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Regular trace formula and base change for $GL(n)$


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Introduction.

Our aim here is to develop the regular trace formula of [F2] from the context of $GL(2)$ to that of a reductive group, and use it to give an elementary proof of the theory of base change for cuspidal automorphic representations of $GL(n)$ which have a supercuspidal component. Our motivation is the belief that a formula as fundamental as the trace formula should be given a simple proof. In [F2] we establish, by means of such a proof, an asymptotic form of the formula which suffices to prove the complete base-change theory for $GL(2)$; in fact the method suffices to establish every lifting theorem involving groups of rank (or twisted rank) one (see [F5] in the case of the symmetric-square).

As in [FK1], the trace formula proven here holds for test functions with a supercusp component. At a second place we choose a spherical function which is sufficiently admissible with respect to the other components of the test function; the notion of admissibility is introduced prior to Theorem 4 below. The second restriction does not restrict the applicability of this trace formula to lifting problems, and indeed it is shown in [FK1]
that the simple trace formula developed there can be used to extend the
metaplectic correspondence of [FK] and the simple algebra (or Deligne-
Kazhdan) correspondence of [BDKV] or [F1; III] from the context of cuspi-
dal representations with two supercuspidal components to that of cuspidal
representations with a single such component.

It is easy to see that the proofs of [FK1] can be adapted to establish
also the base-change lifting for cuspidal $GL(n)$-modules with a supercus-
pidal component, which is proven here by means of a different technique.

However, in [FK1] the test function is chosen to have a regular com-
ponent at a third non-archimedean place (see the definition following Co-
rollary 3 below; a regular function on $GL(n)$ is supported on the conjugacy
classes of regular split elements whose eigenvalues have distinct valuations).
For example, in the proof of the Ramanujan, or purity conjecture for cuspi-
dal $GL(n)$-modules with a supercuspidal component (see [FK2]) we need
a trace formula for a test function whose components are not restricted at
any third place.

Here we develop such a formula on taking the second component to
be a regular – but not spherical – function which is sufficiently admissible
with respect to all other components. We then show that a suitable linear
combination of these regular functions, which annihilates the trace of any
constituent of a reducible unramified principal series representation, has the
same orbital integrals as the corresponding linear combination of spherical
functions. Using the fact that each unramified component of a cuspidal
$GL(n)$-module is an irreducible principal series $GL(n, F_v)$-module, we
derive our applications to the theory of base-change for $GL(n)$.

It will be interesting to develop (by elementary means) a trace
formula for a test function with a sufficiently admissible regular component,
and arbitrary components elsewhere. This may lead to a simple proof of
the base-change lifting for $GL(n)$, and metaplectic and simple algebra
 correspondences, for an arbitrary cuspidal representation. But we have not
done this as yet.

As noted in [FK1], the simple form of the trace formula of [FK1] and
of the present work are analogous to Deligne's conjecture on the Lefschetz
fixed point formula for a finite flat correspondence on a separated scheme of
finite type over a finite field, which is multiplied by a sufficiently high power
of the Frobenius; see [FK2] for more details. The notion of admissibility
is suggested, at least in the case of $GL(2)$, by Drinfeld's use of the trace
formula in [D].
In [F4] we stated a theorem ("fundamental lemma"), asserting the transfer of stable orbital integrals of spherical functions in the case of base-change from $G(F)$ to $G(E), E/F$ cyclic extension. The global-type proof there is incomplete. It is completed in Proposition 24 in a few special cases (e.g. base-change for $GL(n)$ and for $U(3)$). We do not use the Theorem of [F4] here. In fact, Proposition 25 gives a purely local proof of this claim for a sufficiently wide class of spherical functions. Recently Clozel proved the statement claimed in [F4] using his technique of spherical functions (used in [AC] in the case of base-change for $GL(n)$). Labesse also proved that statement, using the technique of Iwahori functions, as in [F4], which he developed independently of [F4].

Finally, we briefly state here the global and local base-change results of Theorems 27 and 32 below. The local and complete global results had been proven by Arthur-Clozel [AC]. Let $E/F$ be a cyclic extension of global fields, of any characteristic. An irreducible $G(A)$-module $\pi$ is the product $\otimes \pi_v$ of local $G_v$-modules $\pi_v$ which are unramified for almost all $v$, where $G_v = G(F_v)$. For $G = GL(n)$, an unramified $\pi_v$ is naturally associated with an unordered $n$-tuple $t = (t_1, \ldots, t_n)$ of non-zero complex numbers, and we write $\pi_v(t)$ for $\pi_v$. If $v$ is a place of $F$ such that $E_v = E \otimes_F F_v$ is an unramified cyclic extension of $F_v$ then we say that the unramified $G_v$-module $\pi_v(t)$ lifts to the unramified $\tilde{G}_v$-module $\tilde{\pi}_v(t')$ if $t' = t^e$, where $\tilde{G}_v = G(E_v)$ and $e = [E_v : F_v]$. If $v$ splits completely in $E$ then we say that the $G_v$-module $\pi_v$ lifts to the $G_v$-module $\tilde{\pi}_v = \pi_1 \otimes \ldots \otimes \pi_e$ if $\pi_i = \pi_v$ for all $i(1 \leq i \leq e)$; here $E_v = F_v \oplus \ldots \oplus F_v$ and $\tilde{G}_v = G_v \times \ldots \times G_v$. If $v$ is a place of $F$ which is unramified in $E$ then the notion of lifting of unramified $G_v$-modules is the obvious combination of the definitions in the totally split and inert unramified cases. Denote by $\sigma$ a generator of the galois group $Gal(E/F)$. A representation $\tilde{\pi}$ of $G(A_E)$ or $\tilde{G}_v$ is called $\sigma$-invariant if $\pi \simeq \sigma \pi$ and $\sigma \pi(g) = \pi(\sigma g)$. An irreducible $G(A)$-module $\pi = \otimes \pi_v$ lifts to an irreducible $\tilde{G}(A) = G(A_E)$-module $\tilde{\pi} = \otimes \tilde{\pi}_v$ (product over all places of $F$) if $\pi_v$ lifts to $\tilde{\pi}_v$ for almost all $v$ (where $\pi_v$ and $\tilde{\pi}_v$ are unramified). Theorem 27 below asserts (cf. [AC]):

**GLOBAL BASE-CHANGE THEOREM FOR $G = GL(n)$.** — Let $w$ be a place of $F$ which splits completely in $E$. Then every cuspidal $G$-module $\pi$ whose component at $w$ is supercuspidal lifts to a unique cuspidal $\tilde{G}$-module $\tilde{\pi}$ which is $\sigma$-invariant and has a supercuspidal component at $w$. Moreover, every $\sigma$-invariant cuspidal $\tilde{G}$-module $\tilde{\pi}$ whose component at
$w$ is supercuspidal if a lift of a cuspidal $G$-module $\pi$, and every other cuspidal $G$-module which lifts to $\tilde{\pi}$ is of the form $\pi \otimes \epsilon$, where $\epsilon$ is a character of $A^\times/F^\times N A_E^\times$; $\pi \otimes \epsilon \not\cong \pi$ if $\epsilon$ is non-trivial.

To state the local result we define, after Lemma 20, the notion of matching functions $f_v$ on $G_v$ and $\tilde{f}_v$ on $\tilde{G}_v$, and say, below Theorem 27, that the irreducible $G_v$-module $\pi_v$ lifts to the irreducible $\sigma$-invariant $\tilde{G}_v$-module $\tilde{\pi}_v$ if $tr\pi_v(f_v) = tr\tilde{\pi}_v(\tilde{f}_v \times \sigma)$ for (a good definition of $\tilde{\pi}_v(\sigma)$ and) all matching $f_v$ on $G_v$ and $\tilde{f}_v$ on $\tilde{G}_v$. Theorem 32 asserts (cf. [AC]):

**Local base-change Theorem for $G = GL(n)$**. — The lifting defines a surjection from the set of equivalence classes of irreducible tempered $G_v$-modules $\pi_v$ to the set of equivalence classes of irreducible tempered $\sigma$-invariant $\tilde{G}_v$-modules $\tilde{\pi}_v$. It maps discrete $G_v$-modules to $\sigma$-discrete $\tilde{G}_v$-modules ($\sigma$-discrete $\tilde{G}_v$-modules are defined prior to Lemma 28).

The lifting commutes with induction and so this local theorem extends to the set of relevant (a notion introduced in [FK]; every component of a cuspidal $G$-module is relevant), not only tempered, $G_v$-modules. It is clear from our proofs that if the cuspidal $\pi$ lifts to the cuspidal $\tilde{\pi}$ then $\pi_v$ lifts to $\tilde{\pi}_v$ for all $v$. Also it is clear that the global theorem holds with any, not necessarily totally split, place $w$, and then it relates $\sigma$-invariant cuspidal $\tilde{\pi}$ with a supercuspidal component at $w$, with cuspidal $\pi$ whose component $\pi_w$ at $w$ lifts to a supercuspidal $\tilde{G}_w$-module. Such $\pi_w$ are obviously supercuspidal. The fibers of the lifting are easily described.

It should also be noted that the base-change theorems are perhaps easier than the metaplectic and simple algebra correspondences. The two groups under comparison admit (as was shown by Arthur-Clozel [AC], Thm III, 3.1; their statement is reproduced as Cor. 6', in [F2], p. 760) a-priori rigidity and multiplicity one theorems by virtue of Proposition 3.6 of [JS] (see the proof of Theorem 27). Consequently to establish our theorems one does not need to know a-priori that given a general $\tilde{f}_v$ (resp. $f_v$ with suitable orbital integrals) there exists a matching $f_v$ (resp. $\tilde{f}_v$). This statement can be deduced from the trace Paley-Wiener theorem of [BDK] (and [F1; I, §7] in the twisted case), and the local base-change theorem, or even purely locally on using the results of [K], which relate orbital integrals with Fourier transforms of invariant distributions on nilpotent orbits in the Lie algebra of $G_v$ (if the characteristic is zero). To prove the metaplectic
and simple-algebra correspondences one gives (in [FK], and [BDKV] or [F1; III]) a local proof that matching functions exist, and one concludes the rigidity and multiplicity one theorems for the metaplectic covers and inner forms of $GL(n)$ from the comparisons of trace formulae and the analogous results for $GL(n)$ itself.

The global base-change theorem for $GL(n)$ had been proven in Arthur-Clozel [AC] in the greater generality of all automorphic $GL(n)$-modules, with no restriction at the place $w$, in the case of characteristic zero. They also prove the local theorem and the global theorem for $\pi$ with two square-integrable components (one of which is supercuspidal) by means of the simple trace formula of Deligne-Kazhdan. In this paper we obtain in particular the strongest global result presently obtainable by elementary means (without using the theory of Eisenstein series, invariant trace formula, correction argument of [F3], cancellation of singularities, orbital integrals of singular classes, etc., which are used in [AC] in a crucial way to deal with general $\pi$). We also derive the transfer of orbital integrals from the local lifting, rather than use it in the proof.

It will be interesting (and I believe possible) to establish the complete global base-change theorem for $GL(n)$ by means of a simple proof.

1. Regular trace formula.

Let $F$ be a global field, $\mathcal{A}$ its ring of adeles and $\mathcal{A}_f$ the ring of finite adeles, $G$ a connected reductive algebraic group over $F$ with center $Z$. The group $G$ of $F$-rational point on $G$ is discrete in the adele group $G(\mathcal{A})$ of $G$. Put $G' = G/Z$ and $G'(\mathcal{A}) = G(\mathcal{A})/Z(\mathcal{A})$. The quotient $G'\backslash G'(\mathcal{A})$ has finite volume with respect to the unique (up to a scalar multiple) Haar measure $dg$ on $G'(\mathcal{A})$. Fix a unitary complex-valued character $\omega$ of $Z\backslash Z(\mathcal{A})$. For any place $v$ of $F$ let $F_v$ be a completion of $F$ at $v$, and $G_v = G(F_v)$ the group of $F_v$-points on $G$. If $F_v$ is non-archimedean, let $R_v$ denote its ring of integers. For almost all $v$ the group $G_v$ is defined over $R_v$, quasi-split over $F_v$, split over an unramified extension of $F_v$, and $K_v = G(R_v)$ is a maximal compact subgroup. For an infinite set of places (of positive density) $u$ of $F$, the group $G_u$ is split (over $F_u$). A fundamental system of open neighborhoods of 1 in $G(\mathcal{A})$ consists of the set of $\prod_{v \in V} H_v \times \prod_{v \not\in V} K_v$, where $V$ is a finite set of places of $F$ and $H_v$ is an open subset of $G_v$, containing 1.
Let $L(G)$ denote the space of all complex-valued functions $\phi$ on $G \backslash G(\mathbb{A})$ which satisfy (1) $\phi(zg) = \omega(z)\phi(g)$ ( $z$ in $Z(\mathbb{A})$, $g$ in $G(\mathbb{A})$), (2) $\phi$ is absolutely square-integrable on $G' \backslash G'(\mathbb{A})$. $G(\mathbb{A})$ acts on $L(G)$ by right translation: $(r(g)\phi)(h) = \phi(hg)$; $L(G)$ is unitary since $\omega$ is unitary. The function $\phi$ in $L(G)$ is called cuspidal if for each proper parabolic subgroup $P$ of $G$ over $F$ with unipotent radical $N$ we have $\int \phi(ng)dn = 0$ ( $n$ in $N \backslash N(\mathbb{A})$) for any $g$ in $G(\mathbb{A})$. Let $L_0(G)$ denote the space of cuspidal functions in $L(G)$, and $r_0$ the restriction of $r$ to $L_0(G)$. The space $L_0(G)$ decomposes as a direct sum with finite multiplicities of invariant irreducible unitary $G(\mathbb{A})$-modules called cuspidal $G$-modules.

Let $f$ be a complex-valued function on $G(\mathbb{A})$ with $f(g) = \omega(z)f(zg)$ for $z$ in $Z(\mathbb{A})$, which is supported on the product of $Z(\mathbb{A})$ and a compact open neighborhood of 1 in $G(\mathbb{A})$, smooth as a function on the archimedean part $G(\mathbb{A}_\infty)$ of $G(\mathbb{A})$, and bi-invariant by an open compact subgroup of $G(\mathbb{A}_f)$. Fix Haar measures $dg_v$ on $G'_v = G_v/Z_v$ for all $v$, such that the product of the volumes $|K_v/Z_v \cap K_v|$ converges. Then $dg = \otimes dg_v$ is a measure on $G'(\mathbb{A})$. The convolution operator $r_0(f) = \int_{G'(\mathbb{A})} f(g)r_0(g)dg$ is of trace class; its trace is denoted by $\text{tr} r_0(f)$. Then

\[ \text{tr} r_0(f) = \sum' m(\pi) \text{tr} \pi(f), \]

where $\sum'$ indicates the sum over all equivalence classes of cuspidal representations $\pi$ of $G(\mathbb{A})$, and $m(\pi)$ denotes the multiplicity of $\pi$ in $L_0(G)$; each $\pi$ here is unitary, and the sum is absolutely convergent.

The Selberg trace formula is an alternative expression for (1). To introduce it we recall the following

DEFINITIONS. — Denote by $Z_\gamma(H)$ the centralizer of an element $\gamma$ in a group $H$. A semi-simple element $\gamma$ of $G$ is called elliptic if $Z_\gamma(G'(\mathbb{A}))/Z_\gamma(G')$ has finite volume.

It is called regular if $Z_\gamma(G'(\mathbb{A}))$ is a torus, and singular otherwise. Let $\gamma$ be an elliptic element of $G$. The orbital integral of $f$ at $\gamma$ is defined to be

\[ \Phi(\gamma, f) = \int_{G'(\mathbb{A})/Z_\gamma(G')} f(\gamma^{-1}g)dg. \]

Similarly, for any place $v$ of $F$ the element $\gamma$ of $G_v$ is called elliptic if $Z_\gamma(G'_v)$ has finite volume, and regular if $Z_\gamma(G_v)$ is a torus. If $\gamma$ is an element of $G$ and there is a place $v$ of $F$ such that $\gamma$ is elliptic (resp. regular) in $G_v$, then $\gamma$ is elliptic (resp. regular). The orbital integral of
$f_v$ at $\gamma$ in $G_v$ is defined to be

$$\Phi(\gamma, f_v) = \Phi(\gamma, f_v; d_\gamma) = \int_{G'_v/\mathbb{Z}(G'_v)} f_v(g\gamma g^{-1}) dg.$$ 

It depends on the choice of a Haar measure $d_\gamma$ on $\mathbb{Z}(G'_v)$.

Let $\{\phi_\alpha\}$ be an orthonormal basis for the space $L_0(G)$. The operator $r_0(f)$ is an integral operator on $G'(A)$ with kernel $K_0^r(x, y) = \sum_{\alpha, \beta} r(f) \phi_\alpha(x) \overline{\phi_\beta(y)}$. The operator $r(f)$ is an integral operator on $G'(A)$ with kernel $K_f(x, y) = \sum_{\gamma} f(x^{-1}\gamma y)$ ($\gamma$ in $G'$). If $G$ is anisotropic (namely $G'/G'(A)$ is compact), then $L_0(G) = L(G)$ and $r = r_0$. Since $K_f^0(x, y) = K_f(x, y)$ is smooth in both $x$ and $y$, we integrate over the diagonal $x = y$ in $G'(A)$, change the order of summation and integration as usual, and obtain the Selberg trace formula in the case of compact quotient, as follows.

1. **Proposition.** — If $G$ is anisotropic then for every function $f$ on $G(A)$ as above we have

$$\sum\prime m(\pi) \text{ tr } \pi(f) = \sum_{\{v\}} \Phi(y, f).$$

The sum on the left is the same as in (1). The sum on the right is finite; it ranges over the conjugacy classes of elements in $G'$.

Remark. — If $G$ is anisotropic, then each element $y$ in $G$ is elliptic.

For a general group $G$ we introduce the following

**Definition.** — The function $f$ is called discrete if for every $x$ in $G(A)$ and $y$ in $G$ we have $f(x^{-1}yx) = 0$ unless $y$ is elliptic regular.

Changing again the order of summation and integration as usual we obtain the

2. **Proposition.** — If $f$ is discrete, then

$$\int_{G'(A)} \left[ \sum_{y \in G'} f(x^{-1}yx) \right] dx = \sum_{\{v\}} \Phi(y, f).$$

The sum on the right is finite. It ranges over the set of conjugacy classes of elliptic regular elements in $G'$.
Remark. — It is well known that the sum on the right is finite; for a proof see [FK], §18 (if $G = GL(n)$), and [F1], Prop. I.3 (in general).

DEFINITION. — The function $f$ is called cuspidal if for every $x, y$ in $G(A)$ and every proper $F$-parabolic subgroup $P$ of $G$, we have $\int_{N(A)} f(xny)dn = 0$, where $N$ is the unipotent radical of $P$.

Let $A^u$ be the ring of $F$-adeles without component at $u$. Put $G^u = G(A^u)$. Write $f = f uf_u$ if $f$ is a function on $G(A)$, $f_u$ on $G_u$, $f^u$ on $G^u$, and $f(x,y) = f_u(x)f^u(y)$ for $x$ in $G_u$ and $y$ in $G^u$. We say that $f^u$ is cuspidal if for every $x, y$ in $G^u$, and $P$ as above, we have $\int_{N(A^u)} f^u(xny)dn = 0$.

If $f^u$ is cuspidal then $f = f^uf_u$ is cuspidal, for any $f_u$.

When $f$ is cuspidal, the convolution operator $r(f)$ factorizes through the projection on $L_0(G)$, $r(f)$ is of trace class, $tr \ r_0(f) = tr \ r(f)$ and $K_f(x,y) = K_f^0(x,y)$, and we obtain

3. COROLLARY. — If $f$ is cuspidal and discrete, then the equality (2) holds. The sum on the left is as in (1). The sum on the right is as in (3).

We shall now construct a useful set of discrete functions. Fix a non-archimedean place $u$ of $F$ such that $G_u$ is split, and the component $\omega_u$ of $\omega$ at $u$ is unramified, namely trivial on the multiplicative group $R_u^\times$ of $R_u$. Put $\nu(x) = |x|$ for $x$ in $A^u / F^u \times$. Multiply $\omega$ by a power of $\nu$ to assume that $\omega_u = 1$. Fix a minimal parabolic subgroup $B_u = A_u U_u$ of $G_u$, where $U_u$ is the unipotent radical of $B_u$ and the Levi subgroup $A_u$ is a maximal (split) torus. Let $X^* = X^*(A_u)$ be the lattice of rational characters on $A_u$, and let $X_* = X^*(A_u)$ be the dual lattice. Let $A_u^0$ be the maximal compact subgroup of $A_u$; then $X_* \simeq A_u / A_u^0$. If $x$ is an element of $X_*$, denote by $a(x)$ an element of $A_u$ which maps to $x$ under this isomorphism. Let $val_u : F^u_\times \to \mathbb{Z}$ be the normalized additive valuation.

DEFINITION. — Consider $x$ in $X_*$ with $val_u(\alpha(a(x))) \neq 0$ for each root $\alpha$ of $A_u$ on $N_u$. A complex-valued locally-constant function $f_u$ on $G_u$ with $f_u(zg) = f_u(g)$ for all $g$ in $G_u$ and $z$ in the center $Z_u$ of $G_u$ which is compactly supported module $Z_u$ is called $x$-regular if $f_u(g)$ is zero unless there is $z$ in $Z_u$ such that $zg$ is conjugate to an element $a$ in $A_u$ whose image in $X_*$ is $x$, in which case the normalized orbital
integral $F(g, f_u) = \Delta_u(g)\Phi(g, f_u)$ is equal to one. An $x$-regular function $f_u$ will be denoted by $f_x$. A regular function is linear combination with complex coefficients of $x$-regular functions.

By definition, a regular function is zero on the non-regular set.

For any maximal (proper) $F_u$-parabolic subgroup $P_u = M_uN_u$ of $G_u$, where $N_u$ is the unipotent radical of $P_u$ and $M_u$ is the Levi subgroup containing $A_u$, let $\alpha_{P_u}$ denote the $F_u^x$-valued character on $M_u$ defined by $\alpha_{P_u}(m) = \det[\text{ad}(m) \mid L(N_u)]$. Here $L(N_u)$ denotes the Lie algebra of $N_u$, and $\text{ad}(m) \mid L(N_u)$ the adjoint action of $m$ in $M_u$ on $L(N_u)$. For any non-negative integer $n$ let $A_u^{(n)}$ be the set of $a$ in $A_u$ such that $|\text{val}_a(\alpha_{P_u}(a))| < n$ for some $P_u$.

**Definition.** — A regular function $f_u$ is called $n$-admissible if the orbital integral $\Phi(a, f_u)$ is zero for every $a$ in $A_u^{(n)}$.

**4. Theorem.** — Let $f^u$ be a function on $G^u$ which is compactly supported modulo $Z^u = Z(A^u)$. Then there exists a positive integer $n_0 = n_0(f^u)$ such that for any $n_0$-admissible regular function $f_u$ the function $f = f_u^u$ is discrete.

**Proof.** — For every maximal $F$-parabolic subgroup $P$ of $G$ and every place $v \neq u$ of $F$ there exists a non-negative integer $C_{v, P}$ which depends on $f^u$, with $C_{v, P} = 0$ for almost all $v$, such that if $\gamma$ lies in a Levi subgroup $M$ of $P$ and $f^u(x^{-1}\gamma x) \neq 0$ for some $x$ in $G^u$, then

$$|\text{val}_v(\alpha_P(\gamma))| \leq C_{v, P}.$$  

Put $C_{u, P} = \sum_{v \neq u} C_{v, P}$. Since $\gamma$ is rational (in $G$), the product formula

$$\sum_v \text{val}_v(\alpha_P(\gamma)) = 0$$  

on $F^x$ implies that the inequality (4) remains valid also for $v = u$. Then the theorem follows for $n_0 > C_{u, P}$ for all of the finitely many conjugacy classes of the subgroups $P$.

**5. Corollary.** — Suppose that $f = f_u^u$ is a cuspidal function, where $f_u$ is a regular $n_0$-admissible function with $n_0 = n_0(f^u)$. Then the equality (2) holds, where the sum on the left is as in (1), while the sum on the right is as in (3).

**Definition.** — (1) A complex-valued compactly supported function $f_u$ on $G_u/Z_u$ is called spherical if it is $K_u$-biinvariant. (2) Let $H_u$
be the convolution algebra of such functions. (3) A spherical function $f_u$ is called $n$-admissible if the orbital integral $\Phi(a, f_u)$ is zero for every regular $a$ in $A_u^{(n)}$.

For applications, we need to replace the regular $n_0$-admissible component $f_u$ in the Corollary by a spherical $n_0$-admissible component. For this purpose, we recall basic results concerning spherical and regular functions. Let $T = X^*(C)$ denote the complex torus $Hom(X_*, C^\times)$. The Weyl group $W$ of $A_u$ in $G_u$ acts on $A_u, X^*, X_*$, and $T$. Each $t$ in $T$ defines a unique $C^\times$-valued character of $B_u$ which is trivial on $N_u$ and on $A_u^0$. The $G_u$-module $I(t) = Ind(e_u^{1/2}t; B_u, G_u)$ normalizedly induced from the character $t$ of $B_u$ is unramified and has a unique unramified irreducible constituent $\pi(t)$. We have $\pi(t) \simeq \pi(t')$ if and only if $t' = wt$ for some $w$ in $W$. The map $t \rightarrow \pi(t)$ is a bijection from the variety $T/W$ to the set of unramified irreducible $G_u$-modules. Put $t(\pi)$ for the $t$ associated with such a $\pi$.

If $f_u$ is a spherical function then the value of the normalized orbital integral $F(a, f_u) = \Delta_u(a)\Phi(a, f_u)$ at a regular $a$ in $A_u$ depends only on the $W$-orbit of the image $x$ of $a$ in $X_*$; it is denoted by $F(x, f_u)$. Let $C[X_*]^W$ be the algebra of $W$-invariant elements in the group ring $C[X_*]$. The Satake transform $f_u \rightarrow f_u^- = \sum_{x \in X_*} F(x, f_u)x$ defines an algebra isomorphism from the convolution algebra $H_u$ of spherical functions, to $C[X_*]^W$. For each $x$ in $X_*$, let $f(x)$ be the element of $H_u$ with $f(x) = \sum_{w \in W} wx$. Then $f(x)$ is $n_0$-admissible if $|val_u(a_P(u)(w(a(x))))| \geq n_0$ for every $w$ in $W$ and parabolic subgroup $P_u$ containing $A_u$; as above, $a(x)$ is an element of $A_u$ which maps to $x$ in $X_*$ under the isomorphism $A_u/A_u^0 \simeq X_*$. A simple computation of the character of an induced representation yields the following

6. LEMMA. — For every $f_u$ in $H_u$ and $t$ in $T$ we have $tr(\pi(t))(f_u) = tr(I(t))(f_u) = f_u^-(t)$, where $f_u^-(t) = \sum_{x \in X_*} F(x, f_u)t(x)$.

To study regular functions, note that the normalized $N_u$-homology module $\pi_{N_*}$ (= module of $N_u$-coinvariants in $[BZ]$) of an admissible $G_u$-module $\pi_u$ with trivial central character, is an $A_u$-module; its character is denoted by $\chi(\pi_{N_*})$. A simple application of the Weyl integration formula and the theorem of Deligne-Casselman [CD] implies the following
7. **Lemma.** — If $f_\chi$ is an $\chi$-regular function, then
\[ tr \pi_u(f_\chi) = [W]^{-1} \int_{A_u/Z_u} (\Delta X(\pi_N))(a) F(a, f_\chi) da. \]
Moreover, if $tr \pi_u(f_\chi)$ is non-zero, then there exists (i) $t$ in $T$ such that $\pi_u$ is a constituent of $I(t)$, and (ii) a subset $W(\pi_u, t)$ of $W$ such that
\[ tr \pi_u(f_\chi) = \sum_w t(wx) \quad (w \text{ in } W(\pi_u, t)). \]
Finally, each constituent of $I(t)$, including $\pi_u$, has a non-zero vector fixed by the action of an Iwahori subgroup.

**Proof.** — (i) follows from Frobenius reciprocity. The final claim is proven in [B], (4.7), in the case of a reductive group, and in [FK], §17, in the case of the metaplectic group considered in [FK].

**Definition.** — (1) The spherical function $f_u$ and the regular function $\phi_u$ are called companions if $f_u = \sum c(x, f_u)f(x)$ and $\phi_u = \sum c(x, f_u)f_\chi(x \text{ in } X_u/W)$.

(2) A spherical, or regular, function $f_u$, is called solid if $tr \pi_u(f_u) = 0$ for every irreducible $G_u$-module $\pi_u$ which is not equivalent to some $I(t)$.

8. **Lemma.** — Suppose that the spherical $f_u$ and the regular $\phi_u$ are solid companions. Then (i) $tr \pi_u(f_u) = tr \pi_u(\phi_u)$ for every irreducible $G_u$-module $\pi_u$, (ii) $\Phi(g, f_u) = \Phi(g, \phi_u)$ for every $g$ in $G_u$.

**Proof.** — (i) follows from Lemmas 2 and 3, which imply that $tr(I(t))(f_u) = tr(I(t))(\phi_u)$ for all $t$; (ii) follows from (i) by the density theorem of [K; Appendix].

9. **Proposition.** — Let $f^u$ be a cuspidal function on $G^u$ which is compactly supported modulo $Z^u$. Then there exists a positive integer $n_0 = n_0(f^u)$ with the following property. For any $n_0$-admissible solid spherical function $f_u$ which has a regular solid companion $\phi_u$, the equality (2) of Proposition 1 holds with the function $f = f_u f^u$, where the sum on the left is as in (1) while the sum on the right is as in (3) (of Proposition 2).

This follows at once from Theorem 4, Lemma 8, and Corollary 5.

We shall use Proposition 9 to study lifting problems related to the group $GL(n)$. For this purpose we now construct solid companions for this
group, using results of [BZ]. This construction can be generalized to the case of any reductive $G$ on using the results of [KL], but we do not do it here. For $G_u = GL(n, F_u)$ the torus $T$ is isomorphic to $C^n$, and $I(t)$ has a trivial central character if $t_1 \ldots t_n = 1$ where $t = (t_1, \ldots, t_n)$. For any $i, j (1 \leq i \neq j \leq n)$, put $\alpha_{i,j}(t) = t_i/t_j$. The result of [BZ] which we need is the following

10. Lemma. — The $G_u$-module $I(t)$ is irreducible if and only if $\alpha_{i,j}(t) \neq q_u$ for all $i \neq j$.

For $G_u = GL(n, F_u)$, $X_\ast$ is isomorphic to $Z^n$; if $t = (t_1, \ldots, t_n)$ lies in $T$ and $x = (x_1, \ldots, x_n)$ in $X_\ast$, then $t(x) = \prod_{i=1}^n t_i^{x_i}$. Let $\alpha_{ij}$ be the element $x$ of $X_\ast$ with $x_k = 0$ for $k \neq i, j$; $x_i = 1$ and $x_j = -1$. For any $x$ in $X_\ast$ we have

$$(q \alpha_{ij}(t) - (q^2 + 1) + q \alpha_{ji}(t)) t(x) = qt(x + \alpha_{ij}) - (q^2 + 1)t(x) + qt(x - \alpha_{ij}).$$

Consequently, we have the following

11. Lemma. — There exist finitely many $x_r$ in $X_\ast$ and integers $c_r$ such that for every $t$ in $T$ and $x$ in $X_\ast$ we have

$$\sum_r c_r t(x + x_r) = \left[ \prod_{i<j} (q \alpha_{ij}(t) - (q^2 + 1) + q \alpha_{ji}(t)) \right] t(x).$$

Proof. — This is obvious on working in the integral group ring $\bar{X} = Z[X_\ast]$ of $X_\ast$. In more detail, let $\bar{X}$ be the quotient of the free abelian ring generated by the objects $a \ast x$ with $a$ in $Z$ and $x$ in $X_\ast$, by the relations $(a \ast x) \cdot (b \ast y) = (ab) \ast (x+y)$ and $(a+b) \ast x = a \ast x + b \ast x$. Then

$$\prod_{i<j} (q \ast \alpha_{ij} - (q^2 + 1) \ast 0 + q \ast \alpha_{ji}) = \sum_r c_r \ast x_r$$

in $\bar{X}$. Given $t$ in $T = Hom(X_\ast, C^\times)$, define $t$ in $Hom(\bar{X}, C)$ by $t(a \ast x + b \ast y) = at(x) + bt(y)$. Then

$$\sum_r c_r t(x + x_r) = t(\sum_r (c_r \ast x_r) \cdot x)$$

$$= t\left( \prod_{i<j} (q \ast \alpha_{ij} - (q^2 + 1) \ast 0 + q \ast \alpha_{ji}) \right) \cdot x$$

$$= \left[ \prod_{i<j} (q \alpha_{ij}(t) - (q^2 + 1) + q \alpha_{ji}(t)) \right] t(x),$$
as required.

Put \( L(f_x) = \sum_r c_r f_{x + x_r} \) and \( L(f(x)) = \sum_r c_r f(x + x_r) \).

12. PROPOSITION. — Suppose that \( f_u = \sum x c(x) f(x) \) is a spherical function and \( \phi_u = \sum x b(x) f_x \) is a regular function such that \( L(f_x) \) is regular if \( b(x) \neq 0 \), and \( b(x) = c(x) \) for every \( x \in X_u \). Then the spherical function \( L(f_u) = \sum x c(x) L(f(x)) \) and the regular function \( L(\phi_u) = \sum x c(x) L(f_x) \) are solid companions.

Proof. — This follows at once from Lemmas 10 and 11.

13. COROLLARY. — Let \( f^u \) be a cuspidal function on \( G^u \) which is compactly supported modulo \( Z^u \). Then there exists a positive integer \( n_0 = n_0(f^u) \) such that for every \( n_0 \)-admissible spherical function \( f_u \), putting \( f = f^u L(f_u) \) we have that \( \sum \Phi(\gamma, f) \) (sum as in (3)) is equal to

\[
\sum \pi' \text{tr} \pi(f) = \sum \pi' b(\pi_u) \text{tr} \pi(f^u f_u),
\]

where

\[
b(\pi_u) = \prod_{i < j} (q_u \alpha_{ij}(t(\pi_u)) - (q_u^2 + 1) + q_u \alpha_{ji}(t(\pi_u))).
\]

The sum \( \sum' \) ranges over the equivalence classes of cuspidal \( G \)-modules \( \pi \) whose component \( \pi_u \) is unramified, equivalent to the irreducible unramified \( G_u \)-module \( I(t) \) with \( t = t(\pi_u) \).

This follows at once from Lemma 11 and Proposition 12. Since \( G = GL(n) \) we have that the \( m(\pi) \) of (1) are equal to one by multiplicity one theorem for the cuspidal spectrum of \( GL(n) \).

Remark. — We refer to the identity (2) for the function \( f \) of Corollary 5, and in particular for the function \( f \) of Proposition 9, as the regular trace formula. Corollary 13 is an explicit form of the regular trace formula in the case of \( G = GL(n) \).

The regular trace formula is a representation theoretic analogue of Deligne's conjecture on the Lefschetz fixed point formula for the trace
of a finite flat correspondence on a separated scheme of finite type over a finite field, which is multiplied by a sufficiently high power of the Frobenius morphism. Its main application is indeed (1) a comparison with the Grothendieck fixed point formula, to deduce the purity theorem (or the Ramanujan conjecture) for cuspidal $GL(n)$-modules, over a function field, which have a supercuspidal component, and (2) a comparison with Deligne's conjectural fixed point formula, to deduce Drinfeld's reciprocity law concerning cuspidal $GL(n)$-modules and irreducible $n$-dimensional $\ell$-adic representations of the Weil group of a function field; for details see our joint work [FK2] with D. Kazhdan.

Here we use the regular trace formula to prove the base-change lifting theorem for cuspidal $GL(n)$-modules with a supercuspidal component. In [FK1] we already gave a simple proof of the form of the trace formula, analogous to Deligne's conjecture, for a test function $f$ with a supercusp component and a sufficiently admissible spherical component. There we used a regular component at a third place to annihilate the singular orbital integrals in the trace formula. This simple trace formula is used in [FK1] to extend the results of [FK] and [BDKV] or [F; III] to the context of cuspidal modules with a supercuspidal component. The metaplectic correspondence of [FK] and the simple-algebra correspondence of [BDKV] or [F; III] are more difficult to prove than the base-change lifting for $GL(n)$, and it is easy to see that the methods of [FK1] apply to establish this lifting in the same context of cuspidal modules with a supercuspidal component.

The proof of the base-change lifting for $GL(n)$ given here is even simpler. The usage of regular functions, for which the transfer of orbital integrals is trivial by definition, eliminates the need to compare explicitly orbital integrals of corresponding spherical functions for the two groups in question, other than the characteristic functions of the maximal compact subgroups of these groups. This comparison of orbital integrals of spherical functions, also referred to as the "fundamental lemma", is not required in our proof of the base-change lifting. But, as the referee pointed out, this transfer is implicit in our arguments, at least for a sufficiently large class of spherical functions.

The regular trace formula of the present work suffices to obtain the local results of [FK] and [BDKV] or [F; III]. However, the passage (in Proposition 12) from regular to spherical function annihilates the constituents of any induced $G_u$-module $I(t)$ which is reducible. Consequently using the present regular trace formula we can obtain the global results of [FK1]
only for cuspidal representations \( \pi \) of the metaplectic group or the multiplicative group of a simple algebra which satisfy the unnatural condition that they have an unramified component \( \pi_u \) which is equivalent to an irreducible \( I(t) \). This condition is automatically held in the case of \( GL(n) \), since by [Z], Theorem 9.7, any component of a cuspidal \( GL(n) \)-module is (non-degenerate hence) of the form \( I(t) \) if it is unramified.

We hope that sufficiently admissible solid companions regular and spherical functions can be used to prove a regular trace formula as in Corollary 13 for \( f \) which is not necessarily cuspidal, but have not done this as yet.

To study the applications of the regular trace formula to the base-change lifting we establish a twisted analogue of the regular trace formula. This is our next aim.

2. Base Change Lifting.

We shall now develop a twisted analogue of the regular trace formula of Chapter I, using the same notations. Thus \( G \) is the group of \( F \)-rational points of a connected reductive group \( G \) over a global field \( F \). Let \( \sigma \) be an automorphism of \( G \) (over \( F \)) of finite order \( e \). Suppose that the character \( \omega \) of \( Z(A)/Z \) satisfies \( \omega(\sigma(z)) = \omega(z) \) for all \( z \). Then (the group generated by) \( \sigma \) acts on \( L_0(G) \) (and \( L(G) \)) by \( (r_0(\sigma)\phi)(g) = \phi(\sigma(g)) \) (resp. \( (r(\sigma))\phi(g) = \phi(\sigma(g)) \)). An irreducible representation \( \pi \) of \( G(A) \) is called \( \sigma \)-invariant if it is equivalent to \( \sigma \pi \), where \( \sigma \pi(g) = \pi(\sigma(g)) \) (the same definition applies to a \( G_v \)-module \( \pi_v \), and \( \pi = \otimes \pi_v \) is \( \sigma \)-invariant if and only if \( \pi_v \) is \( \sigma \)-invariant for all \( v \)). The trace of the operator \( r_0(f \times \sigma) = r_0(f)r_0(\sigma) \) is given by

\[
(5) \quad tr \ r_0(f \times \sigma) = \sum_{m(\pi)} tr \ \pi(f \times \sigma).
\]

Here \( \sum' \) ranges over all \( \sigma \)-invariant cuspidal representations \( \pi \) of \( G(A) \), \( m(\pi) \) is the multiplicity of \( \pi \) in \( L_0(G) \), and \( \pi(f \times \sigma) = \pi(f)\pi(\sigma) \) where \( \pi(\sigma) \) is the restriction of \( r_0(\sigma) \) to the space of \( \pi \). The twisted trace formula is an alternative expression for (5). We shall use the following

**Definitions.** — (1) An element \( \gamma \) of \( G \) (resp. \( G_v \)) is called \( \sigma \)-semi-simple if \( (\gamma \times \sigma)^e = \gamma \sigma(\gamma) \ldots \sigma^{e-1}(\gamma) \times 1 \) is semi-simple as an element of the connected component \( G \) (resp. \( G_v \)) of the identity in the group \( G \times (\sigma) \) (resp. \( G_v \times (\sigma) \)). (2) Denote by \( Z_{\gamma \times \sigma}(H) \) the centralizer of \( \gamma \times \sigma \)
in a group $H(= G'_v, G'$ or $G'(A)$). (3) A $\sigma$-semi-simple element $\gamma$ in $G$ (resp. $G_v$) is called $\sigma$-elliptic if $Z_{\gamma \times \sigma}(G'(A))/Z_{\gamma \times \sigma}(G')$ (resp. $Z_{\gamma \times \sigma}(G'_v)$) has finite volume, and $\sigma$-regular if $Z_{\gamma \times \sigma}(G'(A))$ (resp. $Z_{\gamma \times \sigma}(G'_v)$) is a torus. (4) The $\sigma$-orbital integral of $f$ at a $\sigma$-elliptic $\gamma$ is defined to be

$$\Phi(\gamma \times \sigma, f) = \int_{G'(A)/Z_{\gamma \times \sigma}(G')} f(g \cdot \gamma \cdot \sigma(g^{-1})) dg .$$

The $\sigma$-orbital integral of $f_v$ at a $\sigma$-elliptic $\gamma$ in $G_v$ is defined to be

$$\Phi(\gamma \times \sigma, f_v) = \int_{G'_v/Z_{\gamma \times \sigma}(G'_v)} f_v(g \cdot \gamma \cdot \sigma(g^{-1})) \frac{dg}{d_\gamma} ;$$

it depends on a choice of a Haar measure $d_\gamma$ on $Z_{\gamma \times \sigma}(G'_v)$.

Let $\{\phi_\alpha\}$ be an orthonormal basis for the space $L_0(G)$. The operator $r_0(f \times \sigma)$ is an integral operator on $G'(A)$ with kernel

$$K^0_{f \times \sigma}(x,y) = \sum_{\alpha,\beta} (r(f \times \sigma)\phi_\alpha)(x)\overline{\phi_\beta}(y) .$$

The operator $r(f \times \sigma)$ is an integral operator on $G'(A)$ with kernel

$$K_{f \times \sigma}(x,y) = \sum_\gamma f(x^{-1} \cdot \gamma \cdot \sigma(y)) \quad (\gamma \in G') .$$

If $G$ is anisotropic then $L_0(G) = L(G)$ and $r = r_0$. Since $K^0_{f \times \sigma}(x,y) = K_{f \times \sigma}(x,y)$ is smooth in both $x$ and $y$, we integrate over the diagonal $x = y$ in $G'(A)$, change the order of summation and integration as usual, and obtain

14. **Proposition.** — If $G$ is anisotropic then

$$\sum' m(\pi) tr \pi(f \times \sigma) = \sum_{\{\gamma \times \sigma\}} \Phi(\gamma \times \sigma, f),$$

where the last sum ranges over a set of representatives $\gamma \times \sigma$ for the $\sigma$-conjugacy classes of elements in $G'$.

Here $\gamma$ and $\delta$ of $G'$ (resp. $G'_v$) are called $\sigma$-conjugate if there exists $x$ in $G$ (resp. $G_v$) such that $x\gamma = \delta \sigma(x)$ in $G'$ (resp. $G'_v$). The group $G$ may be any reductive connected $F$-group. Also we say that the function $f$ is $\sigma$-discrete if for every $x$ in $G(A)$ and $\gamma$ in $G$ we have that $f(x \cdot \gamma \cdot \sigma(x^{-1}))$ is zero unless $\gamma$ is $\sigma$-elliptic and $\sigma$-regular. Changing again the order of summation and integration we obtain

15. **Proposition.** — If $f$ is $\sigma$-discrete then

$$\int_{G'(A)} \left[ \sum_{\gamma \in G'} f(x^{-1} \gamma \sigma(x)) \right] dx = \sum_{\{\gamma \times \sigma\}} \Phi(\gamma \times \sigma, f) .$$
The sum on the right ranges over the set of $\sigma$-conjugacy classes of $\sigma$-elliptic $\sigma$-regular elements $\gamma$ in $G'$.

When $f$ is cuspidal the convolution operator $r(f)$ factorizes through the projection on $L_0(G)$, $\text{tr } r_0(f \times \sigma) = \text{tr } r(f \times \sigma)$ and $K_f^0(x, y) = K_{f \times \sigma}(x, y)$. Hence we obtain the following twisted trace formula:

16. COROLLARY. — If $f$ is cuspidal and $\sigma$-discrete then we have

$$
\sum' m(\pi) \text{tr } \pi(f \times \sigma) = \sum_{\{\gamma \times \sigma\}} \Phi(\gamma \times \sigma, f).
$$

On the left the sum is as in (5), while the sum on the right is as in (6).

In this paper we use the twisted trace formula in the special case of base-change. Thus let $E/F$ be a cyclic extension of global fields, and $G$ a reductive connected $F$-group. Let $G^E = \text{Res}_{E/F} G$ be the $F$-group obtained from $G$ by restriction of scalars from $E$ to $F$. Then $G^E(F) \simeq G(E)$, $G^E(A) \simeq G(A_E)$ where $A_E$ is the ring of $E$-adeles, and $G^E(F_v) \simeq G(E_v)$ for every place $v$ of $F$, where $E_v = E \otimes_F F_v$. Fix a generator $\sigma$ of $Gal(E/F)$. Then it acts on $G^E$, and Corollary 16 applies to a function $\tilde{f}$ on $G^E(A)$. We shall also denote $G^E(A)$-modules by $\tilde{\pi}$, the fixed character of the center $Z^E(A) \simeq Z(A_E)$ of $G^E(A)$ by $\tilde{\omega}$, and elements of $G^E$ by $\tilde{\gamma}$. To obtain a twisted trace formula analogous to that of Corollary 5, we fix a non-archimedean place $u$ of $F$ which splits completely in $E$ such that $G_u$ is split and $\tilde{\omega}_u$ is unramified. We may and do assume that $\tilde{\omega}_u = 1$. Then $G^E_u = G_u \times \ldots \times G_u$ ($e$ times), and correspondingly we write $\tilde{\gamma} = (\gamma_1, \ldots, \gamma_e)$ for its elements. Put $\gamma = N\tilde{\gamma} = \gamma_1 \sigma(\gamma_2) \ldots \sigma^{e-1}(\gamma_e)$, and $f_u = f_1 * f_2 * \ldots * f_e$ if $\tilde{f}_u = (f_1, \ldots, f_e)$ is a function on $G^E_u$. A simple computation (see, e.g., [F3], §1.5) shows that $\Phi(\gamma \times \sigma, \tilde{f}_u) = \Phi(N\tilde{\gamma}, f_u)$ for every $\tilde{\gamma}$ in $G^E_u$. We conclude the

17. PROPOSITION. — Suppose that $\tilde{f} = \tilde{f}_u \tilde{f}^u$ is a cuspidal function on $G(A_E)$, where $\tilde{f}_u = (f_u, f_u^0, \ldots, f_u^0)$, $f_u$ is a regular $n_0$-admissible function with $n_0 = n_0(f^u)$, and $f_u^0$ is the characteristic function of an open compact modulo center subgroup of $K_u$ (multiplied by a suitable scalar) such that $f_u = f_u * f_u^0$. Then (7) holds with $f$ replaced by $\tilde{f}$.

It is easy to check that any regular function $f_u$ can be taken to be biinvariant by an Iwahori subgroup; see [F5]. Then $f_u^0$ can be taken to be a scalar multiple of the characteristic function of the product of
Suppose that $G = GL(n)$. Let $\tilde{f}^u$ be a cuspidal function on $\tilde{G}^u$ which is compactly supported modulo $\tilde{Z}^u (= Z(\tilde{A}^E_{K}))$. Then there exists a positive integer $n_0 = n_0(\tilde{f}^u)$ such that for every $n_0$-admissible spherical function $f_u$ on $G_u$, putting $\tilde{f} = \tilde{f}^u L(\tilde{f}_u)$, where $\tilde{f}_u = (f_u, f_0^u, \ldots, f_0^u)$ and $L(\tilde{f}_u) = (L(f_u), f_0^u, \ldots, f_0^u)$, we have that the sum $\sum_{\{\tilde{\gamma} \times \sigma\}} \Phi(\tilde{\gamma} \times \sigma, \tilde{f})$ of (6) is equal to

$$
\sum'_{\#} \ tr \bar{\pi}(\tilde{f}^u \times \sigma) = \sum_{\#} b(\pi_u) \ tr \bar{\pi}(\tilde{f}^u f_u \times \sigma).
$$

Here $\sum'$ ranges over the equivalence classes of cuspidal $\sigma$-invariant $G^E$-modules $\pi$ whose component $\bar{\pi}_u$ is of the form $\pi_u \otimes \ldots \otimes \pi_u$ ($e$ copies), where $\pi_u$ is equivalent to an irreducible unramified $G_u$-module $I(t)$, and $b(\pi_u)$ is as defined in Corollary 13.

Proof. — It remains to explain the last assertion, which concerns $\bar{\pi}_u$. As a $G^E_u$-module, $\bar{\pi}_u$ is of the form $\pi_1 \otimes \ldots \otimes \pi_e$. It is easy to see, as in [F3], §1.5, that $\bar{\pi}_u$ is $\sigma$-invariant if and only if $\pi_i = \pi_1$ for all $i (1 \leq i \leq e)$. Moreover, $tr \bar{\pi}_u(\tilde{f}_u \times \sigma) = tr \bar{\pi}_u(f_u)$ if $\bar{\pi}_u - \pi_u \otimes \ldots \otimes \pi_u$ and $\tilde{f}_u = (f_u, f_0^u, \ldots, f_0^u)$. But then $tr \bar{\pi}_u(L(f_u)) = b(\pi_u) tr \pi_u(f_u)$ by Lemma 11, and this is non-zero only if $\pi_u$ is equivalent to an unramified irreducible $I(t)$, as required.

Remark. — The identity (7) for the function $\tilde{f}$ of Proposition 17 will be referred to as the regular twisted trace formula. Proposition 18 is an explicit form of the regular twisted trace formula in the case of $G = GL(n)$, which we use below to prove the base-change lifting for this group.

From now on $G = GL(n)$. Our aim is to establish the base-change theory for this group on comparing the sides of the regular trace formula (2) and its twisted analogue (7), which involve orbital integrals. We begin with a comparison of the sets $\{\tilde{\gamma}\}$ and $\{\tilde{\gamma} \times \sigma\}$ over which the sums are taken. Let $E/F$ be a cyclic extension of local or global fields; put $G = GL(n, F)$ and $\tilde{G} = GL(n, E)$.

Definition. — Given $\tilde{\gamma}$ in $\tilde{G}$ put

$$
\mathcal{N} \tilde{\gamma} = \tilde{\gamma} \cdot \sigma(\tilde{\gamma}) \cdot \sigma^2(\tilde{\gamma}) \cdot \ldots \cdot \sigma^{e-1}(\tilde{\gamma}).
$$
Since \( \sigma(\gamma) = \gamma^{-1} \cdot N\gamma \cdot \gamma \), the elementary divisors \( p_1(x) | p_2(x) | \ldots | p_j(x) \) of the matrix \( N\gamma \), which a priori have coefficients in \( E \), in fact have coefficients in \( F \). Hence \( N\gamma \) is conjugate to an element \( N\tilde{\gamma} \) of \( G \). Only the conjugacy class of \( N\tilde{\gamma} \) is determined by \( \tilde{\gamma} \), and \( N\tilde{\gamma} \) depends only on the \( \sigma \)-conjugacy class of \( \tilde{\gamma} \).

**19. Lemma.** — The norm map \( N : \tilde{\gamma} \rightarrow N\tilde{\gamma} \) is an injection from the set of \( \sigma \)-conjugacy classes in \( \tilde{G} \) to the set of conjugacy classes in \( G \).

**Proof.** — We have to show that if \( N\tilde{\gamma} \) and \( N\tilde{\delta} \) are conjugate in \( \tilde{G} \) then \( \tilde{\gamma} \) and \( \tilde{\delta} \) are \( \sigma \)-conjugate. To recall the proof of [L], §4, put \( \gamma = N\tilde{\gamma} \). As noted above we may assume that \( \gamma \) lies in \( G \). The centralizer \( Z_\gamma(G) \) of \( \gamma \) in \( G \) is a reductive \( F \)-group; it is the group of invertible elements in \( Z_\gamma(L(G)) \), where \( L(G) = M_n \) denotes the Lie algebra of \( G \). The centralizer \( Z_{\tilde{\gamma} \times \sigma}(\tilde{G}) \) of \( \tilde{\gamma} \times \sigma \) in \( \tilde{G} \) is the group of \( F \)-points of a reductive \( F \)-group \( Z_{\tilde{\gamma} \times \sigma}(\tilde{G}) \). We have \( Z_{\tilde{\gamma} \times \sigma}(\tilde{G}) \subset (Z_\gamma(G))(E) \), and \( Z_{\tilde{\gamma} \times \sigma}(\tilde{G}) \) is an inner form of \( Z_\gamma(G) \) which splits over \( E \) and is defined by the cocycle \( c_\sigma = ad(\tilde{\gamma}) \circ \sigma \); namely the \( F \)-structure on \( Z_{\tilde{\gamma} \times \sigma}(\tilde{G}) \) is given by \( \alpha \mapsto \tilde{\gamma}\sigma(\alpha)\tilde{\gamma}^{-1} \). The same definitions apply to the Lie algebras \( L(G) \) and \( L(\tilde{G}) \); \( Z_{\tilde{\gamma} \times \sigma}(L(\tilde{G})) \) is an \( E/F \)-form of \( Z_\gamma(L(G)) \). By the Hilbert theorem 90 of [S], Ex. 2, p. 160, we have \( H^1(Gal(E/F), Z_{\tilde{\gamma} \times \sigma}(\tilde{G}))(E)) = \{0\} \), and the lemma follows as in [L], §4, from a simple cocycle computation.

We recall (see [AC], §1, or [L], §4) the following basic properties of the norm map.

**20. Lemma.** — Suppose that \( F \) is a global field. Then \( \gamma \) in \( G \) is a norm from \( \tilde{G} \) if and only if it is a norm from \( \tilde{G}_v \) for every place \( v \) of \( F \). (ii) Suppose that \( E/F \) is a cyclic extension of local or global fields, and \( \gamma \) is an elliptic regular element of \( G \). Then \( \gamma \) is a norm from \( \tilde{G} \) if and only if det \( \gamma \) is in \( N_{E/F}E^\times \).

To compare the sums of orbital integrals in (2) and (7) we introduce the following

**Definition.** — Let \( v \) be a place of a global field \( F \). Let \( \omega_v \) be a unitary character of \( N\tilde{Z}_v \simeq N_{E_v/F_v}E_v^\times \) and \( \tilde{\omega}_v(x) = \omega_v(Nx) \) the associated character of \( \tilde{Z}_v = Z(E_v) \). Let \( f_v \) be a smooth (= locally constant if \( v \) is non-archimedean) compactly-supported modulo \( N\tilde{Z}_v \) function on \( G_v \) with \( f_v(zg)\omega_v(z) = f_v(g) \) (\( g \) in \( G_v \), \( z \) in \( N\tilde{Z}_v \)). Let
\( \tilde{f}_v \) be a smooth compactly supported modulo \( \tilde{Z}_v \) function on \( \tilde{G}_v \) with \( \tilde{f}_v(zg)\tilde{\omega}_v(z) = \tilde{f}_v(g) \) (\( g \) in \( \tilde{G}_v \), \( z \) in \( \tilde{Z}_v \)). Then \( f_v \) and \( \tilde{f}_v \) are called matching if for every regular \( \gamma \) in \( G_v \) we have \( \Phi(\gamma, f_v) = \Phi(\gamma \times \sigma, \tilde{f}_v) \) if \( \gamma = N\tilde{\gamma} \), or \( \Phi(\gamma, f_v) = 0 \) if \( \gamma \) is not a norm from \( \tilde{G}_v \).

Remark. — (1) If \( N\tilde{\gamma} = \gamma \) is regular then \( Z_{\tilde{\gamma} \times \sigma}(\tilde{G}) \simeq Z_{\gamma}(G) \) are \( F \)-tori in \( G \), and in the definition of \( \Phi(\tilde{\gamma} \times \sigma, f) \) and \( \Phi(\gamma, f) \) we take the same measure on \( Z_{\tilde{\gamma} \times \sigma}(\tilde{G}) \) and \( Z_{\gamma}(G) \). The definition of the orbital integrals and consequently also of the matching functions depend on a choice of a Haar measure on \( \tilde{G}_v \) and on \( G_v \). (2) It is not hard to see that (i) \( \Phi(\gamma, f_v) \) is a class function on \( G_v \) whose restriction to the regular set of \( G_v \) is smooth, (ii) for any smooth class function \( h \) on \( G_v \) which is supported on the regular set there exists an \( f_v \) supported on the regular set of \( G_v \) with \( \Phi(\gamma, f_v) = h(\gamma) \) for all \( \gamma \) in \( G_v \). The analogous statement holds for \( \tilde{f}_v \). Hence it is clear that for every \( \tilde{f}_v \) (resp. \( f_v \)) which is supported on the regular set of \( \tilde{G}_v \) (resp. \( G_v \), and \( \Phi(\gamma, f_v) = 0 \) if \( \gamma \) is not a norm), there exists a matching \( f_v \) (resp. \( \tilde{f}_v \)). Using the results of [K] it is easy to see that in the case of characteristic zero for every \( \tilde{f}_v \) there exists a matching \( f_v \), and for every \( f_v \) with \( \Phi(\gamma, f_v) = 0 \) if \( \gamma \) is regular but not a norm there exists a matching \( \tilde{f}_v \), but this more refined statement is not needed in our study of base-change for \( GL(n) \); as noted after Theorem 32 below, it is a consequence of [BDK] and the local lifting theorem. The analogous refined statement plays a key role in the study of the deeper metaplectic correspondence (see [FK]) and the simple algebra correspondence (see [BDKV] or [F1; III]).

Definitions. — Let \( v \) be a non-archimedean place of \( F \) which does not ramify in \( E \). Suppose that \( \omega_v \) is unramified. Let \( f^0_v \) be the function on \( G_v \) with \( f^0_v(g) = 0 \) unless \( g = zk \) with \( k \) in \( K_v \) and \( z \) in \( N\tilde{Z}_v \), where \( f^0_v(zk) = \omega_v(z)^{-1}|K_v|^{-1} \). Let \( \tilde{f}^0_v \) be the function on \( \tilde{G}_v \) with \( \tilde{f}^0_v(g) = 0 \) unless \( g = zk \) with \( k \) in \( \tilde{K}_v \) and \( z \) in \( \tilde{Z}_v \) where \( f^0_v(zk) = \tilde{\omega}_v(z)^{-1}|	ilde{K}_v|^{-1} \) (here \( K_v = GL(n, R_v) \), \( \tilde{K}_v = GL(n, R_{E_v}) \)).

21. Proposition. — The functions \( f^0_v \) and \( \tilde{f}^0_v \) are matching.

Proof. — See Kottwitz [Ko].
Let $f = \otimes f_v$ be a smooth compactly supported modulo $N\tilde{Z}(A)$ function on $G(A)$ with $f(zg)\omega(z) = f(g)$ ($g$ in $G(A)$, $z$ in $N\tilde{Z}(A)$), such that $f_v = f_v^0$ for almost all $v$. Let $\tilde{f} = \otimes \tilde{f}_v$ be a smooth compactly supported modulo $\tilde{Z}(A)$ function on $\tilde{G}(A)$ with $\tilde{f}(zg)\tilde{\omega}(z) = \tilde{f}(g)$ ($g$ in $\tilde{G}(A)$, $z$ in $\tilde{Z}(A)$), such that $\tilde{f}_v = \tilde{f}_v^0$ for almost all $v$. Suppose that $f_v$ and $\tilde{f}_v$ are matching for all $v$. At some non-archimedean place $w$ of $F$ which splits completely in $E$ we take $f_w$ to be a supercusp form, and $f_w = (f_w, f_w^0, \ldots, f_w^0)$, where $f_w^0$ is a function supported on the product of $N\tilde{Z}_w$ and a neighborhood of 1, such that $f_w = f_w * f_w^0$. At some non-archimedean place $u \neq w$ of $F$ which splits completely in $E$ we take $f_u = L(f_u')$, where $f_u'$ is any $n_0$-admissible spherical function on $G_u$, and $\tilde{f}_u = (L(f_u'), f_u^0, \ldots, f_u^0)$, as in Proposition 18. Here $n_0$ depends on the support of $f_v$ and $\tilde{f}_v$ for all $v \neq u$. Note that $f$ does not transform under the center $Z(A)$ of $G(A)$ (as in Corollary 13), but only under its subgroup $N\tilde{Z}(A) = NZ(A_E)$. Since $[Z(A) : Z \cdot N\tilde{Z}(A)] = e$, the effect of this change is that (2) takes the form

$$
\sum' \pi f = e \sum \frac{|Z_\gamma(G(A))/Z(G)| \prod_v \Phi(\gamma, f_v)}{\prod_v \Phi(\gamma, f_v)}.
$$

Here $\pi$ ranges over all cuspidal $G$-modules such that the restriction of their central character to $N\tilde{Z}(A)$ is $\omega$, and $\gamma$ ranges over the set of conjugacy classes of regular elliptic elements in $G$. Since $f_v$ matches some $\tilde{f}_v$ for every $v$, by virtue of Lemma 20 (1) each $\gamma$ which yields a non-zero term in the sum is a norm from $\tilde{G}$.

Also we note that $f$ is cuspidal since its component $f_w$ is a supercusp form; hence Corollary 13 applies. The identity (7) takes the form

$$
\sum' \pi \pi(f) = e \sum \frac{|Z_\tilde{\gamma}(\tilde{G}(A))/\tilde{Z}(\tilde{G})(\tilde{G})| \prod_v \Phi(\tilde{\gamma} \times \sigma, \tilde{f}_v)}{\prod_v \Phi(\tilde{\gamma} \times \sigma, \tilde{f}_v)}.
$$

The $\pi$ and $\tilde{\gamma}$ are described in Proposition 18. Since the sums over $\gamma$ in (2') and $\tilde{\gamma}$ in (7') are taken over the same sets, and $f_v, \tilde{f}_v$ are matching for every $v$, and $Z_\gamma(G)$ and $Z_\gamma(G)$ are isomorphic if $\gamma = N\tilde{\gamma}$, we conclude

22. PROPOSITION. — For any matching $f = \otimes f_v$ and $\tilde{f} = \otimes \tilde{f}_v$ as above we have

$$
\sum' \pi f = e \sum' \pi(\tilde{f} \times \sigma).
$$

To derive lifting theorems from this identity we write $\text{tr} \pi(f)$ as a product over all $v$ of $\text{tr} \pi_v(f_v)$; this can be done since each of $\pi = \otimes \pi_v$. 


Let $\pi_u$ be an irreducible unramified unitary $G_u$-module, and $\tilde{\pi}_u = \pi_u \otimes \ldots \otimes \pi_u$. Put $c(\pi_u) = \sum tr \pi_u(f^u)$, where the sum ranges over all irreducible $G^u$-modules $\pi^u$ such that $\pi = \pi^u \otimes \pi_u$ occurs in the sum on the left of (8). Put $c(\tilde{\pi}_u) = \sum tr \tilde{\pi}_u(f^u \times \sigma)$, where the sum ranges over all irreducible $\tilde{G}^u$-modules $\tilde{\pi}^u$ such that $\pi = \tilde{\pi}^u \otimes \tilde{\pi}_u$ occurs in the sum on the right of (8).

23. Proposition. — For every $\pi_u$ we have $c(\pi_u) = c(\tilde{\pi}_u)$.

Proof. — By virtue of Corollary 13 and Proposition 18 the identity (8) can be written in the form

$$\sum_{\pi_u} b(\pi_u) [c(\pi_u) - c(\tilde{\pi}_u)] tr \pi_u(f_u) = 0. \tag{9}$$

The sum ranges over all (equivalence classes of) unitary unramified irreducible $G_u$-modules $\pi_u$. Theorem 2 of [FK1] implies that

$$b(\pi_u) (c(\pi_u) - c(\tilde{\pi}_u)) = 0,$$

since the sum of (9) is absolutely convergent for every spherical $f_u$, and (9) holds for all $n_0$-admissible spherical $f_u$ for some $n_0$. Since each $\pi$ and $\tilde{\pi}$ in (2') and (7') is cuspidal, $\pi_u$ is non-degenerate, hence equivalent to an irreducible induced $G_u$-module of the form $I(t)$. By Lemma 10 we have $b(\pi_u) \neq 0$, hence $c(\pi_u) = c(\tilde{\pi}_u)$, as required.

We need an extension of Proposition 23 from the set which consists of $u$ to the complement of a finite set of places of $F$. To state it we introduce the
DEFINITION. — Let \( E_v / F_v \) be an unramified cyclic extension of degree \( e \) of non-archimedean local fields. Then we say that (1) the \( G_v \)-module \( I(t) \) (resp. \( \bar{\pi}(t) \)) lifts to the \( \tilde{G}_v \)-module \( \tilde{I}(t') \) (resp. \( \tilde{\pi}(t') \)) if \( t' = t_e = (t_1', \ldots, t_n') \) if \( t = (t_1, \ldots, t_n) \); (2) the spherical functions \( f_v \) on \( G_v \) and \( \tilde{f}_v \) on \( \tilde{G}_v \) correspond if \( \text{tr} (\pi(t))(f_v) = \text{tr} (\tilde{\pi}(t'))(\tilde{f}_v) \) for all \( t \) in \( T \) (equivalently, \( \text{tr} (\bar{\pi}(t))(\bar{f}_v) = \text{tr} (\tilde{\pi}(t'))(\tilde{f}_v) \) for all \( t \) in \( T \)).

We also denote by \( \tilde{H}_v \) (resp. \( H_v \)) the Hecke convolution algebra of spherical, namely \( \tilde{K}_v \)-biinvariant, compactly-supported modulo center, functions on \( \tilde{G}_v \) which transform under \( \tilde{Z}_v \) by \( \tilde{\omega}_v^{-1} \) (resp. \( K_v \)-biinvariant, compactly supported modulo \( N\tilde{Z}_v \), functions on \( G_v \) which transform under \( N\tilde{Z}_v \) by \( \omega_v^{-1} \)). Note that the spherical \( f_v \) (resp. \( \tilde{f}_v \)) corresponds to precisely one spherical \( \tilde{f}_v \) (resp. \( f_v \)) by the theory of the Satake transform.

Let \( v \) be a place of \( F \). Then \( E_v = E \otimes_F F_v = F'_v \oplus \ldots \oplus F''_v \) (\( e'' = e/e' \) copies, where \( e' = [F'_v : F'] \)). Suppose that \( F'_v / F_v \) is unramified. Put \( G'_v = GL(n, F'_v) \). Then we say that the unramified \( G_v \)-module \( \pi(t) \) (resp. \( I(t) \)) lifts to the unramified \( \tilde{G}_v \)-module \( \tilde{\pi}(t') = \pi'(t_1) \otimes \ldots \otimes \pi'(t_{e''}) \) (resp. \( \tilde{I}(t') = I'(t_1) \otimes \ldots \otimes I'(t_{e''}) \)) if \( t_i = t_{e'} \) for all \( i \) (\( 1 \leq i \leq e'' \)). The spherical function \( f_v \) on \( G_v \) corresponds to the spherical function \( \tilde{f}_v = f'_1 \otimes \ldots \otimes f''_{e''} \) on \( \tilde{G}_v \) if \( \text{tr} (\pi(t))(f_v) = \text{tr} (\tilde{\pi}(t'))(\tilde{f}_v) \) for all \( t \) in \( T \), where \( \pi(t') = \pi'(t_{e'}) \otimes \ldots \otimes \pi'(t_{e''}) \). For example, if \( f_v \) corresponds to the spherical \( f'_v \) on \( G'_v \), and \( f'_0 \) is the unit element of the Hecke algebra \( \mathcal{H}'_v \) of \( G'_v \), then \( f_v \) corresponds to \( \tilde{f}_v = f'_0 \otimes f'_0 \otimes \ldots \otimes f'_0 \).

The following Proposition 24 is usually considered to be a crucial tool in the extension of Proposition 23 alluded to above. The technique of regular functions permits us below to extend Proposition 23, and complete the proof of the base-change theorems, without ever using Proposition 24. In fact Proposition 24 can be deduced from the local lifting theorem as in Remark (3) in §32 below.

24. PROPOSITION. — If the spherical \( f_v \) on \( G_v \) and \( \tilde{f}_v \) on \( \tilde{G}_v \) are corresponding then they are matching.

**Proof.** — This is the special case of \( G = GL(n) \) of the Theorem of [F4]. The proof of that Theorem is incomplete in the generality stated there. The problem is in the deduction of Lemma 5 from Lemma 4 in §6 of [F4], Contrary to the assertion of [F4], p. 141, l. 4-5, it is not the identity (6.4), but only (6.3), which holds for the unit elements \( \phi^0_u, \alpha^0_u \).

To complete the proof of the Theorem of [F4], we need to prove
Lemma 6.5 of [F4]. By inspection, this Lemma holds in the situation of base-change from $U(3, E/F)$ to $GL(3, E)$, and the analogous case of the symmetric-square from $SL(2)$ to $PGL(3)$. A few general rank cases where this Lemma holds, are the following. Suppose $G = GL(n)$. In this case (of base-change for $GL(n)$) we can work with the Deligne-Kazhdan trace formula. Then all $\pi, \pi'$ of (6.1) are cuspidal. Their local components are non-degenerate. Hence the unramified components are irreducible full-induced $I(\eta_u), I(\eta'_u)$, by [Z], (9.7), and Lemma 6.5 holds.

In the analogous case of the metaplectic correspondence of [FK], the unramified components of the cuspidal $\pi$ and $\pi'$ are unramified irreducible full-induced, as noted above in the case of $GL(n)$, and by virtue of Theorem 16 in [FK] (this theorem establishes a canonical isomorphism of the Iwahori algebras of $GL(n)$ and its covering group considered in [FK]), in the case of the metaplectic group of [FK]. In particular the sentence on lines 3-5, p. 85, of [FK], should be replaced by a reference to Theorem 16 of [FK].

We shall now give a purely local proof, based on Proposition 12, of a special case of Proposition 24 which suffices for our purposes. We first fix some notations. Let $E/F$ be an unramified cyclic extension of non-archimedean local fields, and $\tilde{\omega}$ an unramified character of $\tilde{Z}$. Consider $x = (x_1, \ldots, x_n)$ in $X^*(\tilde{Z})$ with $x_i \neq x_j$ for all $i \neq j$.

DEFINITION. — A locally constant function $\tilde{f}$ on $\tilde{G}$ with $\tilde{f}(zg)\tilde{\omega}(z) = \tilde{f}(g)$ ($z$ in $\tilde{Z}$, $g$ in $\tilde{G}$) which is compactly supported modulo $\tilde{Z}$, is called $x$-regular if $\tilde{f}(g)$ is zero unless there is $z$ in $\tilde{Z}$ such that $zg$ is $\sigma$-conjugate to an element $a$ in $\tilde{A}$ whose image in $\tilde{X}_*$ is $x$, in which case the normalized twisted orbital integral $F(g \times \sigma, \tilde{f}) = \Delta(Ng)\Phi(g \times \sigma, \tilde{f})$ is equal to $\tilde{\omega}(z)$. An $x$-regular function $\tilde{f}$ will be denoted by $\tilde{f}_x$. A regular function $\tilde{f}$ is an element in the span of the $x$-regular functions, for all $x$.

As in the non-twisted case, for each $x$ in $\tilde{X}_*$ let $\tilde{f}(x)$ be the spherical function in $\tilde{H}$ with $\tilde{f}(x) = \sum_{\omega \in W} w x$. As in Proposition 12 we have in the twisted case that : if $\tilde{f} = \sum c(x)f(x)$ is a spherical function, and $\tilde{\phi} = \sum b(x)\tilde{f}_x$ is a regular function such that $L(\tilde{f}_x) = \sum_r c_r f_{x+r}$ is regular if $b(x) \neq 0$, then the following holds. The spherical function $L(\tilde{f}) = \sum c(x)L(\tilde{f}(x))$ and the regular function $L(\tilde{\phi}) = \sum c(x)L(\tilde{f}_x)$ are solid companions.
25. **Proposition.** — Let $n_1$ be the maximum of the absolute values of the entries of the vectors $x_r$ of Lemma 11. If $\tilde{f}$ and $f$ are corresponding $n_1$-admissible spherical functions on $\tilde{G}$ and $G$ then $L(\tilde{f})$ and $L(f)$ are matching.

**Proof.** — Let $b^*: \tilde{H} \to H$ be the linear map defined by $b^*(\tilde{f}(x)) = f(ex)$. It is clear that $\tilde{f}$ and $f$ are corresponding if and only if $f = b^*\tilde{f}$. By definition, the regular functions $\tilde{f}_x$ on $\tilde{G}$ and $f_{ex}$ on $G$ are matching. Hence $L(\tilde{f})$ and $L(b^*\tilde{f})$ are matching, for every $n_1$-admissible spherical $\tilde{f}$ on $\tilde{G}$, as required.

We shall now return to the extension of Proposition 23 alluded to above. Let $E/F$ be a cyclic extension of global fields. Let $V$ be a set of places of $F$ including the archimedean places and those which ramify in $E$. At each place $v$ outside $V$ fix an unramified unitary irreducible $G_v$-module $\pi_v^0$. At some place $w$ in $V$ which splits completely in $E$ fix a supercuspidal $G_w$-module $\pi_w^0$, and put $\tilde{\pi}_w = \pi_w^0 \otimes \cdots \otimes \pi_w^0$.

26. **Proposition.** — For every $v$ in $V$ let $f_v$ and $\tilde{f}_v$ be matching functions on $G_v$ and $\tilde{G}_v$, such that $f_w$ is a matrix coefficient of $\pi_w^0$. Then we have

$$
\sum \prod \text{tr } \pi_v(f_v) = e \sum \prod \text{tr } \tilde{\pi}_v(\tilde{f}_v \times \sigma).
$$

The sum over $\widetilde{\pi}$ ranges over all $\sigma$-invariant cuspidal $\tilde{G}$-modules $\tilde{\pi}$ whose component at $w$ is $\tilde{\pi}_w^0$ and whose component at every $v$ outside $V$ is $\pi_v^0$. The sum over $\pi$ ranges over all cuspidal $G$-modules $\pi$ whose component at $w$ is $\pi_w^0$ and whose component $\pi_v$ at each $v$ outside $V$ lifts to $\pi_v^0$.

**Proof.** — Proposition 23 implies (10) if $V$ is the complement of $u$. By virtue of Proposition 25 we also have (9) where $u$ is replaced by any $v$ outside $V$, as noted in the proof of Proposition 23. By induction (10) holds where $V$ is the complement of a finite set. But then Lemma 3 of [F2], IV, implies the proposition.

**Definition.** — The irreducible $G(A)$-module $\pi = \bigotimes_v \pi_v$ lifts to the irreducible $\tilde{G}(A)$-module $\tilde{\pi} = \bigotimes_v \tilde{\pi}_v$ if $\pi_v$ lifts to $\tilde{\pi}_v$ for almost all $v$.

**Remark.** — For almost all $v$ the components $\pi_v$ and $\tilde{\pi}_v$ are unramified and so the notion of local lifting is indeed defined.

The following is the global base-change theorem for $GL(n)$. 

**Definition.** — The irreducible $G(A)$-module $\pi = \bigotimes_v \pi_v$ lifts to the irreducible $\tilde{G}(A)$-module $\tilde{\pi} = \bigotimes_v \tilde{\pi}_v$ if $\pi_v$ lifts to $\tilde{\pi}_v$ for almost all $v$.
27. Theorem. — Let $E/F$ be a cyclic extension of global fields, and $w$ a place of $F$ which splits completely in $E$. Then every cuspidal $G$-module $\pi$ whose component at $w$ is supercuspidal lifts to a unique cuspidal $\tilde{G}$-module $\tilde{\pi}$; this $\tilde{\pi}$ is $\sigma$-invariant and has a supercuspidal component at $w$. Moreover, every $\sigma$-invariant cuspidal $\tilde{G}$-module $\tilde{\pi}$ whose component at $w$ is supercuspidal is a lift of a cuspidal $G$-module $\pi$; any other cuspidal $G$-module which lifts to $\tilde{\pi}$ is of the form $\pi \otimes \epsilon^i$ ($0 \leq i < e$), where $\epsilon$ is a primitive character of $\mathbb{A}_F^\times$ which is trivial on $F^\times N \mathbb{A}_E^\times$, and $\pi \otimes \epsilon^i \neq \pi$ for all $1 \leq i < e$.

Proof. — On the right of (10) there is at most one $\tilde{\pi}$ by the rigidity theorem for the cuspidal spectrum of $\tilde{G}$ (see [JS]). By Corollary 6' of [F2], $V$, which is an immediate consequence of [JS], Proposition 3.6 (as was noticed first by Arthur-Clozel [AC], III, 3.1), the sum on the left, over $\pi$, consists of at most one orbit $\pi \otimes \epsilon^i$ ($0 \leq i < e$) under tensoring by $\epsilon$ of cuspidal $G$-modules (and each element in the orbit occurs). Since $tr(\pi_v \otimes \epsilon^i_j)(f_v) = tr \pi_v(f_v)$ for any $f_v$ which matches some $\tilde{f}_v$, we conclude that $\pi \otimes \epsilon \neq \pi$, and the theorem follows.

From the global base-change theorem we shall now deduce a local base-change theorem. Let $E/F$ be a cyclic extension of degree $e$ of local non-archimedean fields.

Definition. — The irreducible $G$-module $\pi$ lifts to the irreducible $\sigma$-invariant $\tilde{G}$-module $\tilde{\pi}$ if for some choice of $\tilde{\pi}(\sigma)$ we have $tr \pi(f) = tr \tilde{\pi}(\tilde{f} \times \sigma)$ for all matching $f$ on $G$ and $\tilde{f}$ on $\tilde{G}$.

Remark. — (1) It is clear the $\pi$ lifts to at most one $\tilde{\pi}$, and if $\pi$ lifts to $\tilde{\pi}$ then $\pi \otimes \epsilon$ lifts to $\tilde{\pi}$ for every character $\epsilon$ of $F^\times / N \mathbb{A}_E^\times$.
(2) If $M = \prod_i \tilde{M}_i$ ($\tilde{M}_i = GL(n_i, E)$) is a Levi subgroup of $\tilde{G}$ and $\rho_i$ is an $M_i$-module which lifts to an $\tilde{M}_i$-module $\tilde{\rho}_i$, then $\pi = I(\otimes_i \rho_i; M, G)$ lifts to $\tilde{\pi} = I(\otimes_i \tilde{\rho}_i; \tilde{M}, \tilde{G})$ by virtue of a standard formula for the character of an induced representation. In other words, lifting commutes with induction. (3) If the $\tilde{\rho}_i$ are non-degenerate and $\tilde{\rho}_i(\sigma)$ are the intertwining operators defined using the Whittaker model of $\tilde{\rho}_i$, then the normalized operator $\tilde{\pi}(\sigma)$ is the one obtained by the induction functor from $\otimes \tilde{\rho}_i(\sigma)$.

Definition. — (1) An irreducible tempered $G$-module $\pi$ is called
discrete if it is not equivalent to any properly induced $G$-module. (2) An irreducible tempered $\sigma$-invariant $\tilde{G}$-module $\tilde{\pi}$ is called $\sigma$-discrete if it is not equivalent to any $G$-module properly induced from a $\sigma$-invariant representation of a Levi subgroup.

28. Lemma. — (i) A $G$-module $\pi$ is discrete if and only if it is square-integrable. (ii) A $\tilde{G}$-module $\tilde{\pi}$ is $\sigma$-discrete if and only if $\tilde{\pi} = I(\tilde{\rho} \otimes \sigma \tilde{\rho} \otimes \cdots \otimes \sigma^{a-1} \tilde{\rho})$ where $n = ab$, and $\tilde{\rho}$ is a discrete $G' = GL(b,E)$-module with $\sigma^i \tilde{\rho} \not\cong \tilde{\rho}(1 \leq i < a)$ and $\sigma^a \tilde{\rho} \cong \tilde{\rho}$.

**Proof.** — (i) follows from [BZ]. For (ii), since $\tilde{\pi}$ is tempered it is induced from a representation $(\tilde{\rho}_1, \ldots, \tilde{\rho}_a)$ of a Levi subgroup of type $(b_1, \ldots, b_a)$, where the $\tilde{\rho}_i$ are discrete. Since $\sigma^i \tilde{\pi} \cong \tilde{\pi}$ we have $(\sigma^i \tilde{\rho}_1, \ldots, \sigma^a \tilde{\rho}_a) = (\tilde{\rho}_1, \ldots, \tilde{\rho}_a)$ up to permutation, which must be transitive on $\{1, \ldots, a\}$ since $\tilde{\pi}$ is not induced. Up to reordering the indices $\tilde{\rho}_i = \sigma^{i-1} \tilde{\rho}_1$, as required.

**Definition.** — Let $\tilde{\pi}_0$ be a $\sigma$-discrete $\tilde{G}$-module with central character $\tilde{\omega}$. A locally constant compactly supported modulo $\tilde{Z}$ function $\tilde{f}$ on $\tilde{G}$ with $\tilde{f}(zg)\tilde{\omega}(z) = \tilde{f}(g)$ is called a pseudo-coefficient of $\tilde{\pi}_0$ if $tr \tilde{\pi}_0(\tilde{f} \times \sigma) = 1$ and $tr \tilde{\pi}_0(\tilde{f} \times \sigma) = 0$ for every tempered $\sigma$-invariant irreducible $\tilde{G}$-module $\tilde{\pi}$ with central character $\tilde{\omega}$, which is inequivalent to $\tilde{\pi}_0$.

29. Proposition. — For every $\sigma$-discrete $\tilde{G}$-module there exists a pseudo-coefficient.

**Proof.** — When $E = F$ this is proven in [K], using the trace Paley-Wiener theorem of [BDK]. The proof in the twisted case analogously follows from the twisted variant of the trace PW-theorem (see [F1], I, §7, or [R]; a different, in some sense better, proof in the non-twisted and also twisted case, is now in preparation), which we now recall. Let $\tilde{M}$ be a Levi subgroup of $\tilde{G}$. Let $\text{Irr}_\sigma(\tilde{M})$ be the set of irreducible $\sigma$-invariant $\tilde{G}$-modules $\tilde{\pi}$. Let $X_\sigma(\tilde{M})$ be the group of (necessarily $\sigma$-invariant) unramified characters $\chi$ of $\tilde{M}$. $X_\sigma(\tilde{M})$ has a natural structure of a complex algebraic variety, isomorphic to $\mathbb{C}^d$, where $d = \dim X_\sigma(\tilde{M})$. It acts naturally on $\text{Irr}_\sigma(\tilde{M})$ by $\chi : \tilde{\rho} \mapsto \tilde{\rho} \otimes \chi$. The twisted trace Paley-Wiener theorem asserts, in the terminology of [BDK], that the a "good form" $\lambda$ on $\text{Irr}_\sigma(\tilde{G})$ is a "trace form", namely the following.
30. **Theorem.** — Let $\lambda$ be a complex-valued linear form on the vector space over $\mathbb{C}$ spanned by $\text{Irr}_{\sigma}(\tilde{G})$. Suppose that (1) there exists an open compact $\sigma$-invariant subgroup $K$ of $\tilde{G}$ which dominates $\lambda$, in the sense that $\lambda$ vanishes on each irreducible $\sigma$-invariant $\tilde{G}$-module which has no $\tilde{K}$-fixed vector, and (2) for every proper Levi subgroup $\tilde{M}$ and $\sigma$-invariant irreducible $\tilde{M}$-modules $\tilde{\rho}$, the function $\chi \mapsto \lambda[I(\tilde{\rho} \otimes \chi; \tilde{M}, \tilde{G})]$ is regular on the complex algebraic variety $X_\sigma(\tilde{M})$. Then there exists a function $\tilde{f}$ with $\lambda(\tilde{\pi}) = tr\tilde{\pi}(\tilde{f} \times \sigma)$ for all $\tilde{\pi}$ in $\text{Irr}_\sigma(\tilde{G})$.

Given the existence of pseudo-coefficients, the following is easily proven (see, e.g., [Fl]) using the Deligne-Kazhdan trace formula.

31. **Proposition.** — Let $E'/F'$ be a cyclic extension of local fields and $\tilde{\pi}'$ a $\sigma$-discrete $GL(n,E')$-module. Then there exists a cyclic extension $E/F$ of global fields such that $F$ has a place $w$ with $F_w = F'$ and $E_w = E'$ and every archimedean place of $F$ splits in $E$. Moreover, there exists a cuspidal $\sigma$-invariant $\tilde{G}$-module $\tilde{\pi}$ whose component at $w$ is $\tilde{\pi}'$, with the following properties. There are two finite places $u, u'$ which split completely in $E$ such that $\tilde{\pi}_v$ is unramified for every finite $v \neq u, u'$, and $\tilde{\pi}_u$ is supercuspidal.

**Remark.** — When $E' = F'$ Proposition 31 asserts that given a local $\pi'$ there exists a global $\pi$ with a component $\pi'$ such that $\pi$ has the specified properties.

Given a local $\pi_w$ (resp. $\tilde{\pi}_w$) we apply (10) with the $\pi$ (resp. $\tilde{\pi}$) of Proposition 31. Since $tr\pi_v(f_v) = tr\tilde{\pi}_v(f_v \times \sigma)$ for matching $f_v$, $\tilde{f}_v$ at all $v \neq w$, we conclude the following local base-change theorem for $GL(n)$.

32. **Theorem.** — The lifting defines a surjection from the set of equivalence classes of irreducible tempered $G_v$-modules $\pi_v$ to the set of equivalence classes of irreducible tempered $\sigma$-invariant $\tilde{G}_v$-modules $\tilde{\pi}_v$. It maps discrete $G_v$-modules to $\sigma$-discrete $\tilde{G}_v$-modules, and relevant $G_v$-modules to relevant $\sigma$-invariant $\tilde{G}_v$-modules. The preimage $\pi_v$ of a supercuspidal $\tilde{\pi}_v$ is supercuspidal.

**Remark.** — Relevant $G_v$-modules are defined in [FK], §27. Every component of a cuspidal $G$-module is relevant, and every tempered $G_v$-module is relevant.

**Proof.** — For the first assertion it remains to note that each $\sigma$-invariant tempered $\tilde{G}_v$-module $\tilde{\pi}_v$ is induced from a $\sigma$-discrete one,
and the lifting commutes with induction. The last assertion follows from the Theorem of Deligne [CD], according to which a $G_v$-module $\pi_v$ is supercuspidal if and only if the restriction of its character to any torus in $G_v$ is compactly supported modulo the center $Z_v$ of $G_v$.

Remark. — (1) It is clear that Theorem 27 holds where $w$ is any place of $F$, to assert that the lifting bijects $\sigma$-invariant cuspidal $\tilde{\pi}$ with a supercuspidal component at $w$, and orbits of cuspidal $\pi$ whose component $\pi_w$ at $w$ lifts to a supercuspidal $\tilde{G}_w$-module; $\pi_w$ is necessarily supercuspidal. (2) Using the local base-change theorem and the trace PW-theorem it is easy to deduce (as in [FK], §27) that for every $\tilde{f}_v$ there is a matching $f_v$, and that for every $f_v$ with $\Phi(\gamma, f_v) = 0$ if $\gamma$ is regular but not a norm there is a matching $\tilde{f}_v$. As noted in Remark (2) following Lemma 20 the existence of matching $f_v$ and $\tilde{f}_v$ can be proven purely locally, using [K], in characteristic zero. (3) Using the local base-change theorem it is easy to deduce Proposition 24 (along the lines of [FK], §27.3), namely to show that corresponding spherical functions are matching. This gives an easier proof of the theorem of [F4], but only in the special case of $GL(n)$.

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