Annales de l'institut Fourier

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Annales de l'institut Fourier, tome 40, nº 1 (1990), p. 177-212

<http://www.numdam.org/item?id=AIF_1990__40_1_177_0>

Ann. Inst. Fourier, Grenoble
40, 1 (1990), 177-212

HARMONIC MORPHISMS AND CIRCLE ACTIONS ON 3- AND 4- MANIFOLDS

by Paul BAIRD

0. Introduction.

A continuous mapping $\phi : M \to N$ between Riemannian manifolds of dimensions m, n respectively, is a harmonic morphism if, for every functions f harmonic on an open set V in N, the composition $f \circ \phi$ is harmonic on $\phi^{-1}V$ in M. They are characterized by being harmonic maps and horizontally conformal [19], [23]. In particular if ϕ is non-constant we must have $m \geq n$. If $m = n = 2$, the harmonic morphisms are precisely the weakly conformal mappings. A more detailed account of this class of mappings is given in Section 1.

Certain algebraic equations determine multiple valued harmonic morphisms defined on S^3 (see Section 2). Given such a mapping ψ, by cutting and glueing copies of S^3 we may build up a 3-manifold M regarded naturally as a branched cover of S^3. We also have associated a *single* valued harmonic morphism $\phi : M \to N$ onto some Riemann surface N. When we restrict ϕ to the different sheets of the branched covering $M \to S^3$, ϕ takes on the various values of the multiple valued harmonic morphism ψ. Uniformization in this context amounts to endowing M with a smooth Riemannian structure whilst preserving the property that ϕ be a harmonic morphism.

It is a natural question to ask which 3-manifolds may be constructed by this method. One of our main theorems (Theorem 3.10) asserts that :

Key-words : Harmonic morphism – Seifert fibre space – Circle action.
A.M.S. Classification : 57N10 – 57N13 – 58E20.

If $\phi : M \to N$ is a non-constant harmonic morphism from a closed analytic 3-manifold to a Riemann surface, then M must have the structure of a Seifert fibre space.

In particular a harmonic morphism from a closed hyperbolic 3-manifold with its natural analytic structure to a Riemann surface must be constant. This is in constrast to the situation for harmonic maps where the existence of non-trivial harmonic maps from closed hyperbolic 3-manifolds to Riemann surfaces are well-known (see for example [10]). In the more general case when M is assumed smooth rather than analytic, we are able to show that ϕ determines a C^0 foliation of M. Under the additional assumption that the foliation be C^1 we can again show that M has the structure of a Seifert fibre space (Theorem 3.13).

As a partial conserve we show that (Theorem 3.17) :

Every Seifert fibre space M admits a metric with respect to which the fibres form a conformal foliation by geodesics.

In the case when the leaf space is a smooth surface N, then the leaves are the fibres of a harmonic morphism $\phi : M \to N$.

In Section 4 we establish a fibration criterion for harmonic morphisms (Theorem 4.6). Precisely :

If $\phi : M^m \to N^n$ is a non-constant harmonic morphism (so $m \geq n$) and $(m-2) < 2(n-2)$, then ϕ is a submersion and in particular determines a fibration of M^m.

This restricts the topology of M^m and N^n. For example, if $\phi : S^{n+1} \to S^n$ is a harmonic morphism, $n \geq 4$, then ϕ must be constant. This again contrasts with the case for harmonic maps, where the methods of Smith [33] and Ratto [29] yeld harmonic representatives of the non-trivial class of $\pi_{n+1}(S^n)$ for all n.

In Section 5 we investigate harmonic morphisms $\phi : M^4 \to N^3$. We establish the following (Theorem 5.3) :

If $\phi : M^4 \to N^3$ is a non-constant harmonic morphism from a closed oriented 4-manifold to an oriented 3-manifold, then ϕ determines a locally smooth S^1-action on M^4. There are at most two different orbit types, principal orbits or fixed points.

In the case M^4 is simply connected, the classification theorem of Fintushel [18] tells us that M^4 is a connected sum of the manifolds S^4, $\mathbf{C}P^2$, -$\mathbf{C}P^2$ and $S^2 \times S^2$.

We construct an example of a harmonic morphism $\phi : S^4 \to S^3$ where the corresponding S^1-action has two fixed points. Our method of construction is to take harmonic morphisms from manifolds with boundary, glue along common boundaries and then smooth over the joins.

We may summarize our results in terms of S^1-actions as follows. If $\phi : M^{n+1} \to N^n$, $n \geq 2$, is a non-constant harmonic morphism from a closed oriented manifold M^{n+1} to an oriented manifold N^n with 1-dimensional fibres (where M^{n+1} and N^n are assumed analytic in the case $n = 2$), then ϕ determines a *locally smooth* S^1-action on M^{n+1}. This action has different properties as follows.

(i) If $n = 2$, the action has no fixed points, and is smooth except possibly across isolated "critical" circles, where it is C^0.

(ii) If $n = 3$, the action may have isolated fixed points and is smooth except possibly at the fixed points.

(iii) If $n \geq 4$, the action is smooth and without fixed points.

The ideas of this paper have developed from an example first given in [2], together with classification of harmonic morphisms from domains in S^3 described in [4]. Lemma 4.3 was first stated in [1], but with an incorrect proof.

I would like to express my thanks to I. Aitchison, J. Eells, D.B.A. Epstein, H. Rubinstein and J.C. Wood for their helpfull comments. In particular I. Aitchison provided a crucial ingredient to the proof of Theorem 5.3. I would like to thank the referee for helpful suggestions. I am also grateful to the Science and Engineering Research Council for their financial support.

Notation and Conventions.

We use the notation M^m to denote a manifold of dimension m. Manifolds will be assumed smooth, connected, oriented and without boundary unless otherwise stated.

If M is a Riemannian manifold, we sometimes write the pair (M, g) to indicate that g is the metric of M. If there is no confusion as to the metric, we write $|X|$ for $\sqrt{g(X,X)}$ and $\langle X, Y\rangle$ for $g(X,Y)$.

1. A brief survey of harmonic morphisms.

Let $\phi : M^m \rightarrow N^n$ be a continuous mapping between connected smooth Riemannian manifolds. Then ϕ is called a *harmonic morphism* if for every function f harmonic on an open set $V \subset N$, the composition $f \circ \phi$ is harmonic on $\phi^{-1}(V) \subset M$. It follows by choosing smooth harmonic local co-ordinates on N[21] that any harmonic morphism is necessarily smooth.

The harmonic morphisms are precisely the harmonic maps which are *horizontally weakly conformal* (see [19],[23] and below). For a map $\phi : \mathbf{R}^3 \rightarrow \mathbf{C}$ this is equivalent to ϕ satisfying the equations

$$\sum_{i=1}^{3} \frac{\partial^2 \phi}{\partial x_i^2} = 0 \tag{1.1}$$

$$\sum_{i=1}^{3} \left(\frac{\partial \phi}{\partial x_i} \right)^2 = 0. \tag{1.2}$$

Harmonic morphisms are objects of considerable interest. Their history goes back to Jacobi [24] in 1847, who considered the problem of finding complex-valued (harmonic) functions satisfying (1.1) and (1.2) above. They were independently defined by Constantinescu and Cornea in 1965 in the context of Brelot harmonic spaces.

A Brelot harmonic space is a topological space endowed with a sheaf of (harmonic) functions. A number of axioms must be satisfied (see [8]), one of these asserts that the Dirichlet problem be solvable. The Brelot harmonic spaces were devised as a natural generalization of Riemann surfaces. A harmonic morphism (called harmonic map in [9]) between two Brelot harmonic spaces, is a mapping which pulls back germs of harmonic functions to germs of harmonic functions.

Harmonic morphisms between Riemannian manifolds were considered by Fuglede [19] and Ishihara [23], who established many of their basic properties. More recently they have been studied in the context of stochastic processes, where they are found to be the *Brownian path preserving mappings* (see [6]).

We record some of the fundamental properties of a harmonic morphism below. For more details and proofs, see [19], [23], [1], [3], [12].

Let $\phi : M^m \rightarrow N^n$ be a smooth mapping between Riemannian manifolds. Then ϕ is called *horizontally weakly conformal* if, for every

$x \in M$ where $d\phi_x \neq 0$, the restriction of $d\phi_x$ to the orthogonal complement $(\text{Ker } d\phi_x)^\perp$ is conformal and surjective. Thus there is some number $\lambda(x) > 0$ such that $|d\phi_x(X)| = \lambda(x)|X|$ for each $X \in (\text{Ker } d\phi_x)^\perp$. Setting λ equal to 0 at critical points we obtain a continuous function $\lambda : M \to \mathbf{R}$ called the *dilation* of ϕ.

(1.3) [19], [23] A smooth mapping $\phi : M^m \to N^n$ between Riemannian manifolds is a harmonic morphism if and only if ϕ is both harmonic and horizontally conformal.

(1.4) [19] If $\phi : M^m \to N^n$ is a harmonic morphism then $m \geq n$. If $m = n = 2$, the harmonic morphisms are precisely the weakly conformal mappings. If $m = n \geq 3$, they are the homothetic mappings.

Let $\phi : M^m \to N^n$ be a non-constant harmonic morphism.

(1.5) [19] ϕ is an open mapping.

(1.6) If M is compact, then N is compact and $\phi(M) = N$ (this follows from (1.5), since $\phi(M)$ is both open and closed in N).

(1.7) [3] If $n = 2$, then the fibres over regular points (*i.e.* those $y \in N$ such that $d\phi_x \neq 0$ for all $x \in \phi^{-1}(y)$) are minimal submanifolds of M.

(1.8) The composition of two harmonic morphisms is a harmonic morphism.

(1.9) [19] The set K of all critical points of ϕ is polar in M (see [19] for definition). A consequence is that K cannot disconnect any open ball in M. Further if M and N are real analytic, then so is ϕ and K is a real analytic set of codimension ≥ 2.

In Section 5 we use the concept of an h-harmonic morphism as defined by Fuglede [19].

Let $h : M^m \to \mathbf{R}$ be a smooth function such that $0 < h(x) < \infty$ for all $x \in M$. A C^2-function $f : U \to \mathbf{R}$, defined on an open set $U \subset M$, is h-harmonic if

$$\Delta f + 2g(\nabla \log h, \ \nabla f) = 0,$$

where g denotes the metric of M. A continuous mapping $\phi : M^m \to N^n$ is an *h-harmonic morphism* if, for every harmonic function f defined on an open set $V \subset N$, the composition $f \circ \phi$ is h-harmonic on $\phi^{-1}(V)$. The h-harmonic morphisms are characterized as those mappings which are

(i) h-harmonic, *i.e.*

$$\tau_\phi + 2d\phi(\nabla \log h) = 0 \tag{1.10}$$

and

(ii) horizontally conformal.

It follows that if $\phi : M^m \to N^n$ is an h-harmonic morphism with respect to the metric g on M, then ϕ is a harmonic morphism with respect to the conformally related metric

$$\widetilde{g} = h^{(4/(m-2))} g. \tag{1.11}$$

2. Harmonic morphisms from S^3, the branching set.

We first of all outline the classification of harmonic morphisms from domains in S^3 described in [2], [4].

Let $\phi : M \to N$ be a non-constant harmonic morphism from a domain $M \subset S^3$ onto a Riemann surface N. Suppose in addition that

(i) ϕ is a submersion (equivalently $d\phi_x \neq 0$ for all $x \in M$),

(ii) the fibres of ϕ are connected.

Consider the cone over M in $\mathbf{R}^4$, $P = \mathbf{R}^+ M = \{tx \in \mathbf{R}^4;\ t \in (0,\infty),\ x \in M\}$. Let $\pi : P \to M$ be radial projection $x \to x/|x|$. Then the composition $\Phi = \phi \circ \pi : P \to N$ is also a submersive harmonic morphism with connected fibres.

Since the fibres of ϕ are minimal they are parts of great circles in S^3, so that the fibres of Φ are parts of 2-planes through the origin in $\mathbf{R}^4$. In particular, for each $z \in N$ we have associated an oriented 2-plane in $\mathbf{R}^4$.

The fundamental theorem of [4] asserts that the harmonic morphism ϕ determines a quadruple of meromorphic functions $\xi_j : N \to \mathbf{C} \cup \infty$, $j = 1, 2, 3, 4$ such that

$$\sum_{j=1}^{4} \xi_j^2 = 0. \tag{2.1}$$

The fibre over $z \in N$ is part of the 2-plane in $\mathbf{R}^4$ determined by the equation

$$\xi_1(z)x_1 + \xi_2(z)x_2 + \xi_3(z)x_3 + \xi_4(z)x_4 = 0. \tag{2.2}$$

By a result of Hoffman and Osserman [22], all such $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ may be written in the form

$$\xi = (1 + fg, i(1 - fg), f - g, i(f + g)),$$

where f, g are meromorphic functions on N.

We may remove conditions (i) and (ii) as follows.

If $\phi : M \to N$ is now an arbitrary non-constant harmonic morphism from a domain $M \subset S^3$ onto a Riemann surface N. It is shown in [4] that the fibre through a critical point of ϕ is also a part of a geodesic through that point. Thus in a neighbourhood of a critical point one has determined a foliation by geodesics. This foliation is in fact smooth. (The proof in [4] of this results depends on M having *constant* curvature.) In the case that the leaf space $\widetilde{N}$ of this foliation is Hausdorff, it may be given the structure of a smooth Riemann surface with respect to which the projection $\pi : M \to \widetilde{N}$ is a smooth submersive harmonic morphism with connected fibres. The induced mapping $\zeta : \widetilde{N} \to N$ such that $\phi = \zeta \circ \pi$, is weakly conformal. Critical points of ϕ arise from branch points of ζ. As before ϕ determines a quadruple $(\xi_1, \xi_2, \xi_3, \xi_4)$ of meromorphic functions, now defined on the leaf space $\widetilde{N}$, which satisfy (2.1).

In the case when M equals S^3 then the leaf space $\widetilde{N}$ is automatically Hausdorff and $\pi : S^3 \to \widetilde{N}$ is completely characterized by the meromorphic functions ξ_j. In fact it is shown in [4] that, up to an isometry of S^3, π is the Hopf fibration and $\widetilde{N} = S^2$.

More generally however, suppose we are given a quadruple of meromorphic functions $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ defined on a domain $N \subset \mathbf{C}$ and satisfying (2.1). In what sense does ξ determine a well-defined harmonic morphism?

Suppose each ξ_j is a rational function of z, then the fibre over z is determined by (2.2) which now takes the form

$$P_n(x)z^n + P_{n-1}(x)z^{n-1} + \ldots + P_0(x) = 0, \tag{2.3}$$

where $x = (x_1, x_2, x_3, x_4) \in S^3$. This may be considered as an *algebraic equation* by analogy with the theory of Riemann surfaces. If we are given a point $x \in S^3$, the solution $z = z(x)$ of (2.3) is in general multiple valued and we will think of (2.3) as determining a multiple valued harmonic morphism, taking values in $\mathbf{C} \cup \infty$.

If we write (2.3) as

$$P(x; z) = 0,$$

$x = (x_1, x_2, x_3, x_4)$, then P is linear in the x_j and (2.1) is equivalent to the condition

$$\sum_{j=1}^{4} \left(\frac{\partial P}{\partial x_j}\right)^2 = 0,$$

for each fixed z.

Envelope points in S^3 are those points which are the intersection points of infinitesimally nearby fibres (see [2], [4]). These are obtained by simultaneously solving

$$\text{(2.4)} \qquad \begin{cases} P = 0 \\ \dfrac{\partial P}{\partial z} = 0. \end{cases}$$

As is well known in algebra, the solutions $x = (x_1, \ldots, x_4)$ of (2.4) are obtained by solving a polynomial in the coefficient $P_n, P_{n-1}, \ldots, P_0$, called the *discriminant*. The solution set is precisely the set of the points x at which $P(x; z) = 0$ has a multiple root. The envelope will correspond to the branching set in the theory of analytic functions, and is a real analytic subset.

Example (2.5). — Consider the algebraic equation

$$x_1 + ix_2 - z^r(x_3 + ix_4) = 0,$$

for some positive integer r. This equation determines an r-valued harmonic morphism on S^3. For $r \geq 2$ the envelope points occur when $z = 0, \infty$. These correspond to the great circles $(0, e^{i\eta})$, $(e^{i\xi}, 0) \subset S^3$. We now take r copies of S^3 and cut and paste them in an appropriate way, simultaneously cutting and pasting the codomain $C \cup \infty$. We obtain a single valued harmonic morphism $\phi : M \to N$ from a compact 3-manifold M, in this case the Lens space $L(r, 1)$ to the Riemann surface of $f(z) = z^{1/r} (= S^2)$. The manifold M is an r-fold branched cover of S^3, branched over the envelope circles. In fact $L(r, 1)$ is naturally a circle bundle over S^2 and the single valued harmonic morphism so obtained is the natural projection $L(r, 1) \to S^2$.

3. A harmonic morphism from a closed 3-manifold determines a Seifert fibration.

In Section 2 we saw how to construct a closed 3-manifold from a multiple valued harmonic morphism. The question arises as to which closed 3-manifolds can be obtained in this way. We show that they are the Seifert fibre spaces.

If $\phi : M^3 \to N^2$ is a non-constant harmonic morphism from a closed 3-manifold to a Riemann surface N, then we will show that ϕ determines a C^0-codimension 2 foliation of M. Under the additional assumption that the foliation be C^1 then the leaves are circles and a theorem of Epstein shows that M must be a Seifert fibre space. If we assume M is analytic then we may remove the condition that the foliation be C^1 and again we are able to show that M must be a Seifert fibre space.

Clearly near regular points ϕ determines a smooth foliation. The problem occurs at critical points of ϕ, *i.e.* those $x \in M$ where $d\phi_x = 0$. Then we use a result of Fuglede which asserts that the symbol of ϕ at $x, \sigma_x(\phi) : T_xM \to T_{\phi(x)}N$ is a harmonic polynomial morphism. A result of Baird – Wood gives a precise description of this mapping and some analysis about the point x shows that ϕ determines a (C^0-)foliation in a neighbourhood of x.

Let $\phi : M^m \to N^n$ be a non-constant harmonic morphism between Riemannian manifolds and let Σ denote the critical set of ϕ; $\Sigma = \{x \in M;\ d\phi_x = 0\}$. Fix $x_0 \in \Sigma$. Let $U \subset M$, $V \subset N$ be *geodesically* convex neighbourhoods of $x_0 \in M$, $\phi(x_0) \in N$ respectively, over which normal coordinates $(x^1, \ldots, x^m)$, $(y^1, \ldots, y^n)$ are defined. That is $(x^1, \ldots, x^m)$ are determined by the exponential map as follows :

$$\exp_{x_0} : T_{x_0}M \to M$$

is defined by $\exp_{x_0}(X) = \sigma_X(1)$, where σ_X is the geodesic determined by X. Then $\exp_{x_0}$ is a diffeomorphism in a neighbourhood of x_0. We assume U chosen such that $\exp_{x_0}^{-1}$ is a diffeomorphism when restricted to U. If $e_1, \ldots, e_m$ is an orthonormal basis for $T_{x_0}M$, we may write

$$X = x^1e_1 + \ldots + x^me_m.$$

If $x \in U$ satisfies $\exp_{x_0}^{-1}(x) = X$, we associate the coordinates $(x^1, \ldots, x^m)$ to x.

Let $g = g_{ij}dx^i dx^j$, $h = h_{kl}dy^k dy^l$ denote the metric tensors on M, N respectively, then

$$g_{ij}(0) = \delta_{ij}, h_{kl}(0) = \delta_{kl}.$$

Recall the *order of* ϕ at x_0 is defined to be the smallest integer $p \geq 1$ such that for some $k = 1, \ldots, n$, the k'th component, $\phi^k = y^k \circ \phi$, of ϕ expressed as a function of the coordinates (x^i) has at least one non-vanishing p'th order derivative $\partial^p \phi^k / \partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}$, $|\alpha| = \alpha_1 + \ldots + \alpha_m = p$, at x_0.

If ϕ has order p at $x_0 (p \geq 1)$, the symbol

$$\frac{1}{p!}\, d^p \phi_{x_0} : T_{x_0}M \to T_{\phi(x_0)}N$$

is defined componentwise by

$$\frac{1}{p!}\, d^p \phi^k (a^1 e_1 + \ldots + a^m e_m) = \sum_{|\alpha|=p} \frac{1}{\alpha_1 ! \alpha_2 ! \ldots \alpha_m !} \frac{\partial^p \phi^k}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}} ((a^1)^{\alpha_1} \ldots (a^m)^{\alpha_m},$$

for $k = 1, \ldots, n$. This is well-defined and independent of the choice of coordinates.

Put $\xi^k = \frac{1}{p!}\, d^p \phi^k$, so that $\xi = (\xi^1, \ldots, \xi^n) : T_{x_0}M \to T_{\phi(x_0)}N$ denotes the symbol.

Note : We identify points in U and the tangent space $T_{x_0}M$ via $\exp_{x_0}$ and we assume x_0 and $\phi(x_0)$ correspond to the origin in their respective coordinate systems.

THEOREM 3.1 (Taylor's formula – see for example [19]). — *Near the point* x_0,

(a) $\phi^k(x) = \xi^k(x) + O(r^{p+1}) \qquad (r = |x|)$

(b) $D_i \phi^k(x) = D_i \xi^k(x) + O(r^p)$,

for $i = 1, \ldots, m$; $k = 1, \ldots, n$, *where* $D_i = \partial / \partial x_i$.

THEOREM 3.2 [19]. — *For each* $x \in M$, *the symbol* $\xi_x : T_xM \to T_{\phi(x)}N$ *is a harmonic morphism defined by homogeneous polynomials of degree* p *(where* p *is the order of* ϕ *at* x*).*

THEOREM 3.3 [4]. — *If $\xi : \mathbf{R}^3 \to \mathbf{R}^2$ is a non-constant harmonic morphism, then ξ is an orthogonal projection $\mathbf{R}^3 \to \mathbf{R}^2$ followed by a weakly conformal map $\mathbf{R}^2 \to \mathbf{R}^2$.*

Assume from now on that $m = \dim M = 3$ and $n = \dim N = 2$.

Note that the regular fibres of ϕ carry a natural orientation defined as follows. At each regular point $x \in M$, we may orient the horizontal space $H_x = \ker d\phi_x^\perp$ such that $d\phi|_{H_x} : H_x \to T_{\phi(x)}N$ is orientation preserving. Then we orient the vertical space $V_x = \ker d\phi_x$ such that the orientations on V_x and H_x combine to give the orientation on M.

Let $\gamma : U \setminus \Sigma \to T^1(U \setminus \Sigma)$ denote the Gauss map of ϕ, where $T^1(U \setminus \Sigma)$ denotes the unit tangent bundle over $U \setminus \Sigma$. Thus γ associates to each $x \in U \setminus \Sigma$ the unit tangent vector to the (oriented) fibre of ϕ through x. Under the trivialisation of TU given by the coordinates (x^1, x^2, x^3), we may think of γ as a map $\gamma : U \setminus \Sigma \to S^2 \subset T_{x_0}M \cong \mathbf{R}^3$.

LEMMA 3.4. — *γ extends continuously across Σ and the integral curves of γ determine a C^0- foliation of M by geodesics. The leaves of the foliation are the fibres of ϕ.*

Remark. — This result was established for domains in $\mathbf{R}^3$ using probabilistic arguments, by Bernard, Campbell and Davie [6].

Proof. — Let $x_0 \in \Sigma$. From Theorems 3.2 and 3.3, there is a well-defined direction at x_0, given by the fibre through $0 \in T_{x_0}M \cong \mathbf{R}^3$ of the symbol ξ. We may assume this is e_3. Define $\gamma(x_0) = e_3$.

Without loss of generality we may also assume $\xi : \mathbf{R}^3 \to \mathbf{R}^2 \cong \mathbf{C}$ is given by

$$\xi(x^1, x^2, x^3) = (x^1 + ix^2)^p.$$

That is $\xi(u, x^3) = u^p, u = x^1 + ix^2$. Then the dilation λ of ξ is given by $\lambda = p|u|^{p-1}$, so that

$$|\nabla \xi^k|^2 = p^2 |u|^{2p-2},$$

for $k = 1, 2$.

Let $W \subset U$ be the slice determined by $x^3 = 0$ (W is a smooth surface).

Claim. — Provided U is taken small enough, we may assume $x_0 \in W$ is the only critical point of $\phi|_W$.

Proof of claim. — $\nabla\xi^k(x) = p|u|^{p-1}w^k(x)$, $k = 1, 2$, for $x \neq 0$, where $w^k(x)$ is a unit vector in S^2 perpendicular to e_3. From Theorem 3.1(b)

$$\nabla\phi^k = \nabla\xi^k + O(r^p).$$

Suppose $x \in W$, $x \neq 0$, $\nabla\phi^k(x) = 0$. Then

$$0 = p|u|^{p-1}w^k(x) + O(|u|^p).$$

But $|u| \neq 0$, so dividing by $|u|^{p-1}$ implies

$$0 = pw^k(x) + O(|u|).$$

This is impossible if W is small enough, establishing the claim.

First of all we show that if y_n is a sequence of points in $W, y_n \to x_0$, then $\gamma(y_n) \to \gamma(x_0) = e_3$. For

$$\langle\nabla\phi^k(y_n),\ \gamma(y_n)\rangle = 0,$$

$k = 1, 2$. Put $\gamma(y_n) = \cos\vartheta(y_n)e_3 + \sin\vartheta(y_n)v(y_n) \in S^2$, where $v(y_n) \in S^2$ satisfies $\langle e_3, v(y_n)\rangle = 0$. From Theorem 3.1(b) :

$$\nabla\phi^k(y_n) = \nabla\xi^k(y_n) + O(|u|^p),$$

so

$$\langle\nabla\xi^k(y_n) + O(|u|^p),\ \cos\vartheta(y_n)e_3 + \sin\vartheta(y_n)v(y_n)\rangle = 0,$$

which implies

$$0(|u|^p) = \sin\vartheta(y_n)\langle\nabla\xi^k(y_n), v(y_n)\rangle. \tag{3.5}$$

As before $\nabla\xi^k(y_n) = p|u|^{p-1}w^k(y_n)$, where $w^1(y_n), w^2(y_n) \in S^2$ are perpendicular to e_3 and orthogonal. Thus

$$\frac{1}{|u|^{p-1}}\langle\nabla\xi^k(y_n), v(y_n)\rangle = p\langle w^k(y_n), v(y_n)\rangle \not\to 0 \tag{3.6}$$

as $|u| \to 0$, for one of $k = 1, 2$. (In fact $w^k(y_n)$ approaches any direction as $y_n \to 0$, depending on the choice of sequence y_n.)

From (3.5)

$$\sin\vartheta(y_n)p\langle w^k(y_n), v(y_n)\rangle = O(|u|) \longrightarrow 0,$$

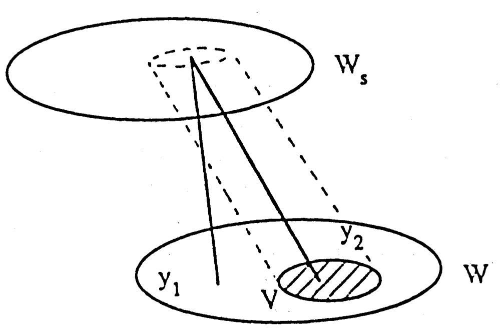
as $|u| \to 0$. Then from (3.6), $\vartheta(y_n) \to 0$ as $|u| \to 0$, which implies $\gamma(y_n) \to e_3 = \gamma(x_0)$.

Thus $\gamma(x)$ varies continuously as x varies over W, further $\gamma(x)$ is transverse to W if W is taken small enough (since $\gamma(x_0) = e_3$ is transverse at $x_0 \in W$).

Consider the geodesic passing through x_0 determined by e_3. Call it $\sigma(t) = (0,0,t)$. Then $\phi(\sigma(t)) =$ constant. For if not, so $\phi(x) \neq \phi(x_0)$, $x = \sigma(t) \in U$, for some t. Let $y_n \in W$ be a sequence of points $y_n \to x_0$. Then $\gamma(y_n) \to \gamma(x_0)$, so the geodesics, which are the fibres of ϕ through y_n, become arbitrarily close to the geodesic $\sigma(t)$, contradicting the continuity of ϕ.

Consider nearby slices $W_t \subset U$ given by $x^3 = t$, $t \in (-\varepsilon, \varepsilon)$ for some small $\varepsilon > 0$. Let $A \subset W$ be a small open disc in W with $x_0 \in A$, chosen so that the mapping $\pi_t : A \to W_t$, obtained by moving along a geodesic determined by $\gamma|_A$ until is hits W_t, is well defined. Note that provided ε is taken small enough by the continuity of $\gamma|_A$ this is possible, and we may assume the geodesics intersect the W_t transversally.

By the continuity of $\gamma|_W$, π_t is a continuous mapping. We claim that π_t is injective. For suppose not. Let $y_1, y_2 \in A$, $y_1 \neq y_2$, be such $\pi_t(y_1) = \pi_t(y_2)$ for some $t \in (-\varepsilon, \varepsilon)$. Then $\phi(y_1) = \phi(y_2)$. Since $x_0 \in A$ is the only critical point of $\phi|_A$ we may assume y_2 is a regular point for $\phi|_A$. Hence by the Inverse Function Theorem, there is a neighbourhood V of y_2 in A with $y_1 \notin V$, such that $\phi|_V$ is a diffeomorphism. Then $\pi_s|_V : V \to W_s$ is injective and by Invariance of Domain [20], Theorem 18.9, is a homeomorphism onto its image. By construction $\pi_s(y_2) \in \pi_s(V)$ and the union of the sets $\pi_s(V)$ forms a tubular neighbourhood of the geodesic passing through y_2.



By continuity there are points $y \in V \setminus \{y_2\}$ such that the geodesic through y intersects the geodesic through y_1. For such y, $\phi(y) = \phi(y_1) = \phi(y_2)$, a contradiction. Hence $\pi_t : A \to W_t$ is injective, and again applying Invariance of Domain we see that the family of sets $\{\pi_t(A)\}_{t\in(-\varepsilon,\varepsilon)}$ fill out a neighbourhood of x_0 foliated by geodesics. It follows that $\gamma(y_n) \to \gamma(x_0) = e_3$ for any sequence $y_n \to x_0$ and that M is foliated by the fibres of ϕ. □

LEMMA 3.7. — *Suppose the foliation determined by ϕ is C^1. Let F be a connected component of a fibre of ϕ and suppose some $x_0 \in F$ is a critical point of ϕ. Then the whole fibre component F is critical.*

Proof. — Take a slice through x_0 as before, and let B be a sufficiently small ball centred at x_0 such that B is foliated by the fibres of ϕ transverse to W. Let π be the projection, $\pi : B \to W$, obtained by projecting down the fibres of ϕ. Then $\phi|_B$ factors as $\phi|_B = \phi|_W \circ \pi$. So for $x \in F \cap B$,

$$\begin{aligned} d\phi(x) &= (d\phi|_W)(\pi(x)) \circ d\pi(x) \\ &= (d\phi|_W)(x_0) \circ d\pi(x) \\ &= 0. \end{aligned}$$

Thus the set of critical points in F is open in F. Since the set of critical points in F is also closed in F, the whole of F must be critical, establishing the lemma. □

LEMMA 3.8. — *If M^3 is compact and either* (a) *M^3 is analytic, or* (b) *the foliation induced by ϕ is C^1, then all fibres of ϕ are compact and so are circles which foliate M.*

Proof. — Let F be a regular fibre component of ϕ. If $x \in F$, take a small slice W through x transverse to F. The derivative of $\phi|_W : W \to N$ has rank 2 at x and so by the Inverse Function Theorem is a local diffeomorphism. Thus provided W is taken small enough we may assume F does not intersect W again. It follows there is a small ball $B_\delta(x)$ centred at x, such that F only passes once through $B_\delta(x)$. Cover F with these balls and extend to a cover of M. Since M is compact there is a finite subcover. Thus F is covered by finitely many such balls and F is compact.

If now the foliation is C^1 and F is a critical fibre component, we may use the Claim of Lemma 3.4 to ensure that F does not return arbitrarily close to itself. Again we can cover F with a finite number af balls and F is also compact.

If on the other hand M^3 is analytic and we assume the foliation is merely C^0, then by a standard property of harmonic mappings ϕ is analytic. Now let $x_0 \in F$ and take a small (analytic) slice W through x_0 as before such that the fibres of ϕ hit W transversally. Then $\phi|_W$ is analytic and if F winds back arbitrarily close to x_0, $\phi|_W$ will be constant on a set having an accumulation point at x_0. Thus $\phi|_W$ is constant, a contradiction. Thus F does not return arbitrarily close to x_0 and we cover F with balls and proceed as before. □

Remark 3.9. — In the case that M^3 has constant curvature it follows by a result in [5] that the foliation by circles is smooth. Since the leaves are compact, by Theorem 4.2 of [16] the leaf space $\widetilde{N}$ is Hausdorff. By results of [5], $\widetilde{N}$ may be given the structure of a Riemann surface such that the projection $\pi : M^3 \to \widetilde{N}$ is a submersive harmonic morphism. Then $\phi = \zeta \circ \pi$, where $\zeta = \widetilde{N} \to N$ is weakly conformal. Critical fibres of ϕ arise from branch points of ζ.

We now apply some results of Epstein to show that M is homeomorphic to one of the standard Seifert fibre spaces (I am grateful to D.B.A. Epstein for pointing this out to me).

Consider the case when M is a closed *analytic* 3-manifold, so that if $\phi : M \to N^2$ is a non-constant harmonic morphism, then ϕ determines a C^0-foliation of M by compact circles. By [17], Theorem 13.1, the volume of the leaves of the foliation is bounded above. Consider a fibre component F. Then by [17] we may construct a small slice W through a point of F which is invariant under the first return map. We may then apply Theorem 2.8 of that paper to put a Riemannian metric of constant curvature on W such that the first return map is rotation by a rational multiple of 2π, or reflexion in a straight line. It now follows that M is a Seifert fibre space (see [15]).

We have therefore established

THEOREM 3.10. — *Let $\phi : M \to N$ be a non-constant harmonic morphism from a closed analytic 3-manifold to a Riemann surface. Then there is a fibre preserving homeomorphism from M to a Seifert fibre space.*

COROLLARY 3.11. — *Let $\phi : M \to N$ be a non-constant harmonic morphism from a closed analytic 3-manifold to a Riemann surface. Then ϕ determines a locally smooth S^1-action on M. The orbits of this action are the fibres of ϕ.*

Proof. — At a regular point x of ϕ, the horizontal space $(\ker d\phi_x)^\perp$ has a natural orientation induced from N. Together with the orientation of M, this gives an orientation on the fibre through x. By continuity the orientation on regular fibres extends to critical fibres, thus determining an S^1-action on M.

Since M is homeomorphic to one of the standard Seifert fibre spaces, it follows that each orbit has an open invariant neighbourhood on which there exists a differentiable structure with respect to which S^1 acts smoothly. Hence the S^1-action is locally smooth [7]. □

COROLLARY 3.12. — *If M^3 is a closed analytic hyperbolic 3-manifold endowed with any analytic metric and $\phi : M^3 \to N^2$ is a harmonic morphism onto a Riemann surface, then ϕ is constant.*

Proof. — No hyperbolic 3-manifold is Seifert fibred (see [31]). □

Remark. — This is in contrast to the case for harmonic maps. For, given a homotopy class of maps from M^3 to N^2, where N has negative curvature, the existence of energy minimizing harmonic maps is well know [10].

Suppose now that M^3 is a *smooth* closed 3-manifold and $\phi : M \to N$ is a non-constant harmonic morphism onto a Riemann surface such that the induced foliation of M^3 is C^1. Then by Lemma 3.8 the fibres are all compact and by [15] M^3 is a Seifert fibre space. Thus

THEOREM 3.13. — *Let $\phi : M \to N$ be a non-constant harmonic morphism from a closed 3-manifold to a Riemann surface. Suppose further that the foliation of M induced by the fibres of ϕ is C^1. Then there is a fibre preserving homeomorphism from M to a Seifert fibre space.*

Among the eight geometrics of Thurston, six of them give rise to the Seifert fibre spaces. They are $\mathbf{R}^3$, $S^2 \times \mathbf{R}$, $\mathbf{H}^2 \times \mathbf{R}$, Nil, $SL_2(\mathbf{R})$, S^3 (leaving Sol and $\mathbf{H}^3$ as the remaining two geometries). Let E denote one of these geometrics, then the Seifert fibre spaces have the form $M = E/\Gamma$, where Γ is a group of isometries acting freely and properly discontinuously. The foliation of M by circles arises from a canonical foliation of E (we refer the reader to [31] for more details).

We end this section by showing that in the canonical metric this foliation gives rise to a harmonic morphism $E \to P^2$ onto a Riemann surface P^2, except when $E = S^3$ and the orbit type $(p,q) \neq (\pm 1, \pm 1)$.

In this latter case, a recent result of Eells and Ratto [13] shows that the corresponding Seifert fibration may be realized by a harmonic morphism from S^3 to S^2, where S^3 is now endowed with a suitable ellipsoidal metric.

(i) $\mathbf{R}^3$: The foliation of $\mathbf{R}^3$ is by parallel lines. Orthogonal projection onto the 2-plane perpendicular to these lines is clearly a harmonic Riemannian submersion $\mathbf{R}^3 \to \mathbf{R}^2$.

(ii) $S^2 \times \mathbf{R}$: The foliation is by the geodesics $t \to (x,t)$, $x \in S^2$. The projection $\mathbf{H}^2 \times \mathbf{R} \to \mathbf{H}^2$ is a harmonic Riemannian submersion.

(iii) $\mathbf{H}^2 \times \mathbf{R}$: As for case (ii) the projection $\mathbf{H}^2 \times \mathbf{R} \to \mathbf{H}^2$ is a harmonic Riemannian submersion.

(iv) Nil : Nil is the Lie group of 3×3 matrices

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ; \ x, y, z \in \mathbf{R} \right\}.$$

It may be identified with $\mathbf{R}^3$ endowed with the natural metric invariant under left multiplication :

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2.$$

The canonical foliation of Nil has as leaves the fibres of the projection π : Nil $\to$ $\mathbf{R}^2$, $\pi(x,y,z) = (x,y)$. These fibres are geodesics with respect to ds^2. Further, at the point (x,y,z), the frame $(1,0,0)$, $(0,1,x)$ is orthonormal and horizontal with respect to π. These map under π_* to $(1,0)$, $(0,1)$ respectively. Thus π is a harmonic Riemannian submersion with respect to the Euclidean metric on $\mathbf{R}^2$.

(v) $\widetilde{SL_2(\mathbf{R})}$: This space is again homeomorphic to $\mathbf{R}^3$ and may be identified with $\{(x,y,z) \in \mathbf{R}^3 \mid y > 0\} \subset \mathbf{R}^3$ with the metric

$$ds^2 = \frac{dx^2}{y^2} + \frac{dy^2}{y^2} + \left(\frac{dx}{y} + dz\right)^2.$$

The canonical foliation of $\widetilde{SL_2(\mathbf{R})}$ has as leaves the fibres of the projection $\pi : \widetilde{SL_2(\mathbf{R})} \to \mathbf{H}^2$, where $\mathbf{H}^2$ denotes the upper half plane and $\pi(x,y,z) = (x,y)$. These fibres are geodesics with respect to ds^2 and π is a harmonic Riemannian submersion with respect to the standard hyperbolic metric on $\mathbf{H}^2$.

(vi) S^3 : Consider the mapping $\pi_{p,q} : S^3 \to S^2$ given by

$$\pi_{p,q}(\cos e^{i\xi},\ \sin e^{i\eta}) = (\cos 2t,\ \sin 2t\, e^{i(p\xi - q\eta)}),$$

where $t \in [0, \pi/2]$, ξ, $\eta \in [0, 2\pi)$, and p, q are coprime integers. The fibres of $\pi_{p,q}$ induce a Seifert fibration of S^3. The exceptional fibres are given by $t = 0, \pi/2$. The orbit type at these fibres is (q,p), (p,q) respectively. Nearby fibres are geodesics with respect to the Euclidean metric only when $p, q = \pm 1$. In this case (up to isometry of S^3), $\pi_{p,q}$ is the Hopf fibration.

Consider the case when at least one of p, q is not equal to ± 1. Let $Q^3_{p,q}$ be the ellipsoid

$$Q^3_{p,q} = \left\{(z_1, z_2) \in \mathbf{C} \times \mathbf{C};\ \frac{|z_1|^2}{p^2} + \frac{|z_2|^2}{q^2} = 1\right\}.$$

Points of $Q^3_{p,q}$ may be parametrized in the form $(\cos tpe^{i\xi}, \sin tqe^{i\eta})$, $t \in [0, \pi/2]$, $\xi, \eta \in [0, 2\pi)$. We assume $Q^3_{p,q}$ is endowed with the Euclidean metric induced from $\mathbf{R}^4 \cong \mathbf{C} \times \mathbf{C}$. Then Eells and Ratto [13] show there is a function $\alpha : [0, \pi/2] \to [0, \pi/2]$, $\alpha(0) = 0$, $\alpha(\pi/2) = \pi/2$, for which the map $\phi : Q^3_{p,q} \to S^2$ given by

$$\phi(\cos tpe^{i\xi}, \sin tqe^{i\eta}) = (\cos 2\alpha(t), \sin 2\alpha(t)e^{i(p\xi - q\eta)}) \tag{3.14}$$

is a harmonic morphism. Clearly the fibres of ϕ induce a Seifert fibration of $Q^3_{p,q}$ which is isomorphic to the one induced by $\pi_{p,q}$ on S^3. (The manifold $Q^3_{p,q}$ has the structure of a Seifert bundle over an orientable base orbifold with cone points whose angles depend on p and q [31]).

Example 3.15. — Let $\pi : E \to P$ be one of the harmonic Riemannian submersions (i) – (v) and (vi) with $|p| = |q| = 1$. Let Π be a group of isometries acting freely and properly discontinuously on E, such that Π induces a free and properly discontinuous action Γ on the leaf space P. Then we have induced a harmonic Riemannian submersion

$$\phi :\ E/\Pi \longrightarrow P/\Gamma.$$

For example if N_k is the subgroup of Nil given by

$$N_k = \left\{ \begin{pmatrix} 1 & m & p \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} ;\ m, n, p \text{ are divisible by } k \right\}.$$

Then Nil/N_k is a circle bundle over the torus T^2 with 1'st Chern number k [27]. The natural projection

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow (x, y) \in \mathbf{R}^2$$

induces a harmonic Riemannian submersion $\mathrm{Nil}/N_k \to T^2$.

Example 3.16. — Consider case (vi). There is a natural action of $S^1 \times S^1$ by isometries on $Q^3_{p,q}$ given by

$$((\mu,\vartheta),(z_1,z_2)) \longrightarrow (e^{i\mu}z_1e^{i\vartheta}, e^{i\mu}z_2e^{-i\vartheta}).$$

This action preserves all the Seifert fibrations given by the various choices of p and q (see [31]). Thus if Π is a finite, and hence cyclic, subgroup of $S^1 \times S^1$, the quotient $M_{p,q} = Q^3_{p,q}/\Pi$ is a manifold which is naturally Seifert fibred. In fact $M_{p,q}$ is one on the Lens spaces $L(r,s)$ endowed with a non-standard metric. The group Π induces an action on the leaf space S^2, thus giving $M_{p,q}$ the structure of a Seifert bundle over an orientable base orbifold.

We may obtain a harmonic morphism $\psi: M_{p,q} \to S^2$ whose fibres are the fibres of the above Seifert fibration as follows.

Let $\phi: Q^3_{p,q} \to S^2$ be the harmonic morphism (3.14). Compose with the n-fold cover $S^2 \to S^2$, $n = l.c.m.\{r,s\}$, given by $z \mapsto z^n$, to obtain a harmonic morphism $\widetilde{\psi}: Q^3_{p,q} \to S^2$ given by

$$\widetilde{\psi}(\cos tpe^{i\xi}, \sin tqe^{i\eta}) = (\cos 2\beta(t), \sin 2\beta(t)e^{in(p\xi-q\eta)}),$$

for some $\beta: [0,\pi/2] \to [0,\pi/2]$, $\beta(0)=0$, $\beta(\pi/2)=\pi/2$. (It may be possible to compose with $z \to z^m$, $m < n$ depending on the values of p,q,r and s). Then $\widetilde{\psi}$ factors through the action of Π to give the required harmonic morphism $\psi: M_{p,q} \to S^2$.

In the case when the right action by S^1 is trivial, $M_{p,q}$ is the Lens space $L(r,1)$, $r \geq 2$. If $p=q=1$, then $Q^3_{p,q} = S^3$, and we have retrieved Example 2.5.

Recall the definition of a conformal foliation [36]. Let $\mathcal{F}$ be a foliation of a smooth Riemannian manifold (M^m,g). Thus $\mathcal{F}$ is represented by a smooth integrable distribution $V \subset TM$ of rank k. Let $H \subset TM$ denote the orthogonal distribution and let $g|_H$ denote the restriction of the metric to H. The foliation $\mathcal{F}$ is *conformal* if

$$(\mathcal{L}_W g|_H)(X,Y) = \sigma(W)g(X,Y)$$

for all $x \in M, X,Y \in H_x$, $W \in V_x$, where $\mathcal{L}_W$ denotes Lie differentiation in the direction W and $\sigma(W)$ is a non-negative number which depends only on W and not on X and Y.

Suppose we are given a *smooth* foliation of M^3 which is induced by a harmonic morphism $\phi: M^3 \to N^2$. Then it is shown in [5] (see also [37]) that the foliation is conformal. Using the above calculations and examples we may establish a partial converse to Theorem 3.13.

THEOREM 3.17. — *Every closed Seifert fibre space admits a metric with respect to which the fibres form a conformal foliation by geodesics.*

Remarks. — If M^3 is a closed orientable Seifert fibre space endowed with the metric of Theorem 3.17, then the leaf space of the associated foliation is in general a 2-dimensional orbifold $\mathcal{O}$, with cone points corresponding to critical fibres. Away from the cone points $\mathcal{O}$ may given a natural conformal structure induced from the orthogonal spaces to the leaves. (This is well defined since the foliation is *conformal*). Letting $K \subset M$ denote the union of critical fibres and $C \subset \mathcal{O}$ the set of corresponding points. Then the projection $M \setminus K \to \mathcal{O} \setminus C$ is a harmonic morphism.

If there is a conformal map $\psi: \mathcal{O} \setminus C \to N$ to a Riemann surface N, which extends continuously across C, then the Seifert fibration is induced by a smooth harmonic morphism $\phi: M \to N$. Indeed ϕ is a harmonic morphism on $M \setminus K$ which extends continuously across K. Since K is a polar set in M it follows that ϕ is a harmonic morphism on M. Example 3.16 illustrates this construction.

Proof (of Theorem). — Consider the foliations determined by cases (i) – (v) and (vi) with $|p| = |q| = 1$. Then in each case the foliation is determined by a harmonic Riemannian submersion $E \to N$, and is a Riemannian (and so conformal) foliation by geodesics. This structure is carried over to a Seifert fibre space of the form E/Γ, where Γ is a group of isometries acting freely and properly discontinuously.

Consider now the foliation determined by the maps $\pi_{p,q}: S^3 \to S^2$, where p, q are coprime integers and one of $|p|$, $|q|$ is not equal to 1. Then the only closed oriented Seifert fibre spaces which arise from $\pi_{p,q}$ are S^3 itself and the Lens spaces $L(r,s)$ [31]. Giving S^3 the ellipsoidal metric of Example 3.16 endows these spaces with a structure with respect to which the foliation is geodesic and conformal.

Since the above accounts for all closed oriented Seifert fibre spaces [31], the theorem is established. $\square$

4. Higher dimensions – a fibration criterion.

Let $\phi : M^m \to N^n$ be a non-constant harmonic morphism and suppose $x \in M$ is a critical point of ϕ. By the theorem of Fuglede (Theorem 3.2), the symbol $\sigma_x(\phi)$ of ϕ at x is a harmonic polynomial morphism defined by homogeneous polynomials of degree $p \geq 2$. In this section we derive conditions on the map $\sigma_x(\phi)$.

Let $\xi : \mathbf{R}^m \to \mathbf{R}^n$ be a harmonic morphism defined by homogeneous polynomials of degree p. Let $\lambda^2 = |d\xi|^2/n$ denote the dilation of ξ. Then

$$\langle \nabla\xi^k, \nabla\xi^l \rangle = \lambda^2 \delta^{kl}. \tag{4.1}$$

Suppose ξ is normalized such that $\sup_{|x|=1}|\xi(x)|^2 = 1$. Let $\Gamma = \{x \in S^{m-1}; |\xi(x)|^2 = 1\}$.

Define $F : \mathbf{R}^m \to \mathbf{R}$ by $F(x) = |\xi(x)|^2$, and let $f = F|_{S^{m-1}}$.

Lemma 4.2. — (a) $\nabla^{S^{m-1}} f = 2\sum_{k=1}^{n} \xi^k \nabla\xi^k - 2p|\xi|^2 x$.

(b) $\Delta^{S^{m-1}} f = 2n\lambda^2 - 2p(2p + m - 2)|\xi|^2$.

Proof. — (a) $\nabla^{S^{m-1}} f = \nabla F|_{S^{m-1}} - \dfrac{\partial F}{\partial r}\dfrac{\partial}{\partial r} \qquad (r^2 = |x|^2)$

$$= 2\sum_{k=1}^{n} \xi^k \nabla\xi^k - 2p|\xi|^2 x.$$

(b) $\Delta^{S^{m-1}} f = \Delta F|_{S^{m-1}} - \dfrac{\partial^2 F}{\partial r^2}|_{S^{m-1}} - (m-1)\dfrac{\partial F}{\partial r}|_{S^{m-1}}$.

Now $\nabla|\xi|^2 = 2\sum_k \xi^k \nabla\xi^k$

and $\nabla \cdot \nabla|\xi|^2 = 2\sum_k \nabla\xi^k \cdot \nabla\xi^k = 2n\lambda^2$ from (4.1).

Thus $\Delta^{S^{m-1}} f = 2n\lambda^2 - 2p(2p + m - 2)|\xi|^2$. □

Lemma 4.3. — $m - 2 \geq p(n-2)$.

Proof. — On Γ, $\nabla^{S^{m-1}} f = 0$. Thus from Lemma 4.2,

$$\sum_{k=1}^{n} \xi^k \nabla \xi^k = p|\xi|^2 x,$$

which from (4.1) implies $|\xi|^2\lambda^2 = p^2|\xi|^4$, so that $\lambda^2 = p^2|\xi|^2 = p^2$.

Since f attains its maximum value on Γ, we have $\Delta^{S^{m-1}} f \leq 0$ on Γ. Thus $2np^2 \leq 2p(2p + m - 2)$, *i.e.* $m - 2 \geq p(n-2)$. □

LEMMA 4.4. — *Suppose $m - 2 = p(n-2)$, then f is identically equal to* 1.

Note. — This result is stated in [1], but with an incorrect proof.

Proof. — If $m - 2 = p(n-2)$, then

$$\begin{aligned} \Delta^{S^{m-1}} f &= 2n\lambda^2 - 2p(2p + p(n-2))|\xi|^2 \\ &= 2n(\lambda^2 - p^2|\xi|^2). \end{aligned}$$

Write $g = \lambda^2 - p^2|\xi|^2 : S^{m-1} \to \mathbf{R}$. Thus $\Delta^{S^{m-1}} f = 2ng$.

Claim. — $g \geq 0$ on S^{m-1}.

Proof of Claim.

$$\begin{aligned} \langle \nabla^{S^{m-1}} f, \nabla^{S^{m-1}} f \rangle &= 4\langle \Sigma\xi^k\nabla\xi^k - p|\xi|^2 x, \Sigma\xi^k\nabla\xi^k - p|\xi|^2 x\rangle \\ &= 4|\xi|^2(\lambda^2 - p^2|\xi|^2) \\ &= 4|\xi|^2 g. \end{aligned}$$

Thus, *either* $|\nabla^{S^{m-1}} f| = 0$ and $|\xi| = 0$, *or* $g \geq 0$. But if $|\xi| = 0$, then $g = \lambda^2 - p^2|\xi|^2 \geq 0$. This establishes the Claim.

But then $\Delta^{S^{m-1}} f \geq 0$, so f is subharmonic and must be constant. Since $f = 1$ on Γ, f must equal 1 everywhere. □

LEMMA 4.5. — *If $m - 2 = p(n-2)$, then $\xi|_{S^{m-1}} : S^{m-1} \to S^{n-1}$ is a harmonic Riemannian submersion defined by homogeneous polynomials of degree p.*

Proof. — *Claim* : for $x \in S^{m-1}, T_x$ (fibre of ξ through x) $\subset T_x S^{m-1}$.

Proof of Claim. — Let $x \in S^{m-1}$ and put $\xi(x) = y \in S^{n-1}$. Suppose $\gamma(u) \subset \xi^{-1}(y)$ is a curve in the fibre over y with $\gamma(0) = x$. Then $\xi(\gamma(u)) = y$. Put $\mu(u) = \gamma(u)/|\gamma(u)| \in S^{m-1}$. Then

$$\xi(\mu(u)) = y/|\gamma(u)|^p,$$

by the homogeneity of ξ. But $\xi|_{S^{m-1}} : S^{m-1} \to S^{n-1}$, so that $|\gamma(u)| = 1$ and $\gamma(u) \subset S^{m-1}$. This establishes the claim.

Now, since $\lambda^2 = p^2$ on S^{m-1}, we have $\nabla\lambda^2 \perp S^{m-1}$ so that $\nabla\lambda^2$ is horizontal with respect to ξ. By Theorem 5.2 of [3], $\nabla\lambda^2$ is proportional to the mean curvature of the fibres. Thus the mean curvature of a fibre of ξ is perpendicular to TS^{m-1} and the fibres of $\xi|_{S^{m-1}}$ are minimal in S^{m-1}.

Clearly $\xi|_{S^{m-1}}$ is horizontally conformal with dilation $\lambda^2 = p^2$ a constant. □

THEOREM 4.6. — *Let $\phi : M^m \to N^n$ be a non-constant harmonic morphism. Either ϕ is a submersion everywhere and in particular is a fibration, or $m-2 \geq 2(n-2)$. Equality is achieved only when $n = 2,3,5,9$.*

Proof. — If ϕ has a critical point $x \in M$, then by Theorem 3.2 the symbol ξ of ϕ at x is a harmonic polynomial morphism $\xi : T_xM \to T_{\phi(x)}N$, defined by homogeneous polynomials of common degree p, where $p \geq 2$. We now apply Lemma 4.3.

In the case when we have equality $m-2 = 2(n-2)$. By Lemma 4.5, $\xi|_{S^{m-1}} : S^{m-1} \to S^{n-1}$ is a quadratic harmonic polynomial morphism between spheres. These have been classified by Yiu [38]. The Hopf fibrations are the only examples, which occur when $n = 2,3,5,9$ and $m = 2(n-1)$. □

Remark. — The inequality of Theorem 4.6 is curiously identical to an inequality obtained by Milnor [26] in a somewhat different context. The inequality of [26] is a condition for the existence of a "non-trivial" fibration and is obtained by purely algebraic topological considerations.

COROLLARY 4.7. — *Let $\phi : M^{n+1} \to N^n$, $n \geq 4$, be a non-constant harmonic morphism from a closed oriented $(n+1)$-dimensional manifold. Then ϕ is a fibration. In particular we must have*

$$\pi_i(M) \cong \pi_i(N),\ i \geq 3.$$

If N is 2-connected, so that $\pi_1(N) = \pi_2(N) = 0$, then $\pi_1(M) = \mathbf{Z}$, $\pi_2(M) = 0$. Futhermore ϕ determines a smooth S^1-action on M^{n+1} without fixed points.

Proof. — By Theorem 4.6, ϕ is a submersion everywhere and so determines a fibration by circles. As in the proof of Corollary 3.11, these circles carry a natural orientation and so determine a smooth S^1-action on

M^{n+1} without fixed points. The conditions on the homotopy groups now follow from the homotopy exact sequence of a fibration [34] :

$$\ldots \to \pi_i(S^1) \to \pi_i(M) \to \pi_i(N) \to \pi_{i-1}(S^1) \to \ldots . \qquad \square$$

COROLLARY 4.8. — *If $\phi : S^{n+1} \to S^n$ is a harmonic morphism, $n \geq 4$, then ϕ is constant.*

Remark. — Again this contrasts with the case for harmonic maps. For the methods of Smith [33] yield a harmonic representative of the non-trivial class of $\pi_{n+1}(S^n) = \mathbf{Z}_2$, for $n = 3, 4, \ldots, 8$ – this with respect to the Euclidean metric on S^{n+1} and S^n. By allowing deformations of the metric, Ratto [29] has shown that the non-trivial class of $\pi_{n+1}(S^n)$ is represented by a harmonic map for all $n \geq 3$.

5. Harmonic morphisms from a closed 4-manifold to a 3-manifold.

As for harmonic morphisms from an $(n+1)$-dimensional manifold to an n-dimensional manifold, where $n = 2, 4, 5, \ldots$, a harmonic morphism from a closed 4-manifold M^4 to a 3-manifold determines an S^1-action on M^4. However, we now find this circle action may have fixed points.

PROPOSITION 5.1. — *If $\phi : M^4 \to N^3$ is a non-constant harmonic morphism from a 4-manifold to a 3-manifold, then ϕ can have only isolated critical points.*

Proof. — Let $x_0 \in M$ be a critical point of ϕ and let $\xi : T_xM \to T_{\phi(x)}N$ denote the symbol of ϕ at x_0. Then ξ is a harmonic polynomial morphism defined by homogeneous polynomials of degree 2. By Lemma 4.5 and the result of Yiu [38], up to isometry ξ is the Hopf map $\xi : \mathbf{R}^4 \to \mathbf{R}^3$ given by

$$\xi(z_1, z_2) = (|z_1|^2 - |z_2|^2,\ 2z_1\overline{z_2}),$$

where $z_1, z_2 \in \mathbf{C} \cong \mathbf{R}^2$ (note that this also follows from Theorem 5.1 of [4] after applying Lemma 4.5). In particular the dilation of ξ is

$$\lambda^2 = p^2|x|^{2p-2} = 4|x|^2,$$

for $x \in \mathbf{R}^4$.

Using normal coordinates in a small neighbourhood U about x_0 as in Section 3, we have

$$\nabla\xi^k(x) = 2|x|w^k(x),$$

for $k = 1,2,3$, where $w^k(x) \in S^3$, $x \in U$. By Theorem 3.1

$$\nabla\phi^k = \nabla\xi^k + O(r^2),\ (r = |x|).$$

Suppose $x \in U$, $x \neq 0$, is such that $\nabla\phi^k(x) = 0$ for $k = 1,2,3$. Then

$$0 = 2|x|w^k(x) + O(r^2).$$

But $|x| \neq 0$, so

$$0 = 2w^k(x) + O(|x|).$$

This is impossible if U is taken small enough. □

COROLLARY 5.2. — *If M^4 is closed, then there are finitely many critical points, and regular fibres are compact (and hence circles). In particular ϕ determines an S^1-action on M. The fixed points of this action correspond to the critical points of ϕ.*

Proof. — Since the critical points are isolated there can only be finitely many of them. The regular fibres are compact by similar arguments to those of Lemma 3.8. The fibres may be oriented as for the proof of Corollary 3.11, so determining an S^1-action on M. □

We recall [7] the definition of a locally smooth action.

Let M be a G-space, where G is a compact Lie group. Let P be an orbit of type G/H and let V be a Euclidean space on which H operates *orthogonally*. Then a *linear tube* about P in M is a tube (G-equivariant homeomorphism onto an open neighbourhood of P) of the form

$$\psi:\ G \times_H V \to M.$$

The G-space M is *locally smooth* if there exists a linear tube about each orbit. If x is a fixed point of the action, the above definition is equivalent to the assertion that a neighbourhood of x in M is equivalent to an orthogonal action (there exists a G-equivariant homeomorphism onto an open invariant set in V upon which G acts orthogonally).

Any smooth action (in the differentiable sense) is locally smooth. Conversely, if M is a locally smooth G-space, then each orbit has an open invariant neighbourhood on which there exists a differentiable structure with respect to which G acts smoothly.

THEOREM 5.3. — *Let $\phi : M^4 \to N^3$ be a non-constant harmonic morphism from a closed oriented 4-manifold to an oriented 3-manifold. Then ϕ induces a locally smooth S^1-action on M.*

Proof. — Let $x \in M$ be a regular point for ϕ, then there is an invariant neighbourhood of x over which the S^1-action induced by the fibres of ϕ is smooth. Since any smooth action is locally smooth [7], it follows that the action is locally smooth at x. It remains to check at a fixed point (equivalently, at a critical point of ϕ).

Let $x_0 \in M$ be a critical point of ϕ. Choosing normal coordinates about $x_0 \in M$ and $y_0 = \phi(x_0) \in N$ as in Section 3, we may approximate ϕ by its Taylor formula (Theorem 3.1) :

$$\phi^a(x) = \xi^a(x) + O(r^3),\ r = |x|, \tag{5.4}$$

for $a = 1,2,3$, where $\xi = (\xi^1, \xi^2, \xi^3) : T_{x_0}M \to T_{\phi(x_0)}N$ is the symbol of ϕ, which, as in the proof of Proposition 5.1 we may assume takes the form

$$\xi(z_1, z_2) = (|z_1|^2 - |z_2|^2,\ 2z_1\overline{z_2}),$$

$z_1, z_2 \in \mathbf{C}$. Note in particular that ξ induces an *orthogonal* S^1-action on $T_xM \cong \mathbf{R}^4$, given explicitly by

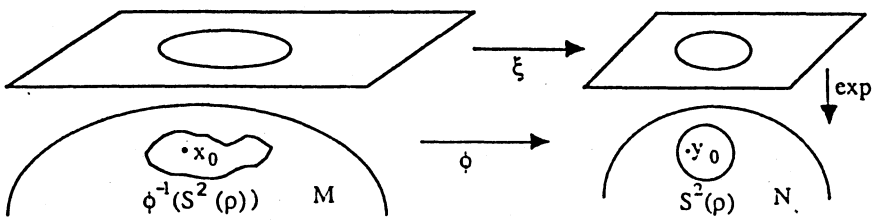
$$e^{i\vartheta}(z_1, z_2) = (e^{i\vartheta}z_1, e^{i\vartheta}z_2).$$

We aim to show that the action induced by ϕ is equivalent to the one induced by ξ, at least in a small enough neighbourhood about x_0.

Let V^3 be a closed 3-ball in N centred at y_0 of radius δ, such that V^3 contains no other critical values of ϕ. Consider the nested sequence of distance spheres $S^2(\rho)$, $0 < \rho \leq \delta$, of radius ρ about y_0, whose union is $V^3 \setminus \{y_0\}$. Now $\phi^{-1}(V^3)$ is a neighbourhood of x_0. Let U^4 denote the connected component containing x_0.

For each $\rho, \phi^{-1}(S^2(\rho))$ is a circle bundle over $S^2(\rho)$. In fact we have a sequence of such filling out a neighbourhood of x_0. Thus U^4 is a cone on some 3-manifold P^3. Then P^3 is simply connected. For any loop in P^3 determines a topological disc in U^4 passing through x_0. Since U^4 is a manifold this disc may be deformed off x_0 and contracted to a point. Thus P^3 is a homotopy 3-sphere. Futhermore P^3 is a circle bundle over

S^2; in particular it is a Seifert fibre space. So P^3 is diffeomorphic to S^3 and regarded as a circle bundle over S^2 it is equivalent to the Hopf bundle.



Since the mapping ϕ and symbol ξ are related by (5.4) and $|\xi(x)| = |x|^2$, we may assume that the radii of U^4 and V^3 are sufficiently small that U^4 and V^3 are contained in the domain of the respective normal coordinate systems.

Now $V^3 \setminus \{y_0\}$ is homeomorphic to the product $S^2 \times I$, where I is the half open interval $(0, \delta]$. So that $U^4 \setminus \{x_0\}$ is a circle bundle over $S^2 \times I$. We denote this circle bundle by E. Letting $B^4 \subset T_{x_0}M$ be the 4-ball of radius $\sqrt{\rho}$, then $B^4 \setminus \{0\}$ is also a circle bundle F over $S^2 \times I$. The S^1-action on F is the orthogonal action induced by the symbol ξ.

Consider the circle bundle $G = S^3 \times I$ over $S^2 \times I$, whose fibres are the Hopf circles induced by the Hopf fibration $S^3 \times \{a\} \to S^2 \times \{a\}$, for each $a \in I$. Then F and G are clearly equivalent as S^1-bundles over $S^2 \times I$.

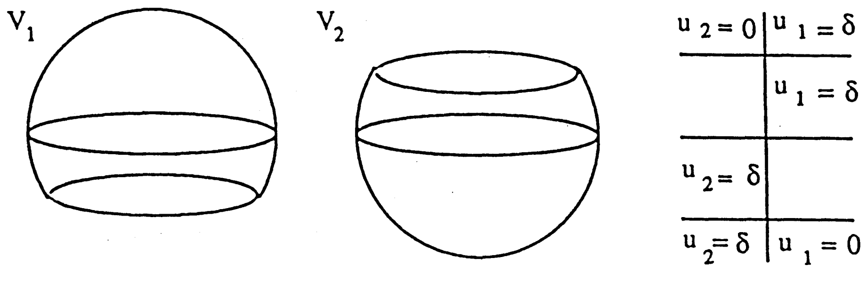
Let $r : \; S^2 \times I \to S^2 \times I$ be the retraction $r(b,t) = (b, \delta)$. Then $G = r^*G$. Further r^*E and r^*G are both equivalent. We show that E and r^*E are equivalent. The following argument is adapted from Milnor (see [25], Lemma 6.9).

Define two coordinate charts V_1, V_2 for S^2 such that $E|_{V_i}$ is trivial for $i = 1, 2$. Let $u_i : \; S^2 \to [0, \delta)$ be continuous functions such that support $u_i \subset V_i$ and

$$\max_{i=1,2} u_i(b) = \delta,$$

for each $b \in S^2$.

We define an S^1-morphism of bundles $(\psi, r) : \; E \to E$. If $p : \; E \to S^2 \times I$ denotes the projection map, let $f_i : \; V_i \times I \times S^1 \to p^{-1}(V_i \times I) \subset E, i = 1, 2$ be trivializations.



For each $i = 1, 2$, define a retraction $r_i : S^2 \times (0, \delta] \to S^2 \times (0, \delta]$ by

$$r_i(b, t) = (b, \max(t, u_i(b))).$$

Note that $r(b, t)$ is the composition $r_2 \circ r_1(b, t)$.

Each r_i is covered by a bundle morphism $\psi_i : E \to E$ as follows. ψ_i is the identity outside $p^{-1}(V_i \times I)$ and

$$\psi_i[f_i(b, t, s)] = f_i(b, \max(U_i(b), t), s),$$

for each $(b, t, s) \in U_i \times I \times S^1$. Then $\psi = \psi_2 \circ \psi_1$.

It now follows that E and r^*E are equivalent. In particular E and F are equivalent as S^1-bundles over $S^2 \times I$. Filling in the point x_0 now gives an S^1-equivariant homeomorphism from U^4 to B^4. Thus the S^1-action on U^4 is equivalent to an orthogonal action and is therefore locally smooth at x_0. □

We quote the classification theorem of Fintushel for locally smooth S^1-actions on a simply connected 4-manifold.

THEOREM 5.5 [18]. — *Let S^1 act locally smoothly on the simply connected 4-manifold M, and suppose the orbit space M^* is not a counter example to the 3-dimensional Poincaré conjecture. Then M is a connected sum of copies of S^4, $\mathbf{C}P^2$, $-\mathbf{C}P^2$, and $S^2 \times S^2$.*

Consider the case when $\phi : S^4 \to N^3$ is a non-constant harmonic morphism. If ϕ has no critical points then ϕ determines a fibration of S^4 by circles. In particular it determines an S^1 action without fixed points. But by the Lefschetz fixed point theorem this is impossible, so the corresponding S^1-action must have at least one fixed point. By a result of Pao [28], there are precisely two fixed points and the orbit space is a homotopy 3-sphere. We construct an example with $N^3 = S^3$ below. It is an interesting question as to whether this is essentially the only example of a harmonic morphism from S^4 to a 3-manifold (up to projection $S^3 \to S^3/\Gamma$ where Γ is a group of isometries acting freely and properly discontinuously).

Example 5.6. — Let B^n denote the closed Euclidean n-ball in $\mathbf{R}^n$. Define a map $\phi_1 : B^4 \to B^3$ by

$$\phi_1(z_1, z_2) = (|z_1|^2 - |z_2|^2,\ 2z_1\overline{z_2}),$$

for $z_1, z_2 \in \mathbf{C}$, $|z_1|^2 + |z_2|^2 \leq 1$. Then ϕ_1 is a harmonic morphism which maps the boundary $\partial B^4 = S^3$ to $\partial B^3 = S^2$ via the Hopf fibration $H : S^3 \to S^2$. Moreover, if we restrict ϕ_1 to a 3-sphere S^3_r of radius r in B^4, then ϕ_1 sends S^3_r to the 2-sphere $S^2_{r^2}$ of radius r^2 in B^3 and $\phi_1|_{S^3_r}$ is the Hopf fibration up to a scaling.

Define $\phi_2 : S^3 \times [1,2] \to S^2 \times [1,3]$ by

$$\phi_2(x, r) = (H(x), 2r - 1).$$

Then ϕ_2 is a harmonic morphism and maps each boundary S^3 to S^2 via the Hopf fibration. Let $\phi_3 : B^4 \to B^3$ be identical to ϕ_1.

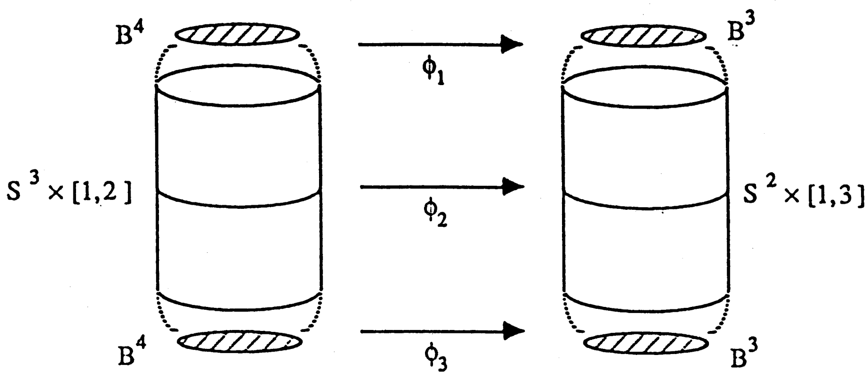
The idea is to glue these three maps together to obtain a map $\phi : M^4 \to N^3$, where

$$M^4 = B^4 \underset{S^3}{\cup} (S^3 \times [1,2]) \underset{S^3}{\cup} B^4$$

is homeomorphic to S^4 and

$$N^3 = B^3 \underset{S^2}{\cup} (S^2 \times [1,3]) \underset{S^2}{\cup} B^3$$

is homeomorphic to S^3. The map ϕ is homotopic to the suspension of the Hopf fibration. Note that in a neighbourhood of 0 in each B^4 and in a neighbourhood of $S^3 \times \{3/2\}$ in $S^3 \times [1,2]$, ϕ is a harmonic morphism. We smooth over the joins and reparametrize ϕ to obtain an h-harmonic morphism between smooth manifolds. A conformal change in the metric then yields the desired harmonic morphism $\phi:\ S^4 \to S^3$.



It suffices to consider one half of the construction only. By symmetry the other half may be glued on.

Points of M^4 may be expressed as pairs (r, x), where $r \in [0,3]$ and $x \in S^3$. Let M_1^4 denote the one half of M^4 given by $0 \le r \le 3/2$. Define a metric on M_1^4 in the form

$$\widetilde{g_1} = dr^2 + \sigma(r)^2 dx^2,$$

where $\sigma(r)$ is a smooth positive function on $(0, 3/2)$ such that

(i) $\sigma(r) = r, \quad r \in [0, 1/2]$.

(ii) $\sigma(r) = 1, \quad r \in [1, 3/2]$.

Since $\widetilde{g_1}$ coincides with the Euclidean metric in a neighbourhood of $r = 0$, $\widetilde{g_1}$ is certainly a smooth metric.

Let $\mu(r)$ be a smooth positive function on $(0, 3/2)$ with the properties

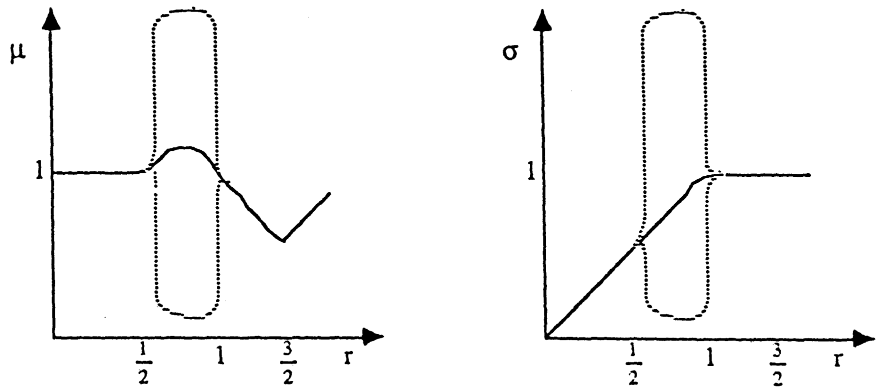
(iii) $\mu(r) = 1$ on $[0, 1/2]$

(iv) $\mu(r) = 1/(2r-1)$ on $[1, 3/2]$.

We may further assume that σ and μ are chosen such that

(v) $\displaystyle\int_{1/2}^{1} \frac{\mu}{\sigma}\, dr = \log 2.$

For the graphs of μ and σ may be adjusted so as to have an arbitrarily large or arbitrarily small area beneath them between 1/2 and 1, as in the sketches below.



Define $\alpha : [0,3/2] \to \mathbf{R}$ by

$$\alpha(r) = \exp\Big(\int_1^r \frac{2\mu(s)}{\sigma(s)}\,ds\Big).$$

Note that $\alpha'(r) > 0$ for all r, so α^{-1} is well-defined on $\alpha([0,3/2])$. Put

$$\rho(u) = u \cdot \mu(\alpha^{-1}(u)),$$

for each $u \in \alpha([0,3/2])$.

Let M_2^4 denote the opposite half of M^4 obtained by reflecting in $r = 3/2$. Extend σ, μ, α and ρ by symmetry to all $r \in [0,3]$, $u \in [0,4]$, and let $\widetilde{g}$ be the extended metric.

Claim. — The map

$$\phi : \ (r,x) \to (\alpha(r), H(x)),$$

defines an h-harmonic morphism (c.f. Section 1) from S^4 to S^3 with respect to the metrics

$$\widetilde{g} = dr^2 + \sigma(r)^2 dx^2$$
$$k = du^2 + \rho(u)^2 dy^2,$$

on S^4 and S^3 respectively, where $r \in [0,3]$, $u \in [0,4]$ and dx^2, dy^2 denote the standard metrics on S^3, S^2 respectively.

LEMMA 5.7. — *The map ϕ coincides with ϕ_1 on $[0,1/2]$, with ϕ_2 on $[1,2]$ and with ϕ_3 on $[5/2,3]$.*

Proof. — For $r \in [0,1/2]$,

$$\begin{aligned}\alpha(r) &= \exp\Big(\int_1^r \frac{2\mu}{\sigma}\,ds\Big)\\ &= \exp\Big(-\int_r^{1/2} \frac{2\mu}{\sigma}\,ds - \int_{1/2}^1 \frac{2\mu}{\sigma}\,ds\Big)\\ &= \exp\Big(-\int_r^{1/2} \frac{2}{s}\,ds - \int_{1/2}^1 \frac{2\mu}{\sigma}\,ds\Big)\\ &= \frac{1}{4}\exp(\log 4 + \log r^2)\\ &= r^2.\end{aligned}$$

For $r \in [1,3/2]$,

$$\begin{aligned}\alpha(r) &= \exp\Big(\int_1^r \frac{2\mu}{\sigma}\,ds\Big)\\ &= \exp\Big(\int_1^r \frac{2}{2s-1}\,ds\Big)\\ &= 2r-1.\end{aligned}$$

By reflected symmetry ϕ coincides with ϕ_2 on $[3/2,2]$ and with ϕ_3 on $[5/2,3]$. □

LEMMA 5.8. — *The metric $k = du^2 + \rho(u)^2 dy^2$ is smooth and coincides with the Euclidean metric for $u \in [0,1/4] \cup [15/4,4]$, and with the standard metric on $S^2 \times [1,3]$ for $u \in [1,3]$.*

Proof. — For $u \in [0,1/4]$ we have $\rho(u) = u$, so that k coincides with the Euclidean metric on a 3-ball of radius $1/4$ in $\mathbf{R}^3$.

For $u \in [1,2]$,

$$\rho(u) = \frac{u}{2(\alpha^{-1}u) - 1} = 1,$$

so k coincides with the standard metric on $S^3 \times [1,2]$. The other parts follow by reflected symmetry. □

PROPOSITION 5.9. — *The map* $\phi : (S^4, \widetilde{g}) \to (S^3, k)$ *is h-harmonic. Thus*

$$\tau_\phi + 2d\phi(\nabla \log h) = 0,$$

where $h : S^4 \to \mathbf{R}$ is the smooth function determined by $h > 0$,

$$h(r)^2 = \frac{\alpha(r) \cdot \mu(r)}{\sigma(r)^2}.$$

Note. — On $[0,1/2] \cup [1,2] \cup [5/2,3]$, $h^2 = 1$ and so h is smooth. Also ϕ is harmonic on these sets.

LEMMA 5.10. — *The tension field of* ϕ *is given by*

$$\tau_\phi = \Big(\alpha''(r) + \frac{3\sigma'}{\sigma}\,\alpha'(r) - \frac{8\rho(\alpha(r))\rho'(\alpha(r))}{\sigma^2}\Big)\,\frac{\partial}{\partial u}.$$

Proof. — The computation is standard, see [1]. □

Proof of Proposition 5.9. — We first establish that ϕ is horizontally conformal. Now

$$\phi_* \frac{\partial}{\partial r} = \alpha'(r)\,\frac{\partial}{\partial u}.$$

If Y_j is a unit tangent to S^3, then Y_j/σ is unit for M, and

$$\phi_*\Big(0, \frac{Y_j}{\sigma}\Big) = \Big(0, \frac{\rho}{\sigma}\, H_*(Y_j)\Big),$$

where H is the Hopf fibration. Since $|H_* Y_j| = 2$ if Y_j is horizontal with respect to H, we require

$$|\alpha'(r)| = 2|\rho/\sigma|$$

for horizontal conformality. But

$$\begin{aligned}\alpha'(r) &= \frac{d}{dr}\exp\Big(\int_1^r \frac{2\mu}{\sigma}\,ds\Big)\\ &= \alpha(r)\cdot\frac{2\mu(r)}{\sigma(r)}\\ &= \frac{2\rho}{\sigma},\end{aligned}$$

as required.

Since $\alpha' = 2\rho/\sigma$, we have

$$\alpha'' = \frac{2(\sigma\rho'\alpha' - \rho\sigma')}{\sigma^2},$$

thus

$$\begin{aligned}\alpha''(r) + \frac{3\sigma'}{\sigma}\,\alpha'(r) - \frac{8\rho\rho'}{\sigma^2} &= \frac{2\sigma\rho'\alpha' - 2\rho\sigma'}{\sigma^2} + \frac{3\sigma'}{\sigma}\,\alpha' - \frac{8\rho\rho'}{\sigma^2}\\ &= \alpha'\Big(\frac{\sigma'}{\sigma} - \frac{\alpha''}{\alpha'}\Big)\\ &= \alpha'\,\frac{\partial}{\partial r}(\log r - \log\alpha').\end{aligned}$$

Assume $\nabla \log h$ is proportional to $\partial/\partial r$. Then, since $2d\phi(\nabla \log h) = -\tau_\phi$ we must have

$$2\,\frac{\partial}{\partial r}\,(\log h) = -\frac{\partial}{\partial r}(\log\sigma - \log\alpha'),$$

i.e.

$$\log h^2 = \log\frac{B\alpha'}{\sigma}$$

for some constant B, and, choosing $B = 1/2$,

$$h^2 = \frac{\rho}{\sigma^2} = \frac{\alpha\mu}{\sigma^2},$$

completing the proof. □

Note. — By the reflected symmetry and the constancy of σ, ρ on the intervals $[1, 3/2]$, $[1, 2]$ respectively, h extends smoothly across $r = 3/2$.

COROLLARY 5.11. — *With respect to the conformally related metric*

$$g = h^2\widetilde{g}$$

the map $\phi: (S^4, g) \to (S^3, k)$ is a smooth harmonic morphism. The map ϕ has two isolated critical points at opposite poles of S^4.

Proof. — This follows from (1.11). Since ϕ coincides with ϕ_1, ϕ_3 in a neighbourhood of $r = 0, 3$ respectively, ϕ has critical points at $r = 0, 3$. □

Remark. — By the same reasoning as for Proposition 5.1, it follows that any harmonic morphism $\phi : M^m \to N^n$, where $(m,n) = (4,3)$, $(8,5)$, $(16,9)$ can have only isolated critical points. Indeed the construction of Example 5.6 may be applied to the other Hopf fibrations $S^7 \to S^4$ and $S^{15} \to S^8$ to yield harmonic morphisms from $S^8 \to S^5$ and $S^{16} \to S^9$ respectively, with isolated critical points at opposite poles.

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Manuscrit reçu le 12 décembre 1988,
révisé le 16 août 1989.

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