m defined in (1.3). Then the operator P, given by (1.1), is non-negative.

In other words, whenever (1.4) is satisfied, we have

$$\sum_{j=1}^{n} \|(D_{x_j} - A_j)u\|^2_{L^2(\mathbb{R}^n)} + \int V(x)|u(x)|^2 dx \geq 0 \ , \tag{1.5}$$

for any $u \in C_0^\infty(\mathbb{R}^n)$.

Remark 1.2. — The magnetic potential $A = (A_1, \ldots, A_n)$ is a one-form $A = \sum_{j=1}^{n} A_j dx_j$. Its exterior derivative

$$dA = \sum_{1\leq k<j\leq n} \left(\frac{\partial A_j}{\partial x_k} - \frac{\partial A_k}{\partial x_j}\right) dx_k \wedge dx_j$$

will be called *curl* A. Note that the quotient norm of A modulo exact forms is equivalent to the norm of *curl* A : if E is the space of I-forms with polynomial coefficients of degree $< m$, F the subspace of $\{d\Phi\}$, where Φ is a polynomial of degree $\leq m$, $G = dE$, we have $\|A\|_{E/F} \sim \|dA\|_G$ where $\| \ \|_{E/F}$ and $\| \ \|_G$ are any norm on the finite dimensional spaces E/F, G.

b. Preliminary lemmas.

The following inequality is proved in a paper by Mohamed and Nourrigat [13]. (See also Helffer-Nourrigat [8] and Nourrigat [14]). It could be seen also as a consequence of [6].

LEMMA 1.3 (Local subelliptic estimates with non-negative potential). — *For any $m \geq 1$, there exists $C_m > 0$ and $\varepsilon_m > 0$ so that, for any polynomials $A_1, \dots, A_n$, V of degree $< m$ with $V(x) \geq 0$ on $|x| \leq 1$, we have*

$$\lambda^{\varepsilon}\|u\|^2 \leq C_m \left\{ \sum_{j=1}^{n} \|(D_j - A_j)u\|^2 + (Vu, u) \right\} \tag{1.6}$$

for any $u \in C_0^\infty(|x| < 1)$, with

$$\lambda^2 = \sup_{|x|\leq 1} (|B(x)|^2 + V(x)) \ , \quad B = curl\, A \ .$$

LEMMA 1.4. — *For any $m \geq 1$, there exists $C_m > 0$, so that (1.4) implies the following inequality; for any $x_0 \in \mathbb{R}^n$ and any $R > 0$:*

$$0 \leq \frac{\delta_m^4 2}{R^2} + C_m \sup_{|x-x_0|\leq R} R^2|B(x)|^2 + \frac{1}{|S^{n-1}|R^n} \int_{|x-x_0|\leq R} V(x)dx \ , \tag{1.7}$$

where $B = curl\, A$.

Proof. — A consequence of (1.4) is, with $\delta = \delta_m$,

$$\iint_{\substack{|y|\leq\delta \\ |\eta|\leq\delta}} (V(x_0 + \lambda y) + |\lambda^{-1}\eta + \nabla\varphi(y) - A(x_0 + \lambda y)|^2) dy\, d\eta \geq 0 \ ,$$

for any $x_0 \in \mathbb{R}^n$, $\lambda > 0$, φ polynomial of degree $\leq m$. Then, we have

$$|B^n|\lambda^{-n}\delta^n \int_{|x-x_0|\leq\lambda\delta} V(x)dx + \iint_{\substack{|y|\leq\delta \\ |\eta|\leq\delta}} |\lambda^{-1}\eta + \nabla\varphi(y)$$

$$-A(x_0 + \lambda y)|^2 dy\, d\eta \geq 0 \ ,$$

and thus

$$0 \leq \lambda^{-n}\delta^n \int_{|x-x_0|\leq\lambda\delta} V(x)dx + |S^{n-1}|\delta^{2n} \sup_{\substack{|y|\leq\delta \\ |\eta|\leq\delta}} |\lambda^{-1}\eta + \nabla\varphi(y) - A(x_0+\lambda y)|^2 ,$$

so

$$0 \leq (\lambda\delta)^{-n}\delta^{2n} \int_{|x-x_0|\leq\lambda\delta} V(x)dx + 2|S^{n-1}|\delta^{2n}\delta^2\lambda^{-2} + 2|S^{n-1}|\delta^{2n} \sup_{|y|\leq\delta} \|\nabla\varphi(y) - A(x_0+\lambda y)\|^2 ;$$

so, with $R = \lambda\delta$, we have

$$0 \leq R^{-n} \int_{|x-x_0|\leq\lambda\delta} V(x)dx + 2|S^{n-1}|\delta^4 R^{-2} + 2|S^{n-1}| \sup_{|y|\leq\delta} \|\nabla\varphi(y) - A(x_0+\lambda y)\|^2 .$$

Remark 1.2 gives then

$$0 \leq R^{-n}|S^{n-1}|^{-1} \int_{|x-x_0|\leq R} V(x)dx + 2\delta^4 R^{-2} + 2C_m R^2 \sup_{|y|\leq\delta} \|(\mathit{curl}\, A)(x_0+\lambda y)\|^2 ,$$

which completes the proof of (1.7).

c. Proof of theorem 1.1.

Let $A_1, \ldots, A_n$, V be polynomials of degree $< m$, $B = \mathit{curl}\, A$. For a given $x \in \mathbb{R}^n$, let's consider the increasing continuous function of R

$$\psi_x(R) = \sup_{|y-x|\leq R} (R^2|B(y)|^2 + V(y)) - \inf_{|y-x|\leq R} V(y) .$$

Assuming that V is not constant, $R \to \psi_x(R)$, is continuous increasing from 0 to $+\infty$ with R. Since $R \to R^{-2}$ is strictly decreasing we can then define, for $\lambda > 1$ given, and for $x \in \mathbb{R}^n$, $R(x)$ to be the unique $R \in (0, +\infty)$ so that $\psi_x(R) = \lambda^2 R^{-2}$, *i.e.*

$$\text{(1.8)} \qquad \sup_{|y-x|\leq R} (R^2|B(y)|^2 + V(y)) - \inf_{|y-x|\leq R} V(y) = \lambda^2 R^{-2} .$$

LEMMA 1.5 (a slowly varying metric on $\mathbb{R}^n$). — *For $x_1, x_2 \in \mathbb{R}^n$, $|x_1 - x_2| \leq \frac{1}{4}R(x_1)$ implies $\frac{1}{2} \leq R(x_1)R(x_2)^{-1} \leq 2$.*

Proof. — Assume $|x_2 - x_1| \leq \frac{1}{2}R_1$, $R_1 = R(x_1)$. The triangle inequality gives

$$\psi_{x_2}\Big(\frac{R_1}{2}\Big) = \sup_{|y-x_2|\leq \frac{R_1}{2}} \Big(\frac{R_1^2}{4}|B(y)|^2 + V(y)\Big) - \inf_{|y-x_2|\leq\frac{R_1}{2}} V(y)$$

$$\leq \sup_{|y-x_1|\leq R_1} \Big(\frac{R_1^2}{4}|B(y)|^2 + V(y)\Big) - \inf_{|y-x_1|\leq R_1} V(y)$$

$$\leq \psi_{x_1}(R_1) = \lambda^2 R_1^{-2} .$$

Consequently, if $\frac{R_1}{2} > R_2 = R(x_2)$, we get $\lambda^2 R_1^{-2} \geq \psi_{x_2}(\frac{R_1}{2}) \geq \psi_{x_2}(R_2) = \lambda^2 R_2^{-2}$ so $R_2 \geq R_1 > 2R_2 > 0$ which is impossible. Thus we have $\frac{R_1}{2} \leq R_2$.

As in [11] (1.4.5)', $|x_1 - x_2| \leq \frac{1}{4}R(x_1) \leq \frac{1}{2}R(x_1)$ implies $R(x_1) \leq 2R(x_2)$ and thus $|x_1 - x_2| \leq \frac{1}{2}R(x_2)$ which gives $R(x_2) \leq 2R(x_1)$, and the lemma.

The metric $g_x(t) = \frac{|t|^2 2^4}{R(x)^2}$ is slowly varying in $\mathbb{R}^n$ (*cf.* definition 1.4.7 in [11]) thus the theorem 1.4.10 in [11] implies the existence of $(\varphi_v)_{v\in\mathbb{N}} \in C_0^\infty(\mathbb{R}^n)$, $(x_v)_{v\in\mathbb{N}} \in \mathbb{R}^n$ so that ·

(a) $\operatorname{supp}\varphi_v \subset \{x, |x - x_v| \leq R_v = R(x_v)\} = Q_v$

(b) $\sum_v \varphi_v^2 = 1$ identically,

(1.9) (c) $\sum_v |\partial_x^\alpha \varphi_v(x)|^2 \leq R(x)^{-2|\alpha|}\gamma_{|\alpha|}$,

where $\gamma_{|\alpha|}$ depends only on $|\alpha|$ and the dimension n, but is independent of λ.

For $v \in \mathbb{N}$, let's define

$$m_v = \inf_{Q_v} V(y) \ , \quad V_v(t) = R_v^2(V(x_v + tR_v) - m_v) \geq 0$$

if $|t| \leq 1$, $A_v^j(t) = R_v A_j(x_v + tR_v)$, and thus $curl\, A_v = B_v = R_v^2 B(x_v + tR_v)$, $B = curl\, A$.

From (1.8), we have $\lambda^2 = \sup_{|t|\leq 1} (|B_v(t)|^2 + V_v(t))$. Consequently, from (1.6), $\lambda^\varepsilon \|u\|^2 \leq C_m\{\sum_j \|(D_j - A_v^j)u\|^2 + (V_v u, u)\}$ for any $u \in C_0^\infty(|x| \leq 1)$ (the constant C_m depends only on the dimension and on the degree of A,

V). By translation and dilation we get, for any $u_v \in C_0^\infty(Q_v)$,

(1.10) $$\lambda^\varepsilon R_v^{-2}\|u_v\|^2 \le C_m\{\sum_j \|(D_j - A^j)u_v\|^2 + ((V - m_v)u_v, u_v)\} .$$

Note that C_m does *not* depend on λ, v.

The inequality (1.10) implies, for $u \in C_0^\infty(\mathbb{R}^n)$,

(1.11) $$\sum_v \lambda^\varepsilon R_v^{-2}\|\varphi_v u\|^2 \le C_m \sum_v \sum_{j=1}^n \|(D_j - A_j)\varphi_v u\|^2 + \sum_v ((V - m_v)\varphi_v u, \varphi_v u) ,$$

with φ_v as in (1.9).

Moreover, from (1.9)(c) and the lemma 1.5, we have

$$\sum_v \|\nabla\varphi_v(x)\|^2 \le n\gamma_1 R(x)^{-2} \le \gamma_1' \sum_v R_v^{-2}\varphi_v(x)^2 ,$$

where γ_1' depends only on n. We then get from (1.11)

(1.12) $$\lambda^\varepsilon \sum_v R_v^{-2}\|\varphi_v u\|^2 \le 2C_m \left\{ \sum_{j=1}^n \|(D_j - A_j)u\|^2 + (Vu, u) \right\} + \sum_v C_m(2\gamma_2' R_v^{-2} - m_v)\|\varphi_v u\|^2 .$$

Then, we choose λ so that $\lambda^\varepsilon \ge 4C_m\gamma_2'$.

The parameter λ is fixed in the subsequent computations. We have

(1.13) $$\sum_v (\lambda^\varepsilon R_v^{-2} + 2C_m m_v)\|\varphi_v u\|^2 \le 4C_m \Big(\sum_{j=1}^n \|(D_j - A_j)u\|^2 + (Vu, u)\Big) .$$

The following lemma implies the theorem 1.1 since $\lambda^\varepsilon \ge 1$.

LEMMA 1.6 (Choice of δ_m in (1.4) in terms of λ). — *For any $\lambda > 1$, if $\delta_m = \delta$ is chosen small enough, (1.4) implies $2C_m m_v + R_v^{-2} \ge 0$ for any $v \in \mathbb{N}$.*

Proof. — Let y_v be a point in Q_v at which V is minimum,

$$m_v = V(y_v) = \inf_{y \in Q_v} V(y) , \quad |y_v - x_v| \le R_v .$$

Let's denote by S_v the ball with center y_v and radius βR_v, $\beta \in (0, 1)$ to be chosen later. We have,

(1.14) $$S_v \subset Q(x_v, 2R_v) = Q_v^*$$

($Q(x, R)$ is the closed ball with center x, and radius R).

Moreover, we have, using V is polynomial with degree $\leq m$,

$$(1.15) \qquad |S_v|^{-1} \int_{S_v} V(y)dy \leq m_v + C_m^{(0)} \beta \{ \sup_{Q_v} V - \inf_{Q_v} V \} \ :$$

in fact

$$\begin{aligned} |S_v|^{-1} \int_{S_v} V(y) - m_v = |S_v|^{-1} \int_{S_v} (V(y) - m_v)dy &\leq \sup_{S_v} (V(y) - m_v) \\ &\leq \beta R_v \sup_{S_v} \|(\nabla V)(y)\| \leq \beta R_v \sup_{Q_v^*} \|\nabla V(y)\| \\ &\leq C_m \beta R_v \sup_{Q_v} \|\nabla V(y)\| \leq C'_m \beta \sup_{Q_v} (V(y) - m_v) \end{aligned}$$

which gives (1.15).

Moreover, we have also, using B polynomial with degree $\leq m$,

$$(1.16) \qquad \sup_{S_v} \|B(y)\| \leq \sup_{Q_v^*} \|B(y)\| \leq C_m^{(1)} \sup_{Q_v} \|B(y)\| \ .$$

So the inequality (1.7) with $x_0 = y_v$, $R = \beta R_v$ gives, together with (1.15), (1.16),

$$(1.17) \quad 0 \leq \frac{2\delta^4}{\beta^2 R_v^2} + C''_m \beta^2 R_v^2 \sup_{Q_v} \|B(y)\|^2 + m_v + C'_m \beta (\sup_{Q_v} V - \inf_{Q_v} V) \ .$$

But $\lambda^2 R_v^{-2} = \sup_{Q_v} \{ \|B(y)\|^2 R_v^2 + V(y) - m_v \}$, thus

$$\begin{aligned} \lambda^2 R_v^{-2} \geq |Q_v|^{-1} \int_{Q_v} \|B(y)\|^2 dy R_v^2 + |Q_v|^{-1} \int_{Q_v} (V(y) - m_v)dy \\ \geq C_m^{-1} \{ \sup_{Q_v} \|B(y)\|^2 R_v^2 + \sup_{Q_v} V - \inf_{Q_v} V \} \end{aligned}$$

and consequently $C_m \lambda^2 R_v^{-2} \geq \sup_{Q_v} \|B(y)\|^2 R_v^2 + \sup_{Q_v} V - \inf_{Q_v} V$.

From (1.17), we thus get,

$$0 \leq \frac{2\delta^4}{\beta^2 R_v^2} + \frac{C'''_m \beta^2 \lambda^2}{R_v^2} + m_v + C'_m \beta \frac{\lambda^2}{R_v^2} \ .$$

Since $\beta \in (0,1)$, we get

$$0 \leq \frac{2\delta^4}{\beta^2 R_v^2} + C_m^{(4)} \beta \frac{\lambda^2}{R_v^2} + m_v \ .$$

Then, we choose β so that $C_m^{(4)} \beta \lambda^2 \leq \dfrac{1}{4C_m}$, then δ, $\dfrac{2\delta^4}{\beta^2} \leq \dfrac{1}{4C_m}$ and we get the lemma 1.6.

2. SCHRÖDINGER EQUATION IN ONE DIMENSION

a. Statement of the result.

We study here a one-dimensional Schrödinger equation with very little assumptions on the potential. This result will be useful in the analysis of pseudo-differential operators.

THEOREM 2.1. — *Let $V \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that, for any interval Q, with length $|Q|$, $|Q| \leq 2$,*

$$(2.1) \qquad |Q|^{-1} \int_Q V_+(t)dt - 9|Q|^{-1} \int_Q V_-(t)dt + |Q|^{-2} \geq 0 \ ,$$

with $V_\pm = \max(\pm V, 0)$.

Then

$$(2.2) \qquad -\frac{d^2}{dx^2} + V(x) \geq 2^{-4}\mu - 1 \ ,$$

with

$$(2.3) \qquad \mu = \inf_{Q \text{ interval}} \{|Q|^{-1} \int_Q V(t)dt + |Q|^{-2}\} \ .$$

Moreover,

$$(2.4) \qquad \mu \geq \inf_{J \in \mathcal{F}} \iint_J (|\xi|^2 + V(x))dx\, d\xi \ ,$$

where $\mathcal{F}$ is the following family of "symplectic cubes",

$$\mathcal{F} = \{J_{a,b}\}_{a<b \text{ real}} \ , \qquad J_{a,b} = \{(x,\xi), a \leq x \leq b, \ |\xi| \leq \frac{1}{2(b-a)}\} \ .$$

We can remark here that a minimal regularity (L^1_{loc}) is required for the potential and that, on the other hand, no derivative is lost in the inequality, which appears as a Gårding inequality with gain of 2 full derivatives (see [6]) when V behaves like a symbol. As a matter of fact, if V is a C^∞ function such that $|V^{(k)}(x)| \leq C_k M^2$, where M is a fixed constant, the non-negativity condition (2.1) ensures the non-negativity of the second order operator $-\frac{d^2}{dx^2} + V(x) + 1$. Moreover the non-negativity condition in (2.1) is a very weak one and yields potential taking negative values, even singular ones. Note also that the condition is local ($|Q| \leq 2$) (see also remark 2.2 below).

b. Proof.

Replacing V by $V+1$, we have, for

$$Q = Q(x_0, \delta) = \{x, x_0 \leq x \leq x_0 + \delta\},$$

$$\delta^{-1} \int_Q V(x)dx + \delta^{-2} \geq 1, \quad \delta \leq 2, \tag{2.5}$$

as a consequence of (2.1). So we get from (2.5),

$$\delta \int_Q V(x)dx + 1 \geq \delta^2 \ , \quad \text{and thus}$$

$$\delta \int_Q V(x)dx > 1 \ , \quad \text{if } 4 \geq \delta^2 \geq 3 \ . \tag{2.6}$$

On the other hand, with $V \in L^1_{\text{loc}}$,

$$\lim_{\delta \to 0} \delta \int_{Q(x_0,\delta)} V(x)dx = 0 \ , \tag{2.7}$$

and, for x_0 fixed, $\delta \to \delta \int_{Q(x_0,\delta)} V(x)dx$ is a continuous function. So we can pick-up (using (2.6), (2.7) and the continuity) the largest $\delta(x_0) = \delta_0$ such that

$$\delta(x_0) \int_{Q(x_0,\delta(x_0))=Q_0} V(x)dx = 1 \ . \tag{2.8}$$

Consequently, we get, with μ defined in (2.3)

$$\delta_0^{-1} \int_{Q_0} V(x)dx = \delta_0^{-2} \geq 2^{-1}\mu \ . \tag{2.9}$$

We have now, for $u \in C^\infty(\mathbb{R})$ (*without* assumption on the support of u),

$$\begin{aligned} (Vu,u)_{Q_0} = \int_{Q_0} V(x)|u(x)|^2 dx = \delta_0^{-1} &\int_{Q_0} V_+(x)dx\, \delta_0 |u(x_+)|^2 \\ &-\delta_0^{-1} \int_{Q_0} V_-(x)dx\delta_0|u(x_-)|^2 \ , \end{aligned} \tag{2.10}$$

for some x_+, x_- in Q_0 .

Thus, from (2.10), we obtain

$$\begin{aligned} (Vu,u)_{Q_0} = &\big(\delta_0^{-1} \int_{Q_0} V(x)dx\big)\delta_0|u(x_+)|^2 \\ &+\delta_0^{-1} \int_{Q_0} V_-(x)dx\{\delta_0|u(x_+)|^2 - \delta_0|u(x_-)|^2\} \ . \end{aligned} \tag{2.11}$$

From (2.1), (2.9), we get

$$8\delta_0^{-1}\int_{Q_0} V_-(x)dx \le \delta_0^{-1}\int_{Q_0} V(x)dx + \delta_0^{-2} \le 2\delta_0^{-2}\,. \tag{2.12}$$

So, from (2.11), (2.9) and (2.12) we obtain (using $|u(x_+)|^2 - |u(x_-)|^2 = \int_{x_-}^{x_+} 2\mathbb{R}e\, u'(s)\overline{u}(s)ds$),

$$(Vu,u)_{Q_0} \ge \delta_0^{-1}|u(x_+)|^2 - \delta_0^{-1}2^{-1}\|u'\|_{Q_0}\|u\|_{Q_0}\,, \tag{2.13}$$

where $\|v\|_{Q_0} = \|v\|_{L^2(Q_0)}$.

On the other hand, we know (using $u(x) = u(x_+) + \int_{x_+}^{x} u'(t)dt$)

$$\|u\|_{Q_0}^2 \le 2\delta_0|u(x_+)|^2 + 2\delta_0^2\|u'\|_{Q_0}^2\,. \tag{2.14}$$

So, from (2.14), (2.13) we get,

$$\begin{aligned}\frac{1}{4}\delta_0^{-2}\|u\|_{Q_0}^2 &\le \frac{1}{2}\delta_0^{-1}|u(x_+)|^2 + \frac{1}{2}\|u'\|_{Q_0}^2 \le \delta_0^{-1}|u(x_+)|^2 + \frac{1}{2}\|u'\|_{Q_0}^2\\ &\le (Vu,u)_{Q_0} + \frac{1}{2}\delta_0^{-1}\|u'\|_{Q_0}\|u\|_{Q_0} + \frac{1}{2}\|u'\|_{Q_0}^2\\ &\le (Vu,u)_{Q_0} + \frac{1}{2}\|u'\|_{Q_0}^2 + \frac{1}{2}\|u'\|_{Q_0}^2 + \frac{2}{16\delta_0^2}\|u\|_{Q_0}^2\,.\end{aligned} \tag{2.15}$$

Consequently, we have from (2.15), (2.9),

$$(Vu,u)_{Q_0} + \|u'\|_{Q_0}^2 \ge \frac{1}{8}\delta_0^{-2}\|u\|_{Q_0}^2 \ge \frac{\mu}{16}\|u\|_{Q_0}^2\,. \tag{2.16}$$

Let's consider now a compact interval K of $\mathbb{R}$, $x_0 = \inf K$. Let's define (cf. (2.6), (2.7), (2.8))

$$\delta_0 = \delta(x_0) = \sup\left\{\delta, \delta\int_{x_0}^{x_0+\delta} V(x)dx = 1\right\}\,,$$

$$\delta_{k+1} = \delta(x_0 + \delta_0 + \cdots + \delta_k)\,,\quad k \ge 0\,.$$

In order to prove that (2.16) implies (2.2) (for $u \in C_0^\infty(\mathbb{R})$) we need only to prove $x_0 + \Sigma\delta_j \ge \sup K$.

Let's remark that for $\delta > 0$ and $[x, x+\delta] \subset K$

$$\delta\int_x^{x+\delta} V(t)dt \le \delta\int_x^{x+\delta}|V(t)|dt \le \delta\int_K |V(t)|dt\,.$$

So $\delta\int_x^{x+\delta} V(t)dt < 1$ if $\delta < \delta_K = \dfrac{1}{\int_K |V(t)|dt + 1}$ and $[x, x+\delta] \subset K$.

So, from the definition of $\delta(x)$, we get $\delta(x) > \delta_K$ if $[x, x+\delta(x)] \subset K$, which completes the proof.

Remark 2.2. — Note that, if (2.1) is satisfied for all intervals Q, $|Q| \leq 2$ so that $Q \subset [a, b+2]$ we get (2.16) for any $u \in C_0^\infty(a,b)$ and $Q = (x_k, x_{k+1})$, $x_0 = a$, $x_{k-1} = x_k + \delta(x_k)$ as long as $x_k \in (a,b)$.

In particular, we get $(Vu, u) + \|u'\|^2 \geq -\|u\|^2$ whenever (2.1) is satisfied for intervals Q, $|Q| \leq 2$, $Q \subset [a, b+2]$ and $u \in C_0^\infty(a,b)$.

3. MISCELLANEOUS PROPERTIES OF THE PROPER CLASS

a. Preliminary remarks.

The main goal of this section is to prove that a very mild non-negativity condition for a symbol still ensures that the Calderón-Zygmund procedure used by C. Fefferman and D.H. Phong ([3] - [7]) leads to the same trilogy. More precisely, we intend to show that *non-negativity of averages* on special "boxes" of volume 1 implies that, in a conformal class of pseudo-differential operators, the symbol is either elliptic positive, or bounded, or non-degenerate *i.e.* can be written after a canonical transformation

$$\xi_1^2 + V(x_1, x', \xi') ,$$

where V is a pseudo-differential potential.

Let's begin with a simple algebraic lemma.

LEMMA 3.1. — *Let $\mathcal{A}$ be a real commutative algebra. For any integer $k \geq 1$, there exists an integer $N(k)$ and real numbers $(\lambda_{ij})_{\substack{1\le i\le k\\ 1\le j\le N(k)}}$ $(\mu_j)_{1\leq j\leq N(k)}$ with* $\max(|\lambda_{ij}, |\mu_j|) \leq A(k)$ *such that for any $T_1, \ldots, T_k \in \mathcal{A}$*

$$T_1 \ldots T_k = \sum_{1\leq j\leq N(k)} \mu_j \Big(\sum_{1\leq i\leq k} \lambda_{ij} T_i \Big)^k . \tag{3.1}$$

Proof. — Induction on k. While $4T_1T_2 = (T_1+T_2)^2 - (T_1-T_2)^2$ we have to check

$$T_1 \ldots T_k\, T_{k+1} = \sum \mu_j \Big(\sum_{1\leq i\leq k} \lambda_{ij} T_i \Big)^k T_{k+1} . \tag{3.2}$$

It is then enough to write S^kT as a sum of $(k+1)$ th power of linear forms in S,T :

$$
\begin{aligned}
\sum_{1\le j\le k} \beta_j (S+jT)^{k+1} &= \sum_{\substack{0\le \ell\le k+1\\ 1\le j\le k}} C_{k+1}^{\ell} S^{k+1-\ell}\beta_j j^{\ell} T^{\ell} \\
&= \Big(\sum_{1\le j\le k} \beta_j\Big) S^{k+1} + (k+1)\Big(\sum_{1\le j\le k} j\beta_j\Big) S^k T \\
&\quad + \sum_{2\le \ell\le k} C_{k+1}^{\ell}\Big(\sum_{1\le j\le k} j^{\ell}\beta_j\Big) S^{k+1-\ell}T^{\ell} \\
&\quad + \Big(\sum_{1\le j\le k} \beta_j j^{k+1}\Big) T^{k+1}\ .
\end{aligned} \tag{3.3}
$$

Let's solve the non singular $k\times k$ linear system with unknowns β_j :

$$
\sum_{1\le j\le k} j\beta_j = \frac{1}{k+1}
$$

$$
\sum_{1\le j\le k} j^{\ell}\beta_j = 0\ , \quad 2\le \ell\le k\ .
$$

Then (3.3) gives

$$
S^kT = \sum_{1\le j\le k} \beta_j (S+jT)^{k+1} - \Big(\sum_{1\le j\le k} \beta_j j^{k+1}\Big) T^{k+1} - \Big(\sum_{j=1}^{k} \beta_j\Big) S^{k+1}\ ,
$$

which gives the result.

Remark 3.2. — An immediate consequence of this lemma is that for any integer k, there exists $\alpha(k)>0$, so that for any k–multilinear symmetric form A on $\mathbb{R}^n$, and any norm $\|\cdot\|$ on $\mathbb{R}^n$

$$
1\le \sup_{\substack{T_j\in\mathbf{R}^n\\ \|T_j\|=1}} |A(T_1,\dots,T_k)| \Big[\sup_{\substack{T\in\mathbf{R}^n\\ \|T\|=1}} |A(T,\dots,T)|\Big]^{-1} \le \alpha(k)\ . \tag{3.5}
$$

b. The proper class of a symbol.

Let's now recall the definition of an Hörmander metric on $\mathbb{R}^{2n}$ (see [11] section 18.5). For each $X\in\mathbb{R}^{2n}$, G_X is a positive definite quadratic form on $\mathbb{R}^{2n}$ such that the three following properties are satisfied.

(3.6) There exists $C>0$ such that for any $X,Y,T\in\mathbb{R}^{2n}$, $G_X(Y-X)\le C^{-1}$ implies $C^{-1}G_Y(T)\le G_X(T)\le CG_Y(T)$.

(3.7) For any $X, T \in \mathbb{R}^{2n}$, $G_X(T) \leq G^\sigma_X(T)$ where
$G^\sigma_X(T) = \inf_{G_X(U)=1} \sigma(T,U)^2$, σ the symplectic form on $\mathbb{R}^{2n}$.

There exists $C > 0$, N such that, for any $X, Y, T \in \mathbb{R}^{2n}$,

$$G_X(T) \leq C G_Y(T)(1 + G^\sigma_X(X - Y))^N \ . \tag{3.8}$$

Let's also define the reciprocal Planck function

$$\Lambda_G(X) = \inf_T \left(\frac{G^\sigma_X(T)}{G_X(T)}\right)^{\frac{1}{2}} \quad \text{(note that (3.7) implies } \Lambda \geq 1). \tag{3.9}$$

A function $a \in C^\infty(\mathbb{R}^{2n})$ belongs to $S^m(G)$ if for any k, there exists C_k such that

$$|a^k(X)T^k| \leq C_k \Lambda_G(X)^m G_X(T)^{k/2} \ , \tag{3.10}$$

any $X, T \in \mathbb{R}^{2n}$.

The semi-norms of a are the best constants

$$\gamma_{k,G}(a) = \sup_{\substack{X,T \in \mathbb{R}^2 \\ G_X(T)=1}} |a^k(X)T^k| \Lambda_G(X)^{-m} \ . \tag{3.11}$$

Note that, from (3.5), we have

$$\sup_{\substack{X, T_1, \dots, T_k \in \mathbb{R}^{2n} \\ G_X(T_j)=1}} |a^{(k)}(X)T_1 \dots T_k| \Lambda_G(X)^{-m} \leq \alpha(k)\gamma_{k,G}(a) \ .$$

For a given in $S^2(G)$ (see 3.10), let's consider

$$g_X = \Lambda_G(X)\lambda(X)^{-1}G_X \ , \tag{3.12}$$

with

$$\lambda^2(X) = \max_{0 \leq k \leq 3} \left(1, \|a^{(k)}(X)\|_{G_X}^{\frac{4}{4-k}} \Lambda_G(X)^{-\frac{2k}{4-k}}\right) \ , \tag{3.13}$$

where $\|a^{(k)}(X)\|_{G_X} = \sup_{T, G_X(T)=1} |a^{(k)}(X)T^k|$.

The next proposition summarizes the properties of a Calderón-Zygmund decomposition of a symbol (see [3], [6]).

PROPOSITION 3.3. — (a) *The metric g defined by (3.12) is an Hörmander metric, i.e. satisfies (3.6), (3.7), (3.8). The constants in (3.6) for g depend only on the constants in (3.6) for G and on $(\gamma_{kG}(a))_{0 \leq k \leq 4}$.*

(b) *We have $\lambda_g(X) = \lambda(X)$, according to (3.12) and (3.9).*

(c) *The symbol $a \in S^2(g)$ and*

$$\gamma_{kg}(a) \leq 1 \ \text{ if } k \leq 3$$

$$\gamma_{kg}(a) \le \gamma_{kG}(a) \max_{0\le \ell \le 3} \left(1, (\gamma_{\ell G}(a))^{\frac{k-4}{4-\ell}}\right) \text{ if } k \ge 4 \ .$$

(d) *We have* $\lambda(X) \le \Lambda(X) \max_{0\le k\le 3} \left(1, (\gamma_{kG}(a))^{\frac{2}{4-k}}\right)$.

Proof. — The points (b) and (a) (3.7) are obvious from the definitions. Let's prove (c) : if $k \le 3$, from (3.13) we get

$$(3.14) \quad \|a^{(k)}(X)\|_{g_X} = \|a^{(k)}(X)\|_{G_X} \Lambda(X)^{-\frac{k}{2}} \lambda(X)^{\frac{k}{2}} \le \Lambda^{-\frac{k}{2}} \lambda^{\frac{k}{2}} \left(\lambda^2 \Lambda^{\frac{2k}{4-k}}\right)^{\frac{4-k}{4}} = \lambda(X)^2 \ .$$

Moreover, if $\ell \ge 4$, we have

$$(3.15) \quad \|a^{(\ell)}(X)\|_{g_X} = \|a^{(\ell)}(X)\|_{G_X} \Lambda(X)^{-\frac{\ell}{2}} \lambda(X)^{\frac{\ell}{2}} \le \Lambda^{2-\frac{\ell}{2}} \lambda^{\frac{\ell}{2}-2} \lambda^2 \gamma_{\ell G}(a) = \lambda^2 \gamma_{\ell G}(a) \left(\frac{\lambda^2}{\Lambda^2}\right)^{\frac{\ell-4}{4}} .$$

But, from (3.13), we have

$$(3.16) \quad \Lambda(X)^{-2}\lambda(X)^2 \le \max_{0\le k\le 3} \left(1, \gamma_{kG}(a)^{\frac{4}{4-k}}\right) \max_{0\le k\le 3} \left(\Lambda(X)^{\frac{8-2k}{4-k}-2}, 1\right) ,$$

so we obtain, from (3.15), (3.16) and $\ell \ge 4$,

$$(3.17) \quad \|a^{(\ell)}(X)\|_{g_X} \le \lambda(X)^2 \gamma_{\ell G}(a) \max_{0\le k\le 3} \left(1, \gamma_{kG}(a)^{\frac{\ell-4}{4-k}}\right) .$$

So (3.14) and (3.17) gives (c) in proposition 3.3.

Note also that (3.16) gives

$$(3.18) \quad \frac{\lambda(X)}{\Lambda(X)} \le \max_{0\le k\le 3} \left(1, \gamma_{kG}(a)^{\frac{2}{4-k}}\right) ,$$

that is (d) in proposition 3.3. Then it suffices to prove (a) (3.6) since the proposition 18.5.6 in Hörmander's book [11] can be applied (g is conformal to G), to get (a) (3.8). As pointed out in [11] (1.4.5)', it is enough to prove the existence of δ, C such that

$$(3.19) \quad g_X(Y-X) \le \delta \text{ implies } g_Y \le C g_X \ .$$

Let us first remark that

$$(3.20) \quad \lambda^2(X) = \max_{0\le k\le 3} (1, \|a^{(k)}(X)\|_{g_X}) \ .$$

As a matter of fact, we get

$$(3.21) \quad \max_{0\le k\le 3} (1, \|a^{(k)}(X)\|_{g_X}) \le \lambda^2(X) ,$$

from the proposition 3.3 (c) (already proved!) and $\lambda \geq 1$. Conversely, if $\max_{0\leq k\leq 3} \left(\|a^{(k)}(X)\|_{g_X}^{\frac{4}{4-k}} \Lambda(X)^{-\frac{2k}{4-k}}\right) > 1$, then, from (3.13), we obtain

$$\lambda^2(X) = \|a^{(k_0)}(X)\|_{g_X}^{\frac{4}{4-k_0}} \Lambda(X)^{-\frac{2k_0}{4-k_0}} ,$$

for some k_0, $0 \leq k_0 \leq 3$ (X is fixed). Thus, (3.12) gives

$$\lambda^2(X) = \|a^{(k_0)}(X)\|_{g_X}^{\frac{4}{4-k_0}} \Lambda(X)^{\frac{k_0}{2}(\frac{4}{4-k_0})} \lambda(X)^{-\frac{k_0}{2}\frac{4}{4-k_0}} \Lambda(X)^{-\frac{2k_0}{(4-k_0)}} , \quad i.e.$$

$$\lambda^2(X) = \|a^{(k_0)}(X)\|_{g_X} . \tag{3.22}$$

Now, if $\max_{0\leq k\leq 3} \left(\|a^{(k)}(X)\|_{G_X}^{\frac{4}{4-k}} \Lambda(X)^{-\frac{2k}{4-k}}\right) \leq 1$, we have, from (3.13),

$$\lambda^2(X) = 1 . \tag{3.23}$$

So we get (3.20) from (3.21), (3.22), (3.23). Let us assume that G is slowly varying (*i.e.* satisfies (3.6)) with a constant C_0, and take δ such that

$$\delta C_0 \max_{0\leq k\leq 3} \left(1, \gamma_{kG}(a)^{\frac{2}{4-k}}\right) \leq 1 . \tag{3.24}$$

If $g_X(Y-X) \leq \delta$, then $G_X(Y-X) \leq \dfrac{\lambda(X)}{\Lambda(X)}\delta$, and thus, using proposition 3.3 (d) (already proved) we obtain,

$$G_X(Y-X) \leq \max_{0\leq k\leq 3} \left(1, \gamma_{kG}(a)^{\frac{2}{4-k}}\right)\delta \leq C_0^{-1} \text{ by (3.24)} .$$

Consequently $C_0^{-1}G_X \leq G_Y \leq C_0 G_X$. We need thus only to estimate from above the ratio $\dfrac{\lambda(X)}{\lambda(Y)}$ to get the conclusion of (3.19).

From Taylor's formula, $0 \leq k \leq 3$, we have

$$\begin{aligned} \|a^{(k)}(X)\|_{g_X} \leq & \sum_{k\leq \ell\leq 3} \frac{1}{(\ell-k)!} \|a^{(\ell)}(Y)\|_{g_X} g_X(Y-X)^{\frac{\ell-k}{2}} \\ & + \frac{1}{(4-k)!} \sup_{Z\in[X,Y]} \|a^{(4)}(Z)\|_{g_X} g_X(Y-X)^{\frac{4-k}{2}} \\ \leq & \sum_{k\leq \ell\leq 3} \|a^{(\ell)}(Y)\|_{g_Y} \left(\frac{\lambda(X)}{\lambda(Y)}\right)^{\frac{\ell}{2}} C_1 \delta^{\frac{\ell-k}{2}} + \delta^{\frac{4-k}{2}} \gamma_{4G}(a) C_1 \lambda(X)^2 , \end{aligned} \tag{3.25}$$

where C_1 depends only on C_0, since $\|a^{(4)}(Z)\|_{g_X} = \|a^{(4)}(Z)\|_{\frac{\Lambda(X)}{\lambda(X)}G_X}$, and thus

$$\|a^{(4)}(Z)\|_{g_X} = \|a^{(4)}(Z)\|_{G_X} \frac{\lambda(X)^2}{\Lambda(X)^2} \leq \gamma_{4,G}(a) C_1 \Lambda(X)^2 \frac{\lambda(X)^2}{\Lambda(X)^2} ;$$

we have in fact $Z=(1-\theta)X+\theta Y$, $\theta\in[0,1]$ and consequently $G_X(Z-X)\le G_X(Y-X)\le C_0^{-1}$, so that $C_0^{-1}G_X\le G_Z\le C_0G_X$, and $\Lambda(X)$ is equivalent to $\Lambda(Z)$.

Now, using (3.20), we may assume $\lambda^2(X)=\|a^{(k)}(X)\|_{g_X}$, for some k, $0\le k\le 3$ (otherwise $\lambda(X)=1$ and since $\lambda(Y)\ge 1$ we get $\dfrac{\lambda(X)}{\lambda(Y)}\le 1$).

We obtain, from (3.20) and (3.25),

$$\lambda^2(X)\le\lambda(Y)^2\sum_{k\le\ell\le 3}C_1\Big(\frac{\lambda(X)}{\lambda(Y)}\Big)^{\frac{\ell}{2}}\delta^{\frac{\ell-k}{2}}+\delta^{\frac{4-k}{2}}\gamma_{4G}(a)C_1\lambda(X)^2\,. \tag{3.26}$$

Consequently, if δ satisfies (3.24) and

$$\sup_{0\le k\le 3}\delta^{\frac{4-k}{2}}\gamma_{4G}(a)C_1\le\frac{1}{2}\,, \tag{3.27}$$

we get

$$\Big(\frac{\lambda(X)}{\lambda(Y)}\Big)^2\le 2C_1\sum_{k\le\ell\le 3}\Big(\frac{\lambda(X)}{\lambda(Y)}\Big)^{\frac{\ell}{2}}\delta^{\frac{\ell-k}{2}}\,,$$

so the term $\dfrac{\lambda(X)}{\lambda(Y)}$ must be bounded from above by a constant depending on δ and C_1, that is on C_0 (in (3.6) for G) and $(\gamma_{kG}(a))_{0\le k\le 4}$. The proof of proposition 3.3 is complete.

c. Fefferman-Phong's classification.

The rest of section 4 is devoted to the proof of the following proposition, ensuring that Fefferman-Phong's classification is still valid under a positivity assumption on averages.

PROPOSITION 3.4. — *Let G be an Hörmander metric on $\mathbb{R}^{2n}$ (i.e. satisfying (3.6), (3.7) and (3.8)), a a symbol in $S^2(G)$ (see (3.10)) and g the proper metric of a defined by (3.12) and proposition 3.3.*

(3.28) *Let us assume that the averages of a on $G-$balls of symplectic volume 1 are non-negative.*

Then, there exists positive constants C, ρ depending only on a finite number of semi-norms of a such that the proper metric of a is made with three types of "boxes" : for any X_0 in $\mathbb{R}^{2n}$ and any X in the g_{X_0} ball of radius ρ and center X_0,

(1) *Either* $\lambda(X) \leq C$,

(2) or $a(X)\lambda(X)^{-2} \geq C^{-1}$

(3) or $a(X) = e_0(X)(X_1 - \alpha(X'))^2 + b(X')$.

Here X_1, X' *is a set of linear symplectic coordinates, and the functions* $\lambda(X_0)^{-1}e_0$, $\lambda(X_0)^{-2}b$, $\lambda(X_0)^{-1/2}\alpha$ *satisfy the estimates of* $S^0(g)$ *with semi-norms controlled by those of* a *and* $\lambda^{-1}e_0 \geq C^{-1} > 0$.

Let's give a

DEFINITION 3.5. — *Given* δ, ε *positive numbers,* $a \in \mathcal{A}(\delta, \varepsilon)$ *means that* a *is a* C^∞ *function on* $|X| \leq 1$ *so that*

(i) $|a^{(k)}(X)| \leq 1, \ 0 \leq k \leq 4$

(ii) $\max\limits_{0 \leq k \leq 3} |a^{(k)}(0)| \geq \delta$

(iii) *The averages of* a *on balls of radius* ε *(included in the unit ball) are non-negative.*

The proposition 3.4 is a consequence of the following lemma by rescaling, using (3.20).

LEMMA 3.6. — *Let* δ *be a positive number. There exist* $r(\delta)$, $\varepsilon(\delta)$ *and* $\omega(\delta)$ *positive so that, if* $a \in \mathcal{A}(\delta, \varepsilon)$ *with* $\varepsilon \leq \varepsilon(\delta)$, *then, on* $|X| \leq r(\delta)$,

(1) *Either* $a(X) \geq \omega(\delta)$

(2) *Or, for some choice of euclidean coordinates,*

$$a(X) = a(\alpha(X'), X') + e_0(X)(X_1 - \alpha(X'))^2$$

with $e_0(X) \geq \omega(\delta)$ *and, for all* k,

$$\max_{|X| \leq r(\delta)} (|e_0^{(k)}(X) + |\alpha^{(k)}(X)|) \leq C(k)F(\max_{\substack{|X| \leq r(\delta) \\ \ell \leq k+2}} |a^{(\ell)}(X)|) \ .$$

Proof. — Let $\mu_0 \in]0, 1]$.

(1) Assume $|a(0)| \geq \mu_0 > 0$.

If $a(0) \leq -\mu_0$, $a(X) \leq -\mu_0/2$ on $|X| \leq \mu_0/2$, which contradicts (iii) if $\varepsilon \leq \mu_0/2$, $\mu_0 \leq 1$.

Consequently, we have in that case $|a(X)| \geq \mu_0/2$ on $|X| \leq \mu_0/2$.

(2) Assume $|a(0)| < \mu_0$, $|a'(0)| \geq \mu_1 > 0$.

Then, for some choice of euclidean coordinates,

$$\frac{\partial a}{\partial X_1}(X_1, X') \geq \frac{\mu_1}{2} \quad \text{on} \quad |X| \leq \frac{\mu_1}{2} \leq 1 ,$$

and thus

$$a(X_1, 0) = a(0,0) + e_0(X_1)X_1 \ , \quad e_0(X_1) \geq \mu_1/2 \ , \quad \text{if } |X_1| \leq \mu_1/2 \ .$$

Thus

$$a(X_1, 0) \leq \mu_0 - \frac{\mu_1^2}{2^3} \leq -\frac{\mu_1^2}{2^4} \ , \quad \text{if } -\frac{\mu_1}{2} \leq X_1 \leq -\frac{\mu_1}{2^2} \quad \text{and} \quad \mu_0 \leq \frac{\mu_1^2}{2^4} \ .$$

Consequently

$$a(X_1, X') \leq a(X_1, 0) + |X'| \leq -\frac{\mu_1^2}{2^5} \ , \ \text{if } |X'| \leq \frac{\mu_1^2}{2^5} \leq 1, \ -\frac{\mu_1}{2} \leq X_1 \leq -\frac{\mu_1}{4},$$

which contradicts (iii) if $\varepsilon \leq \dfrac{\mu_1^2}{32} , \dfrac{\mu_1}{4} \leq 1$.

(3) Assume $|a(0)| < \mu_0$, $|a'(0)| < \mu_1$, $|a''(0)| \geq \mu_2 > 0$. Then, for some choice of euclidean coordinates (see lemma 3.1),

$$\left|\frac{\partial^2 a}{\partial X_1^2}(X)\right| \geq \frac{\mu_2}{2} \quad \text{on} \quad |X| \leq \frac{\mu_2}{2} \leq 1 \ .$$

If $\left|\dfrac{\partial^2 a}{\partial X_1^2}(0)\right| \leq -\mu_2$, $\dfrac{\partial^2 a}{\partial X_1^2}(X) \leq -\dfrac{\mu_2}{2}$ on $|X| \leq \dfrac{\mu_2}{2} \leq 1$. Thus

$$a(X_1, 0) \leq \mu_0 + \mu_1|X_1| - \frac{\mu_2}{4}X_1^2 \quad \text{on} \quad \frac{\mu_2}{4} \leq X_1 \leq \frac{\mu_2}{2} \leq 1 \ .$$

Consequently, there, we have

$$a(X_1, 0) \leq \mu_0 + \frac{\mu_1\mu_2}{2} - \frac{\mu_2^3}{2^6} \leq -\frac{\mu_2^3}{2^7} \quad \text{if} \quad \mu_0 + \frac{\mu_1\mu_2}{2} \leq \frac{\mu_2^3}{2^7}$$

e.g. if

$$\mu_0 \leq \frac{\mu_2^3}{2^8} \ , \quad \frac{\mu_1}{2} \leq \frac{\mu_2^2}{2^8} \ .$$

Then, we get $a(X_1, X') \leq -\dfrac{\mu_2^3}{2^8}$ if $|X'| \leq \dfrac{\mu_2^3}{2^8}$, which contradicts (iii) if

$$\varepsilon < \frac{\mu_2^3}{2^8} \ , \quad \mu_2 \leq 1 \ .$$

(4) From case (3) above, if $|a(0)| \leq \mu_0$, $|a'(0)| < \mu_1$, $|a''(0)| \geq \mu_2$, we must have $\dfrac{\partial^2 a}{\partial X_1^2}(X_1, X') \geq \dfrac{\mu_2}{2}$ on $|X| \leq \dfrac{\mu_2}{2}$.

Moreover, $\dfrac{\partial a}{\partial X_1}(X_1, X') = \dfrac{\partial a}{\partial X_1}(0, X') + e_0(X)X_1$, with $e_0(X) \geq \dfrac{\mu_2}{2}$, if $|X| \leq \dfrac{\mu_2}{2}$. Thus

$$\Big|\frac{\partial a}{\partial X_1}(0, X')\Big| \leq \mu_1 + |X'| \leq 2\mu_1 \ , \quad \text{if} \quad |X'| \leq \mu_1 \ .$$

Thus

$$\frac{\partial a}{\partial X_1}\big(\frac{\mu_2}{2}, X'\big) \geq -2\mu_1 + \frac{\mu_2^2}{4} > 0 \ , \quad \text{if} \quad \frac{\mu_2^2}{4} > 2\mu_1 \ .$$

Also

$$\frac{\partial a}{\partial X_1}\big(- \frac{\mu_2}{2}, X'\big) \leq 2\mu_1 - \frac{\mu_2^2}{4} < 0 \ .$$

So for any X', $|X'| \leq \inf(\mu_1, \frac{\mu_2}{2}) \leq 1$, there exists $a(X') \in (-\frac{\mu_2}{2}, \frac{\mu_2}{2})$, $\dfrac{\partial a}{\partial X_1}(\alpha(X'), X') = 0$.

By the implicit function theorem, α is a smooth function and its K^{th} derivatives are controlled by fixed polynomials of the $(k+1)^{th}$ derivatives of a and $\big(\dfrac{\partial^2 a}{\partial X_1^2}\big)^{-1}$. We get then

$$a(X_1, X') = a(\alpha(X'), X') + e(X)(X_1 - \alpha(X'))^2 \ , \quad e(X) \geq \frac{\mu_2}{2} \ ,$$

on $|X| \leq \inf(\mu_1, \frac{\mu_2}{2}) \leq 1$.

(5) If $|a(0)| < \mu_0$, $|a'(0)| < \mu_1$, $|a''(0)| < \mu_2$, $|a'''(0)| \geq \mu_3$.

We get, from lemma 3.1, $\dfrac{\partial^3 a}{\partial X_1^3}(X) \geq \dfrac{\mu_3}{2}$ if $|X| \leq \dfrac{\mu_3}{2} \leq 1$.

Then on $-\dfrac{\mu_3}{2} \leq X_1 \leq -\dfrac{\mu_3}{4}$

$$a(X_1, 0) \leq \mu_0 + \mu_1 \frac{\mu_3}{2} + \frac{1}{2}\mu_2\big(\frac{\mu_3}{2}\big)^2 - \frac{1}{6}\frac{\mu_3}{2}\big(\frac{\mu_3}{4}\big)^3 \leq -\frac{1}{2}\frac{1}{6}\frac{\mu_3^4}{2^7}$$

if $\mu_0 < \dfrac{1}{3}\dfrac{1}{2}\dfrac{1}{6}\dfrac{\mu_3^4}{2^7}$, and $\mu_1 < \dfrac{1}{3}\dfrac{1}{6}\dfrac{\mu_3^3}{2^7}$, and $\mu_2 < \dfrac{1}{3}\dfrac{1}{6}2^2\dfrac{\mu_3^2}{2^7}$.

Thus, we have,

$$a(X_1, X') \leq -\frac{1}{4}\frac{1}{6}\frac{\mu_3^4}{2^7} \ , \quad \text{if} \quad |X'| \leq \frac{1}{4}\frac{1}{6}\frac{\mu_3^4}{2^7} \ , \quad -\frac{\mu_3}{2} \leq X_1 \leq -\frac{\mu_3}{4} \ ,$$

which contradicts (iii) if $\varepsilon \leq \min\big(\frac{1}{4}\frac{1}{6}\frac{\mu_3^4}{2^7}, \frac{\mu_3}{4}\big)$.

(6) Assume $|a(0)| < \mu_0$, $|a'(0)| < \mu_1$, $|a''(0)| < \mu_2$, $|a'''(0)| \leq \mu_3$. This contradicts (ii) if $\max(\mu_0, \mu_1, \mu_2, \mu_3) < \delta$.

Eventually, we have to perform the following successive (and compatible!) choices with $\delta > 0$ given.

First choose

$$\mu_3 \ , \qquad 0 < \mu_3 < \min\left(\frac{\delta}{2}, 1\right)$$

then

$$\mu_2 \ , \qquad 0 < \mu_2 \leq \min\left(\frac{1}{3}\frac{1}{6}\frac{\mu_3^2}{2^6}, \frac{\delta}{2}, 1\right)$$

then

$$\mu_1 \ , \qquad 0 < \mu_1 \leq \min\left(\frac{1}{3}\frac{1}{6}\frac{\mu_3^3}{2^7}, \frac{\delta}{2}, \frac{\mu_2^2}{2^7}, 1\right)$$

then

$$\mu_0 \ , \qquad 0 < \mu_0 \leq \min\left(\frac{1}{3}\frac{1}{2}\frac{1}{6}\frac{\mu_3^4}{2^7}, \frac{\delta}{2}, \frac{\mu_2^3}{2^8}, 1, \frac{\mu_1^2}{16}\right) .$$

Then $\varepsilon(\delta)$ can be taken as

$$\varepsilon(\delta) = \min\left(\frac{1}{4}\frac{1}{6}\frac{\mu_3^4}{2^7}, \frac{\mu_2^3}{2^8}, \frac{\mu_1^2}{2^5}, \frac{\mu_0}{2}\right)$$

and

$$r(\delta) = \min\left(\frac{\mu_0}{2}, \mu_1, \frac{\mu_2}{2}\right) \ , \quad \omega(\delta) = \mu_2 \ .$$

The lemma 3.6 is proved.

End of the proof of proposition 3.4.

Let's now consider $X_0 \in \mathbb{R}^{2n}$. We can assume (see lemma 18.6.4 in[11]) that

$$g_{X_0} = g = \sum_{j=1}^{n} \lambda_j^{-1}(dx_i^2 + d\xi_j^2) \ , \quad \text{with} \quad \inf \lambda_j = \lambda = \lambda(X_0) \ . \tag{3.29}$$

We are going to apply lemma 3.6 to

$$A(t_1, \ldots, t_n, \tau_1, \ldots, \tau_n) = \lambda^{-2} a\big(X_0 + \sum_{j=1}^{n} t_j \lambda_j^{1/2} e_j + \tau_j \lambda_j^{1/2} \varepsilon_j\big) \ ,$$

where (e_j, ε_j) is the "canonical" basis of $\mathbb{R}^{2n}$ in the coordinates corresponding to the diagonalization of g in (3.29). The case (1) and (2) in proposition 3.4 are easily obtained from (1) of lemma 3.6 or by assuming $\lambda \leq C$. The case (3) is obtained as follows : we get, assuming $X_0 = 0$,

$$(\eta_1 - \alpha(y, \eta'))^2 e_0(y, \eta)\lambda^2 + b(y, \eta')\lambda^2 = a\Big(\sum_{j=1}^{n} t_j \lambda_j^{1/2} e_j + \tau_j \lambda_j^{1/2} \varepsilon_j\Big) \ ,$$

where $\begin{pmatrix}\eta\\ y\end{pmatrix} = \Omega\begin{pmatrix}\tau\\ t\end{pmatrix}$, where the $2n \times 2n$ matrix Ω is orthogonal.

So, we obtain, setting-up $x_j = t_j\lambda_j^{1/2}$, $\xi_j = t_j\lambda_j^{1/2}$,

$$a(x,\xi) = \lambda\Big(\lambda^{1/2}\eta_1\Big(\frac{x_j}{\lambda_j^{1/2}}, \frac{\xi_j}{\lambda_j^{1/2}}\Big) - \lambda^{1/2}\alpha\Big(y\Big(\frac{x_j}{\lambda_j^{1/2}}, \frac{\xi_j}{\lambda_j^{1/2}}\Big), \eta'\Big(\frac{x_j}{\lambda_j^{1/2}}, \frac{\xi_j}{\lambda_j^{1/2}}\Big)\Big)\Big)^2$$

$$e_0\Big(y\Big(\frac{x_j}{\lambda_j^{1/2}}, \frac{\xi_j}{\lambda_j^{1/2}}\Big), \eta\Big(\frac{x_j}{\lambda_j^{1/2}}, \frac{\xi_j}{\lambda_j^{1/2}}\Big)\Big) + \lambda^2 b\Big(y\Big(\frac{x_j}{\lambda_j^{1/2}}, \frac{\xi_j}{\lambda_j^{1/2}}\Big), \eta'\Big(\frac{x_j}{\lambda_j^{1/2}}, \frac{\xi_j}{\lambda_j^{1/2}}\Big)\Big) .$$

Through a linear symplectic change of coordinates we have, on the $g-$ball with center 0 and radius $r(\delta)$

$$a(x,\xi) = \lambda(\eta_1 - \beta(y,\eta'))^2 e_0(y,\eta) + b_0(y,\eta')\lambda^2 \ , \tag{3.30}$$

that is the result of proposition 3.4.

d. Sharp Egorov principle.

The problem can be microlocalized. If $\{\chi_v(x,\xi)^2\}_v$ is a partition of unity related to the metric g (see [11] section 18.5)

$$a^w = \sum_v (\chi_v^2 a)^w = \sum_v (\chi_v \sharp a \sharp \chi_v)^w + r^w \ , \quad \text{where } r^w \text{ is } L^2 \text{ bounded.}$$

Moreover, if $\operatorname{supp}\chi_v \subset Q_v(a\, g_v-\text{ball})$, and if $a(x,\xi) = a(x,\xi)$ on Q_v^* (the g_v-ball with same center and double radius), a satisfying the estimates of g_v, we have

$$\chi_v \sharp a \sharp \chi_v = \chi_v \sharp a \sharp \chi_v + \widetilde{r}_v \ , \quad \Sigma \widetilde{r}_v^{\,w}\ L^2 \text{ bounded.}$$

We are consequently reduced to look at

$$a = (\eta_1 - \alpha(y,\eta'))^2 \lambda e_0(y,\eta) + b(y,\eta') \tag{3.3.1}$$

that can be easily extended to $\mathbb{R}^{2n}$ (and still satisfy the estimates).

For the reader's convenience, we state a version of the "Sharp Egorov Principle", proved by Fefferman and Phong in [6].

THEOREM 3.7. — *Let g be a quadratic form on $\mathbb{R}^{2n}$ such that $g = \lambda^{-1}\Gamma$, where $\lambda \geq 1$ and Γ is a quadratic form such that $\Gamma = \Gamma^\sigma$ (see (3.7)).*

Let $a \in S^2(g)$ real valued (i.e. $|a^{(k)}(X)T^k| \leq \gamma_{kg}(a)\lambda^{2-\frac{k}{2}}$) supported in Q, a $g-$ball of radius 1. Let χ be a canonical transformation $\chi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\chi^*(\sigma) = \sigma$ and

$$\begin{aligned} &|\chi^{(k)}(X)|_\Gamma \leq \gamma_k(\chi)\lambda^{\frac{1}{2}-\frac{k}{2}} \\ &|\chi'(X)^{-1}|_\Gamma \leq \gamma_1(\chi) \ . \end{aligned} \tag{3.31}$$

Then, there exists a Fourier integral operator U, bounded on $L^2(\mathbb{R}^n)$ and $r \in S^0(g)$ so that the operator

$$a^w = U^*(a \circ \chi)^w U + r^w \ . \tag{3.32}$$

4. PSEUDO-DIFFERENTIAL OPERATORS IN ONE DIMENSION

a. Preliminaries.

We wish to prove here a one-dimensional version of Fefferman and Phong's conjecture ♯5 in section 7 of [6]. As a matter of fact, we can prove here an inequality with gain of two full derivatives ($\varepsilon = 0$ with Fefferman-Phong's notation). Let's set-up our framework. Let G be an Hörmander metric (*i.e.* (3.6), (3.7), (3.8) are satisfied) such that $G_X = \Lambda(X)^{-1}\Gamma_X$, where $\Gamma_X = \Gamma_X^\sigma$ (see (3.7)) and $\Lambda(X) \geq 1$. Note that this is the case for the classical metric

$$dx^2 + \frac{d\xi^2}{\langle\xi\rangle^2} = \langle\xi\rangle^{-1}(\langle\xi\rangle dx^2 + \langle\xi\rangle^{-1}d\xi^2) \ , \quad \langle\xi\rangle^2 = 1 + |\xi|^2 \ .$$

We shall denote $g_X = \lambda(X)^{-1}\Gamma_X$ the proper conformal metric of a symbol a (λ is defined in (3.13)). Our first assumption will be

$$\int_{X, \pi\Gamma_Y(X-Y)\leq 1} a(X)dX \geq 0 \quad \text{for any} \quad Y \in \mathbb{R}^{2n} \ . \tag{4.1}$$

Moreover, using propositions 3.4 and 3.5, since (4.1) implies (3.28), we are reduced to consider $g-$balls on which ($g = \lambda^{-1}\Gamma$, $\Gamma = dy^2 + d\eta^2$)

$$a = (\eta - \lambda^{1/2}\alpha(y))^2 e_0(y,\eta)\lambda + W(y)\lambda^2 \ , \tag{4.2}$$

with

$$|a^{(k)}(y)| + |e_0^{(k)}(y,\eta)| + |W^{(k)}(y)| \leq C_k\lambda^{-k/2} \ , \tag{4.3}$$

and

$$e_0(y,\eta) \geq C_1^{-1} > 0 \ . \tag{4.4}$$

Note also that, dividing the symbol a by one of its semi-norm, we can assume that $\gamma_{kG}(a) \leq 1$ for $k \leq N_0$, with an arbitrary N_0, to be chosen.

Consequently, all the constants C_k in (4.3) up to $k = \theta(N_0)$ ($\theta(N) \to \infty$ with N) are "universal constants".

It is an easy matter to extend the functions α, e_0, W to the whole $\mathbb{R}^{2n}$, keeping their properties :

$$\widetilde{\alpha}(y) = \alpha(y)\psi(y\lambda^{-1/2}) ,$$
$$\psi \in C_0^\infty(\mathbb{R}) , \quad \psi \equiv 1 \text{ on } (-2,+2)$$
$$0 \le \psi \le 1 , \quad \psi \equiv 0 \text{ outside } (-3,+3) .$$

We define $\widetilde{e}_0(y,\eta) = e_0(y,\eta)\psi(\frac{y^2+\eta^2}{\lambda}) + \big(1-\psi(\frac{y^2+\eta^2}{\lambda})\big)$ and $W(y) = W(y)\psi(y\lambda^{-1/2})$.

We get then that (4.2) is satisfied on $y^2+\eta^2 \le \lambda$, but with a right-hand side so that (4.3) and (4.4) are still satisfied. In what follows we'll keep using the notation α, e_0, W dropping the $\sim$.

Let's use now the Egorov theorem (th. 3.7). We consider the canonical transformation χ^{-1} on $\mathbb{R}^2$

$$\text{(4.5)} \qquad \begin{cases} \tau = \eta - \alpha(y) \\ t = y . \end{cases}$$

The estimates (3.31) are satisfied, and we get that

$$a(\chi(t,\tau)) = \lambda\tau^2 e_0 + \lambda^2 W(t) \quad \text{on} \quad t^2+\tau^2 \le \lambda .$$

Consequently, from (3.32)

$$((a\circ\chi)(\psi\circ\lambda^{-1}\Gamma\circ\chi)^2)^w \equiv U^*(a(\psi\circ\lambda^{-1}\Gamma)^2)^w U \quad \text{modulo} \quad \mathcal{L}(L^2) .$$

Moreover $a(\psi\circ\lambda^{-1}\Gamma)^2 = \psi\circ\lambda^{-1}\Gamma\,\sharp\, a\,\sharp\,\psi\circ\lambda^{-1}\Gamma + r$, $r^w \in \mathcal{L}(L^2)$. So in order to check the non-negativity of a^w, we need only to check the one of

$$\big[(\lambda\tau^2 e_0 + \lambda^2 W(t))(\psi\circ\lambda^{-1}\Gamma\circ\chi)^2\big]^w ,$$

and thus eventually, because of the symbolic calculus, the one of $\lambda\tau^2 e_0 + \lambda^2 W(t)$.

b. Statement of the result.

Let's introduce the following family of canonical transformations : $\chi \in \Phi$ if χ is C^∞, canonical, and satisfies

$$\text{(4.6)} \qquad \begin{cases} |\chi^{(k)}(X)|_{\Gamma_X} \le C_k(\chi)\lambda(X)^{\frac12-\frac k2} , \\ g_X(\chi(X)-X) \le \delta_0 , \end{cases}$$

where $\delta_0 \le C^{-1}$ in (3.6) for g.

We'll denote by Q any symplectic unit cube of

$$\mathbb{R}^2 : Q = \{(t,\tau), \max(|t|,|\tau|) \leq 1\}$$

for some linear symplectic coordinates.

THEOREM 4.1. — *There exists $C_{00} > 0$ such that, if (4.1) is satisfied for $a \in S^2(G)$ such that*

$$\int_Q (a \circ \chi) dX + C_{00} \inf_Q (a \circ \chi) \geq 0 \tag{4.7}$$

for any symplectic cube Q and any $\chi \in \Phi$, then

$$a^w + C \geq 0 \,, \tag{4.8}$$

where C depends only on a fixed finite number of semi-norms of a.

Proof. — Using the preliminaries and a linear rescaling, it is enough to prove the non-negativity of $C_1^{-1}\tau^2 + V(t)$ on functions supported in $|t| \leq 1$.

Using theorem 2.1 and remark 2.2 we need to check for Q interval,

$$|Q| \leq 2 \,, \quad Q \subset (0,3) : |Q|^{-1} \int_Q C_1 V(t) dt + |Q|^{-2} \geq 8|Q|^{-1} \int_Q C_1 V_-(t) dt \,.$$

Since we can assume $a \circ \chi = e_0 \tau^2 + V(t)$ on $|t| \leq 4$, $|\tau| \leq \lambda$, we have from (4.7)

$$\delta^{-2} + \delta^{-1} \int_Q V(t) dt \geq C_{00} \max_{t \in Q} V_-(t) \,,$$

and so

$$C_1 \delta^{-2} + \delta^{-1} \int_Q C_1 V_+(t) dt \geq C_{00} \max_{t \in Q} C_1 V_-(t) \,.$$

If $C_{00} \geq 9C_1$, $C_1 \geq 1$, we obtain

$$\delta^{-2} + \delta^{-1} \int_Q C_1 V_+(t) dt \geq 9 \max_{t \in Q} C_1 V_-(t),$$

so (2.1) is satisfied.

Since (4.7) and (4.1) are preserved by multiplication by a positive number, we get (4.8) for $\dfrac{a^w}{\gamma_N(a)}$ with a universal constant C. In particular C_{00} can be chosen independently of the semi-norms of a.

The proof is complete.

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