ANNALES DE L'INSTITUT FOURIER

ALLAN GREENLEAF
GUNTHER UHLMANN

Composition of some singular Fourier integral operators and estimates for restricted X-ray transforms

Annales de l'institut Fourier, tome 40, nº 2 (1990), p. 443-466
<http://www.numdam.org/item?id=AIF_1990__40_2_443_0>

Ann. Inst. Fourier, Grenoble
40, 2 (1990), 443-466.

COMPOSITION OF SOME SINGULAR FOURIER INTEGRAL OPERATORS AND ESTIMATES FOR RESTRICTED X-RAY TRANSFORMS

by A. GREENLEAF (*) and G. UHLMANN (**)

0. Introduction.

Let X and Y be C^∞ manifolds of dimension n and $C \subset (T^*X\backslash 0) \times (T^*Y\backslash 0)$ a canonical relation; that is, C is conic, smooth of dimension $2n$ and the product symplectic form $\rho^*\omega_X - \pi^*\omega_Y$ vanishes identically on TC. (Here, ω_X, ω_Y are the canonical symplectic forms on T^*X, T^*Y, respectively, and $\rho : T^*X \times T^*Y \to T^*X$, $\pi : T^*X \times T^*Y \to T^*Y$ are the projections onto the first and second factors.) To C is associated the class $I^m(C;X,Y)$ of Fourier integral operators (FIOs) of order m from $\mathscr{E}'(Y)$ to $\mathscr{D}'(X)$ ([18].) Composition calculi and sharp L^2 estimates for FIOs are only known under certain geometric conditions on the canonical relation(s). Most importantly, the transverse intersection calculus of Hörmander [18] implies that if $A_1 \in I^{m_1}(C_1;X,Y)$, $A_2 \in I^{m_2}(C_2;Z,X)$ with C_1 and C_2 local canonical graphs, then $A_2A_1 \in I^{m_1+m_2}(C_2 \circ C_1; Z, Y)$. In particular, if C_1 is a canonical graph, $A_1^*A_1 \in I^{2m_1}(\Delta_{T^*Y}; Y, Y)$ is a pseudodifferential operator and thus $A_1 : H^s_{\text{comp}}(Y) \to H^{s-m_1}_{\text{loc}}(X)$ continuously, $\forall s \in \mathbf{R}$. Later, this composition calculus was extended by Duistermaat and Guillemin [9] and Weinstein [32] to the case of clean intersection.

(*) Partially supported by NSF Grant DMS-8601534.
(**) Partially supported by NSF Grant DMS-8800153.

Key-words : Fourier integral operators - Integral geometry.
A.M.S. Classification : 58G15 - 53G65.

For L^2 estimates, the following more general result holds ([18]). If the differentials of the mappings ρ and π drop rank by at most k, for some $k < n$, there is an estimate with a loss of $k/2$ derivatives: $A : H^s_{\text{comp}}(Y) \to H^{s-m-\frac{k}{2}}_{\text{loc}}(X)$. This can be refined in the following way ([19], p. 30). Since C is a canonical relation, on C we have a closed 2-form $\omega_C = \rho^*\omega_X = \pi^*\omega_Y$, which is nondegenerate (i.e ; symplectic) iff C is a local canonical graph. If r is the co-rank of $C(=2n-\text{rank}\ \omega_C \leqslant 2k)$, then $A : H^s_{\text{comp}}(Y) \to H^{s-m-\frac{r}{4}}_{\text{loc}}(X)$. These results are sharp in that there are examples, such as the case when $C \subset (T^*X\backslash 0) \times (T^*Y\backslash 0)$ is the flowout of a codimension k involutive submanifold of $T^*Y\backslash 0$, where one cannot do better. For canonical relations C for which π and ρ become singular in specific ways, however, one expects there to be a sharp value $0 < s_0 = s_0(C) \leqslant \frac{r}{4} \leqslant \frac{k}{2}$ such that $A : H^s_{\text{comp}}(Y) \to H^{s-m-s_0}_{\text{loc}}(X)$, $\forall s \in \mathbf{R}$. A result of this nature is contained in the work of Melrose and Taylor [25] on folding canonical relations, for which π and ρ have at most Whitney folds, so that $k = 1$, $r = 2$ and ω_C is a folded symplectic form. Via canonical transformations of $T^*X\backslash 0$ and $T^*Y\backslash 0$, C can be conjugated (microlocally) to a single normal form ; on the operator level, A can be conjugated by elliptic FIOs to an Airy operator on $\mathbf{R}^n$, from which the sharp boundedness $A : H^s_{\text{comp}}(Y) \to H^{s-m-1/6}_{\text{loc}}(X)$ can be read off.

The purpose of the present work is to establish a composition calculus and obtain sharp L^2 estimates, with a loss of $\frac{1}{4}$ derivative, for a somewhat more singular class of canonical relations, the fibered folding canonical relations (FFCRs), for which again π is a Whitney fold and ω_C is a folded symplectic form but for which ρ is a « blow-down » ($\simeq$ polar coordinates in two variables). These canonical relations arise naturally in integral geometry and were described independently in Greenleaf and Uhlmann [12] and Guillemin [15]. A specific canonical relation of this type had already been analyzed in considerable detail by Melrose [23]. Related operators are in Boutet de Monvel [3]. An unfortunate feature of FFCRs is that they cannot be conjugated to a single normal form. There are already obstructions to a formal power series attempt to derive a normal form (cf. [12]). Alternatively, as shown in [15], the canonical involution of $T^*X\backslash\rho(L)$, where $L \subset C$ is the fold hypersurface for π, induced by the $2-1$ nature of π near L, may or may not extend smoothly past $\rho(L)$. In any event, it is not possible to give exactly a phase function ϕ that parametrizes a general FFCR. A somewhat remarkable fact is that this difficulty disappears when one composes an $A \in I^m(C;X,Y)$ with its adjoint. Our main result is

THEOREM 0.1. – *Let* $C \subset (T^*X\backslash 0) \times (T^*Y\backslash 0)$ *be a nonradial fibered folding canonical relation and* $A \in I^m(C;X,Y)$, $B \in I^{m'}(C^t;Y,X)$ *properly supported Fourier integral operators. Then* $BA \in I^{m+m',0}(\Delta_{T^*Y},\Lambda_{\pi(L)})$.

Here, Δ_{T^*Y} is the diagonal of $(T^*Y\backslash 0) \times (T^*Y\backslash 0)$, $\pi(L) \subset T^*Y\backslash 0$ is the image of the fold hypersurface and $\Lambda_{\pi(L)}$ its flowout, and $I^{p,\ell}(\Delta,\Lambda)$ is the space of oscillatory integrals (« pseudodifferential operators with singular symbols ») associated to the intersecting Lagrangians Δ and Λ by Melrose and Uhlmann [26] and Guillemin and Uhlmann [16]. Using the estimates for elements of $I^{p,\ell}(\Delta,\Lambda)$ given in Greenleaf and Uhlmann [13], we obtain

COROLLARY 0.2. – *For A as above,* $A : H^s(Y) \to H^{s-m-\frac{1}{4}}_{\text{loc}}(X)$ *continuously,* $\forall s \in \mathbf{R}$.

It should be remarked that the composition AB is of a completely different nature, with the absence of a normal form for C introducing serious analytical difficulties; this is discussed in Guillemin [15].

A special case of the theorem and corollary was proved in [13] for the restricted X-ray transform. If (M,g) is an n-dimensional riemannian manifold for which the space $\mathcal{M}$ of (oriented) geodesics is a smooth $(2n-2)$-dimensional manifold (e.g., $\mathbf{R}^n$ with the standard metric or a sufficiently small ball in any riemannian manifold), then the X-ray transform $\mathcal{R} : \mathcal{E}'(M) \to \mathcal{D}'(\mathcal{M})$ is given by

$$(0.3) \qquad \mathcal{R}f(\gamma) = \int_{\mathbf{R}} f(\gamma(s))\, ds, \quad \gamma \in \mathcal{M},$$

$\gamma(s)$ any unit-velocity parametrization of γ. In the absence of conjugate points, $\mathcal{R}$ is an FIO of order $-\frac{n}{4}$ associated with a canonical relation satisfying the Bolker condition [14] and so $\mathcal{R} : H^s_{\text{comp}}(M) \to H^{s-\frac{1}{2}}_{\text{loc}}(\mathcal{M})$, generalizing (locally) the result of Smith and Solmon [28] on $\mathbf{R}^n$. (See also Strichartz [30] for the case of hyperbolic space.) Following Gelfand, one is also interested in the restriction of $\mathcal{R}f$ to n-dimensional submanifolds $\mathscr{C} \subset \mathscr{M}$ (geodesic complexes); denote $\mathscr{R}f|_{\mathscr{C}}$ by $\mathscr{R}_{\mathscr{C}}f$. Of particular interest are those $\mathscr{C}$'s which are admissible for reconstruction of f from $\mathscr{R}_{\mathscr{C}}f$ in that they satisfy a generalization of Gelfand's criterion [11]; in [12] it was shown that, with appropriate curvature assumptions, for such a $\mathscr{C}$, $\mathscr{R}_{\mathscr{C}}$ is an FIO of order $-\frac{1}{2}$ associated with a FFCR. In this case the Schwartz kernel of $\mathscr{R}_{\mathscr{C}}^*\mathscr{R}_{\mathscr{C}}$ is quite explicit and was shown in [13] to belong to $I^{-1,0}(\Delta_{T^*M}, \Lambda_{\pi(L)})$, yielding the boundedness of $\mathscr{R}_{\mathscr{C}} : H^s_{\text{comp}}(M) \to H^{s+\frac{1}{4}}_{\text{loc}}(\mathscr{C})$, $s \geqslant -\frac{1}{4}$.

To prove local L^p estimates for admissible geodesic complexes, we extend $\mathscr{R}_{\mathscr{C}}$ to an analytic family $R^\alpha \in I^{-\text{Re}(\alpha)-\frac{1}{2}}(C;\mathscr{C},M)$; application of analytic interpolation then requires L^2 estimates for general elements of $I(C;\mathscr{C},M)$, for which the argument of [13] is insufficient. We prove

THEOREM 0.4. — *Let $\mathscr{C} \subset \mathscr{M}$ be an admissible geodesic complex and let $P(x,D)$ be a zeroth order pseudodifferential operator on M such that $\mathscr{R}_{\mathscr{C}}P \in I(C;\mathscr{C},M)$ with C a fibered folding canonical relation. Then $\mathscr{R}_{\mathscr{C}}P : L^p_{\text{comp}}(M) \to L^q_{\text{loc}}(\mathscr{M})$ for p, q satisfying either of the following conditions:*

(a) $1 < p \leqslant \dfrac{4n-3}{2n-1}, \ \dfrac{1}{q} \geqslant \dfrac{2n+1}{2np} - \dfrac{1}{2n}$;

(b) $\dfrac{4n-3}{2n-1} \leqslant p < \infty, \ \dfrac{1}{q} \geqslant \dfrac{2n-1}{2np}$.

For the full X-ray transform in $\mathbf{R}^n$, global L^p estimates have been proven by Drury [6] [7] and refined by Christ [5] to mixed $L^p - L^q$ norms (see also [30], Oberlin and Stein [27]); however, even in $\mathbf{R}^n$ our estimates do not seem to be retrievable from theirs because of the high codimension of $\mathscr{C}$ in $\mathscr{M}$. Wang [31], using variations of the techniques of [5] [6] [7], has established global L^p estimates for some special line complexes in $\mathbf{R}^n$.

There is a gap between the estimates in (0.4) and the expected optimal ones. Furthermore, one expects that, just as for the L^2 estimates [13], for general (nonadmissible) $\mathscr{C} \subset \mathscr{M}$, *better* estimates hold, reflecting the more singular way in which C sits in $T^*\mathscr{C} \times T^*M$ when $\mathscr{C}$ is admissible. This is confirmed below for a particularly nice class of inadmissible $\mathscr{C}$'s, for which C is a folding canonical relation.

The paper is organized as follows. In § 1 we give a precise definition of FFCRs and recall the symplectic geometry needed to conjugate a FFCR into a position where it has a generating function $S(x, y_n, \eta')$. The geometry of C then allows us to put a S in a weak normal form. The relevant facts concerning $I^{p,\ell}(\Delta,\Lambda)$, including the iterated regularity characterization given in [13], are recalled in § 2. In § 3 we prove (0.1) by computing BA, simplifying the phase, and then applying first order pseudodifferential operators to verify the iterated regularity condition. The applications to the restricted X-ray transform are given in § 4.

The first author would like to thank the Mathematical Sciences Research Institute, Berkeley, where a portion of this work was done, for its hospitality and support.

1. Weak normal form and phase functions.

Consider on $\mathbb{R}^n \times \mathbb{R}^n \times (\mathbb{R}^{n-1}\backslash 0)$ the phase function

$$\phi_0(x,y,\theta') = (x'-y').\theta' + \frac{x_n^2 y_n}{2}\theta_1, \qquad |\theta_1| \geqslant c|\theta|, \, y_n \neq 0, \tag{1.1}$$

where we write $x = (x',x_n) = (x_1,x'',x_n) \in \mathbb{R}^n$. Calculating the critical set $\{(x,y,\theta') : d_\theta'\phi_0=0\}$ and computing the map

$$(x,y,\phi') \to (x,d_x\phi_0\,;\, y, -d_y\phi_0),$$

we find that ϕ_0 parametrizes the canonical relation

$$(1.2)\quad C_0 = \left\{\left(y_1 - \frac{x_n^2 y_n}{2}, y'', x_n, \eta', x_n y_n \eta_1; y, \eta', -\frac{x_n^2\eta_1}{2}\right) : \right.$$
$$\left. (x_n, y, \eta') \in \mathbb{R}^{2n}, |\eta_1| \geqslant c|\eta'|, y_n \neq 0\right\}$$
$$= \left\{\left(x, \xi', x_n y_n \xi_1; x_1 + \frac{x_n^2 y_n}{2}, x'', y_n, \xi', -\frac{x_n^2}{2}\xi_1\right) : \right.$$
$$\left. (x,\xi',y_n) \in \mathbb{R}^{2n}, |\xi_1| \geqslant c|\xi'|, y_n \neq 0\right\}.$$

Denoting, as before, the projections $C_0 \to T^*\mathbb{R}^n\backslash 0$ onto the first and second factors by ρ and π, respectively, one sees immediately that C_0 is a local canonical graph away from $L = \{x_n=0\}$, where π has a Whitney fold (defined below) ; $\pi(L) = \{\eta_n=0\} \subset T^*\mathbb{R}^n\backslash 0$ is an embedded hypersurface. At L, ρ is more singular: $\rho(L) = \{x_n = \xi_n = 0\} \subset T^*\mathbb{R}^n \backslash 0$ is embedded, codimension 2, and symplectic (i.e. $\sum_1^n d\xi_j \wedge dx_j|_{\rho(L)}$ is nondegenerate), and ρ « blows up » $\rho(L)$, having 1-dimensional fibers with tangents $\frac{\partial}{\partial y_n}\cdot$ C_0 is an example of a fibered folding canonical relation; we recall from [12] and [15] the general definition of a FFCR and then show that any such can be conjugated sufficiently close to C_0 so that it has a phase similar to ϕ_0

DEFINITION 1.3. — *Let M and N be n-dimensional manifolds ; f: $M \to N$ C^∞.*

a) *f is a Whitney fold if near each $m_0 \in M$, f is either a local diffeomorphism or df drops rank simply by 1 at m_0, so that $L = \{m \in M :$* rank $(df(m)) = n-1\}$ *is a smooth hypersurface through m_0, and* ker $(df(m_0)) \not\subset T_{m_0}L$.

b) *f is a blow-down along a smooth hypersurface $K \subset M$ if f is a local diffeomorphism away from K, while df drops rank simply by 1 at K, where* Hess $f \equiv 0$ *and* ker $(df) \subset TK$, *so that $f|_K$ has 1-dimensional fibers ; furthermore, letting, for $m_0 \in K$,*

$$\overline{df} : f^{-1}(f(m_0)) \to G_{n-1,n}(T_{f(m_0)}N)$$

be the map sending m to the hyperplane $df(m)(T_mM) \subset T_{f(m_0)}N$, we demand that $d(\overline{df})(v) \neq 0$, $v \in$ ker $(df(m_0))\backslash 0$.

Remark. — In [12], a blow-down was called a fibered fold. Since this terminology is apparently not standard, we have dropped it.

DEFINITION 1.4. — *Let X and Y be n-dimensional C^∞ manifolds and $C \subset (T^*X\backslash 0) \times (T^*Y\backslash 0)$ a canonical relation. C is a (nonradial) fibered folding canonical relation if*

a) *$\pi : C \to T^*Y\backslash 0$ is a Whitney fold, with fold hypersurface L, and $\pi(L)$ an embedded nonradial hypersurface;*

b) *$\rho : C \to T^*X\backslash 0$ is a blow-down (necessarily along L), with $\rho(L)$ embedded, nonradial and symplectic, and $\rho : C\backslash L \to T^*X\backslash 0$* is $1-1$.

In [12], an additional compatibility condition was imposed; namely, that the fibers $\rho|_L$ be the lifts by π of the bicharacteristic curves of $\pi(L)$. It was shown by Guillemin [15] that this is automatically satisfied.

By suitable choice of coordinate systems, the projections π and ρ may each be put into normal form ; the lack of a normal form for FFCRs stems from the inability to reconcile these coordinate systems in general. We recall

PROPOSITION 1.5 (Melrose, [20]). — *Let M and N be conic manifolds of dimension $2n$, with N symplectic. Suppose $f: M \to N$ has a Whitney fold along $L \ni m_0$ and $f(L)$ is non radial at $f(m_0)$.*

Then there exist canonical coordinates on N near $f(m_0)$ and coordinates (s,σ) near m_0 on M, homogeneous of degrees 0 *and* 1, *respectively, with* $s_j(m_0) = \delta_{nj}, \sigma_j(m_0) = \delta_{1j}, \forall j$, such that $f(s,\sigma) = \left(s,\sigma', -\frac{\sigma_n^2}{2\sigma_1}\right)$.

PROPOSITION 1.6. — *Let M and N be as above. Suppose $g : M \to N$ is a blow-down along $L \ni m_0$ and $g(L)$ is nonradial and symplectic near $g(m_0)$. Then there exist canonical coordinates on N near $g(m_0)$ and coordinates (t,τ) near m_0 on M, homogeneous of degrees* 0 *and* 1, *respectively, with $t_j(m_0) = 0$, $\tau_j(m_0) = \delta_{1j} + \delta_{nj}$, $\forall j$, such that $g(t,\tau) = (t,\tau',t_n\tau_n)$.*

Proof. — Without the homogeneity, this is Theorem 4.5 of [12] ; the proof there is easily adapted to the conic setting using the version of Darboux' theorem in [21].

Now let C be a FFCR and apply (1.5), (1.6) to $f = \pi$, $g = \rho$, respectively, to obtain canonical coordinates on $T^*Y\backslash 0$, $T^*X\backslash 0$ and homogeneous coordinates (s,σ), (t,τ) near $c_0 \in L \subset C$. Let

$$T_1 = s_1 - \frac{\sigma_n^2 s_n^2}{2\sigma_1^2}, \quad T_n = \frac{\sigma_n}{\sigma_1} \quad \text{and} \quad S_n = \frac{\tau_n}{\tau_1},$$

so that with respect to the homogeneous coordinate systems $(T_1, T_n, s'', s_n, \sigma')$ and $(t,\tau' S_n)$ near c_0,

$$\pi(T_1, T_n, s'', s_n, \sigma') = \left(T_1 + \frac{T_n^2}{2} s_n, s'', s_n, \sigma', -\frac{T_n^2}{2}\sigma_1\right); \tag{1.7}$$

$$\rho(t,\tau',S_n) = (t,\tau',S_n t_n \tau_1); \tag{1.8}$$

$$\begin{aligned}\omega_C &= d\sigma_1 \wedge dT_1 + d\sigma'' \wedge ds'' + T_n(s_n d\sigma_1 + \sigma_1 ds_n) \wedge dT_n \\ &= d\tau' \wedge dt' + t_n(S_n d\tau_1 + \tau_1 dS_n) \wedge dt_n ;\end{aligned} \tag{1.9}$$

and

$$L = \{T_n = 0\} = \{t_n = 0\}. \tag{1.10}$$

A function $f \in C^\infty(C)$ has a (singular) Hamiltonian vector field with respect to the folded symplectic form ω_c, which expressed in the $(T_1, T_n, s'', s_n, \sigma')$ coordinates is

$$(1.11)\quad H_f^C = \left(\frac{\partial f}{\partial \sigma_1} - \frac{s_n}{\sigma_1}\frac{\partial f}{\partial s_n}\right)\frac{\partial}{\partial T_1} + \frac{1}{T_n\sigma_1}\frac{\partial f}{\partial s_n}\frac{\partial}{\partial T_n} + \sum_{j=2}^{n-1} \frac{\partial f}{\partial \sigma_j}\frac{\partial}{\partial s_j} - \frac{\partial f}{\partial s_j}\frac{\partial}{\partial \sigma_j} + \left(\frac{s_n}{\sigma_1}\frac{\partial f}{\partial T_1} - \frac{1}{T_n\sigma_1}\frac{\partial f}{\partial T_n}\right)\frac{\partial}{\partial s_n} - \frac{\partial f}{\partial T_1}\frac{\partial}{\partial \sigma_1}.$$

On L, $\{S_n = 1\}$ has the form $\{s_n = 1 + F(T_1, s'', \sigma')\}$, so we let

$$f(T_1, T_n, s'', s_n, \sigma') = -\sigma_1 F(T_1, s'', \sigma')\frac{T_n^2}{2}.$$

Then there is a smooth function on $T^*Y\backslash 0$, which we denote by $\pi_* f$, such that $\pi^*(\pi_* f) = f$; of course, $H_{\pi * f}$ is a C^∞ vector field on $T^*Y\backslash 0$, with $\chi_{\pi * f} = \exp(H_{\pi * f})$ a canonical transformation. On the other hand, $H_f^C = F\dfrac{\partial}{\partial s_n} + O(T_n^2)$ and is C^∞ by (1.11), and the ω_c-morphism $\chi_f^C = \exp(H_f^C)$ is of the form

$$\chi_f^C(T_1, T_n, s'', s_n, \sigma') = (T_1, T_n, s'', s_n + F(T_1, s'', \sigma'), \sigma') + O(T_n^2).$$

Changing variables on C and $T^*Y\backslash 0$ simultaneously, we retain (1.7) and (1.9), but now have $\{T_n = s_n - 1 = 0\} = \{t_n = S_n - 1 = 0\}$ near c_0; denote this smooth $(2n-2)$- dimensional manifold by L_0 and let i: $L_0 \hookrightarrow C$ be the inclusion map. From (1.9), we have

$$i^*\omega_C = d\sigma_1 \wedge dT_1 + d\sigma'' \wedge ds'' = d\tau' \wedge dt'.$$

By Darboux we can find a canonical transformation χ_0 of $\mathbb{R}^{2n-2}$ such that $\chi_0^*(T_1, s'', \sigma') = (t', \tau')$. Extending χ_0 to be independent of T_n and s_n, we obtain an ω_C-morphism χ such that

$$\chi^*(T_1, s'', \sigma') = (t', \tau') + O(t_n) + O(S_n - 1),\ \chi^* s_n = 1 + aS_n + O((S_n-1)^2) + O(t_n)$$

and $\chi^* T_n = b t_n$ with $a \neq 0$, $b \neq 0$ near c_0. On the other hand, by simultaneously applying χ_0 in the (y', η') variables, we preserve (1.7). Thus, we have $\rho^*(x) = t$, $\pi^*(y_n) = s_n$ and $\pi^*(\eta') = \sigma'$ forming local coordinates on C near c_0; furthermore, $L = \{x_n = 0\}$ in these coordinates, $\pi(L) = \{(y, \eta) : \eta_n = 0\}$ and $\rho(L) = \{(x, \xi) : x_n = \xi_n = 0\}$, and $d\rho^*(d\xi_n) \neq 0$.

Since (x, y_n, η') form coordinates on C, there exists a generating function $S(x, y_n, \eta')$ for C ([18]): S is C^∞, homogeneous of degree 1 in η', and

$$C = \{(x, d_x S; d'_\eta S, y_n, \eta', d_{y_n} S) : (x, y_n, \eta') \in U\} \tag{1.12}$$

near c_0, where U is a conic neighborhood of $x = 0$, $y_n = 1$, $\eta' = dy_1$, and $\phi(x, y, \eta') = S(x, y_n, \eta') - y' . \eta'$ parametrizes C near c_0. The fact that C is a FFCR imposes several conditions on S, which we next derive.

That $\pi(L) = \{\eta_n = 0\}$ implies that $\dfrac{\partial S}{\partial y_n}(x', 0, y_n, \eta') = 0$, whence $S|_{\{x_n = 0\}}$ is independent of y_n: $S(x', 0, y_n, \eta') = S_0(x', \eta')$ for some smooth, homogeneous S_0. Since $\rho(L) = \{x_n = \xi_n = 0\}$, we have $\dfrac{\partial S}{\partial x_n}(x', 0, y_n, \eta') = 0$, so that

$$S(x, y_n, \eta') = S_0(x', \eta') + \frac{x_n^2}{2} S_2(x, y_n, \eta'), \tag{1.13}$$

where S_2 is smooth and homogeneous of degree 1 in η'. The matrix representing $d\pi$ is

$$d\pi = \begin{bmatrix} d^2_{\eta' x} S & d^2_{\eta' y_n} S & d^2_{\eta' \eta'} S \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-1} \\ d^2_{y_n x} S & d^2_{y_n y_n} S & d^2_{y_n \eta'} S \end{bmatrix}. \tag{1.14}$$

By the above comments, at $x_n = 0$ the y_n-row and the x_n-column vanish; but since π is a fold, $d\pi|_{dx_n = 0}$ has rank $2n - 1$, and thus $\det\,(d^2_{\eta' x'} S) \neq 0$ at $x_n = 0$, i.e.,

(1.15) $S_0(x', \eta')$ is a nondegenerate generating function, in $(n-1)$ variables. Also, $\ker(d\pi) = \mathbf{R}\frac{\partial}{\partial y_n}$ at $x_n = 0$. Additionally,

$$(1.16)\qquad d\rho = \begin{bmatrix} I_{n-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & O \\ 0\cdots 0 & 1 & 0 & \\ d^2_{x'x'}S & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & d^2_{x'y_n}S & d^2_{x'\eta'}S \\ 0\cdots 0 & d^2_{x_nx_n}S & 0 & 0 \end{bmatrix}.$$

The nondegeneracy of $d^2_{x'\eta'}S$ yields (at $x_n=0$)

$$(1.17)\qquad \mathrm{Im}\,(d\rho) = \mathrm{span}\left\{\left\{\frac{\partial}{\partial x_j}\right\}_{j=1}^{n-1}, \frac{\partial}{\partial x_n} + \frac{\partial^2 S}{\partial x_n^2}\frac{\partial}{\partial \xi_n}, \left\{\frac{\partial}{\partial \xi_j}\right\}_{j=1}^{n-1}\right\}.$$

From $d\rho^*(d\xi_n) \neq 0$ it follows that

$$(1.18)\qquad \frac{\partial^2 S}{\partial x_n^2}(x',0,y_n,\eta') = S_2(x',0,y_n,\eta') \neq 0\,;$$

on the other hand, the nondegeneracy of the blow-down implies that

$$(1.19)\qquad \frac{\partial^3 S(x',0,y_n,\eta')}{\partial y_n\,\partial x_n^2} = \frac{\partial S_2}{\partial y_n}(x',0,y_n,\eta') \neq 0\,.$$

Conversely, one can easily show that any generating function of the form $S_0(x',\eta') + \frac{x_n^2}{2}S_2(x,y_n,\eta')$, with S_0 satisfying (1.15) and S_2 satisfying (1.18) and (1.19) gives rise to a FFCR. We have now proven

THEOREM 1.20. — *A canonical relation* $C \subset (T^*X\backslash 0) \times (T^*Y\backslash 0)$ *is a fibered folding canonical relation near a point* (x_0,ξ_0,y_0,η_0), *critical for* π (*or* ρ), *iff there exist canonical transformations* $\chi_1 : T^*\mathbf{R}^n\backslash 0 \to T^*Y\backslash 0$, $\chi_2 : T^*X\backslash 0 \to T^*\mathbf{R}^n\backslash 0$, *with* $\chi_1((0,\dots,0,1),\ (1,0,\dots,0)) = (y_0,\eta_0)$, $\chi_2(x_0,\xi_0) = ((0,\dots,0),\ (1,0,\dots,0,1))$, *such that* $Gr(\chi_2)\circ C\circ Gr(\chi_1)$ *is parametrized by a phase function of the form*

$$(1.21)\qquad \phi(x,y,\eta') = S_0(x',\eta') - y'\cdot\eta' + \frac{x_n^2}{2}S_2(x,y_n,\eta')$$

with S_0 *and* S_2 *satisfying* (1.15), (1.18) *and* (1.19).

2. $I^{p,\ell}(\Delta,\Lambda)$ and iterated regularity.

We now review the spaces of distributions associated with two cleanly intersecting Lagrangians [26], [16]; their characterization by means of iterated regularity [13]; and the L^2 estimates for operators whose Schwartz kernels are of this type [13]. Since only codimension 1 intersection is relevant to this paper, we will restrict our attention to that case. In the model case $\tilde{\Delta} = \Delta_{T^*\mathbf{R}^n}$, $\tilde{\Lambda} = \{(x', x_n, \xi', 0; x', y_n, \xi', 0) : x \in \mathbf{R}^n, \xi' \in \mathbf{R}^{n-1}\backslash 0, y_n \in \mathbf{R}\}$ = the flowout of $\{\xi_n = 0\}$, $I^{p,\ell}(\tilde{\Delta}', \tilde{\Lambda}')$ is defined to be the space of all sums of C^∞ functions and distributions on $\mathbf{R}^n \times \mathbf{R}^n$ of the form

$$(2.1)\quad u(x,y) = \int e^{i((x'-y').\xi' + (x_n - y_n - s).\xi_n + s.\sigma)} a(x,y,s;\xi;\sigma)\, d\sigma\, ds\, d\xi$$

where a is a product type symbol of order $p' = p - \frac{n}{2} + \frac{1}{2}$, $\ell' = \ell - \frac{1}{2}$, satisfying

$$(2.2)\qquad |\partial_\xi^\alpha \partial_\sigma^\beta \partial_{x,y,s}^\gamma a| \leqslant C_{\alpha\beta\gamma K}(1+|\xi|)^{p'-|\alpha|}(1+|\sigma|)^{\ell'-|\beta|}$$

on each compact $K \subset \mathbf{R}_x^n \times \mathbf{R}_y^n \times \mathbf{R}_s$. In general, for a canonical relation $\Lambda \subset (T^*Y\backslash 0) \times (T^*Y\backslash 0)$ that intersects Δ_{T^*Y} cleanly in codimension 1, one can find microlocally a canonical transformation $\chi : (T^*Y\backslash 0) \times (T^*Y\backslash 0) \to (T^*\mathbf{R}^n\backslash 0) \times (T^*\mathbf{R}^n\backslash 0)$ taking the pair (Δ,Λ) to $(\tilde{\Delta},\tilde{\Lambda})$; $I^{p,\ell}(\Delta',\Lambda')$ is defined as the space of all microlocally finite sums of distributions $F_i u_i$, with u_i of the form (2.1) and $F_i \in I^0(Gr(\chi); \mathbf{R}^n \times \mathbf{R}^n, Y \times Y)$ for such a χ. $I^{p,\ell}(\Delta,\Lambda)$ is then the class of operators with Schwartz kernel in $I^{p,\ell}(\Delta',\Lambda')$; microlocally if $T \in I^{p,\ell}(\Delta,\Lambda)$, $T \in I^{p+\ell}(\Delta\backslash\Lambda; Y)$ and $T \in I^p(\Lambda\backslash\Delta; Y)$. Furthermore, the principal symbol of T on $\Delta\backslash\Lambda$ lies in the space $R^{\ell-\frac{1}{2}}$ defined in [16] and has a conormal singularity of order $\ell - \frac{1}{2}$ at Λ. The leading term of this singularity belongs to the space $S^{p,\ell}(Y \times Y; \Delta, \Delta \cap \Lambda)$ of [16] and is denoted by $\sigma_0(T)$, the principal symbol of T as an element of $I^{p,\ell}(\Delta,\Lambda)$.

The oscillatory representation (2.1) can be difficult to verify directly. Instead, we make use of the following characterization of $I^{p,\ell}(\Delta',\Lambda')$ from [13], which is a variant of the iterated regularity characterizations given by Melrose [22], [24] for various classes of distributions.

PROPOSITION 2.3. — *Let* $\Lambda \subset (Y^*Y\backslash 0) \times (T^*Y\backslash 0)$ *be a canonical relation cleanly intersecting the diagonal* Δ *in codimension* 1. *Then* $u \in I^{p,\ell}(\Delta',\Lambda')$ *for some* p, $\ell \in \mathbf{R}$ *iff for some* $s_0 \in \mathbf{R}$ *and all* $k \geqslant 0$, *and all first order pseudodifferential operators* $P_1(z,D_z,y,D_y)$, $P_2(z,D_z,y,D_y)$, ..., *whose principal symbols vanish on* $\Delta' \cup \Lambda'$,

$$P_1 \dots P_k u \in H^{s_0}_{\text{loc}}(Y \times Y). \tag{2.4}$$

In the model case $(\tilde{\Delta},\tilde{\Lambda})$, the principal symbol of a first order $P(z,D_z,y,D_y)$, characteristic for $\tilde{\Delta}' \cup \tilde{\Lambda}'$, can be written (via the preparation theorem)

$$p(z,\zeta\ y,\eta) = \sum_{j=1}^{n} p_j(\zeta_j+\eta_j) + \sum_{j=1}^{n-1} q_j(z_j-y_j) + q_n(\zeta_n-\eta_n)(z_n-y_n) \tag{2.5}$$

where the p_j, q_j and q_n are homogeneous of degrees 0,1 and 0, respectively.

Finally, the following estimates are proven in [13], using the functional calculus of Antoniano and Uhlmann [1] and Jiang and Melrose (unpublished).

THEOREM 2.6. — *Let* $\Sigma \subset T^*Y\backslash 0$ *be a smooth, conic, codimension* 1 *submanifold and* $\Lambda \subset (T^*Y\backslash 0) \times (T^*Y\backslash 0)$ *its flowout. Then, if* $T \in I^{p,\ell}(\Delta,\Lambda)$, $T : H^s_{\text{comp}}(Y) \to H^{s+s_0}_{\text{loc}}(Y)$, $\forall s \in \mathbf{R}$, *if*

$$\max\left(p+\frac{1}{2}, p+\ell\right) \leqslant s_0. \tag{2.7}$$

3. Composition and loss of $\frac{1}{4}$ – derivative.

Let $C \subset (T^*X\backslash 0) \times (T^*Y\backslash 0)$ be a FFCR and $A \in I^m(C;X,Y)$, $B \in I^{m'}(C^t;Y,X)$ properly supported FIOs.

Let $\Lambda = \Lambda_{\pi(L)}$ be the flowout of $\pi(L)$ in $(T^*Y\backslash 0) \times (T^*Y\backslash 0)$. By a microlocal partition of unity, we may write A and B as locally finite sums of operators $A = \sum A_i$, $B = \sum B_j$, such that on each $WF(A_i)'$ or $WF(B_j)'$, either C is a canonical graph or Theorem 1.20 applies.

Furthermore, if $WF(B_j)' \circ WF(A_i)' \subset \Lambda$ (i.e., there is no contribution from the diagonal), then the clean intersection calculus of [9] and [32] applies, with excess $e = 0$, to give $B_jA_i \in I^{m+m'}(\Lambda; Y, Y) \subset$

$I^{m+m'0}(\Delta,\Lambda;Y,Y)$. We may thus restrict our attention to a composition BA, where $A\in I^m(C;\mathbf{R}^n,\mathbf{R}^n)$, $B\in I^{m'}(C^t;\mathbf{R}^n,\mathbf{R}^n)$, with $C\subset (T^*\mathbf{R}^n\backslash 0)\times(T^*\mathbf{R}^n\backslash 0)$ parametrized by a phase function $\phi(x,y,\theta')=S_0(x',\theta')-y'\cdot\theta'+\frac{x_n^2}{2}S_2(x,y_n,\theta')$, S_0 and S_2 satisfying (1.15), (1.18) and (1.19) in a conic neighborhood of $x=0$, $y_n=1$, $\theta'=(1,0,\dots,0)$. By Hörmander's theorem [18], A has an oscillatory representation

$$(3.1)\quad Af(x)=\int e^{i(S_0(x',\theta')-y'.\theta'+\frac{x_n^2}{2}S_2(x,y_n,\theta'))}a(x,y,\theta')f(y)\,d\theta'\,dy$$

modulo a smoothing operator, where $a\in S^{m-\frac{1}{2}}_{1,0}(\mathbf{R}^n\times\mathbf{R}^n\times(\mathbf{R}^{n-1}\backslash 0))$ is supported on a suitably small conic neighborhood of $x=(0,\dots,0)$, $y=(0,\dots,0,1)$, $\theta'=(1,0,\dots,0)$. $S_0(x',\theta')$ is, by (1.15), the generating function of a canonical transformation χ^0: $T^*\mathbf{R}^{n-1}\backslash 0\to T^*\mathbf{R}^{n-1}\backslash 0$, which we denote by $(\chi^0_{x'}(x',\xi'),\ \chi^0_{\xi'}(x',\xi'))$; we may assume that $\chi^0(0,e_1^*)=(0,e_1^*)$. Then $\chi=\chi^0\otimes\mathrm{Id}: T^*\mathbf{R}^n\backslash 0\to T^*\mathbf{R}^n\backslash 0$ is a canonical transformation. Let F be a zeroth order FIO associated with χ^{-1}, elliptic on $\rho(C)$. F has the representation

$$Ff(w)=\int e^{i(-S_0(x',\omega')+w'\cdot\omega'+(w_n-x_n)\cdot\omega_n)}c(x,w,\omega)f(x)\,dw\,dx,$$
$$c\in S^0_{1,0}(\mathbf{R}^n\times\mathbf{R}^n\times(\mathbf{R}^n\backslash 0)).$$

We compute the composition FA, applying as usual stationary phase in the x, ω variables. The critical points are given by $\omega'=\theta'+\frac{x_n^2}{2}g_1(w',x_n,y_n,\theta')$, g_1 smooth $\mathbf{R}^{n-1}$-valued and homogeneous of degree 1, $\omega_n=0$, $x_n=w_n$, and x' determined by $w'=d_{\omega'}S_0(x',\omega')$, so that $x'=\chi^0_{x'}(w',\theta')+\frac{x_n^2}{2}g_0(w',x_n,y_n,\theta')$, g_0 smooth and homogeneous of degree 0. We thus have an oscillatory expression for FA with symbol of order $m-\frac{1}{2}$ and phase

$$(3.3)\quad (w'-y')\cdot\theta'+\frac{w_n^2}{2}(S_2(x',w_n,y_n,\theta')+g_1\cdot(-d_{\theta'}S_0(x',\theta')+w')).$$

Since both $d_{\theta'}S_0$ and w' vanish at $w=0$, $y=(0,\dots,0,1)$, $\theta'=(1,0,\dots,0)$, conditions (1.18) and (1.19) are still satisfied (if the conic support of A has been chosen suitably small to start with). Relabeling w by x, one obtains

$$(3.4) \quad FAf(x) = \int e^{i((x'-y')\cdot\theta' + \frac{x_n^2}{2}\tilde{S}_2(x,y_n,\theta'))}\tilde{a}(x,y,\theta')f(y)\, d\theta'\, dy,$$

with $\tilde{S}_2$ satisfying (1.18) and (1.19) and $\tilde{a} \in S^{m-\frac{1}{2}}_{1,0}$, a refinement on the operator level of (1.21).

F^*F is a zeroth order pseudodifferential operator P, elliptic on $\rho(C)$; let Q be a property supported parametrix, so that $QP = I \bmod C^\infty$ on distributions with wave-front set in $\rho(C)$. Then $BQ \in I^{m'}(C^t; \mathbf{R}^n, \mathbf{R}^n)$ and by repeating the above argument we obtain for BQF^* an oscillatory representation adjoint to (3.4), with symbol $\tilde{b} \in S^{m'-\frac{1}{2}}_{1,0}$. Hence, modulo a smooth kernel, (cf. [8] [18]) the Schwartz kernel of BA has the following representation as an oscillatory integral:

$$(3.5) \quad K_{BA}(z,y) = \int e^{i((x'-y')\cdot\theta' - (x'-z')\sigma' + \frac{x_n^2}{2}(\tilde{S}_2(x,y_n,\theta') - \tilde{S}_2(x,z_n,\sigma')))} c\, d\theta'\, d\sigma'\, dx,$$

where $c \in S^{m+m'-1}_{1,0}$ is $\tilde{a}\cdot\tilde{b}$ cutoff to be supported in $\{|\theta'| \simeq |\sigma'|\}$.

Now, since the gradient of the phase $\Phi(z,y,x,\theta',\sigma') = (x'-y')\cdot\theta' - (x'-z')\cdot\sigma' + \frac{x_n^2}{2}(\tilde{S}_2(x,y_n,\theta') - \tilde{S}_2(x,z_n,\sigma'))$ in all the variables is $\neq 0$, integration by parts a finite number of times shows that all expressions of the form (3.5), with amplitude in $S^{m+m'-1}_{1,0}$, lie in a fixed Sobolev space $H^{s_0}_{\mathrm{loc}}(\mathbf{R}^n \times \mathbf{R}^n)$; in fact, we may take s_0 to be any number $< -(3n+m+m'-4)$ (cf., [18], p. 90).

PROPOSITION 3.6. — *For x_n sufficiently small, there are smooth functions $C(y,z,x,\theta',\sigma')$ and $D(y,z,x,\theta',\sigma')$, taking values in $\mathbf{R}^n$ and* Hom $(\mathbf{R}^{n^*}, \mathbf{R}^{n-1})$ *and homogeneous of degrees -1 and 0, respectively, such that*

$$(3.7) \qquad x_n(z_n - y_n)e^{i\Phi} = C\cdot d_x(e^{i\Phi})$$

and

$$(3.8) \qquad (\sigma'-\theta')e^{i\Phi} = D(d_x e^{i\Phi}).$$

Proof. — Vanishing as it does at $\{z_n = y_n, \sigma' = \theta'\}$, $\tilde{S}_2(x,y_n,\theta') - \tilde{S}_2(x,z_n,\sigma')$ may be written as $(z_n - y_n)A(z,y,x,\theta',\sigma')$

$+ B(z,y,x,\theta',\sigma')\cdot(\sigma'-\theta')$, where A and B are smooth, $\mathbf{R}$-and $\mathbf{R}^{n-1}$-valued and homogeneous of degrees 1 and 0, respectively. By (1.19), $A \neq 0$ near $z = y$, $x_n = 0$, $\theta' = \sigma'$. Then we have

$$(3.9) \quad d_{x_n}\Phi = x_n\left((z_n-y_n)\left(A+\frac{x_n}{2}d_{x_n}A\right) + (\sigma'-\theta')\cdot\left(B+\frac{x_n}{2}d_{x_n}B\right)\right),$$

and

$$(3.10) \quad d_{x'}\Phi = \theta' - \sigma' + \frac{x_n^2}{2}((z_n-y_n)d_{x'}A + (\sigma'-\theta')\cdot d_{x'}B).$$

Solving (3.10), we have

$$(3.11) \quad \left(I-\frac{x_n^2}{2}d_{x'}B\right)(\sigma'-\theta') = - d_{x'}\Phi + \frac{x_n^2}{2}(z_n-y_n)d_{x'}A,$$

and combining this with (3.9) we have, for x_n small,

$$(3.12) \quad x_n(z_n-y_n) = \frac{1}{\tilde{A}}\left(x_n\left(I-\frac{x_n^2}{2}d_{x'}B\right)^{-1*}\left(B+\frac{x_n^2}{2}d_{x_n}B\right)\cdot d_{x'}\Phi + d_{x_n}\Phi\right).$$

where

$$\tilde{A} = A + \frac{x_n^2}{2}d_{x_n}A + \frac{x_n^2}{2}\left(I-\frac{x_n^2}{2}d_{x'}B\right)^{-1*}\left(B+\frac{x_n}{2}d_{x_n}B\right)\cdot d_{x'}A \neq 0,$$

implying (3.7). From this and the step following (3.11) we obtain (3.8).

We are now in a position to verify that $K_{BA} \in I^{p,\ell}(\Delta',\Lambda')$, for some p, $\ell \in \mathbf{R}$, using iterated regularity. Given a first order $P(z,D_z,y,D_y)$, characteristic for $\Delta' \cup \Lambda'$, we recall from (2.5) that its principal symbol may be written

$$p(z,\zeta,y,\eta) = \sum_1^n p_j(\zeta_j+\eta_j) + \sum_1^{n-1} q_j(z_j-y_j) + q_n(z_n-y_n)(\zeta_n-\eta_n).$$

By (3.5), we have (cf. [8])

$$(3.13) \quad PK_{BA}(z,y) = \int e^{i\varphi(z,y,x,\theta',\sigma')}(p(z,d_z\Phi,y,d_y\Phi)c+d)d\theta' d\sigma' dx,$$

with $d \in S^{m+m'-1}_{1,0}$. Since $d_{z'}\Phi + d_{y'}\Phi = \sigma' - \theta'$, if we let $p' = (p_1, \ldots, p_{n-1})$, the $p' \cdot (\zeta' + \eta')$ term of PK_{BA} is

$$\int e^{i\Phi} p' \cdot (\sigma' - \theta') c\, d\theta'\, d\sigma'\, dx = \int D(d_x e^{i\Phi}) \cdot p' c\, d\theta'\, d\sigma'\, dx$$
$$= \int e^{i\Phi} d_x^t D^*(p'c)\, d\theta'\, d\sigma'\, dx$$

by (3.8); but because D is homogeneous of degree 0, $d_x^t D^*(p'c) \in S^{m+m'-1}_{1,0}$ and this is of the form (3.5). For the $p_n(\zeta_n + \eta_n)$ term, note that

$$d_{z_n}\Phi + d_{y_n}\Phi = \frac{x_n^2}{2}\left((z_n - y_n) d_{z_n}A + d_{y_n}A\right) + (\sigma' - \theta') \cdot (d_{z_n}B + d_{y_n}B)),$$

leading to

$$\int e^{i\Phi}\, d_x^t \cdot \left(C^* \left(\frac{x_n p_n c}{2} (d_{z_n}\mathrm{A} + d_{y_n}A) \right) + D^*(p_n c(d_{y_{z_n}}B + d_{y_n}B)) \right) d\theta'\, d\sigma'\, dx,$$

which is again of the form (3.5). Similarly, noting

$$d_{\sigma'}\Phi + d_{\theta'}\Phi = z' - y' + \frac{x_n^2}{2}\left((z_n - y_n)(d_{\sigma'}A + d_{\theta'}A) + (\sigma' - \theta') \cdot d_{\sigma'}B + d_{\theta'}B)\right),$$

we find that

$$(3.14) \quad (z' - y')e^{i\Phi} = i^{-1}(d_{\sigma'} + d_{\theta'})e^{i\Phi} - \frac{x_n}{2}(d_{\sigma'}A + d_{\theta'}A) C \cdot d_x e^{i\Phi}$$
$$- \frac{x_n^2}{2} D^*(d_{\sigma'}B + d_{\theta'}B) \cdot d_x e^{i\Phi}$$

and thus the $\sum_1^{n-1} q_j(z_j - y_j)$ term of PK_{BA} is of the form (3.5). Finally,

$$d_{z_n}\Phi - d_{y_n}\Phi$$
$$= x_n\left(x_n(x_n A + \frac{x_n}{2}(z_n - y_n) d_{z_n}A - d_{y_n}A) + \frac{x_n}{2}(\sigma' - \theta') \cdot (d_{z_n}B - d_{y_n}B) \right),$$

so that the $q_n(z_n - y_n)(\zeta_n - \eta_n)$ term of PK_{BA} is

$$\int e^{i\Phi}\, d_x^t \cdot C^*(x_n A + \ldots)\, d\theta'\, d_{\sigma'} dx,$$

again an oscillatory integral of the form (3.5) with symbol in $S_{1,0}^{m+m'-1}$. By induction, for any first order operators $P_1, \ldots, P_k$, characteristic for $\Delta' \cup \Lambda'$, $P_1, \ldots, P_k K_{BA}$ is of this form, and hence in $H^{s_0}_{\text{loc}}(\mathbf{R}^n \times \mathbf{R}^n)$ by the comment above.

Prop. 2.3 yields $K_{BA} \in I^{p,\ell}(\Delta', \Lambda')$ and hence $BA \in I^{p,\ell}(\Delta, \Lambda)$, for some $p, \ell \in \mathbf{R}$.

To determine the orders p and ℓ, note that away from L the composition is covered by Hörmander's calculus and hence $BA \in I^{m+m'}(\Delta \backslash \Lambda; Y, Y)$ microlocally so that $p + \ell = m + m'$. Furthermore, the calculation of the principal symbol of BA in [18] is still valid away from $\pi(L)$. If a is the principal symbol of A, considered as a $\frac{1}{2}$-density on C, we may express a as $\alpha \cdot |\pi^* \omega_Y^n|^{1/2}$. Since $\pi^* \omega_Y = \omega_C$ is folded sympletic, $\pi^* \omega_Y^n$ vanishes to first order at L and thus α has a conormal singularity of order $-\frac{1}{2}$ at L.

Similarly, the principal symbol of B is $b = \beta \cdot |\pi^* \omega_{Y'}^n|^{1/2}$ with β having a conormal singularity of order $-\frac{1}{2}$ at L^t (here Y' denotes the second copy of Y). Thus $\beta \cdot \alpha|_{T_* Y' \times \Delta_{T_* X} \times T^* Y}$ has a conormal singularity of order -1 above $\pi(L)$; when pushed down by the Whitney fold π, this gives rise to a conormal singularity of order $-\frac{1}{2}$ at L, in the principal symbol $b \times a$ of BA (cf. [12]). Hence, $\ell - \frac{1}{2} = -\frac{1}{2}$, and $p = m + m'$, $\ell = 0$, finishing the proof of Theorem 0.1. In addition, we see that the principal symbol $\sigma_0(BA)$ is the image of $b \times a$ in $S^{m+m',0}(Y \times Y; \Delta, \pi(L))$.

To prove Corollary 0.2, suppose $A \in I^m(C; X, Y)$ is properly supported, with C a FFCR. Then $A^* A \in I^{2m,0}(\Delta, \Lambda_{\pi(L)}; Y, Y)$ and is properly supported and so maps $H^s_{\text{comp}}(Y) \to H^{s-2m-1/2}_{\text{loc}}(Y)$ by Theorem 2.6. This yields Corollary 0.2 for $s = m + \frac{1}{4}$. For general $s \in \mathbf{R}$, we simply apply this result to PAQ, where P and Q are elliptic pseudodifferential operators on X and Y of orders $-s + m + \frac{1}{4}$ and $s - m - \frac{1}{4}$, respectively. As shown by an example in [13], one does not lose less than $\frac{1}{4}$ derivative in general.

It is also possible to give sharp estimates for A in terms of nonisotropic Sobolev spaces. Let $\Psi^m(Z)$ denote the pseudodifferential operators of order m and type 1,0 on a manifold Z. Then, for $s \in \mathbf{R}$,

$$\text{(3.15)} \quad H^{s,k}_{\text{loc}}(X) = \{v \in \mathscr{D}'(X) : Q_1 \ldots Q_k v \in H^s_{\text{loc}}(X) \text{ for all } Q_j \in \Psi^1(X) \text{ with } \sigma_{\text{prin}}(Q_j)|_{\rho(L)} = 0, \forall j\}$$

is the nonisotropic Sobolev space of [3]; defined initially for $k \in \mathbf{Z}_+$, one uses interpolation and duality to extend the definition to $k \in \mathbf{R}$. Since $\rho(L)$ is symplectic, we have $H^{s,k}_{\text{loc}}(X) \hookrightarrow H^{s+k/2}_{\text{loc}}(X)$; microlocally away from $\rho(L)$, of course, $H^{s,k}_{\text{loc}}(X) \hookrightarrow H^{s+k}_{\text{loc}}(X)$. For $s \in \mathbf{R}$, set

$$\text{(3.16)} \quad H^{s,k}_{\text{loc}}(Y) = \{u \in \mathscr{D}'(Y) : P_1 \ldots P_k u \in H^s_{\text{loc}}(Y) \text{ for all } P_j \in \Psi^1(Y) \text{ with } \sigma_{\text{prin}}(P_j)|_{\pi(L)} = 0, \forall j\},$$

again extended to $k \in \mathbf{R}$ by interpolation and duality. (For $\pi(L)$ the characteristic variety of the wave operator, this space has been widely used in the study of nonlinear problems.) One can then show that if $A \in I^m(C; X, Y)$ is properly supported, with C a FFCR,

$$\text{(3.17)} \qquad A : H^{s,k}_{\text{loc}}(Y) \to H^{s-k-m-1/2,\,2k+1/2}_{\text{loc}}(X),$$

giving a sharper form of (0.2). The main point in the proof is to show that if Q_1, Q_2, $\in \Psi^1(X)$ are characteristic for $\rho(L)$, then there are operators P_1, $P_2 \in \Psi^1(Y)$ characteristic for $\pi(L)$ and A_1, A_2, $A_3 \in I^{m+1}(C; X, Y)$ such that $Q_1 Q_2 A = A_1 P_1 + A_2 P_2 + A_3$. This is done by splitting $\rho^*(\sigma_{\text{prin}}(Q_1)\sigma_{\text{prin}}(Q_2))$ into its even and odd components with respect to the fold involution of C. The details are left to the reader.

4. L^p estimates for restricted X-ray transforms.

Let (M,g) be an n-dimensional riemannian manifold. The hamiltonian function $H(x,\xi) = g(x,\xi)^{1/2}$ generates the geodesic flow on $T^*M\backslash 0$, which preserves $S^*M = \{(x,\xi) : H(x,\xi) = 1\}$. Suppose M is such that

S^*M modded out by this flow is a smooth, $(2n-2)$-dimensional manifold, $\mathscr{M}$. This holds, for example, if the action of $\mathbf{R}$ on S^*M given by the geodesic flow is free and proper, as is the case if M is geodesically convex (e.g., $\mathbf{R}^n$ with the standard metric). $\mathscr{M}$ is also smooth if M is a compact, rank one symmetric space [2]. One identifies $\mathscr{M}$ with the space of oriented geodesics on M and then defines the X-ray transform (cf. Helgason [27])

$$(4.1) \qquad \mathscr{R}f(\gamma) = \int_{\mathbf{R}} f(\gamma(s))\, ds, \qquad f \in C_0^\infty(M),\ \gamma \in \mathscr{M},$$

where $\gamma(s)$ is any unit-velocity parametrization of γ. $\mathscr{R}$ is a generalized Radon transform in the sense of Guillemin, satisfying the Bolker condition, and hence the clean intersection calculus applies, yielding that $\mathscr{R}^*\mathscr{R}$ is a pseudodifferential operator of order -1 on M [14]. Thus, $\mathscr{R} : H^s_{\mathrm{comp}}(M) \to H^{s+1/2}_{loc}(\mathscr{M})$, generalizing (locally) the result of Smith and Solmon [28] for the X-ray transform in $\mathbf{R}^n$.

One now considers the restriction of $\mathscr{R}f$ to n-dimensional submanifolds (geodesic complexes) $\mathscr{C} \subset \mathscr{M}$, and the question of reconstructing f from $\mathscr{R}_{\mathscr{C}}f = \mathscr{R}f|_{\mathscr{C}}$. (The following is a summary of the discussion in [12], to which the reader is referred for more details.) To even define $\mathscr{R}_{\mathscr{C}}f$ for $f \in \mathscr{E}'(M)$, we have to impose a restriction on the wave-front set of f. Let

$$(4.2) \qquad Z_{\mathscr{C}} = \{(\gamma, x) \in \mathscr{C}M : x \in \gamma\}$$

be the point-geodesic relation of $\mathscr{C}$; the Schwartz kernel of $\mathscr{R}_{\mathscr{C}}$ is a smooth multiple of the delta function on $Z_{\mathscr{C}}$. Let $\mathrm{Crit}(\mathscr{C})$ be the critical values of the projection from $Z_{\mathscr{C}}$ to M; by Sard's theorem, this is nowhere dense and of measure 0. There is a closed conic set $K_0 \subset T^*M\backslash 0$, whose complement sits over $\mathrm{Crit}(\mathscr{C})$, such that for

$$f \in \mathscr{E}'_{K_0}(M) = \{f \in \mathscr{E}'(M) : \ WF(f) \subset K_0\},\ \mathscr{R}_{\mathscr{C}}f \in \mathscr{D}(\mathscr{C})$$

is well-defined. Shrinking K_0 to a somewhat smaller K in order to avoid the nonfold critical points of $\pi : C = N^*Z'_{\mathscr{C}} \to T^*M\backslash 0$, in [12] it was shown that if $\mathscr{C}$ satisfies a generalization of Gelfand's admissibility criterion [11], then, over K, C is a FFCR and we have $\mathscr{R}_{\mathscr{C}} \in I^{-1/2}(C; \mathscr{C}, M)$. Using an explicit description of the integral kernel of $\mathscr{R}^*_{\mathscr{C}}\mathscr{R}_{\mathscr{C}}$, it was also shown that $\mathscr{R}^*_{\mathscr{C}}\mathscr{R}_{\mathscr{C}} \in I^{-1,0}(\Delta_{T_*M}, \Lambda_{\pi(L)})$, where $\pi(L)$

is the boundary of the support of the Crofton symbol, allowing the construction of a relative left-parametrix for $\mathscr{R}_{\mathscr{C}}$. From Theorem 2.6 it then followed that

$$(4.3) \qquad \|\mathscr{R}_{\mathscr{C}}f\|_{H^{s+1/4}(\mathscr{C})} \leqslant C_s\|f\|_{H^s(M)},\quad f \in \mathscr{E}'_K\, s \geqslant -\frac{1}{4},$$

C_s depending on s and the support of f. It now follows directly from (0.2) that (4.3) holds for all $s \in \mathbf{R}$; furthermore, by (3.17), $\mathscr{R}_{\mathscr{C}}: H^{s,k}_{\text{loc}}(M) \to H^{s-k+1/4,2k}_{\text{loc}}(\mathscr{C})$. Moreover, (0.2) can be applied to an analytic continuation of $\mathscr{R}_{\mathscr{C}}$ to obtain Theorem 0.4.

First, we derive necessary conditions for local boundedness

$$(4.4) \qquad \mathscr{R}_{\mathscr{C}}: L^p_{\text{comp}}(M) \to L^q_{\text{loc}}(\mathscr{C})$$

by considering, in $\mathbf{R}^n$, the following two families of functions. If $x \in \mathbf{R}^n\backslash\text{Crit}(\mathscr{C})$, i.e., the projection from $Z_{\mathscr{C}}$ to $\mathbf{R}^n$ is a submersion at x_0, then if we set $f_\varepsilon = \chi_{B(x_0;\varepsilon)}$, we have $\|f_\varepsilon\|_{L^p} \sim \varepsilon^{n/p}$ while $\mathscr{R}_{\mathscr{C}}f_\varepsilon \geqslant c\varepsilon$ on a rectangle in $\mathscr{C}$ of dimensions $\sim 1 \times \varepsilon \times \varepsilon^{n-1}$, so that $\|\mathscr{R}_{\mathscr{C}}f_\varepsilon\|_{L^q} \geqslant c\varepsilon^{1+\frac{n-1}{q}}$; (4.4) then implies that $\frac{1}{q} \geqslant (n/n-1)\frac{1}{p} - \frac{1}{n-1}$. If $0 = x_0 \in \gamma_0 = x_1 -$ axis and $T_{\gamma_0}\Sigma = x_1 - x_2$ plane, where

$$\sum_{x_0} = \bigcup_{\{\gamma \in \mathscr{C} : x_0 \in \gamma\}}$$

is a two-dimensional cone with vertex at x_0 and $T_{\gamma_0}\sum_{x_0}$ is its tangent plane along γ_0, we may set $f_\varepsilon = \chi_{[-1,1]\times[-\varepsilon,\varepsilon]\times[-\varepsilon^2,\varepsilon^2]\times\ldots\times[-\varepsilon^2,\varepsilon^2]}$, obtaining $\|f_\varepsilon\|_{L^p} \sim \varepsilon\frac{2n-3}{p}$ while $\|\mathscr{R}_{\mathscr{C}}f\|_{L^q} \geqslant c\varepsilon\frac{2n-2}{q}$, so that (4.4) implies that $\frac{1}{q} \geqslant (2n-3)/(2n-2).\frac{1}{q}\cdot$ Thus, a necessary condition for (4.4) to hold is that $\left(\frac{1}{p},\frac{1}{q}\right)$ lie in the convex hull of (0,0), (1,1) and $\left(\frac{2}{3},(2n-3)/(3n-3)\right)\cdot$ Our positive results, (0.4 *a*) and (0.4 *b*), are only for $\left(\frac{1}{p},\frac{1}{q}\right)$ lying in a proper subset of this region and so are probably not sharp.

The proof of Theorem 0.4 is straightforward, given Theorem 0.2. Let $\rho_1(\gamma,x), \ldots, \rho_{n-1}(\gamma x) \in \mathscr{C}^\infty(\mathscr{C} \times M)$ be defining functions for $Z_{\mathscr{C}}$. Consider the entire, distribution-valued family

$$(4.5)\quad K^\alpha(\gamma,x) = \Gamma\left(\frac{\alpha}{2}\right)^{-1} |\vec{\rho}(\gamma,x)|^{\alpha-(n-1)}\psi(\gamma,x), \alpha \in \mathbf{C},$$

where $\vec{\rho} = (\rho_1, \ldots, \rho_{n-1})$ and $\psi \in C_0^\infty(\mathscr{C} \times M)$ is $\equiv 1$ on $Z_{\mathscr{C}}$ over the support of f and supported close to $Z_{\mathscr{C}}$. If we denote the operator with Schwartz kernel K^α by $\mathscr{R}^\alpha$, then $\mathscr{R}^\alpha \in I^{-1/2-\mathrm{Re}(\alpha)}(C;\mathscr{C},M)$. Furthermore, if $P(x,D)$ is a zeroth order pseudodifferential operator on M, elliptic on a subcone $K_1 \subset K$ and smoothing outside of K, then $\mathscr{R}^0 = \mathscr{R}_{\mathscr{C}}P$ acting on $\mathscr{E}'_{K_1}$. By (0.2), we have $\mathscr{R}^\alpha P : L^2_{\mathrm{comp}}(M) \to L^2_{\mathrm{loc}}(\mathscr{C})$ for $\mathrm{Re}(\alpha) = -\frac{1}{4}\cdot$ On the other hand, for $\mathrm{Re}(\alpha) = n-1$, we clearly have $\mathscr{R}^\alpha P : H^1 \to L^\infty_{\mathrm{loc}}$, where H^1 is the Hardy space on M ([29]). By the Fefferman-Stein interpolation theorem [10],

$$\mathscr{R}^0 : L^{p_0}_{\mathrm{comp}} \to L^{q_0}_{\mathrm{comp}} \left(\frac{1}{p_0}, \frac{1}{q_0}\right) = \left(\frac{2n-1}{4n-3}, \frac{2n-2}{4n-3}\right).$$

(A word is needed about the dependence of the L^2 bounds on $\mathrm{Im}(\alpha)$ for $\mathrm{Re}(\alpha) = -\frac{1}{4}\cdot$ To obtain estimates on any finite number of derivatives of the product-type symbol of $\mathscr{R}^{\alpha*}\mathscr{R}^\alpha \in I^{-1-2\mathrm{Re}(\alpha),0}(\Delta,\Lambda)$, only a finite number of applications of first order pseudodifferential operators (as in (2.3)) have to be made. However, the dependence of L^2 bounds for elements of $I^{-1/2,0}(\Delta,\Lambda)$ on only a finite number of derivatives of the product-type symbols is not clear in the proof presented in [13], § 3, since that proof uses the full functional calculus for $I(\Delta,\Lambda)$. An alternate proof may be given, though, in which this dependence is clear. There are fixed elliptic FIOs F_1, F_2 such that $T^\alpha = F_2\mathscr{R}^{\alpha*}\mathscr{R}^\alpha F_1 \in I^{-1/2,0}(\tilde{\Delta},\tilde{\Lambda})$ has the representation (cf. [13], § 1).

$$T^\alpha f(z) = \int e^{i((z'-y')\cdot\zeta' + (z_n-y_n)\zeta_n)} a_\alpha(z,y\,;\zeta';\zeta_n) f(y',y_n) d\zeta'\, d\zeta_n\, dy'\, dy_n$$

where a_α is a symbol-valued symbol of order $M = 0$, $M' = 0$. We may consider this as a pseudodifferential operator, of order 0 and type 1,0, acting on $L^2(\mathbf{R}^{n-1}; (L^2(\mathbf{R}))$, whose symbol is the pseudodifferential operator on $\mathbf{R}$ with symbol $a_\alpha(z',\cdot,y',\cdot;\zeta';\cdot)$, which is of order 0 and type 1,0. By the standard proofs of L^2 boundedness for operators of type 1,0, we only need the $S^0_{1,0}$ estimates for a finite number (say, n) of derivatives. Thus, the L^2 bounds for $\mathscr{R}^\alpha$ grow at most exponentially in $|\mathrm{Im}(\alpha)|$ for $\mathrm{Re}(\alpha) = -\frac{1}{4}$).

On compact sets away from $\mathrm{Crit}(\mathscr{C})$, $\sup_x \|K_{\mathscr{R}_\mathscr{C}}(\cdot,x)\|$ and $\sup_\gamma \|K_{\mathscr{R}_\mathscr{C}}(\gamma,\cdot)\|$ are bounded, where $\|d\mu\|$ is the total variation of a complex measure $d\mu$, and hence $\mathscr{R}_\mathscr{C} : L^p_{\mathrm{comp}} \to L^p_{\mathrm{loc}}$ $1 \leqslant p \leqslant \infty$, acting on functions supported away from $\mathrm{Crit}(\mathscr{C})$, and hence $\mathscr{R}_\mathscr{C} P : L^p_{\mathrm{comp}} \to L^p_{\mathrm{loc}}$ $1 < p \leqslant \infty$. Interpolating between these estimates, we obtain Theorem 0.4. Of course, if we can take $K = T^*M \backslash 0$, then the microlocalization $P(x,D)$ is unnecessary and (0.4) holds for $p = 1$, $p = \infty$ as well.

Just as with the L^2 estimates in [13], one expects the estimates for $\mathscr{R}_\mathscr{C}$ for a general $\mathscr{C}$ to be better than those in (0.4). For instance, it was shown in [13] that for an open set of $\mathscr{C}$'s in three variables, $N^*Z'_\mathscr{C}$ is a folding canonical relation in the sense of Melrose and Taylor [25], so that there is a loss of only $\frac{1}{6}$, rather than $\frac{1}{4}$, derivatives on L^2. Incorporating the L^2 estimates of [25] into the above interpolation argument, one obtains

THEOREM 4.6. — *Let $\mathscr{C} \subset \mathscr{M}$ be a geodesic complex and let $P(x,D)$ be a zeroth order pseudodifferential operator on M such that $C = N^*Z'_\mathscr{C}$ is a folding canonical relation over the conic support of P. Then $\mathscr{R}_\mathscr{C} P : L^p_{\mathrm{comp}}(M) \to L^q_{\mathrm{loc}}(\mathscr{M})$ for p, q satisfying either of the following conditions :*

(a) $$\frac{1}{q} \geqslant \frac{3n-1}{3n-3}\left(\frac{1}{p} - \frac{1}{2(3n-2)}\right), \qquad 1 < p \leqslant \frac{2(3n-2)}{3n-1}\,;$$

(b) $$\frac{1}{q} \geqslant \frac{3n-3}{3n-1}\frac{1}{p}, \qquad \frac{2(3n-2)}{3n-1} \leqslant p < \infty .$$

As described in [13], examples of $\mathscr{C}$'s to which Theorem 4.6 applies are given by equipping $\mathbf{R}^3$ with the Heisenberg group structure with Planck's constant $\varepsilon \neq 0$ suitably small and taking $\mathscr{C}_\varepsilon$ to be all light rays through the origin and their left translates. Because of the stability of Whitney folds, Theorem 4.6 also applies to small perturbations of these in the C^∞ topology.

BIBLIOGRAPHY

[1] J. Antoniano and G. Uhlmann, A functional calculus for a class of pseudodifferential operators with singular symbols, Proc. Symp. Pure Math, 43 (1985), 5-16.

[2] A. Besse, Manifolds all of Whose Geodesics are Closed, Springer-Verlag, New York, 1978.

[3] L. Boutet de Monvel, Hypoelliptic equations with double characteristics and related pseudodifferential operators, Comm. Pure Appl. Math., 27 (1974), 585-639.

[4] A. P. Calderón and R. Vaillancourt, A class of bounded pseudodifferential operators, Proc. Nat. Acad. Sci. USA, 69 (1972), 1185-1187.

[5] M. Christ, Estimates for the k-plane transform, Indiana Univ. Math. Jour., 33 (1984), 891-910.

[6] S. Drury, L^p estimates for the X-ray transform, Illinois Jour. Math., 27 (1983), 125-129.

[7] S. Drury, Generalizations of Riesz potentials and L^p estimates for certain k-plane transforms, Illinois Jour. Math., 28 (1984), 495-512.

[8] J. J. Duistermaat, Fourier Integral Operators, Courant Institute, New York, 1973.

[9] J. J. Duistermaat and V. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, Inv. Math., 29 (1975), 39-79.

[10] C. Fefferman and E. M. Stein, H^p spaces of several variables, Acta Math., 129 (1972), 137-193.

[11] I. M. Gelfand, M. I. Graev and N. Ya. Vilenkin, Generalized Functions, V, Academic Press, New York, 1966.

[12] A. Greenleaf and G. Uhlmann, Nonlocal inversion formulas for the X-ray transform, Duke Math. Jour., 58 (1989), 205-240.

[13] A. Greenleaf and G. Uhlmann, Estimates for singular Radon transforms and pseudodifferential operators with singular symbols, Jour. Func. Anal., 89 (1990), 202-232.

[14] V. Guillemin, On some results of Gelfand in integral geometry, Proc. Symp. Pure Math., 43 (1985), 149-155.

[15] V. Guillemin, Cosmology in $(2+1)$-dimensions, Cyclic Models, and Deformations, Princeton University Press, Princeton, 1989.

[16] V. Guillemin and G. Uhlmann, Oscillatory integrals with singular symbols, Duke Math. Jour., 48 (1981), 251-267.

[17] S. Helgason, The Radon Transform, Birkhaüser, Boston, 1980.

[18] L. Hörmander, Fourier integral operators, I, Acta Math., 127 (1971), 79-183.

[19] L. Hörmander, The Analysis of Linear Partial Differential Operators, IV, Springer-Verlag, New York, 1985.

[20] R. MELROSE, Equivalence of glancing hypersurfaces, Inv. Math., 37 (1976), 165-191.

[21] R. MELROSE, Equivalence of glancing hypersurfaces, II, Math. Ann., 255 (1981), 159-198.

[22] R. MELROSE, Transformation of boundary problems, Acta Math., 147 (1981), 149-236.

[23] R. MELROSE, The wave equation for a hypoelliptic operator with symplectic characteristics of codimension two, Jour. d'Analyse Math., 44 (1984-1985), 134-182.

[24] R. MELROSE, Marked lagrangians, notes of lectures at Max Planck Institut, 1987, paper in preparation.

[25] R. MELROSE and M. TAYLOR, Near peak scattering and the corrected Kirchhoff approximation for a convex obstacle, Adv. in Math., 55 (1985), 242-315.

[26] R. MELROSE and G. UHLMANN, Lagrangian intersection and the Cauchy problem, Comm. Pure Appl. Math., 32 (1979), 482-519.

[27] D. OBERLIN and E. M. STEIN, Mapping Properties of the Radon transform, Indiana Univ. Math. Jour., 31 (1982), 641-650.

[28] K. T. SMITH and D. C. SOLMON, Lower dimensional integrability of L^2 functions, Jour. Math. Anal. Appl., 51 (1975), 539-549.

[29] R. STRICHARTZ, The Hardy space H^1 on manifolds and submanifolds, Canad. Jour. Math., 24 (1972), 915-925.

[30] R. STRICHARTZ, L^p estimates for Radon transforms in euclidian and non-euclidian spaces, Duke Math. Jour., 48 (1981), 699-737.

[31] H.-T. WANG, L^p estimates for the restricted X-ray transform, Ph. D. thesis, Univ. of Rochester, June 1987.

[32] A. WEINSTEIN, On Maslov's quantization condition, in Fourier Integral Operators and Partial Differential Equations, J. Chazarain, ed., Springer-Verlag, 1975.

Manuscrit reçu le 25 octobre 1988.

A. GREENLEAF,
Department of Mathematics
University of Rochester
Rochester N.Y. 14627 (USA).

G. UHLMANN,
Department of Mathematics
University of Washington
Seattle, WA 98195 (USA).