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REGULAR TRACE FORMULA AND BASE CHANGE FOR $GL(n)$

by Yuval Z. FLICKER (*)

Introduction.

Our aim here is to develop the regular trace formula of [F2] from the context of $GL(2)$ to that of a reductive group, and use it to give an elementary proof of the theory of base change for cuspidal automorphic representations of $GL(n)$ which have a supercuspidal component. Our motivation is the belief that a formula as fundamental as the trace formula should be given a simple proof. In [F2] we establish, by means of such a proof, an asymptotic form of the formula which suffices to prove the complete base-change theory for $GL(2)$; in fact the method suffices to establish every lifting theorem involving groups of rank (or twisted rank) one (see [F5] in the case of the symmetric-square).

As in [FK1], the trace formula proven here holds for test functions with a supercusp component. At a second place we choose a spherical function which is sufficiently admissible with respect to the other components of the test function; the notion of admissibility is introduced prior to Theorem 4 below. The second restriction does not restrict the applicability of this trace formula to lifting problems, and indeed it is shown in [FK1]

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that the simple trace formula developed there can be used to extend the metaplectic correspondence of [FK] and the simple algebra (or Deligne-Kazhdan) correspondence of [BDKV] or [F1; III] from the context of cuspidal representations with two supercuspidal components to that of cuspidal representations with a single such component.

It is easy to see that the proofs of [FK1] can be adapted to establish also the base-change lifting for cuspidal $GL(n)$-modules with a supercuspidal component, which is proven here by means of a different technique.

However, in [FK1] the test function is chosen to have a regular component at a third non-archimedean place (see the definition following Corollary 3 below; a regular function on $GL(n)$ is supported on the conjugacy classes of regular split elements whose eigenvalues have distinct valuations). For example, in the proof of the Ramanujan, or purity conjecture for cuspidal $GL(n)$-modules with a supercuspidal component (see [FK2]) we need a trace formula for a test function whose components are not restricted at any third place.

Here we develop such a formula on taking the second component to be a regular – but not spherical – function which is sufficiently admissible with respect to all other components. We then show that a suitable linear combination of these regular functions, which annihilates the trace of any constituent of a reducible unramified principal series representation, has the same orbital integrals as the corresponding linear combination of spherical functions. Using the fact that each unramified component of a cuspidal $GL(n)$-module is an irreducible principal series $GL(n, F_v)$-module, we derive our applications to the theory of base-change for $GL(n)$.

It will be interesting to develop (by elementary means) a trace formula for a test function with a sufficiently admissible regular component, and arbitrary components elsewhere. This may lead to a simple proof of the base-change lifting for $GL(n)$, and metaplectic and simple algebra correspondences, for an arbitrary cuspidal representation. But we have not done this as yet.

As noted in [FK1], the simple form of the trace formula of [FK1] and of the present work are analogous to Deligne's conjecture on the Lefschetz fixed point formula for a finite flat correspondence on a separated scheme of finite type over a finite field, which is multiplied by a sufficiently high power of the Frobenius; see [FK2] for more details. The notion of admissibility is suggested, at least in the case of $GL(2)$, by Drinfeld's use of the trace formula in [D].

In [F4] we stated a theorem ("fundamental lemma"), asserting the transfer of stable orbital integrals of spherical functions in the case of base-change from $G(F)$ to $G(E), E/F$ cyclic extension. The global-type proof there is incomplete. It is completed in Proposition 24 in a few special cases (e.g. base-change for $GL(n)$ and for $U(3)$). We do not use the Theorem of [F4] here. In fact, Proposition 25 gives a purely local proof of this claim for a sufficiently wide class of spherical functions. Recently Clozel proved the statement claimed in [F4] using his technique of spherical functions (used in [AC] in the case of base-change for $GL(n)$). Labesse also proved that statement, using the technique of Iwahori functions, as in [F4], which he developed independently of [F4].

Finally, we briefly state here the global and local base-change results of Theorems 27 and 32 below. The local and complete global results had been proven by Arthur-Clozel [AC]. Let E/F be a cyclic extension of global fields, of any characteristic. An irreducible $G(\mathbb{A})$-module π is the product $\otimes\pi_v$ of local G_v-modules π_v which are unramified for almost all v, where $G_v = G(F_v)$. For $G = GL(n)$, an unramified π_v is naturally associated with an unordered n-tuple $t = (t_1, \dots, t_n)$ of non-zero complex numbers, and we write $\pi_v(t)$ for π_v. If v is a place of F such that $E_v = E \otimes_F F_v$ is an unramified cyclic extension of F_v then we say that the unramified G_v-module $\pi_v(t)$ *lifts* to the unramified $\tilde{G}_v$-module $\tilde{\pi}_v(t')$ if $t' = t^e$, where $\tilde{G}_v = G(E_v)$ and $e = [E_v : F_v]$. If v splits completely in E then we say that the G_v-module π_v *lifts* to the $\tilde{G}_v$-module $\tilde{\pi}_v = \pi_1 \otimes \dots \otimes \pi_e$ if $\pi_i = \pi_v$ for all $i (1 \leq i \leq e)$; here $E_v = F_v \oplus \dots \oplus F_v$ and $\tilde{G}_v = G_v \times \dots \times G_v$. If v is a place of F which is unramified in E then the notion of lifting of unramified G_v-modules is the obvious combination of the definitions in the totally split and inert unramified cases. Denote by σ a generator of the galois group $\mathrm{Gal}(E/F)$. A representation $\tilde{\pi}$ of $G(\mathbb{A}_E)$ or $\tilde{G}_v$ is called *σ-invariant* if $\pi \simeq {}^\sigma\pi$ and ${}^\sigma\pi(g) = \pi(\sigma g)$. An irreducible $G(\mathbb{A})$-module $\pi = \otimes\pi_v$ *lifts* to an irreducible $\tilde{G}(\mathbb{A}) = G(\mathbb{A}_E)$-module $\tilde{\pi} = \otimes_v \tilde{\pi}_v$ (product over all places of F) if π_v lifts to $\tilde{\pi}_v$ for almost all v (where π_v and $\tilde{\pi}_v$ are unramified). Theorem 27 below asserts (cf. [AC]):

GLOBAL BASE-CHANGE THEOREM FOR $G = GL(n)$. — *Let w be a place of F which splits completely in E. Then every cuspidal G-module π whose component at w is supercuspidal lifts to a unique cuspidal $\tilde{G}$-module $\tilde{\pi}$ which is σ-invariant and has a supercuspidal component at w. Moreover, every σ-invariant cuspidal $\tilde{G}$-module $\tilde{\pi}$ whose component at*

w is supercuspidal is a lift of a cuspidal G-module π, and every other cuspidal G-module which lifts to $\tilde{\pi}$ is of the form $\pi \otimes \epsilon$, where ϵ is a character of $\mathbb{A}^\times / F^\times N\mathbb{A}_E^\times$; $\pi \otimes \epsilon \not\simeq \pi$ if ϵ is non-trivial.

To state the local result we define, after Lemma 20, the notion of *matching* functions f_v on G_v and $\tilde{f}_v$ on $\tilde{G}_v$, and say, below Theorem 27, that the irreducible G_v-module π_v *lifts* to the irreducible σ-invariant $\tilde{G}_v$-module $\tilde{\pi}_v$ if $tr\ \pi_v(f_v) = tr\ \tilde{\pi}_v(\tilde{f}_v \times \sigma)$ for (a good definition of $\tilde{\pi}_v(\sigma)$ and) all matching f_v on G_v and $\tilde{f}_v$ on $\tilde{G}_v$. Theorem 32 asserts (cf. [AC]):

LOCAL BASE-CHANGE THEOREM FOR $G = GL(n)$. — *The lifting defines a surjection from the set of equivalence classes of irreducible tempered G_v-modules π_v to the set of equivalence classes of irreducible tempered σ-invariant $\tilde{G}_v$-modules $\tilde{\pi}_v$. It maps discrete G_v-modules to σ-discrete $\tilde{G}_v$-modules (σ-descrete $\tilde{G}_v$-modules are defined prior to Lemma 28).*

The lifting commutes with induction and so this local theorem extends to the set of relevant (a notion introduced in [FK]; every component of a cuspidal G-module is relevant), not only tempered, G_v-modules. It is clear from our proofs that if the cuspidal π lifts to the cuspidal $\tilde{\pi}$ then π_v lifts to $\tilde{\pi}_v$ for all v. Also it is clear that the global theorem holds with any, not necessarily totally split, place w, and then it relates σ-invariant cuspidal $\tilde{\pi}$ with a supercuspidal component at w, with cuspidal π whose component π_w at w lifts to a supercuspidal $\tilde{G}_w$-module. Such π_w are obviously supercuspidal. The fibers of the lifting are easily described.

It should also be noted that the base-change theorems are perhaps easier then the metaplectic and simple algebra correspondences. The two groups under comparison admit (as was shown by Arthur-Clozel [AC], Thm III, 3.1; their statement is reproduced as Cor. $6'$, in [F2], p. 760) a-priori rigidity and multiplicity one theorems by virtue of Proposition 3.6 of [JS] (see the proof of Theorem 27). Consequently to establish our theorems one does not need to know a-priori that given a general $\tilde{f}_v$ (resp. f_v with suitable orbital integrals) there exists a matching f_v (resp. $\tilde{f}_v$). This statement can be deduced from the trace Paley-Wiener theorem of [BDK] (and [F1; I, §7] in the twisted case), and the local base-change theorem, or even purely locally on using the results of [K], which relate orbital integrals with Fourier transforms of invariant distributions on nilpotent orbits in the Lie algebra of G_v (if the characteristic is zero). To prove the metaplectic and simple-algebra correspondences one gives (in [FK], and [BDKV] or [F1; III]) a local proof that matching functions exist, and one concludes the rigidity and multiplicity one theorems for the metaplectic covers and inner forms of $GL(n)$ from the comparisons of trace formulae and the analogous results for $GL(n)$ itself.

The global base-change theorem for $GL(n)$ had been proven in Arthur-Clozel [AC] in the greater generality of all automorphic $GL(n)$-modules, with no restriction at the place w, in the case of characteristic zero. They also prove the local theorem and the global theorem for π with two square-integrable components (one of which is supercuspidal) by means of the simple trace formula of Deligne-Kazhdan. In this paper we obtain in particular the strongest global result presently obtainable by elementary means (without using the theory of Eisenstein series, invariant trace formula, correction argument of [F3], cancellation of singularities, orbital integrals of singular classes, etc., which are used in [AC] in a crucial way to deal with general π). We also derive the transfer of orbital integrals from the local lifting, rather than use it in the proof.

It will be interesting (and I believe possible) to establish the complete global base-change theorem for $GL(n)$ by means of a simple proof.

1. Regular trace formula.

Let F be a global field, $\mathbb{A}$ its ring of adeles and $\mathbb{A}_f$ the ring of finite adeles, $\mathbf{G}$ a connected reductive algebraic group over F with center $\mathbf{Z}$. The group G of F-rational point on $\mathbf{G}$ is discrete in the adele group $G(\mathbb{A})$ of $\mathbf{G}$. Put $G' = G/Z$ and $G'(\mathbb{A}) = G(\mathbb{A})/Z(\mathbb{A})$. The quotient $G'\backslash G'(\mathbb{A})$ has finite volume with respect to the unique (up to a scalar multiple) Haar measure dg on $G'(\mathbb{A})$. Fix a *unitary* complex-valued character ω of $Z\backslash Z(\mathbb{A})$. For any place v of F let F_v be a completion of F at v, and $G_v = G(F_v)$ the group of F_v-points on $\mathbf{G}$. If F_v is non-archimedean, let R_v denote its ring of integers. For almost all v the group G_v is defined over R_v, quasi-split over F_v, split over an unramified extension of F_v, and $K_v = G(R_v)$ is a maximal compact subgroup. For an infinite set of places (of positive density) u of F, the group G_u is split (over F_u). A fundamental system of open neighborhoods of 1 in $G(\mathbb{A})$ consists of the set of $\prod_{v\in V} H_v \times \prod_{v\notin V} K_v$, where V is a finite set of places of F and H_v is an open subset of G_v, containing 1.

Let $L(G)$ denote the space of all complex-valued functions ϕ on $G\backslash G(\mathbb{A})$ which satisfy (1) $\phi(zg) = \omega(z)\phi(g)$ (z in $Z(\mathbb{A})$, g in $G(\mathbb{A})$), (2) ϕ is absolutely square-integrable on $G'\backslash G'(\mathbb{A})$. $G(\mathbb{A})$ acts on $L(G)$ by right translation: $(r(g)\phi)(h) = \phi(hg)$; $L(G)$ is unitary since ω is unitary. The function ϕ in $L(G)$ is called *cuspidal* if for each proper parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ over F with unipotent radical $\mathbf{N}$ we have $\int \phi(ng)dn = 0$ (n in $N\backslash N(\mathbb{A})$) for any g in $G(\mathbb{A})$. Let $L_0(G)$ denote the space of cuspidal functions in $L(G)$, and r_0 the restriction of r to $L_0(G)$. The space $L_0(G)$ decomposes as a direct sum with finite multiplicities of invariant irreducible unitary $G(\mathbb{A})$-modules called *cuspidal* G-modules.

Let f be a complex-valued function on $G(\mathbb{A})$ with $f(g) = \omega(z)f(zg)$ for z in $Z(\mathbb{A})$, which is supported on the product of $Z(\mathbb{A})$ and a compact open neighborhood of 1 in $G(\mathbb{A})$, smooth as a function on the archimedean part $G(F_\infty)$ of $G(\mathbb{A})$, and bi-invariant by an open compact subgroup of $G(\mathbb{A}_f)$. Fix Haar measures dg_v on $G'_v = G_v/Z_v$ for all v, such that the product of the volumes $|K_v/Z_v \cap K_v|$ converges. Then $dg = \otimes dg_v$ is a measure on $G'(\mathbb{A})$. The convolution operator $r_0(f) = \int_{G'(\mathbb{A})} f(g)r_0(g)dg$ is of trace class; its trace is denoted by $tr\ r_0(f)$. Then

$$tr\ r_0(f) = \sum\nolimits' m(\pi)\ tr\ \pi(f), \tag{1}$$

where $\sum'$ indicates the sum over all equivalence classes of cuspidal representations π of $G(\mathbb{A})$, and $m(\pi)$ denotes the multiplicity of π in $L_0(G)$; each π here is unitary, and the sum is absolutely convergent.

The Selberg trace formula is an alternative expression for (1). To introduce it we recall the following

DEFINITIONS. — *Denote by $Z_\gamma(H)$ the centralizer of an element γ in a group H. A semi-simple element γ of G is called* elliptic *if $Z_\gamma(G'(\mathbb{A}))/Z_\gamma(G')$ has finite volume.*

It is called regular *if $Z_\gamma(G'(\mathbb{A}))$ is a torus, and singular otherwise. Let γ be an elliptic element of G. The* orbital integral *of f at γ is defined to be*

$$\Phi(\gamma, f) = \int_{G'(\mathbb{A})/Z_\gamma(G')} f(g\gamma g^{-1})dg\ .$$

Similarly, for any place v of F the element γ of G_v is called elliptic *if $Z_\gamma(G'_v)$ has finite volume, and* regular *if $Z_\gamma(G_v)$ is a torus. If γ is an element of G and there is a place v of F such that γ is elliptic (resp. regular) in G_v, then γ is elliptic (resp. regular). The* orbital integral *of*

f_v at γ in G_v is defined to be

$$\Phi(\gamma, f_v) = \Phi(\gamma, f_v; d_\gamma) = \int_{G'_v/Z_\gamma(G'_v)} f_v(g\gamma g^{-1})dg \,.$$

It depends on the choice of a Haar measure d_γ on $Z_\gamma(G'_v)$.

Let $\{\phi_\alpha\}$ be an orthonormal basis for the space $L_0(G)$. The operator $r_0(f)$ is an integral operator on $G'(\mathbb{A})$ with kernel $K_f^0(x,y) = \sum_{\alpha,\beta} r(f)\phi_\alpha(x)\overline{\phi}_\beta(y)$. The operator $r(f)$ is an integral operator on $G'(\mathbb{A})$ with kernel $K_f(x,y) = \sum_\gamma f(x^{-1}\gamma y)$ (γ in G'). If $\mathbf{G}$ is anisotropic (namely $G'\backslash G'(\mathbb{A})$ is compact), then $L_0(G) = L(G)$ and $r = r_0$. Since $K_f^0(x,y) = K_f(x,y)$ is smooth in both x and y, we integrate over the diagonal $x = y$ in $G'(\mathbb{A})$, change the order of summation and integration as usual, and obtain the Selberg trace formula in the case of compact quotient, as follows.

1. PROPOSITION. — *If G is anisotropic then for every function f on $G(\mathbb{A})$ as above we have*

$$\sum{}' m(\pi)\ tr\ \pi(f) = \sum_{\{y\}} \Phi(y, f) \,. \tag{2}$$

The sum on the left is the same as in (1). The sum on the right is finite; it ranges over the conjugacy classes of elements in G' .

Remark. — If $\mathbf{G}$ is anisotropic, then each element y in G is elliptic.

For a general group $\mathbf{G}$ we introduce the following

DEFINITION. — *The function f is called discrete if for every x in $G(\mathbb{A})$ and y in G we have $f(x^{-1}yx) = 0$ unless y is elliptic regular.*

Changing again the order of summation and integration as usual we obtain the

2. PROPOSITION. — *If f is discrete, then*

$$\int_{G'(\mathbb{A})} \left[\sum_{y\in G'} f(x^{-1}yx)\right] dx = \sum_{\{y\}} \Phi(y, f) \,. \tag{3}$$

The sum on the right is finite. It ranges over the set of conjugacy classes of elliptic regular elements in G' .

Remark. — It is well known that the sum on the right is finite; for a proof see [FK], §18 (if $G = GL(n)$), and [F1], Prop. I.3 (in general).

DEFINITION. — *The function* f *is called* cuspidal *if for every* x, y *in* $G(\mathbb{A})$ *and every proper* F*-parabolic subgroup* $\mathbf{P}$ *of* $\mathbf{G}$, *we have* $\int_{N(\mathbb{A})} f(xny)dn = 0$, *where* $\mathbf{N}$ *is the unipotent radical of* $\mathbf{P}$.

Let $\mathbb{A}^u$ be the ring of F-adeles without component at u. Put $G^u = G(\mathbb{A}^u)$. Write $f = f_u f^u$ if f is a function on $G(\mathbb{A})$, f_u on G_u, f^u on G^u, and $f(x, y) = f_u(x)f^u(y)$ for x in G_u and y in G^u. We say that f^u is cuspidal if for every x, y in G^u, and $\mathbf{P}$ as above, we have $\int_{N(\mathbb{A}^u)} f^u(xny)dn = 0$.

If f^u is cuspidal then $f = f^u f_u$ is cuspidal, for any f_u.

When f is cuspidal, the convolution operator $r(f)$ factorizes through the projection on $L_0(G)$, $r(f)$ is of trace class, $tr\ r_0(f) = tr\ r(f)$ and $K_f(x, y) = K_f^0(x, y)$, and we obtain

3. COROLLARY. — *If* f *is cuspidal and discrete, then the equality* (2) *holds. The sum on the left is as in* (1). *The sum on the right is as in* (3).

We shall now construct a useful set of discrete functions. Fix a non-archimedean place u of F such that G_u is split, and the component ω_u of ω at u is unramified, namely trivial on the multiplicative group $R_u^\times$ of R_u. Put $\nu(x) = |x|$ for x in $\mathbb{A}^\times/F^\times$. Multiply ω by a power of ν to assume that $\omega_u = 1$. Fix a minimal parabolic subgroup $B_u = A_u U_u$ of G_u, where U_u is the unipotent radical of B_u and the Levi subgroup A_u is a maximal (split) torus. Let $X^* = X^*(A_u)$ be the lattice of rational characters on A_u, and let $X_* = X_*(A_u)$ be the dual lattice. Let A_u^0 be the maximal compact subgroup of A_u; then $X_* \simeq A_u/A_u^0$. If $\mathbf{x}$ is an element of X_*, denote by $a(\mathbf{x})$ an element of A_u which maps to $\mathbf{x}$ under this isomorphism. Let $val_u : F_u^\times \to \mathbb{Z}$ be the normalized additive valuation.

DEFINITION. — *Consider* $\mathbf{x}$ *in* X_* *with* $val_u(\alpha(a(\mathbf{x}))) \neq 0$ *for each root* α *of* A_u *on* N_u. *A complex-valued locally-constant function* f_u *on* G_u *with* $f_u(zg) = f_u(g)$ *for all* g *in* G_u *and* z *in the center* Z_u *of* G_u *which is compactly supported module* Z_u *is called* $\mathbf{x}$-regular *if* $f_u(g)$ *is zero unless there is* z *in* Z_u *such that* zg *is conjugate to an element* a *in* A_u *whose image in* X_* *is* $\mathbf{x}$, *in which case the normalized orbital* *integral* $F(g, f_u) = \Delta_u(g)\Phi(g, f_u)$ *is equal to one. An* $\mathbf{x}$*-regular function* f_u *will be denoted by* $f_{\mathbf{x}}$. *A* regular *function is linear combination with complex coefficients of* $\mathbf{x}$*-regular functions.*

By definition, a regular function is zero on the non-regular set.

For any maximal (proper) F_u-parabolic subgroup $P_u = M_u N_u$ of G_u, where N_u is the unipotent radical of P_u and M_u is the Levi subgroup containing A_u, let α_{P_u} denote the $F_u^\times$-valued character on M_u defined by $\alpha_{P_u}(m) = det[ad(m) \mid L(N_u)]$. Here $L(N_u)$ denotes the Lie algebra of N_u, and $ad(m) \mid L(N_u)$ the adjoint action of m in M_u on $L(N_u)$. For any non-negative integer n let $A_u^{(n)}$ be the set of a in A_u such that $|val_u(\alpha_{P_u}(a))| < n$ for some P_u.

DEFINITION. — *A regular function* f_u *is called* n-admissible *if the orbital integral* $\Phi(a, f_u)$ *is zero for every* a *in* $A_u^{(n)}$.

4. THEOREM. — *Let* f^u *be a function on* G^u *which is compactly supported modulo* $Z^u = Z(\mathbb{A}^u)$. *Then there exists a positive integer* $n_0 = n_0(f^u)$ *such that for any* n_0*-admissible regular function* f_u *the function* $f = f_u f^u$ *is discrete.*

Proof. — For every maximal F-parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ and every place $v \neq u$ of F there exists a non-negative integer $C_{v,P}$ which depends on f^u, with $C_{v,P} = 0$ for almost all v, such that if γ lies in a Levi subgroup M of P and $f^u(x^{-1}\gamma x) \neq 0$ for some x in G^u, then

$$(4) \qquad |val_v(\alpha_P(\gamma))| \leq C_{v,P} \,.$$

Put $C_{u,P} = \sum_{v \neq u} C_{v,P}$. Since γ is rational (in G), the product formula $\sum_v val_v(\alpha_P(\gamma)) = 0$ on $F^\times$ implies that the inequality $(4)_v$ remains valid also for $v = u$. Then the theorem follows for $n_0 > C_{u,P}$ for all of the finitely many conjugacy classes of the subgroups P.

5. COROLLARY. — *Suppose that* $f = f_u f^u$ *is a cuspidal function, where* f_u *is a regular* n_0*-admissible function with* $n_0 = n_0(f^u)$. *Then the equality* (2) *holds, where the sum on the left is as in* (1), *while the sum on the right is as in* (3).

DEFINITION. — (1) *A complex-valued compactly supported function* f_u *on* G_u/Z_u *is called* spherical *if it is* K_u*-biinvariant.* (2) *Let* $\mathbb{H}_u$

be the convolution algebra of such functions. (3) *A spherical function* f_u *is called* n-admissible *if the orbital integral* $\Phi(a, f_u)$ *is zero for every regular* a *in* $A_u^{(n)}$.

For applications, we need to replace the regular n_0-admissible component f_u in the Corollary by a spherical n_0-admissible component. For this purpose, we recall basic results concerning spherical and regular functions. Let $T = X^*(\mathbb{C})$ denote the complex torus $Hom(X_*, \mathbb{C}^\times)$. The Weyl group W of A_u in G_u acts on A_u, X^*, X_*, and T. Each t in T defines a unique $\mathbb{C}^\times$-valued character of B_u which is trivial on N_u and on A_u^0. The G_u-module $I(t) = Ind(\delta_u^{1/2} t; B_u, G_u)$ normalizedly induced from the character t of B_u is unramified and has a unique unramified irreducible constituent $\pi(t)$. We have $\pi(t) \simeq \pi(t')$ if and only if $t' = wt$ for some w in W. The map $t \to \pi(t)$ is a bijection from the variety T/W to the set of unramified irreducible G_u-modules. Put $t(\pi)$ for the t associated with such a π.

If f_u is a spherical function then the value of the normalized orbital integral $F(a, f_u) = \Delta_u(a)\Phi(a, f_u)$ at a regular a in A_u depends only on the W-orbit of the image $\mathbf{x}$ of a in X_*; it is denoted by $F(\mathbf{x}, f_u)$. Let $\mathbb{C}[X_*]^W$ be the algebra of W-invariant elements in the group ring $\mathbb{C}[X_*]$. The Satake transform $f_u \to f_u\check{} = \sum_{\mathbf{x}\in X_*} F(\mathbf{x}, f_u)\mathbf{x}$ defines an algebra isomorphism from the convolution algebra $\mathbf{H}_u$ of spherical functions, to $\mathbb{C}[X_*]^W$. For each $\mathbf{x}$ in X_*, let $f(\mathbf{x})$ be the element of $\mathbf{H}_u$ with $f(\mathbf{x})\check{} = \sum_{w\in W} w\mathbf{x}$. Then $f(\mathbf{x})$ is n_0-admissible if $|val_u(\alpha_{P_u}(w(a(\mathbf{x}))))| \geq n_0$ for every w in W and parabolic subgroup P_u containing A_u; as above, $a(\mathbf{x})$ is an element of A_u which maps to $\mathbf{x}$ in X_* under the isomorphism $A_u/A_u^0 \simeq X_*$. A simple computation of the character of an induced representation yields the following

6. LEMMA. — *For every* f_u *in* $\mathbf{H}_u$ *and* t *in* T *we have* $tr(\pi(t))(f_u) = tr(I(t))(f_u) = f_u\check{}(t)$, *where* $f_u\check{}(t) = \sum_{\mathbf{x}\in X_*} F(\mathbf{x}, f_u)t(\mathbf{x})$.

To study regular functions, note that the normalized N_u-homology module π_{N_u} (= module of N_u-coinvariants in [BZ]) of an admissible G_u-module π_u with trivial central character, is an A_u-module; its character is denoted by $\chi(\pi_{N_u})$. A simple application of the Weyl integration formula and the theorem of Deligne-Casselman [CD] implies the following

7. LEMMA. — *If $f_{\mathbf{x}}$ is an* **x**-*regular function, then*

$$tr\ \pi_u(f_{\mathbf{x}}) = [W]^{-1} \int_{A_u/Z_u} (\Delta\chi(\pi_{N_u}))(a)\ F(a, f_{\mathbf{x}})da\,.$$

Moreover, if $tr\ \pi_u(f_{\mathbf{x}})$ is non-zero, then there exists (i) *t in T such that π_u is a constituent of $I(t)$, and* (ii) *a subset $W(\pi_u, t)$ of W such that*

$$tr\ \pi_u(f_{\mathbf{x}}) = \sum_w t(w\mathbf{x}) \qquad (w \text{ in } W(\pi_u, t))\,.$$

Finally, each constituent of $I(t)$, including π_u, has a non-zero vector fixed by the action of an Iwahori subgroup.

Proof. — (i) follows from Frobenius reciprocity. The final claim is proven in [B], (4.7), in the case of a reductive group, and in [FK], §17, in the case of the metaplectic group considered in [FK].

DEFINITION. — (1) *The spherical function f_u and the regular function ϕ_u are called* companions *if $f_u = \sum_{\mathbf{x}} c(\mathbf{x}, f_u)f(\mathbf{x})$ and $\phi_u = \sum_{\mathbf{x}} c(\mathbf{x}, f_u)f_{\mathbf{x}}$ ($\mathbf{x}$ in X_*/W).* (2) *A spherical, or regular, function f_u, is called* solid *if $tr\ \pi_u(f_u) = 0$ for every irreducible G_u-module π_u which is not equivalent to some $I(t)$.*

8. LEMMA. — *Suppose that the spherical f_u and the regular ϕ_u are solid companions. Then* (i) *$tr\ \pi_u(f_u) = tr\ \pi_u(\phi_u)$ for every irreducible G_u-module π_u,* (ii) *$\Phi(g, f_u) = \Phi(g, \phi_u)$ for every g in G_u.*

Proof. — (i) follows from Lemmas 2 and 3, which imply that $tr(I(t))(f_u) = tr(I(t))(\phi_u)$ for all t; (ii) follows from (i) by the density theorem of [K; Appendix].

9. PROPOSITION. — *Let f^u be a cuspidal function on G^u which is compactly supported modulo Z^u. Then there exists a positive integer $n_0 = n_0(f^u)$ with the following property. For any n_0-admissible solid spherical function f_u which has a regular solid companion ϕ_u, the equality* (2) *of Proposition 1 holds with the function $f = f_u f^u$, where the sum on the left is as in* (1) *while the sum on the right is as in* (3) *(of Proposition 2).*

This follows at once from Theorem 4, Lemma 8, and Corollary 5.

We shall use Proposition 9 to study lifting problems related to the group $GL(n)$. For this purpose we now construct solid companions for this group, using results of [BZ]. This construction can be generalized to the case of any reductive G on using the results of [KL], but we do not do it here. For $G_u = GL(n, F_u)$ the torus T is isomorphic to $\mathbb{C}^{\times n}$, and $I(t)$ has a trivial central character if $t_1 \dots t_n = 1$ where $t = (t_1, \dots, t_n)$. For any i, $j (1 \leq i \neq j \leq n)$, put $\alpha_{i,j}(t) = t_i/t_j$. The result of [BZ] which we need is the following

10. LEMMA. — *The G_u-module $I(t)$ is irreducible if and only if $\alpha_{i,j}(t) \neq q_u$ for all $i \neq j$.*

For $G_u = GL(n, F_u)$, X_* is isomorphic to $\mathbb{Z}^n$; if $t = (t_1, \dots, t_n)$ lies in T and $\mathbf{x} = (x_1, \dots, x_n)$ in X_*, then $t(\mathbf{x}) = \prod_{i=1}^{n} t_i^{x_i}$. Let $\boldsymbol{\alpha}_{ij}$ be the element $\mathbf{x}$ of X_* with $x_k = 0$ for $k \neq i, j$; $x_i = 1$ and $x_j = -1$. For any $\mathbf{x}$ in X_* we have
$(q\alpha_{ij}(t) - (q^2+1) + q\alpha_{ji}(t))t(\mathbf{x}) = qt(\mathbf{x} + \boldsymbol{\alpha}_{ij}) - (q^2+1)t(\mathbf{x}) + qt(\mathbf{x} - \boldsymbol{\alpha}_{ij})$.
Consequently, we have the following

11. LEMMA. — *There exist finitely many $\mathbf{x}_r$ in X_* and integers c_r such that for every t in T and $\mathbf{x}$ in X_* we have*

$$\sum_r c_r \, t(\mathbf{x} + \mathbf{x}_r) = \left[\prod_{i<j} (q\alpha_{ij}(t) - (q^2+1) + q\alpha_{ji}(t)) \right] t(\mathbf{x}) .$$

Proof. — This is obvious on working in the integral group ring $\tilde{X} = \mathbb{Z}[X_*]$ of X_*. In more detail, let $\tilde{X}$ be the quotient of the free abelian ring generated by the objects $a * \mathbf{x}$ with a in $\mathbb{Z}$ and $\mathbf{x}$ in X_*, by the relations $(a * \mathbf{x}) \cdot (b * \mathbf{y}) = (ab) * (\mathbf{x} + \mathbf{y})$ and $(a + b) * \mathbf{x} = a * \mathbf{x} + b * \mathbf{x}$. Then

$$\prod_{i<j} (q * \boldsymbol{\alpha}_{ij} - (q^2+1) * \mathbf{0} + q * \boldsymbol{\alpha}_{ji}) = \sum_r c_r * \mathbf{x}_r$$

in $\tilde{X}$. Given t in $T = Hom(X_*, \mathbb{C}^\times)$, define t in $Hom(\tilde{X}, \mathbb{C})$ by $t(a * \mathbf{x} + b * \mathbf{y}) = at(\mathbf{x}) + bt(\mathbf{y})$. Then

$$\begin{aligned} \sum_r c_r \, t(\mathbf{x} + \mathbf{x}_r) &= t(\sum_r (c_r * \mathbf{x}_r) \cdot \mathbf{x}) \\ &= t(\left[\prod_{i<j} (q * \boldsymbol{\alpha}_{ij} - (q^2+1) * \mathbf{0} + q * \boldsymbol{\alpha}_{ji}) \right] \cdot \mathbf{x}) \\ &= \left[\prod_{i<j} (q\alpha_{ij}(t) - (q^2+1) + q\alpha_{ji}(t)) \right] t(\mathbf{x}), \end{aligned}$$

as required.

Put $L(f_{\mathbf{x}}) = \sum_r c_r f_{\mathbf{x}+\mathbf{x}_r}$ and $L(f(\mathbf{x})) = \sum_r c_r f(\mathbf{x}+\mathbf{x}_r)$.

12. PROPOSITION. — *Suppose that* $f_u = \sum_{\mathbf{x}} c(\mathbf{x}) f(\mathbf{x})$ *is a spherical function and* $\phi_u = \sum_{\mathbf{x}} b(\mathbf{x}) f_{\mathbf{x}}$ *is a regular function such that* $L(f_{\mathbf{x}})$ *is regular if* $b(\mathbf{x}) \neq 0$, *and* $b(\mathbf{x}) = c(\mathbf{x})$ *for every* $\mathbf{x}$ *in* X_*. *Then the spherical function* $L(f_u) = \sum_{\mathbf{x}} c(\mathbf{x}) L(f(\mathbf{x}))$ *and the regular function* $L(\phi_u) = \sum_{\mathbf{x}} c(\mathbf{x}) L(f_{\mathbf{x}})$ *are solid companions.*

Proof. — This follows at once from Lemmas 10 and 11.

13. COROLLARY. — *Let* f^u *be a cuspidal function on* G^u *which is compactly supported modulo* Z^u. *Then there exists a positive integer* $n_0 = n_0(f^u)$ *such that for every* n_0*-admissible spherical function* f_u, *putting* $f = f^u L(f_u)$ *we have that* $\sum_{\{\gamma\}} \Phi(\gamma, f)$ *(sum as in ((3)) is equal to*

$$\sum_{\pi}{}' \, tr\ \pi(f) = \sum_{\pi}{}' \, b(\pi_u)\ tr\ \pi(f^u f_u)\ ,$$

where

$$b(\pi_u) = \prod_{i<j} (q_u \alpha_{ij}(t(\pi_u)) - (q_u^2+1) + q_u \alpha_{ji}(t(\pi_u)))\ .$$

The sum $\sum'$ *ranges over the equivalence classes of cuspidal* G*-modules* π *whose component* π_u *is unramified, equivalent to the irreducible unramified* G_u*-module* $I(t)$ *with* $t = t(\pi_u)$.

This follows at once from Lemma 11 and Proposition 12. Since $G = GL(n)$ we have that the $m(\pi)$ of (1) are equal to one by multiplicity one theorem for the cuspidal spectrum of $GL(n)$.

Remark. — We refer to the identity (2) for the function f of Corollary 5, and in particular for the function f of Proposition 9, as the *regular* trace formula. Corollary 13 is an explicit form of the regular trace formula in the case of $G = GL(n)$.

The regular trace formula is a representation theoretic analogue of Deligne's conjecture on the Lefschetz fixed point formula for the trace of a finite flat correspondence on a separated scheme of finite type over a finite field, which is multiplied by a sufficiently high power of the Frobenius morphism. Its main application is indeed (1) a comparison with the Grothendieck fixed point formula, to deduce the purity theorem (or the Ramanujan conjecture) for cuspidal $GL(n)$-modules, over a function field, which have a supercuspidal component, and (2) a comparison with Deligne's conjectural fixed point formula, to deduce Drinfeld's reciprocity law concerning cuspidal $GL(n)$-modules and irreducible n-dimensional ℓ-adic representations of the Weil group of a function field; for details see our joint work [FK2] with D. Kazhdan.

Here we use the regular trace formula to prove the base-change lifting theorem for cuspidal $GL(n)$-modules with a supercuspidal component. In [FK1] we already gave a simple proof of the form of the trace formula, analogous to Deligne's conjecture, for a test function f with a supercusp component and a sufficiently admissible spherical component. There we used a regular component at a third place to annihilate the singular orbital integrals in the trace formula. This simple trace formula is used in [FK1] to extend the results of [FK] and [BDKV] or [F; III] to the context of cuspidal modules with a supercuspidal component. The metaplectic correspondence of [FK] and the simple-algebra correspondence of [BDKV] or [F; III] are more difficult to prove then the base-change lifting for $GL(n)$, and it is easy to see that the methods of [FK1] apply to establish this lifting in the same context of cuspidal modules with a supercuspidal component.

The proof of the base-change lifting for $GL(n)$ given here is even simpler. The usage of regular functions, for which the transfer of orbital integrals is trivial by definition, eliminates the need to compare explicitly orbital integrals of corresponding spherical functions for the two groups in question, other than the characteristic functions of the maximal compact subgroups of these groups. This comparison of orbital integrals of spherical functions, also referred to as the "fundamental lemma", is not required in our proof of the base-change lifting. But, as the referee pointed out, this transfer is implicit in our arguments, at least for a sufficiently large class of spherical functions.

The regular trace formula of the present work suffices to obtain the local results of [FK] and [BDKV] or [F; III]. However, the passage (in Proposition 12) from regular to spherical function annihilates the constituents of any induced G_u-module $I(t)$ which is reducible. Consequently using the present regular trace formula we can obtain the global results of [FK1] only for cuspidal representations π of the metaplectic group or the multiplicative group of a simple algebra which satisfy the unnatural condition that they have an unramified component π_u which is equivalent to an irreducible $I(t)$. This condition is automatically held in the case of $GL(n)$, since by [Z], Theorem 9.7, any component of a cuspidal $GL(n)$-module is (non-degerate hence) of the form $I(t)$ if it is unramified.

We hope that sufficiently admissible solid companions regular and spherical functions can be used to prove a regular trace formula as in Corollary 13 for f which is not necessarily cuspidal, but have not done this as yet.

To study the applications of the regular trace formula to the base-change lifting we establish a twisted analogue of the regular trace formula. This is our next aim.

2. Base Change Lifting.

We shall now develop a twisted analogue of the regular trace formula of Chapter I, using the same notations. Thus G is the group of F-rational points of a connected reductive group $\mathbf{G}$ over a global field F. Let σ be an automorphism of $\mathbf{G}$ (over F) of finite order e. Suppose that the character ω of $Z(\mathbb{A})/Z$ satisfies $\omega(\sigma(z)) = \omega(z)$ for all z. Then (the group generated by) σ acts on $L_0(G)$ (and $L(G)$) by $(r_0(\sigma)\phi)(g) = \phi(\sigma(g))$ (resp. $(r(\sigma)\phi)(g) = \phi(\sigma(g))$). An irreducible representation π of $G(\mathbb{A})$ is called *σ-invariant* if it is equivalent to ${}^\sigma\pi$, where ${}^\sigma\pi(g) = \pi(\sigma(g))$ (the same definition applies to a G_v-module π_v, and $\pi = \otimes\pi_v$ is σ-invariant if and only if π_v is σ-invariant for all v). The trace of the operator $r_0(f \times \sigma) = r_0(f)f_0(\sigma)$ is given by

$$tr\ r_0(f \times \sigma) = \sum{}' \ m(\pi)\ tr\ \pi(f \times \sigma)\ . \tag{5}$$

Here $\sum'$ ranges over all σ-invariant cuspidal representations π of $G(\mathbb{A})$, $m(\pi)$ is the multiplicity of π in $L_0(G)$, and $\pi(f \times \sigma) = \pi(f)\pi(\sigma)$ where $\pi(\sigma)$ is the restriction of $r_0(\sigma)$ to the space of π. The twisted trace formula is an alternative expression for (5). We shall use the following

DEFINITIONS. — (1) *An element γ of G (resp. G_v) is called* σ-semi-simple *if $(\gamma\times\sigma)^e = \gamma\sigma(\gamma)\dots\sigma^{e-1}(\gamma)\times 1$ is semi-simple as an element of the connected component G (resp. G_v) of the identity in the group $G\times\langle\sigma\rangle$ (resp. $G_v\times\langle\sigma\rangle$). (2) Denote by $Z_{\gamma\times\sigma}(H)$ the centralizer of $\gamma\times\sigma$*

in a group $H(=G'_v\,,\ G'$ *or* $G'(\mathbb{A})\,)$. (3) *A* σ-*semi-simple element* γ *in* G *(resp.* G_v *) is called* σ-elliptic *if* $Z_{\gamma\times\sigma}(G'(\mathbb{A}))/Z_{\gamma\times\sigma}(G')$ *(resp.* $Z_{\gamma\times\sigma}(G')$ *) has finite volume, and* σ-regular *if* $Z_{\gamma\times\sigma}(G'(\mathbb{A}))$ *(resp.* $Z_{\gamma\times\sigma}(G'_v)$ *) is a torus.* (4) *The* σ-orbital integral *of* f *at a* σ-*elliptic* γ *is defined to be*

$$\Phi(\gamma\times\sigma,f)=\int_{G'(\mathbb{A})/Z_{\gamma\times\sigma}(G')} f(g\cdot\gamma\cdot\sigma(g^{-1}))dg\ .$$

The σ-orbital integral *of* f_v *at a* σ-*elliptic* γ *in* G_v *is defined to be*

$$\Phi(\gamma\times\sigma,f_v)=\int_{G'_v/Z_{\gamma\times\sigma}(G'_v)} f_v(g\cdot\gamma\cdot\sigma(g^{-1}))\frac{dg}{d_\gamma}\ ;$$

it depends on a choice of a Haar measure d_γ *on* $Z_{\gamma\times\sigma}(G'_v)$.

Let $\{\phi_\alpha\}$ be an orthonormal basis for the space $L_0(G)$. The operator $r_0(f\times\sigma)$ is an integral operator on $G'(\mathbb{A})$ with kernel

$$K^0_{f\times\sigma}(x,y)=\sum_{\alpha,\beta}(r(f\times\sigma)\phi_\alpha)(x)\overline{\phi}_\beta(y)\,.$$

The operator $r(f\times\sigma)$ is an integral operator on $G'(\mathbb{A})$ with kernel

$$K_{f\times\sigma}(x,y)=\sum_\gamma f(x^{-1}\cdot\gamma\cdot\sigma(y))\qquad(\gamma\ \text{in}\ G')\,.$$

If $\mathbf{G}$ is anisotropic then $L_0(G)=L(G)$ and $r=r_0$. Since $K^0_{f\times\sigma}(x,y)=K_{f\times\sigma}(x,y)$ is smooth in both x and y , we integrate over the diagonal $x=y$ in $G'(\mathbb{A})$, change the order of summation and integration as usual, and obtain

14. Proposition. — *If* $\mathbf{G}$ *is anisotropic then* $\sum' m(\pi)\ tr\ \pi(f\times\sigma)$ $=\sum_{\{\gamma\times\sigma\}}\Phi(\gamma\times\sigma,f)$, *where the last sum ranges over a set of representatives* $\gamma\times\sigma$ *for the* σ-*conjugacy classes of elements in* G' .

Here γ and δ of G' (resp. G'_v) are called σ-*conjugate* if there exists x in G (resp. G_v) such that $x\gamma=\delta\sigma(x)$ in G' (resp. G'_v). The group $\mathbf{G}$ may be any reductive connected F-group. Also we say that the function f is σ-*discrete* if for every x in $G(\mathbb{A})$ and γ in G we have that $f(x\cdot\gamma\cdot\sigma(x^{-1}))$ is zero unless γ is σ-elliptic and σ-regular. Changing again the order of summation and integration we obtain

15. Proposition. — *If* f *is* σ-*discrete then*

$$\text{(6)}\qquad \int_{G'(\mathbb{A})}\left[\sum_{\gamma\in G'} f(x^{-1}\gamma\sigma(x))\right]dx=\sum_{\{\gamma\times\sigma\}}\Phi(\gamma\times\sigma,f)\ .$$

The sum on the right ranges over the set of σ-conjugacy classes of σ-elliptic σ-regular elements γ in G'.

When f is cuspidal the convolution operator $r(f)$ factorizes through the projection on $L_0(G)$, $tr\ r_0(f \times \sigma) = tr\ r(f \times \sigma)$ and $K^0_{f\times\sigma}(x,y) = K_{f\times\sigma}(x,y)$. Hence we obtain the following *twisted trace formula* :

16. COROLLARY. — *If f is cuspidal and σ-discrete then we have*

$$\sum{}' m(\pi)\ tr\ \pi(f \times \sigma) = \sum_{\{\gamma\times\sigma\}} \Phi(\gamma \times \sigma, f)\ . \tag{7}$$

On the left the sum is as in (**5**), *while the sum on the right is as in* (6).

In this paper we use the twisted trace formula in the special case of base-change. Thus let E/F be a cyclic extension of global fields, and $\mathbf{G}$ a reductive connected F-group. Let $\mathbf{G}^E = Res_{E/F}\mathbf{G}$ be the F-group obtained from $\mathbf{G}$ by restriction of scalars from E to F. Then $G^E(F) \simeq G(E)$, $G^E(\mathbb{A}) \simeq G(\mathbb{A}_E)$ where $\mathbb{A}_E$ is the ring of E-adeles, and $G^E(F_v) \simeq G(E_v)$ for every place v of F, where $E_v = E \otimes_F F_v$. Fix a generator σ of $Gal(E/F)$. Then it acts on $\mathbf{G}^E$, and Corollary 16 applies to a function $\tilde{f}$ on $G^E(\mathbb{A})$. We shall also denote $G^E(\mathbb{A})$-modules by $\tilde{\pi}$, the fixed character of the center $Z^E(\mathbb{A}) \simeq Z(\mathbb{A}_E)$ of $G^E(\mathbb{A})$ by $\tilde{\omega}$, and elements of G^E by $\tilde{\gamma}$. To obtain a twisted trace formula analogous to that of Corollary 5, we fix a non-archimedean place u of F which splits completely in E such that G_u is split and $\tilde{\omega}_u$ is unramified. We may and do assume that $\tilde{\omega}_u = 1$. Then $G^E_u = G_u \times \ldots \times G_u$ (e times), and correspondingly we write $\tilde{\gamma} = (\gamma_1, \ldots, \gamma_e)$ for its elements. Put $\gamma = N\tilde{\gamma} = \gamma_1\sigma(\gamma_2)\ldots\sigma^{e-1}(\gamma_e)$, and $f_u = f_1 * f_2 * \ldots * f_e$ if $\tilde{f}_u = (f_1, \ldots, f_e)$ is a function on G^E_u. A simple computation (see, e.g., [F3], §1.5) shows that $\Phi(\tilde{\gamma} \times \sigma, \tilde{f}_u) = \Phi(N\tilde{\gamma}, f_u)$ for every $\tilde{\gamma}$ in G^E_u. We conclude the

17. PROPOSITION. — *Suppose that $\tilde{f} = \tilde{f}_u\tilde{f}^u$ is a cuspidal function on $G(\mathbb{A}_E)$, where $\tilde{f}_u = (f_u, f^0_u, \cdots, f^0_u)$, f_u is a regular n_0-admissible function with $n_0 = n_0(f^u)$, and f^0_u is the characteristic function of an open compact modulo center subgroup of K_u (multiplied by a suitable scalar) such that $f_u = f_u * f^0_u$. Then* (7) *holds with f replaced by $\tilde{f}$.*

It is easy to check that any regular function f_u can be taken to be biinvariant by an Iwahori subgroup; see [F5]. Then f^0_u can be taken to be a scalar multiple of the characteristic function of the product of

Z_u with the Iwahori subgroup. In particular f_u^0 can be chosen to satisfy $L(f_u) * f_u^0 = L(f_u)$. Proposition 12 then implies the following twisted analogue of Corollary 13.

18. PROPOSITION. — *Suppose that* $\mathbf{G} = GL(n)$. *Let* $\tilde{f}^u$ *be a cuspidal function on* $\tilde{G}^u$ *which is compactly supported modulo* $\tilde{Z}^u(= Z(\mathbb{A}_E^u))$. *Then there exists a positive integer* $n_0 = n_0(\tilde{f}^u)$ *such that for every* n_0*-admissible spherical function* f_u *on* G_u, *putting* $\tilde{f} = \tilde{f}^u L(\tilde{f}_u)$, *where* $\tilde{f}_u = (f_u, f_u^0, \dots, f_u^0)$ *and* $L(\tilde{f}_u) = (L(f_u), f_u^0, \dots, f_u^0)$, *we have that the sum* $\sum_{\{\tilde{\gamma}\times\sigma\}} \Phi(\tilde{\gamma}\times\sigma, \tilde{f})$ *of* (6) *is equal to*

$$\sum_{\tilde{\pi}}{}' \, tr\ \tilde{\pi}(\tilde{f}\times\sigma) = \sum_{\tilde{\pi}}{}' \, b(\pi_u)\ tr\ \tilde{\pi}(\tilde{f}^u\tilde{f}_u\times\sigma) .$$

Here $\sum'$ *ranges over the equivalence classes of cuspidal* σ*-invariant* G^E*-modules* π *whose component* $\tilde{\pi}_u$ *is of the form* $\pi_u\otimes\ldots\otimes\pi_u$ *(* e *copies), where* π_u *is equivalent to an irreducible unramified* G_u*-module* $I(t)$, *and* $b(\pi_u)$ *is as defined in Corollary 13.*

Proof. — It remains to explain the last assertion, which concerns $\tilde{\pi}_u$. As a G_u^E-module, $\tilde{\pi}_u$ is of the form $\pi_1\otimes\ldots\otimes\pi_e$. It is easy to see, as in [F3], §1.5, that $\tilde{\pi}_u$ is σ-invariant if and only if $\pi_i = \pi_1$ for all $i (1\le i\le e)$. Moreover, $tr\ \tilde{\pi}_u(\tilde{f}_u\times\sigma) = tr\ \pi_u(f_u)$ if $\tilde{\pi}_u - \pi_u\otimes\ldots\otimes\pi_u$ and $\tilde{f}_u = (f_u, f_u^0, \dots, f_u^0)$. But then $tr\ \pi_u(L(f_u)) = b(\pi_u) tr\ \pi_u(f_u)$ by Lemma 11, and this is non-zero only if π_u is equivalent to an unramified irreducible $I(t)$, as required.

Remark. — The identity (7) for the function $\tilde{f}$ of Proposition 17 will be referred to as the *regular twisted trace formula.* Proposition 18 is an explicit form of the regular twisted trace formula in the case of $G = GL(n)$, which we use below to prove the base-change lifting for this group.

From now on $\mathbf{G} = GL(n)$. Our aim is to establish the base-change theory for this group on comparing the sides of the regular trace formula (2) and its twisted analogue (7), which involve orbital integrals. We begin with a comparison of the sets $\{\gamma\}$ and $\{\tilde{\gamma}\times\sigma\}$ over which the sums are taken. Let E/F be a cyclic extension of local or global fields; put $G = GL(n, F)$ and $\tilde{G} = GL(n, E)$.

DEFINITION. — *Given* $\tilde{\gamma}$ *in* $\tilde{G}$ *put*

$$\mathsf{N}\tilde{\gamma} = \tilde{\gamma}\cdot\sigma(\tilde{\gamma})\cdot\sigma^2(\tilde{\gamma})\cdot\ldots\cdot\sigma^{e-1}(\tilde{\gamma}) .$$

Since $\sigma(\mathbf{N}\tilde{\gamma}) = \tilde{\gamma}^{-1} \cdot \mathbf{N}\tilde{\gamma} \cdot \tilde{\gamma}$, the elementary divisors $p_1(x) | p_2(x) | \dots | p_j(x)$ of the matrix $\mathbf{N}\tilde{\gamma}$, which *a priori* have coefficients in E, in fact have coefficients in F. Hence $\mathbf{N}\tilde{\gamma}$ is conjugate to an element $N\tilde{\gamma}$ of G. Only the conjugacy class of $N\tilde{\gamma}$ is determined by $\tilde{\gamma}$, and $N\tilde{\gamma}$ depends only on the σ-conjugacy class of $\tilde{\gamma}$.

19. LEMMA. — *The norm map $N : \tilde{\gamma} \to N\tilde{\gamma}$ is an injection from the set of σ-conjugacy classes in $\tilde{G}$ to the set of conjugacy classes in G.*

Proof. — We have to show that if $\mathbf{N}\tilde{\gamma}$ and $\mathbf{N}\tilde{\delta}$ are conjugate in $\tilde{G}$ then $\tilde{\gamma}$ and $\tilde{\delta}$ are σ-conjugate. To recall the proof of [L], §4, put $\gamma = \mathbf{N}\tilde{\gamma}$. As noted above we may assume that γ lies in G. The centralizer $Z_\gamma(\mathbf{G})$ of γ in $\mathbf{G}$ is a reductive F-group; it is the group of invertible elements in $Z_\gamma(L(\mathbf{G}))$, where $L(\mathbf{G}) = \mathbf{M}_n$ denotes the Lie algebra of $\mathbf{G}$. The centralizer $Z_{\tilde{\gamma}\times\sigma}(\tilde{G})$ of $\tilde{\gamma} \times \sigma$ in $\tilde{G}$ is the group of F-points of a reductive F-group $Z_{\tilde{\gamma}\times\sigma}(\tilde{\mathbf{G}})$. We have $Z_{\tilde{\gamma}\times\sigma}(\tilde{G}) \subset (Z_\gamma(\mathbf{G}))(E)$, and $Z_{\tilde{\gamma}\times\sigma}(\tilde{\mathbf{G}})$ is an inner form of $Z_\gamma(\mathbf{G})$ which splits over E and is defined by the cocycle $c_\sigma = ad(\tilde{\gamma}) \circ \sigma$; namely the F-structure on $Z_{\tilde{\gamma}\times\sigma}(\tilde{\mathbf{G}})$ is given by $\alpha \mapsto \tilde{\gamma}\sigma(\alpha)\tilde{\gamma}^{-1}$. The same definitions apply to the Lie algebras $L(\mathbf{G})$ and $L(\tilde{\mathbf{G}})$; $Z_{\tilde{\gamma}\times\sigma}(L(\tilde{\mathbf{G}}))$ is an E/F-form of $Z_\gamma(L(\mathbf{G}))$. By the Hilbert theorem 90 of [S], Ex. 2, p. 160, we have $H^1(Gal(E/F), Z_{\tilde{\gamma}\times\sigma}(\tilde{\mathbf{G}}))(E)) = \{0\}$, and the lemma follows as in [L], §4, from a simple cocycle computation.

We recall (see [AC], §1, or [L], §4) the following basic properties of the norm map.

20. LEMMA. — *Suppose that F is a global field. Then γ in G is a norm from $\tilde{G}$ if and only if it is a norm from $\tilde{G}_v$ for every place v of F.* (ii) *Suppose that E/F is a cyclic extension of local or global fields, and γ is an elliptic regular element of G. Then γ is a norm from $\tilde{G}$ if and only if det γ is in $N_{E/F}E^\times$.*

To compare the sums of orbital integrals in (2) and (7) we introduce the following

DEFINITION. — *Let v be a place of a global field F. Let ω_v be a unitary character of $N\tilde{Z}_v \simeq N_{E_v/F_v}E_v^\times$ and $\tilde{\omega}_v(x) = \omega_v(Nx)$ the associated character of $\tilde{Z}_v = Z(E_v)$. Let f_v be a smooth (= locally constant if v is non-archimedean) compactly-supported modulo $N\tilde{Z}_v$ function on G_v with $f_v(zg)\omega_v(z) = f_v(g)$ (g in G_v, z in $N\tilde{Z}_v$). Let*

$\tilde{f}_v$ be a smooth compactly supported modulo $\tilde{Z}_v$ function on $\tilde{G}_v$ with $\tilde{f}_v(zg)\tilde{\omega}_v(z) = \tilde{f}_v(g)$ (g in $\tilde{G}_v$, z in $\tilde{Z}_v$). Then f_v and $\tilde{f}_v$ are called matching *if for every regular γ in G_v we have $\Phi(\gamma, f_v) = \Phi(\tilde{\gamma}\times\sigma, \tilde{f}_v)$ if $\gamma = N\tilde{\gamma}$, or $\Phi(\gamma, f_v) = 0$ if γ is not a norm from $\tilde{G}_v$.*

Remark. — (1) If $N\tilde{\gamma} = \gamma$ is regular then $Z_{\tilde{\gamma}\times\sigma}(\tilde{G}) \simeq Z_\gamma(G)$ are F-tori in G, and in the definition of $\Phi(\tilde{\gamma}\times\sigma, \tilde{f})$ and $\Phi(\gamma, f)$ we take the same measure on $Z_{\tilde{\gamma}\times\sigma}(\tilde{G})$ and $Z_\gamma(G)$. The definition of the orbital integrals and consequently also of the matching functions depend on a choice of a Haar measure on $\tilde{G}_v$ and on G_v. (2) It is not hard to see that (i) $\Phi(\gamma, f_v)$ is a class function on G_v whose restriction to the regular set of G_v is smooth, (ii) for any smooth class function h on G_v which is supported on the regular set there exists an f_v supported on the regular set of G_v with $\Phi(\gamma, f_v) = h(\gamma)$ for all γ in G_v. The analogous statement holds for $\tilde{f}_v$. Hence it is clear that for every $\tilde{f}_v$ (resp. f_v) which is supported on the regular set of $\tilde{G}_v$ (resp. G_v, and $\Phi(\gamma, f_v) = 0$ if γ is not a norm), there exists a matching f_v (resp. $\tilde{f}_v$). Using the results of [K] it is easy to see that in the case of characteristic zero for every $\tilde{f}_v$ there exists a matching f_v, and for every f_v with $\Phi(\gamma, f_v) = 0$ if γ is regular but not a norm there exists a matching $\tilde{f}_v$, but this more refined statement is not needed in our study of base-change for $GL(n)$; as noted after Theorem 32 below, it is a consequence of [BDK] and the local lifting theorem. The analogous refined statement plays a key role in the study of the deeper metaplectic correspondence (see [FK]) and the simple algebra correspondence (see [BDKV] or [F1; III]).

DEFINITIONS. — *Let v be a non-archimedean place of F which does not ramify in E. Suppose that ω_v is unramified. Let f_v^0 be the function on G_v with $f_v^0(g) = 0$ unless $g = zk$ with k in K_v and z in $N\tilde{Z}_v$, where $f_v^0(zk) = \omega_v(z)^{-1}|K_v|^{-1}$. Let $\tilde{f}_v^0$ be the function on $\tilde{G}_v$ with $\tilde{f}_v^0(g) = 0$ unless $g = zk$ with k in $\tilde{K}_v$ and z in $\tilde{Z}_v$ where $f_v^0(zk) = \tilde{\omega}_v(z)^{-1}|\tilde{K}_v|^{-1}$ (here $K_v = GL(n, R_v)$, $\tilde{K}_v = GL(n, R_{E_v})$).*

21. PROPOSITION. — *The functions f_v^0 and $\tilde{f}_v^0$ are matching.*

Proof. — See Kottwitz [Ko].

We are now ready to compare the trace formulae of (2) and (7). Let ω be a unitary character of $N\tilde{Z}(\mathbb{A})/N\tilde{Z} \simeq N_{E/F}\mathbb{A}_E^\times/N_{E/F}E^\times$, and put $\tilde{\omega}(z) = \omega(Nz)$; $\tilde{\omega}$ is a character of $\tilde{Z}(\mathbb{A})/\tilde{Z} = Z(\mathbb{A}_E)/Z(E)$.

Let $f = \otimes f_v$ be a smooth compactly supported modulo $N\tilde{Z}(\mathbb{A})$ function on $G(\mathbb{A})$ with $f(zg)\omega(z) = f(g)$ (g in $G(\mathbb{A})$, z in $N\tilde{Z}(\mathbb{A})$), such that $f_v = f_v^0$ for almost all v. Let $\tilde{f} = \otimes \tilde{f}_v$ be a smooth compactly supported modulo $\tilde{Z}(\mathbb{A})$ function on $\tilde{G}(\mathbb{A})$ with $\tilde{f}(zg)\tilde{\omega}(z) = \tilde{f}(g)$ (g in $\tilde{G}(\mathbb{A})$, z in $\tilde{Z}(\mathbb{A})$), such that $\tilde{f}_v = \tilde{f}_v^0$ for almost all v. Suppose that f_v and $\tilde{f}_v$ are matching for all v. At some non-archimedean place w of F which splits completely in E we take f_w to be a supercusp form, and $\tilde{f}_w = (f_w, f_w^0, \dots, f_w^0)$, where f_w^0 is a function supported on the product of $N\tilde{Z}_w$ and a neighborhood of 1, such that $f_w = f_w * f_w^0$. At some non-archimedean place $u \neq w$ of F which splits completely in E we take $f_u = L(f'_u)$, where f'_u is any n_0-admissible spherical function on G_u, and $\tilde{f}_u = (L(f'_u), f_u^0, \dots, f_u^0)$, as in Proposition 18. Here n_0 depends on the support of f_v and $\tilde{f}_v$ for all $v \neq u$. Note that f does not transform under the center $Z(\mathbb{A})$ of $G(\mathbb{A})$ (as in Corollary 13), but only under its subgroup $N\tilde{Z}(\mathbb{A}) = NZ(\mathbb{A}_E)$. Since $[Z(\mathbb{A}) : Z \cdot N\tilde{Z}(\mathbb{A})] = e$, the effect of this change is that (**2**) takes the form

$$(2') \qquad \sum_{\pi}{}' \, tr\; \pi(f) = e \sum_{\{\gamma\}} |Z_\gamma(G(\mathbb{A}))/Z(\mathbb{A})Z_\gamma(G)| \prod_v \Phi(\gamma, f_v) .$$

Here π ranges over all cuspidal G-modules such that the restriction of their central character to $N\tilde{Z}(\mathbb{A})$ is ω, and γ ranges over the set of conjugacy classes of regular elliptic elements in G. Since f_v matches some $\tilde{f}_v$ for every v, by virtue of Lemma 20 (1) each γ which yields a non-zero term in the sum is a norm from $\tilde{G}$.

Also we note that f is cuspidal since its component f_w is a supercusp form; hence Corollary 13 applies. The identity (7) takes the form

$$(7') \; \sum_{\pi}{}' \, tr\; \tilde{\pi}(\tilde{f} \times \sigma) = \sum_{\{\tilde{\gamma}\}} |Z_{\tilde{\gamma}\times\sigma}(\tilde{G}(\mathbb{A}))/Z(\mathbb{A})Z_{\tilde{\gamma}\times\sigma}(\tilde{G})| \prod_v \Phi(\tilde{\gamma} \times \sigma, \tilde{f}_v) .$$

The $\tilde{\pi}$ and $\tilde{\gamma}$ are described in Proposition 18. Since the sums over γ in (2') and $\tilde{\gamma}$ in (7') are taken over the same sets, and $f_v, \tilde{f}_v$ are matching for every v, and $Z_{\tilde{\gamma}\times\sigma}(\tilde{G})$ and $Z_\gamma(G)$ are isomorphic if $\gamma = N\tilde{\gamma}$, we conclude

22. PROPOSITION. — *For any matching $f = \otimes f_v$ and $\tilde{f} = \otimes \tilde{f}_v$ as above we have*

$$(8) \qquad \sum_{\pi}{}' \, tr\; \pi(f) = e \sum_{\tilde{\pi}}{}' \, tr\; \tilde{\pi}(\tilde{f} \times \sigma) .$$

To derive lifting theorems from this identity we write $tr\; \pi(f)$ as a product over all v of $tr\; \pi_v(f_v)$; this can be done since each of $\pi = \otimes \pi_v$,

$f = \otimes f_v$ and the global measure $dg = \otimes dg_v$ is a product of local objects. To obtain such factorization in the twisted case we need to factorize the operator $\tilde{\pi}(\sigma)$. Since $\tilde{\pi}$ is cuspidal, it is generic (or non-degenerate), and can be realized in a space of Whittaker functions W on $G(\mathbb{A})$ with respect to an additive character $\psi = \psi' \circ N_{E/F}$ of $\mathbb{A}_E$, where ψ' is a non-trivial character of $\mathbb{A}$ *mod* F. Thus each W satisfies $W(ng) = \psi(\sum_{1 \leq i < n} n_{i,i+1})W(g)$ for any upper triangular unipotent matrix $n = (n_{ij})$ in $\tilde{G}(\mathbb{A}) = G(\mathbb{A}_E)$ and any g in $G(\mathbb{A}_E)$. Here $\tilde{\pi}$ acts by $(\tilde{\pi}(g)W)(h) = W(hg)$, and $\tilde{\pi}(\sigma)$ acts by $(\tilde{\pi}(\sigma)W)(g) = W(\sigma(g))$. Now each component $\tilde{\pi}_v$ of $\tilde{\pi}$ is also non-degenerate, and can be realized in the space of Whittaker functions W_v with respect to the component $\psi_v = \psi'_v \circ N_{E_v/F_v}$ of ψ. The normalization $(\tilde{\pi}_v(\sigma)W_v)(g) = W_v(\sigma(g))$ of $\tilde{\pi}_v(\sigma)$ satisfies $\tilde{\pi}(\sigma) = \otimes_v \tilde{\pi}_v(\sigma)$, hence $tr\ \tilde{\pi}(\tilde{f} \times \sigma) = \prod_v tr\ \tilde{\pi}_v(\tilde{f}_v \times \sigma)$.

Let π_u be an irreducible unramified unitary G_u-module, and $\tilde{\pi}_u = \pi_u \otimes \ldots \otimes \pi_u$. Put $c(\pi_u) = \sum tr\ \pi^u(f^u)$, where the sum ranges over all irreducible G^u-modules π^u such that $\pi = \pi^u \otimes \pi_u$ occurs in the sum on the left of (8). Put $c(\tilde{\pi}_u) = \sum tr\ \tilde{\pi}^u(\tilde{f}^u \times \sigma)$, where the sum ranges over all irreducible $\tilde{G}^u$-modules $\tilde{\pi}^u$ such that $\tilde{\pi} = \tilde{\pi}^u \otimes \tilde{\pi}_u$ occurs in the sum on the right of (8).

23. PROPOSITION. — *For every π_u we have $c(\pi_u) = c(\tilde{\pi}_u)$.*

Proof. — By virtue of Corollary 13 and Proposition 18 the identity (8) can be written in the form

$$\sum_{\pi_u} b(\pi_u)[c(\pi_u) - c(\tilde{\pi}_u)]\ tr\ \pi_u(f_u) = 0\,. \tag{9}$$

The sum ranges over all (equivalence classes of) unitary unramified irreducible G_u-modules π_u. Theorem 2 of [FK1] implies that

$$b(\pi_u)(c(\pi_u) - c(\tilde{\pi}_u)) = 0,$$

since the sum of (9) is absolutely convergent for every spherical f_u, and (9) holds for all n_0-admissible spherical f_u for some n_0. Since each π and $\tilde{\pi}$ in (2') and (7') is cuspidal, π_u is non-degenerate, hence equivalent to an irreducible induced G_u-module of the form $I(t)$. By Lemma 10 we have $b(\pi_u) \neq 0$, hence $c(\pi_u) = c(\tilde{\pi}_u)$, as required.

We need an extension of Proposition 23 from the set which consists of u to the complement of a finite set of places of F. To state it we introduce the

DEFINITION. — *Let* E_v/F_v *be an unramified cyclic extension of degree* e *of non-archimedean local fields. Then we say that (1) the* G_v*-module* $I(t)$ *(resp.* $\pi(t)$ *)* lifts *to the* $\tilde{G}_v$*-module* $\tilde{I}(t')$ *(resp.* $\tilde{\pi}(t')$ *) if* $t' = t^e (= (t_1^e, \dots, t_n^e)$ *if* $t = (t_1, \dots, t_n)$ *); (2) the spherical functions* f_v *on* G_v *and* $\tilde{f}_v$ *on* $\tilde{G}_v$ correspond *if* $tr\ (\pi(t))(f_v) = tr\ (\tilde{\pi}(t^e))(\tilde{f}_v)$ *for all* t *in* T *(equivalently,* $tr\ (I(t))(f_v) = tr\ (\tilde{I}(t^e))(\tilde{f}_v)$ *for all* t *in* T *).*

We also denote by $\tilde{\mathsf{H}}_v$ (resp. H_v) the Hecke convolution algebra of spherical, namely $\tilde{K}_v$-biinvariant, compactly-supported modulo center, functions on $\tilde{G}_v$ which transform under $\tilde{Z}_v$ by $\tilde{\omega}_v^{-1}$ (resp. K_v-biinvariant, compactly supported modulo $N\tilde{Z}_v$, functions on G_v which transform under $N\tilde{Z}_v$ by ω_v^{-1}). Note that the spherical f_v (resp. $\tilde{f}_v$) corresponds to precisely one spherical $\tilde{f}_v$ (resp. f_v) by the theory of the Satake transform.

Let v be a place of F. Then $E_v = E \otimes_F F_v = F'_v \oplus \dots \oplus F'_v$ ($e'' = e/e'$ copies, where $e' = [F'_v : F_v]$). Suppose that F'_v/F_v is unramified. Put $G'_v = GL(n, F'_v)$. Then we say that the unramified G_v-module $\pi(t)$ (resp. $I(t)$) *lifts* to the unramified $\tilde{G}_v$-module $\tilde{\pi}(t') = \pi'(t_1) \otimes \dots \otimes \pi'(t_{e''})$ (resp. $\tilde{I}(t') = I'(t_1) \otimes \dots \otimes I'(t_{e''})$) if $t_i = t^{e'}$ for all $i (1 \le i \le e'')$. The spherical function f_v on G_v *corresponds* to the spherical function $\tilde{f}_v = f'_1 \otimes \dots \otimes f'_{e''}$ on $\tilde{G}_v$ if $tr\ (\pi(t))(f_v) = tr\ (\tilde{\pi}(t'))(\tilde{f}_v \times \sigma)$ for all t in T, where $\tilde{\pi}(t') = \pi'(t^{e'}) \otimes \dots \otimes \pi'(t^{e'})$. For example, if f_v corresponds to the spherical f'_v on G'_v, and $f_v'^0$ is the unit element of the Hecke algebra H'_v of G'_v, then f_v corresponds to $\tilde{f}_v = f'_v \otimes f_v'^0 \otimes \dots \otimes f_v'^0$.

The following Proposition 24 is usually considered to be a crucial tool in the extension of Proposition 23 alluded to above. The technique of regular functions permits us below to extend Proposition 23, and complete the proof of the base-change theorems, without ever using Proposition 24. In fact Proposition 24 can be deduced from the local lifting theorem as in Remark (3) in §32 below.

24. PROPOSITON. — *If the spherical* f_v *on* G_v *and* $\tilde{f}_v$ *on* $\tilde{G}_v$ *are corresponding then they are matching.*

Proof. — This is the special case of $G = GL(n)$ of the Theorem of [F4]. The proof of that Theorem is incomplete in the generality stated there. The problem is in the deduction of Lemma 5 from Lemma 4 in §6 of [F4]. Contrary to the assertion of [F4], p. 141, l. 4-5, it is not the identity (6.4), but only (6.3), which holds for the unit elements f_u^0, ϕ_u^0.

To complete the proof of the Theorem of [F4], we need to prove

Lemma 6.5 of [F4]. By inspection, this Lemma holds in the situation of base-change from $U(3, E/F)$ to $GL(3, E)$, and the analogous case of the symmetric-square from $SL(2)$ to $PGL(3)$. A few general rank cases where this Lemma holds, are the following. Suppose $G = GL(n)$. In this case (of base-change for $GL(n)$) we can work with the Deligne-Kazhdan trace formula. Then all π, π' of (6.1) are cuspidal. Their local components are non-degenerate. Hence the unramified components are irreducible full-induced $I(\eta_u), I(\eta'_u)$, by [Z], (9.7), and Lemma 6.5 holds.

In the analogous case of the metaplectic correspondence of [FK], the unramified components of the cuspidal π and π' are unramified irreducible full-induced, as noted above in the case of $GL(n)$, and by virtue of Theorem 16 in [FK] (this theorem establishes a canonical isomorphism of the Iwahori algebras of $GL(n)$ and its covering group considered in [FK]), in the case of the metaplectic group of [FK]. In particular the sentence on lines 3-5, p. 85, of [FK], should be replaced by a reference to Theorem 16 of [FK].

We shall now give a purely local proof, based on Proposition 12, of a special case of Proposition 24 which suffices for our purposes. We first fix some notations. Let E/F be an unramified cyclic extension of non-archimedean local fields, and $\tilde{\omega}$ an unramified character of $\tilde{Z}$. Consider $\mathbf{x} = (x_1, \ldots, x_n)$ in $\tilde{X}_*(\simeq \mathbb{Z}^n)$ with $x_i \neq x_j$ for all $i \neq j$.

DEFINITION. — *A locally constant function $\tilde{f}$ on $\tilde{G}$ with $\tilde{f}(zg)\tilde{\omega}(z) = \tilde{f}(g)$ (z in $\tilde{Z}$, g in $\tilde{G}$) which is compactly supported modulo $\tilde{Z}$, is called* $\mathbf{x}$-regular *if $\tilde{f}(g)$ is zero unless there is z in $\tilde{Z}$ such that zg is σ-conjugate to an element a in $\tilde{A}$ whose image in $\tilde{X}_*$ is $\mathbf{x}$, in which case the normalized twisted orbital integral $F(g \times \sigma, \tilde{f}) = \Delta(Ng)\Phi(g \times \sigma, \tilde{f})$ is equal to $\tilde{\omega}(z)$. An $\mathbf{x}$-regular function $\tilde{f}$ will be denoted by $\tilde{f}_{\mathbf{x}}$.* A regular *function $\tilde{f}$ is an element in the span of the $\mathbf{x}$-regular functions, for all $\mathbf{x}$.*

As in the non-twisted case, for each $\mathbf{x}$ in $\tilde{X}_*$ let $\tilde{f}(\mathbf{x})$ be the spherical function in $\tilde{\mathsf{H}}$ with $\tilde{f}(\mathbf{x})^{\vee} = \sum_{w \in W} w\mathbf{x}$. As in Proposition 12 we have in the twisted case that : if $\tilde{f} = \sum c(\mathbf{x}) f(\mathbf{x})$ is a spherical function, and $\tilde{\phi} = \sum b(\mathbf{x}) \tilde{f}_{\mathbf{x}}$ is a regular function such that $L(\tilde{f}_{\mathbf{x}}) = \sum_r c_r f_{\mathbf{x}+\mathbf{x}_r}$ is regular if $b(\mathbf{x}) \neq 0$, then the following holds. The spherical function $L(\tilde{f}) = \sum c(\mathbf{x}) L(\tilde{f}(\mathbf{x}))$ and the regular function $L(\tilde{\phi}) = \sum c(\mathbf{x}) L(\tilde{f}_{\mathbf{x}})$ are solid companions.

25. PROPOSITION. — *Let n_1 be the maximum of the absolute values of the entries of the vectors $\mathbf{x}_r$ of Lemma 11. If $\tilde{f}$ and f are corresponding n_1-admissible spherical functions on $\tilde{G}$ and G then $L(\tilde{f})$ and $L(f)$ are matching.*

Proof. — Let $b^* : \tilde{\mathbf{H}} \to \mathbf{H}$ be the linear map defined by $b^*(\tilde{f}(\mathbf{x})) = f(e\mathbf{x})$. It is clear that $\tilde{f}$ and f are corresponding if and only if $f = b^*\tilde{f}$. By definition, the regular functions $\tilde{f}_{\mathbf{x}}$ on $\tilde{G}$ and $f_{e\mathbf{x}}$ on G are matching. Hence $L(\tilde{f})$ and $L(b^*\tilde{f})$ are matching, for every n_1-admissible spherical $\tilde{f}$ on $\tilde{G}$, as required.

We shall now return to the extension of Proposition 23 alluded to above. Let E/F be a cyclic extension of global fields. Let V be a set of places of F including the archimedean places and those which ramify in E. At each place v outside V fix an unramified unitary irreducible $\tilde{G}_v$-module $\tilde{\pi}_v^0$. At some place w in V which splits completely in E fix a supercuspidal G_w-module π_w^0, and put $\tilde{\pi}_w^0 = \pi_w^0 \otimes \ldots \otimes \pi_w^0$.

26. PROPOSITION. — *For every v in V let f_v and $\tilde{f}_v$ be matching functions on G_v and $\tilde{G}_v$, such that f_w is a matrix coefficient of π_w^0. Then we have*

$$(10) \qquad \sum_{\pi} \prod_{v \in V} tr\ \pi_v(f_v) = e \sum_{\tilde{\pi}} \prod_{v \in V} tr\ \tilde{\pi}_v(\tilde{f}_v \times \sigma).$$

The sum over $\tilde{\pi}$ ranges over all σ-invariant cuspidal $\tilde{G}$-modules $\tilde{\pi}$ whose component at w is $\tilde{\pi}_w^0$ and whose component at every v outside V is $\tilde{\pi}_v^0$. The sum over π ranges over all cuspidal G-modules π whose component at w is π_w^0 and whose component π_v at each v outside V lifts to $\tilde{\pi}_v^0$.

Proof. — Proposition 23 implies (10) if V is the complement of u. By virtue of Proposition 25 we also have (9) where u is replaced by any v outside V, as noted in the proof of Proposition 23. By induction (10) holds where V is the complement of a finite set. But then Lemma 3 of [F2], IV, implies the proposition.

DEFINITION. — *The irreducible $G(\mathbb{A})$-module $\pi = \otimes_v \pi_v$ lifts to the irreducible $\tilde{G}(\mathbb{A})$-module $\tilde{\pi} = \otimes_v \tilde{\pi}_v$ if π_v lifts to $\tilde{\pi}_v$ for almost all v.*

Remark. — For almost all v the components π_v and $\tilde{\pi}_v$ are unramified and so the notion of local lifting is indeed defined.

The following is the *global base-change theorem* for $GL(n)$.

27. THEOREM. — *Let E/F be a cyclic extension of global fields, and w a place of F which splits completely in E. Then every cuspidal G-module π whose component at w is supercuspidal lifts to a unique cuspidal $\tilde{G}$-module $\tilde{\pi}$; this $\tilde{\pi}$ is σ-invariant and has a supercuspidal component at w. Moreover, every σ-invariant cuspidal $\tilde{G}$-module $\tilde{\pi}$ whose component at w is supercuspidal is a lift of a cuspidal G-module π; any other cuspidal G-module which lifts to $\tilde{\pi}$ is of the form $\pi \otimes \epsilon^i$ $(0 \leq i < e)$, where ϵ is a primitive character of $\mathbb{A}^\times$ which is trivial on $F^\times N\mathbb{A}_E^\times$, and $\pi \otimes \epsilon^i \not\simeq \pi$ for all $i (1 \leq i < e)$.*

Proof. — On the right of (10) there is at most one $\tilde{\pi}$ by the rigidity theorem for the cuspidal spectrum of $\tilde{G}$ (see [JS]). By Corollary 6′ of [F2], V, which is an immediate consequence of [JS], Proposition 3.6 (as was noticed first by Arthur-Clozel [AC], III, 3.1), the sum on the left, over π, consists of at most one orbit $\pi \otimes \epsilon^i$ $(0 \leq i < e)$ under tensoring by ϵ of cuspidal G-modules (and each element in the orbit occurs). Since $tr(\pi_v \otimes \epsilon_v^i)(f_v) = tr\ \pi_v(f_v)$ for any f_v which matches some $\tilde{f}_v$, we conclude that $\pi \otimes \epsilon \not\simeq \pi$, and the theorem follows.

From the global base-change theorem we shall now deduce a local base-change theorem. Let E/F be a cyclic extension of degree e of local non-archimedean fields.

DEFINITION. — *The irreducible G-module π lifts to the irreducible σ-invariant $\tilde{G}$-mdoule $\tilde{\pi}$ if for some choice of $\tilde{\pi}(\sigma)$ we have $tr\ \pi(f) = tr\ \tilde{\pi}(\tilde{f} \times \sigma)$ for all matching f on G and $\tilde{f}$ on $\tilde{G}$.*

Remark. — (1) It is clear the π lifts to at most one $\tilde{\pi}$, and if π lifts to $\tilde{\pi}$ then $\pi \otimes \epsilon$ lifts to $\tilde{\pi}$ for every character ϵ of $F^\times/NE^\times$. (2) If $\tilde{M} = \prod_i \tilde{M}_i$ $(\tilde{M}_i = GL(n_i, E))$ is a Levi subgroup of $\tilde{G}$ and $M = \prod_i M_i$ $(M_i = GL(n_i, F))$, and ρ_i is an M_i-module which lifts to an $\tilde{M}_i$-module $\tilde{\rho}_i$, then $\pi = I(\otimes_i \rho_i; M, G)$ lifts to $\tilde{\pi} = I(\otimes_i \tilde{\rho}_i; \tilde{M}, \tilde{G})$ by virtue of a standard formula for the character of an induced representation. In other words, lifting commutes with induction. (3) If the $\tilde{\rho}_i$ are nondegenerate and $\tilde{\rho}_i(\sigma)$ are the intertwining operators defined using the Whittaker model of $\tilde{\rho}_i$, then the normalized operator $\tilde{\pi}(\sigma)$ is the one obtained by the induction functor from $\otimes \tilde{\rho}_i(\sigma)$.

DEFINITION. — (1) *An irreducible tempered G-module π is called* discrete *if it is not equivalent to any properly induced G-module.* (2) *An irreducible tempered σ-invariant $\tilde{G}$-module $\tilde{\pi}$ is called* σ-discrete *if it is not equivalent to any G-module properly induced from a σ-invariant representation of a Levi subgroup.*

28. LEMMA. — (i) *A G-module π is discrete if and only if it is square-integrable.* (ii) *A $\tilde{G}$-module $\tilde{\pi}$ is σ-discrete if and only if $\tilde{\pi} = I(\tilde{\rho}\otimes{}^{\sigma}\tilde{\rho}\otimes\ldots\otimes{}^{\sigma^{a-1}}\tilde{\rho})$ where $n = ab$, and $\tilde{\rho}$ is a discrete $G' = GL(b,E)$-module with ${}^{\sigma^i}\tilde{\rho} \not\simeq \tilde{\rho}(1 \leq i < a)$ and ${}^{\sigma^a}\tilde{\rho} \simeq \tilde{\rho}$.*

Proof. — (i) follows from [BZ]. For (ii), since $\tilde{\pi}$ is tempered it is induced from a representation $(\tilde{\rho}_1,\ldots,\tilde{\rho}_a)$ of a Levi subgroup of type $(b_1,\ldots,b_a)$, where the $\tilde{\rho}_i$ are discrete. Since ${}^{\sigma}\tilde{\pi} \simeq \tilde{\pi}$ we have $({}^{\sigma}\tilde{\rho}_1,\ldots, {}^{\sigma}\tilde{\rho}_a) = (\tilde{\rho}_1,\ldots,\tilde{\rho}_a)$ up to permutation, which must be transitive on $\{1,\ldots,a\}$ since $\tilde{\pi}$ is not induced. Up to reordering the indeces $\tilde{\rho}_i = {}^{\sigma^{i-1}}\tilde{\rho}_1$, as required.

DEFINITION. — *Let $\tilde{\pi}_0$ be a σ-discrete $\tilde{G}$-module with central character $\tilde{\omega}$. A locally constant compactly supported modulo $\tilde{Z}$ function $\tilde{f}$ on $\tilde{G}$ with $\tilde{f}(zg)\tilde{\omega}(z) = \tilde{f}(g)$ is called a* pseudo-coefficient *of $\tilde{\pi}_0$ if $tr\ \tilde{\pi}_0(\tilde{f} \times \sigma) = 1$ and $tr\ \tilde{\pi}(\tilde{f} \times \sigma) = 0$ for every tempered σ-invariant irreducible $\tilde{G}$-module $\tilde{\pi}$ with central character $\tilde{\omega}$, which is inequivalent to $\tilde{\pi}_0$.*

29. PROPOSITION. — *For every σ-discrete $\tilde{G}$-module there exists a pseudo-coefficient.*

Proof. — When $E = F$ this is proven in [K], using the trace Paley-Wiener theorem of [BDK]. The proof in the twisted case analogously follows from the twisted variant of the trace PW-theorem (see [F1], I, §7, or [R]; a different, in some sense better, proof in the non-twisted and also twisted case, is now in preparation), which we now recall. Let $\tilde{M}$ be a Levi subgroup of $\tilde{G}$. Let $Irr_\sigma(\tilde{M})$ be the set of irreducible σ-invariant $\tilde{G}$-modules $\tilde{\pi}$. Let $X_\sigma(\tilde{M})$ be the group of (necessarily σ-invariant) unramified characters χ of $\tilde{M}$. $X_\sigma(\tilde{M})$ has a natural structure of a complex algebraic variety, isomorphic to $\mathbb{C}^{\times d}$, where $d = dim\ X_\sigma(\tilde{M})$. It acts naturally on $Irr_\sigma(\tilde{M})$ by $\chi : \tilde{\rho} \to \tilde{\rho}\otimes\chi$. The twisted trace Paley-Wiener theorem asserts, in the terminology of [BDK], that the a "good form" λ on $Irr_\sigma(\tilde{G})$ is a "trace form", namely the following.

30. THEOREM. — *Let λ be a complex-valued linear form on the vector space over $\mathbf{C}$ spanned by $Irr_\sigma(\tilde{G})$. Suppose that (1) there exists an open compact σ-invariant subgroup $\tilde{K}$ of $\tilde{G}$ which dominates λ, in the sense that λ vanishes on each irreducible σ-invariant $\tilde{G}$-module which has no $\tilde{K}$-fixed vector, and (2) for every proper Levi subgroup $\tilde{M}$ and σ-invariant irreducible $\tilde{M}$-modules $\tilde{\rho}$, the function $\chi \mapsto \lambda[I(\tilde{\rho}\otimes\chi;\tilde{M},\tilde{G})]$ is regular on the complex algebraic variety $X_\sigma(\tilde{M})$. Then there exists a function $\tilde{f}$ with $\lambda(\tilde{\pi}) = tr\ \tilde{\pi}(\tilde{f}\times\sigma)$ for all $\tilde{\pi}$ in $Irr_\sigma(\tilde{G})$.*

Given the existence of pseudo-coefficients, the following is easily proven (see, e.g., [F1]) using the Deligne-Kazhdan trace formula.

31. PROPOSITION. — *Let E'/F' be a cyclic extension of local fields and $\tilde{\pi}'$ a σ-discrete $GL(n,E')$-module. Then there exists a cyclic extension E/F of global fields such that F has a place w with $F_w = F'$ and $E_w = E'$ and every archimedean place of F splits in E. Moreover, there exists a cuspidal σ-invariant $\tilde{G}$-module $\tilde{\pi}$ whose component at w is $\tilde{\pi}'$, with the following properties. There are two finite places u, u' which split completely in E such that $\tilde{\pi}_v$ is unramified for every finite $v \neq u, u'$, and $\tilde{\pi}_u$ is supercuspidal.*

Remark. — When $E' = F'$ Proposition 31 asserts that given a local π' there exists a global π with a component π' such that π has the specified properties.

Given a local π_w (resp. $\tilde{\pi}_w$) we apply (10) with the π (resp. $\tilde{\pi}$) of Proposition 31. Since $tr\ \pi_v(f_v) = tr\ \tilde{\pi}_v(\tilde{f}_v\times\sigma)$ for matching f_v, $\tilde{f}_v$ at all $v \neq w$, we conclude the following *local base-change theorem* for $GL(n)$.

32. THEOREM. — *The lifting defines a surjection from the set of equivalence classes of irreducible tempered G_v-modules π_v to the set of equivalence classes of irreducible tempered σ-invariant $\tilde{G}_v$-modules $\tilde{\pi}_v$. It maps discrete G_v-modules to σ-discrete $\tilde{G}_v$-modules, and relevant G_v-modules to relevant σ-invariant $\tilde{G}_v$-modules. The preimage π_v of a supercuspidal $\tilde{\pi}_v$ is supercuspidal.*

Remark. — Relevant G_v-modules are defined in [FK], §27. Every component of a cuspidal G-module is relevant, and every tempered G_v-module is relevant.

Proof. — For the first assertion it remains to note that each σ-invariant tempered $\tilde{G}_v$-module $\tilde{\pi}_v$ is induced from a σ-discrete one, and the lifting commutes with induction. The last assertion follows from the Theorem of Deligne [CD], according to which a G_v-module π_v is supercuspidal if and only if the restriction of its character to any torus in G_v is compactly supported modulo the center Z_v of G_v.

Remark. — (1) It is clear that Theorem 27 holds where w is any place of F, to assert that the lifting bijects σ-invariant cuspidal $\tilde{\pi}$ with a supercuspidal component at w, and orbits of cuspidal π whose component π_w at w lifts to a supercuspidal $\tilde{G}_w$-module; π_w is necessarily supercuspidal. (2) Using the local base-change theorem and the trace PW-theorem 30 it is easy to deduce (as in [FK], §27) that for every $\tilde{f}_v$ there is a matching f_v, and that for every f_v with $\Phi(\gamma, f_v) = 0$ if γ is regular but not a norm there is a matching $\tilde{f}_v$. As noted in Remark (2) following Lemma 20 the existence of matching f_v and $\tilde{f}_v$ can be proven purely locally, using [K], in characteristic zero. (3) Using the local base-change theorem it is easy to deduce Proposition 24 (along the lines of [FK], §27.3), namely to show that corresponding spherical functions are matching. This gives an easier proof of the theorem of [F4], but only in the special case of $GL(n)$.

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