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VARIATIONS OF COMPLEX STRUCTURES ON AN OPEN RIEMANN SURFACE

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1. Introduction.

The purpose of this paper is to study the local properties of variations of complex structures on a relatively compact subdomain of an open Riemann surface.

Let M be an open Riemann surface and M_1 a relatively compact subdomain of M. Let $\mathcal{S}(t)$ be a family of complex structures on M (or on a neighbourhood of M_1) which depends holomorphically on t, t being in a neighbourhood U_1 of t_0 in $\mathbf{C}^m$. We suppose that $\mathcal{S}(t_0)$ is identical with the given structure on M. Consider the family of complex structures $\mathcal{S}(t, M_1)$ induced on M_1 by $\mathcal{S}(t)$. The family $\mathcal{S}(t, M_1)$ defines a complex analytic structure on $M_1 \times U_1$; we denote by $\mathcal{S}(M_1 \times U_1)$ the complex analytic manifold (or structure) thus defined. The projection $\pi_1 : M_1 \times U_1 \to U_1$ defines a family of deformations of complex structures in the sense of Kodaira-Spencer.

We first prove that for every sufficiently small Stein neighbourhood U of t_0, $\mathcal{S}(M_1 \times U)$ is a Stein manifold (Theorem 1). We then show that the restriction of the family

$$\pi_1 : \mathcal{S}(M_1 \times U_1) \to U_1$$

to a sufficiently small neighbourhood U of t_0 is complex analytically homeomorphic to the family $\pi: \Omega \rightarrow \pi(\Omega) \subset C^m$, where Ω is an open Stein submanifold of the product complex manifold $M \times \mathbf{C}^m$ and $\pi: M \times \mathbf{C}^m \rightarrow \mathbf{C}^m$ is the canonical projection of $M \times \mathbf{C}^m$ onto $\mathbf{C}^m$ (Theorem 2). This result may be viewed as a sort of local triviality (« semi-triviality ») or a local imbedding theorem.

We prove also an analogue of Theorem 2 for differentiable variations of complex structures (Theorem 3).

The proofs use the theory of linear elliptic partial differential equations and some tools from functional analysis.

We now give a rough sketch of the proofs. We show that there exists a sufficiently small neighbourhood U_2 of t_0 such that any functions which is holomorphic (upto the boundary) on any fibre over a point of U_2 can be extended to a holomorphic function on the whole fibre system restricted to U_2. From this it follows easily that we can separate points on the fibre system by holomorphic functions and that there exist $(m + 1)$ holomorphic functions which form a local coordinate system at a given point. To prove the holomorph-convexity, we first prove, by considering variations of complex structures on a disc, that the fibre system, restricted to a small Stein neighbourhood of t_0, is « locally holomorphically convex ». Then, by solving a problem analogous to the first Cousin problem with the help of currents, the holomorph-convexity is proved.

Once Theorem 1 is proved, Theorem B on Stein manifolds assures the vanishing of certain cohomology groups; we then prove theorem 2, adopting a method of Kodaira-Spencer.

Theorem 3 (differentiable case) is proved by solving the following problem : given Cousin data on $\mathcal{Y}(t, \overline{M}_1)$ which depend differentiably on the parameter, to find solutions of the (first) Cousin problem such that the solutions also depend differentiably on the parameter. The proof is inspired by a proof (unpublished) by L. Schwartz of some results concerning Cousin problems on a compact Riemann surface with varying complex structures and by some considerations in Kodaira-Spencer [2].

The author is thankful to Professor L. Schwartz for suggesting the use of Lemma 1, which simplifies the earlier demonstration of the author using power series expansions.

2. Statement of the theorems.

Let M be an open Riemann surface. Let Θ be the holomorphic tangent bundle of M. Let $\mathcal{E}(\Theta \otimes \overline{\Theta}^*)$ denote the space of $C^\infty(0, 1)$ forms with coefficients in Θ, endowed with the natural topology [5]. If $\tilde{\mu} \in \mathcal{E}(\Theta \otimes \overline{\Theta}^*)$ and z a local coordinate system then $\tilde{\mu}$ is of the form $\mu(z)\, d\bar{z} \otimes \partial/\partial z$. If we define $|\tilde{\mu}| = |\mu|$ locally, then $|\tilde{\mu}|$ is intrinsically defined as a function on M. If $\tilde{\mu} \in \mathcal{E}(\Theta \otimes \overline{\Theta}^*)$ with $|\tilde{\mu}| < 1$ then locally the forms $dz + \mu(z)\, d\bar{z}$ define a $(1, 0)$ form for a complex structure and thus $\tilde{\mu}$ defines a complex structure on M.

Let $t_0 \in \mathbf{C}^m$ and U_0 be an open set in $\mathbf{C}^m$ containing t_0. 't' will denote a point in U_0.

For our purposes a holomorphic family $\mathcal{S}(t)$ of complex structures on M will be, by definition, a holomorphic function $\tilde{\mu}(t)$ defined in U_0 with values in $\mathcal{E}(\Theta \otimes \overline{\Theta}^*)$ such that $|\tilde{\mu}(t)| < 1$ and $\tilde{\mu}(t_0) = 0$. We then have on $M \times U_0$ an almost complex structure defined locally by the forms $dz + \mu(t, z)\, d\bar{z}$, $dt^1, \ldots, dt^m$ where $t^1, \ldots, t^m$ are the coordinate function in $\mathbf{C}^m$. This almost complex structure is integrable since $\tilde{\mu}(z, t)$ is holomorphic in t. Hence we have a complex structure on $M \times U_0$ (see also proposition 1). We denote $M \times U_0$ endowed with this complex structure by $\mathcal{S}(M \times U_0)$. The projection $\pi_1 : S(M \times U_0) \to U_0$ is holomorphic and we have a holomorphic familly of deformations of complex structures in the sense of Kodaira-Spencer [2].

If M_1 is a subdomain of M and V a neighbourhood of t_0 in $\mathbf{C}^m$ with $V \subset U_0$, we denote the manifold $M_1 \times V$ with the complex structure induced from $\mathcal{S}(M \times U_0)$ by $\mathcal{S}(M_1 \times V)$. We denote by $\mathcal{S}(t)$ the complex analytic structure on M defined by $\tilde{\mu}(t)$.

We have

THEOREM 1. — *Let $\mathcal{S}$ be a holomorphic family of complex structures on an open Riemann surface* M. *Let* M_1 *be a relatively compact subdomain of* M. *Then there exists a neighbourhood* V *of* t_0 *such that for every Stein neighbourhood* U *of* t_0 *contained in* V, $\mathcal{S}(M_1 \times U)$ *is a Stein manifold.*

THEOREM 2. — *Let $\mathcal{S}$ be a holomorphic family of complex structures on an open Riemann surface* M *and* M_1 *a relatively compact subdomain of* M. *Then there exist a neighbourhood* U *of* t_0, *an open Stein submanifold* Ω *of the product manifold* $M \times \mathbf{C}^m$, *a complex analytic homeomorphism* Φ *of* $\mathcal{S}(M_1 \times U)$ *onto* Ω *and a complex analytic homeomorphism* φ *of* U *onto* $\pi(\Omega)$ (π *denoting the projection* $M \times \mathbf{C}^m \to \mathbf{C}^m$) *such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{S}(M_1 \times U) & \xrightarrow{\Phi} & \Omega \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{\varphi} & \pi(\Omega). \end{array}$$

REMARK. — In Theorems 1 and 2 and as well in Theorem 3, if the boundary of M_1 is smooth it is sufficient to assume that the variation is given only upto the boundary of M_1.

Let U_0 be an open subset in $\mathbf{R}^m$ and $t_0 \in U_0$. A differentiable family of complex structures we mean differentiable function $\tilde{\mu}(t)$ defined in U_0 with values in $\mathcal{E}(\Theta \otimes \overline{\Theta}^*)$ such that $|\tilde{\mu}(t)| < 1$ and $\tilde{\mu}(t_0) = 0$. (By differentiable we always mean « indefinitely differentiable ».) For a subdomain M_1 of M we denote by $\mathcal{S}(t, M_1)$, $t \in U_0$, the surface M_1 endowed with the complex structure defined by $\tilde{\mu}(t)$.

We have then

THEOREM 3. — *Let $\mathcal{S}(t)$ be a family of complex structures on* M *depending differentiably on t, t being in a neighbourhood of* t_0 *in* $\mathbf{R}^m$. *Let* M_1 *be a relatively compact subdomain of* M. *Then there exist a neighbourhood* U *of* t_0 *and a differentiable map* Φ *of* $M_1 \times U$ *into* M *which maps each fibre* $\mathcal{S}(t, M_1)$, $t \in U$, *biholomorphically into* M.

3. Some lemmas in functional analysis and potential theory.

Some of the lemmas stated in this section are more or less well-known. We state them here for convenience of reference.

We denote by U_0 an open set in $\mathbf{C}^m$ or $\mathbf{R}^m$ according as we consider holomorphic or differentiable variations. t_0 is a point of U_0.

Let E and F be two complete barrelled locally convex topological vector spaces. We shall say that a family of continuous linear operators $T_t: E \to F$, $t \in U_0$, depends holomorphically (resp. differentiably) on $t \in U_0$ if $t \to T_t$ is a holomorphic (resp. differentiable) function of U_0 with values in $\mathcal{L}_s(E, F)$, where $\mathcal{L}_s(E, F)$ denotes the space of continuous linear operators of E into F endowed with the topology simple convergence. We remark that if T_t depends holomorphically (resp. differentiably, differentiably) on t and $f(t)$ is a holomorphic (resp. differentiable) function with values in E then $t \to T_t f(t)$ is a holomorphic (resp. differentiable) function with values in F.

Lemma 1. — *Let* E *and* F *be two Banach spaces and* T_t: $E \to F$ *depend holomorphically (resp. differentiably) on t. Assume that* T_{t_0} *is an isomorphism. Then there exists a neighbourhood* U_0' *of* t_0 *such that* T_t *is an isomorphism for each* $t \in U_0'$ *and the operators* $T_t^{-1}: F \to E$ *depend holomorphically (resp. differentiably) on* $t \in U_0'$.

This lemma is a special case of implicit function theorem in Banach spaces and is proved easily.

Lemma 2. — *Let* E *and* F *be two Banach spaces and* T_t: $E \to F$ *depend holomorphically (resp. differentiably) on t. Assume that* T_{t_0} *admits of a right inverse. Then there exists a neighbourhood* U_0' *of* t_0 *such that for* $t \in U_0'$, T_t *admits of a right inverse depending holomorphically (resp. differentiably) on t.*

Proof. — We recall that a right inverse for T_{t_0} is a continuous linear map $S_{t_0}: F \to E$ such that $T_{t_0} \circ S_{t_0}$ is the identity map of F. Now we apply Lemma 1 to the operators

$$T_t \mid S_{t_0}(F) : S_{t_0}(F) \to F$$

and Lemma 2 follows.

Let D be a relatively compact open subset of $\mathbf{C}$. Let α be a fixed real number with $0 < \alpha < 1$. Let f be a complex valued function satisfying a Hölder condition of order α on $\overline{D}$. Put

$$\|f\|_{0,\alpha,D} = \operatorname*{Sup}_{\overline{D}} |f| + \operatorname*{Sup}_{\substack{z_1, z_2 \in \overline{D} \\ z_1 \neq z_2}} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha}.$$

We denote the space of these functions by $H_\alpha(D)$.

If f is a function which is once differentiable such that its partial derivatives satisfy in $\overline{D}$ a Hölder condition of order α put

$$||f||_{1,\alpha,D} = \operatorname{Sup}_{\overline{D}} |f| + \left\|\frac{\partial f}{\partial \bar{z}}\right\|_{0,\alpha,D} + \left\|\frac{\partial f}{\partial z}\right\|_{0,\alpha,D}.$$

Let $H_{1,\alpha}(D)$ denote the space of such functions.

Let now D be a disc $|z| < R$, $0 < R < \infty$ in the plane. The operator $\frac{\partial}{\partial \bar{z}}$ is a continuous linear operator from the Banach space $H_{1,\alpha}(D)$ (with the norm $||f||_{1,\alpha}$) to the Banach space $H_\alpha(D)$ (with the norm $||f||_{0,\alpha}$).

LEMMA 3. — *Let* D *be a disc in the plane. The operator*

$$\frac{\partial}{\partial \bar{z}} : H_{1,\alpha}(D) \rightarrow H_\alpha(D)$$

admits of a right inverse.

This lemma is classical. For instance convolution with $\frac{1}{\pi z}$ yields a right inverse [1].

Let M_0 be a relatively compact subdomain of an open Riemann surface M such that M_0 is bounded by a finite number of disjoint analytic Jordan curves. We shall say, for brevity, that M_0 has an analytic boundary. We shall denote by ∂M_0 boundary of M_0 in M.

Let $D_1, \ldots, D_k, D_{k+1}, \ldots, D_n$ be a covering of $\overline{M}_0$ by coordinate discs D_i in M with $\overline{D}_i$ compact and contained in a coordinate disc such that the following conditions are satisfied :

i) $\overline{D}_i$ is contained in M_0 for $i = 1, \ldots, k$

ii) if z_j is the coordinate function in D_j mapping D_j onto $|z| < \varepsilon$, then z_j maps for $j = k+1, \ldots, n$, $D_j \cap \overline{M}_0$ onto the « semi-disc » $\{|z| < \varepsilon, \operatorname{Im} z > 0\}$ and $D_j \cap \partial M_0$ onto $\{-\varepsilon < Rlz < \varepsilon\}$. Let D'_i denote the covering of $\overline{M}_0$ formed by $D_1, \ldots, D_k$, $D_{k+1} \cap \overline{M}_0, \ldots, D_n \cap M_0$. Let $\{D''_i\}$ be a shrinking of the covering $\{D'_i\}$.

Let $H_{1,\alpha}(M_0)$ denote the Banach space of complex valued functions in $\overline{M}_0$ which are once differentiable in M_0 and whose first partial derivatives satisfy a Hölder condition of order α

in every compact subset contained in a coordinate neighbourhood of $\overline{M}_0$ (e.g. D'_i) with the norm

$$\|f\|_{1,\alpha} = \operatorname*{Sup}_{i=1,\dots,n} \|f\|_{1,\alpha,D''_i}.$$

Let $\overset{0,1}{H}_\alpha(M_0)$ denote the space of (0, 1) forms whose coefficients satisfy a Hölder condition of order α in every compact set contained in a coordinate neighbourhood of $\overline{M}_0$. If $f \in \overset{0,1}{H}_\alpha(M_0)$ and $f = f_i\, d\bar{z}^i$ in D' (z^i being the coordinate function in D'_i) define

$$\|f\|_{0,\alpha,M_0} = \operatorname*{Sup}_i \|f_i\|_{0,\alpha,D''_i};$$

with this norm $\overset{0,1}{H}_\alpha(M_0)$ becomes a Banach space.

Lemma 4. — *Let* M_0 *be a relatively compact subdomain of* M *with analytic boundary. Then the operator*

$$d_{\bar{z}}:\ H_{1,\alpha}(M_0) \to \overset{0,1}{H}_\alpha(M_0)$$

admits of a right inverse.

Proof. — We give a sketch of the proof of this lemma. Let M_1 be a relatively compact subdomain of M, with analytic boundary, containing $\bigcup_{i=1,\dots,n} \overline{D}_i$. We first remark that we can find a continuous linear map $\rho : \overset{0,1}{H}_\alpha(M_0) \to \overset{0,1}{H}_\alpha(M_1)$ such that $\gamma \circ \rho =$ identity map of $\overset{0,1}{H}_\alpha(M_0)$, where $\gamma : \overset{0,1}{H}_\alpha(M_1) \to \overset{0,1}{H}_\alpha(M_0)$ denotes the restriction map. [The question being local at the boundary, locally the extension is given by reflection at the x-axis. For details see e.g. [4, Th. 2. 4]]. On $M_1 \times M_1$ there exists (H. Behnke-K. Stein, Math. Ann. 120, p. 436) a meromorphic differential $K(z, d\zeta)$, holomorphic for $z \neq \zeta$ such that in a coordinate disc around $z = \zeta$ we have,

$$K(z, d\zeta) = \left\{\frac{-1}{4\pi(z-\zeta)} + \text{regular function}\right\} d\zeta.$$

We may then estimate the potential

$$T_1\tilde{f} = 2i \int_{M_1} K(z, d\zeta) \wedge \tilde{f}(\zeta), \qquad \tilde{f} \in \overset{0,1}{H}_\alpha(M_1)$$

on compact subsets of D_i using the estimate on a disc for the potential with the kernel $\frac{1}{\pi(z-\zeta)}$ [1]. If $f \in \overset{0,1}{H}_\alpha(M_0)$ let Tf denote the restriction of $T_1(\rho(f))$ to M_0. Then $d_{\bar{z}}Tf = f$ and

$$\|Tf\|_{1,\alpha,M_0} \leqslant C_1\|\rho(f)\|_{0,\alpha,M_1} \leqslant C_2\|f\|_{0,\alpha,M_0}$$

with positive constants C_1 and C_2. This proves Lemma 4.

The next lemma will be required only for the holomorphic tangent bundle of M. But we shall prove it for a general holomorphic line bundle.

Let L be a holomorphic line bundle on M. Let $H_{1,\alpha}(M_0, L)$ denote the Banach space of sections of L in $\overline{M}_0$ which are once differentiable in M_0 and whose first partial derivatives satisfy a Hölder condition of order α. Let $\overset{0,1}{H}_\alpha(M_0, L)$ denote the Banach space of Hölder continuous (0, 1) forms in M_0 with coefficients in L (we introduce norms on $H_{1,\alpha}(M_0, L)$ and $\overset{0,1}{H}_\alpha(L)$ as on $H_{1,\alpha}(M_0)$ and $\overset{0,1}{H}_\alpha(M_0)$.

Lemma 5. — *Let* L *be a holomorphic line bundle on* M. *Let* M_0 *be a relatively compact subdomain with analytic boundary. Then the operator*

$$d_{\bar{z}}: H_{1,\alpha}(M_0, L) \rightarrow \overset{0,1}{H}_\alpha(M_0, L)$$

admits of a right inverse.

Proof. — Since M is an open Riemann surface, every holomorphic line bundle on M is holomorphically trivial. This follows for example from the exact sequence.

$$H^1(M, O) \rightarrow H^1(M, O^*) \rightarrow H^2(M, Z)$$

remarking that $H^1(M, O) = 0$, $H^2(M, Z) = 0$. (Here O denotes the sheaf of germs of holomorphic functions and O^* the sheaf of germs of non-vanishing holomorphic functions.) Since L is holomorphically trivial on M there exist topological isomorphisms

$$\psi_1 : H_{1,\alpha}(M_0, L) \rightarrow H_{1,\alpha}(M_0)$$
$$\psi_2 : \overset{0,1}{H}_\alpha(M_0, L) \rightarrow \overset{0,1}{H}_\alpha(M_0)$$

such that the following diagram is commutative:

$$\begin{array}{ccc} H_{1,\alpha}(M_0, L) & \xrightarrow{\psi_1} & H_{1,\alpha}(M_0) \\ \downarrow d_{\bar z} & & \downarrow d_{\bar z} \\ \overset{0,1}{H}_\alpha(M_0, L) & \xrightarrow{\psi_2} & \overset{0,1}{H}_\alpha(M_0) \end{array}$$

Since $d_{\bar z}: H_{1,\alpha}(M_0) \to \overset{0,1}{H}_\alpha(M_0)$ admits of a right inverse it follows thats $d_{\bar z}: H_{1,\alpha}(M_0, L) \to \overset{0,1}{H}_\alpha(M_0, L)$ admits of a right inverse.

4. Variation of complex structures on a disc.

PROPOSITION 1. — *Let D be a disc in the plane. Let $\mu(t) = \mu(z, t)$ be a holomorphic function defined in a neighbourhood of t_0 in $\mathbf{C}^m$ with values in $H_\alpha(D)$ with $\mu(t_0) = 0$. Then there exist a neighbourhood U' of t_0 and a C^1 function $\zeta(z, t)$ defined in $D \times U'$ such that*

i) $$\begin{cases} \dfrac{\partial\zeta(z, t)}{\partial\bar z} - \mu(z, t)\dfrac{\partial\zeta(z, t)}{\partial\bar z} = 0, \\ \dfrac{\partial\zeta(z, t)}{\partial\bar t^i} = 0, \quad i = 1, \ldots, m. \end{cases}$$

ii) *there exist positive constants K_1, and K_2 such that one has* $K_1|z_1 - z_2| \leqslant |\zeta(z_1, t) - \zeta(z_2, t)| \leqslant K_2|z_1 - z_2|$ for $z_1, z_2 \in \bar D$ *and all $t \in U'$.*

iii) *If* $F(t) = F(z, t) = \dfrac{1}{\zeta(z, t) - \zeta(z_0, t)}$, $z_0 \in D$, *the function $t \to F(t)$ is a holomorphic function in U' with values in $\mathcal{D}'(D)$, where $\mathcal{D}'(D)$ denotes the space of distributions in D; moreover for each fixed t, $F(z, t)$ is holomorphic outside z_0 for the complex structure defined by $dz + \mu(z, t)\,d\bar z\,(|\mu| < 1)$.*

Proof. — There exists a constant $C_1 > 0$ such that for $f, g \in H_\alpha(D)$ one has $\|fg\|_{0,\alpha} \leqslant C_1\|f\|_{0,\alpha}\|g\|_{0,\alpha}$. Hence the operator of multiplication by $\mu(z, t)$ is a holomorphic function of t with values in $\mathcal{L}_s(H_\alpha, H_\alpha)$. It follows that the operators

$$T_t = \frac{\partial}{\partial\bar z} - \mu(z, t)\frac{\partial}{\partial z} : H_{1,\alpha}(D) \to H_\alpha(D)$$

depend holomorphically on t. Now $T_{t_0} = \dfrac{\partial}{\partial\bar z}\cdot$ By Lemma 3

T_{t_0} admits a right inverse. Hence by lemma 2 there exists a neighbourhood U″ of t_0 and continuous linear operators

$$S_t : H_\alpha \to H_{1,\alpha}$$

depending holomorphically on $t \in U''$ such that $T_t \circ S_t =$ Identity map of H_α. Now $\mu(t)$ is a holomorphic function with values in H_α. Hence $f(t) = S_t(\mu(t))$ is a holomorphic function with values in $H_{1,\alpha}$. Let

$$\zeta(z, t) = z + f(z, t).$$

$\zeta(z, t)$ is of class C^1. Moreover

$$\begin{aligned}\frac{\partial \zeta}{\partial \bar{z}} = \frac{\partial f(z, t)}{\partial \bar{z}} &= \mu(z, t)\frac{\partial f}{\partial z} + \mu(z, t)\\ &= \mu(z, t)\left(1 + \frac{\partial f}{\partial z}\right)\\ &= \mu(z, t)\frac{\partial \zeta}{\partial z},\end{aligned}$$

so that $\zeta(z, t)$ satisfies

$$\text{i)} \qquad \begin{cases} \dfrac{\partial \zeta(z, t)}{\partial \bar{z}} - \mu(z, t)\dfrac{\partial \zeta}{\partial z} = 0, \\ \dfrac{\partial \zeta(z, t)}{\partial \bar{t}^i} = 0, \quad i = 1, \ldots, m. \end{cases}$$

To prove ii) we remark that there exists a constant $k > 0$ (depending only on D) such that for each $f \in H_{1,\alpha}$ one has

$$|f(z_1) - (z_2)| \leqslant k|z_1 - z_2|\, ||f||_{1,\alpha}, \qquad z_1, z_2 \in \bar{D}.$$

(This is proved easily applying the mean value theorem.) Since $f(t_0) = 0$ we can choose a relatively compact neighbourhood U′ of t_0 with $\overline{U'} \subset U''$ such that for $t \in U'$, $||f(t)||_{1,\alpha} \leqslant \frac{\varepsilon}{k}$, given ε with $0 < \varepsilon < 1$. It is evident that there exists a constant K_2 such that

$$|\zeta(z_1, t) - \zeta(z_2, t)| \leqslant K_2|z_1 - z_2|, \qquad t \in U', \qquad z_1, z_2 \in \bar{D}.$$

On the other hand

$$\begin{aligned}|\zeta(z_1, t) - \zeta(z_2, t)| &= |\{z_1 + f(z_1, t)\} - \{z_2 + f(z_2, t)\}|\\ &\geqslant |z_1 - z_2| - |f(z_1, t) - f(z_2, t)|\\ &\geqslant (1 - \varepsilon)|z_1 - z_2|.\end{aligned}$$

This completes the proof of ii).

To prove iii), we note that for t fixed $1/\zeta(z, t)-\zeta(z_0, t)$ is a locally summable function in D (see ii); and since

$$|F(z, t)| \leqslant K_1^{-1}/|z-z_0|,$$

$t \in U'$, we see, by Lebesgue's dominated convergence theorem, that $t \rightarrow F(t)$ is a continuous function with values in $\mathfrak{D}'(D)$. To prove that $F(t)$ is a holomorphic function with values in $\mathfrak{D}'(D)$ it is sufficient to prove that $h(t) = \langle F(t), \varphi\rangle$ is a holomorphic function of t for each $\varphi \in \mathfrak{D}(D)$. [$\mathfrak{D}(D)$ denotes the space of C^∞ functions with compact supports in D; $\langle F(t), \varphi\rangle$ denotes the scalar product between $F(t)$ and φ]. As was noted earlier $h(t)$ is a continuous function. Let $t_1 = (t_1^1, \ldots, t_1^m) \in U'$. We shall show that $h(t^1, t_1^2, \ldots, t_1^m)$ is differentiable at t_1^1 as a function of t^1.

Let

$$\psi(t^1) = \{h(t^1, t_1^2, \ldots, t_1^m) - h(t_1^1, t_1^2, \ldots, t_1^m)\}/(t^1 - t_1^1).$$

Then

$$\psi(t^1) = \int_K \frac{1}{t^1 - t_1^1} \times \left\{\frac{[\zeta(z, t_1) - \zeta(z_0, t_1)] - [\zeta(z, t^1, t_1^2, \ldots, t_1^m) - \zeta(z_0, t^1, \ldots, t_1^m)]}{[\zeta(z, t^1, t_1^2, \ldots, t_1^m) - \zeta(z_0, t^1, \ldots, t_1^m)] - [\zeta(z, t_1) - \zeta(z_0, t_1)]}\right\} \varphi \, dx\, dy$$

where K is the support of φ.

We assert that there exists a constant K_3 such that for t^1 in a sufficiently small neighbourhood of t_1^1 we have

(A)

$$\left|\frac{[\zeta(z, t_1^1, \ldots, t_1^m) - \zeta(z_0, t_1^1, \ldots, t_4^m)] - [\zeta(z, t^1, t_1^2, \ldots, t_1^m) - \zeta(z_0, t', t_1^2, \ldots, t_1^m)]}{t^1 - t_1^1}\right| \leqslant K_3/|z-z_0|.$$

In fact, consider the function with values in $H_{1,\alpha}$ defined in a neighbourhood of t_1^1:

$$g(t^1) = \begin{cases} \dfrac{\zeta(t_1^1, \ldots, t_1^m) - \zeta(t^1, t_2^2, \ldots, t_1^m)}{t^1 - t_1^1} & \text{for } t^1 \neq t_1^1 \\ \left\{\dfrac{d}{dt^1}\zeta(t^1, t_1^2, \ldots, t_1^m)\right\}_{t^1 = t_1^1} & \text{for } t^1 = t_1^1 \end{cases}$$

Since $\zeta(t)$ is a holomorphic function with values in $H_{1,\alpha}$,

$g(t^1)$ is a continuous function and hence in a neighbourhood of t_1^1, $||g(t^1)||_{1,\alpha} \leqslant K_4$. Using the inequality

$$|f(z_1) - f(z_2)| \leqslant k|z_1 - z_2|\,||f||_{1,\alpha}$$

we obtain (A). From (A) and the first inequality in ii) we see that the integrand is majorised by $K_5/|z - z_0|$ for all t^1 in a sufficiently small neighbourhood of t_1^1. By Lebesgue's theorem we see that $\lim_{t_1 \to t_1^1} \psi(t^1)$ exists and is equal to

$$-\int_K \frac{\frac{d}{dt^1}(\zeta(t^1, \ldots, t_1^m)(z))_{t^1=t_1^1} - \left(\frac{d}{dt^1}\zeta(t^1, \ldots, t_1^m)(z_0)\right)_{t^1=t_1^1}}{\{\zeta(z, t_1^1, \ldots, t_1^m) - \zeta(z_0, t_1^1, \ldots, t_1^m)\}^2}\, \varphi\, dx\, dy.$$

Similary we show that the other derivatives exist. This proves that $h(t)$ is holomorphic.

From the first inequality in ii) we see that $\zeta(z, t) - \zeta(z_0, t) \neq 0$ for $z \neq z_0$, $t \in U'$. The second assertion in iii) follows immediately from this fact. This completes the proof of Proposition 1.

REMARK 2. — Using i) and ii) we can show easily that if U is a polydisc contained in U′ the map $(t, z) \to (t, \zeta(t, z))$ maps $\mathcal{S}(U \times D)$ (endowed with the complex structure defined by $dz + \mu(z, t)\, d\bar{z}, dt^1, \ldots, dt^m$, $|\mu(z, t)| < 1$), biholomorphically onto a bounded domain of holomorphy in $\mathbf{C}^{m+1}$. Proposition 1 is also valid if we replace the disc by a bounded plane domain with a smooth boundary. Thus Theorems 1 and 2 are immediate consequences of Proposition 1 in the case of plane domains.

5. Elementary kernels for elliptic differential operators depending holomorphically on a parameter.

Let M be an open Riemann surface and let $\overset{p,q}{\mathfrak{D}}(M)$, $\overset{p,q}{\mathcal{E}}(M)$, $\overset{p,q}{\mathfrak{D}'}(M)$, $\overset{p,q}{\mathcal{E}'}(M)$ $(p = 0, 1,\ q = 0, 1)$ denote the space of C^∞ forms of type (p, q) with compact supports, C^∞ forms of type (p, q), currents of type (p, q) and currents of type (p, q) with compact supports respectively ,each endowed with the usual topology [5].

Let $\mathcal{S}(t, M)$ be a family of complex structures depending holomorphically on t. Define $T_t : \overset{0,0}{\mathfrak{D}'}(M) \to \overset{0,1}{\mathfrak{D}'}(M)$ by

$$T_t f = d_{\bar{z}} f - \langle \tilde{\mu}(t), d_z f \rangle, \qquad f \in \overset{0,0}{\mathfrak{D}'}(M)$$

where $\langle \tilde{\mu}(t), d_z f \rangle$ denotes the current of type (0, 1) obtained by contracting $\tilde{\mu}(t)$ (which is of type (— 1, 1)) and $d_z f$ (which is of type (1, 0)). We remark that a function f is holomorphic for the structure $\mathcal{S}(t, M)$ if and only if $T_t f = 0$. A function f defined on $U_0 \times M$ is holomorphic for the structure $\mathcal{S}(U_0 \times M)$ if and only if it satisfies the system of differential equations:

$$\begin{cases} T_t f(z, t) = 0, \\ \dfrac{\partial f(z, t)}{\partial \bar{t}^i} = 0, \quad i = 1, \ldots, m. \end{cases}$$

Proposition 2. — *Let* M_0 *be a relatively compact subdomain of* M, *with analytic boundary. Then there exists a neighbourhood* U_3 *of* t_0 *and for* $t \in U_3$ *continuous linear operators* $S_t : \overset{0,1}{\mathcal{E}}(M_0) \to \overset{0,0}{\mathfrak{D}'}(M_0)$ *depending* holomorphically *on* t *such that for* $f \in \overset{0,1}{\mathcal{E}'}(M_0)$ *one has* $T_t S_t f = f$ $(T_t = d_{\bar{z}} - \langle \tilde{\mu}, d_z \rangle)$.

Proof. — Let $H_{1,\alpha}(M_0)$ and $\overset{0,1}{H}_\alpha(M_0)$ have the meaning given in § 3. By Lemma 4, $T_{t_0} : H_{1,\alpha} \to \overset{0\,1}{H}_\alpha$ has a right inverse. Let $S_t : \overset{0,1}{H}_\alpha \to H_{1,\alpha}$ be right inverses defined in a neighbourhood U_3 of t_0 depending holomorphically on t (Lemma 2). S_t maps $\overset{0,1}{\mathfrak{D}}(M_0)$ into $H_{1,\alpha}$. We shall show that $S_t : \overset{0,1}{\mathfrak{D}} \to H_{1,\alpha}$ can be extended to a continuous linear map, still denoted by S_t, of $\overset{0,1}{\mathcal{E}'}$ into $\overset{0,0}{\mathfrak{D}'}$ and $S_t : \overset{0,1}{\mathcal{E}'} \to \overset{0\,0}{\mathfrak{D}'}$ depends holomorphically on t. This will prove Proposition 2, as is easy to see.

Now each T_t is a linear elliptic operator. By the hypo-ellipticity of T_t, S_t maps $\overset{0,1}{\mathfrak{D}}$ into $\overset{0,0}{\mathcal{E}}$ and $S_t : \overset{0,1}{\mathfrak{D}} \to \overset{0,0}{\mathcal{E}}$ is continuous by Banach's closed graph theorem. We prove that $S_t : \overset{0,1}{\mathfrak{D}} \to \overset{0,0}{\mathcal{E}}$ depends holomorphically on t. Let $\varphi \in \overset{0,1}{\mathfrak{D}}$. Then the current $F(z, t) = S_t \varphi$ satisfies the system of differential equations

$$\begin{cases} T_t F = \varphi \\ \dfrac{\partial F}{\partial \bar{t}^i} = 0, \qquad i = 1, \ldots, m. \end{cases}$$

Since this system is elliptic, $F(z, t)$ is a C^∞ function in $M_0 \times U_3$. It follows that $t \to S_t\varphi$ is a holomorphic function with values in $\overset{0,0}{\mathcal{E}}$.

Let $T_t' : \overset{1,0}{\mathcal{D}'} \to \overset{1,1}{\mathcal{D}'}$ be the transpose of the differential operator T_t. Then T_t is a linear elliptic differential operator with C^∞ coefficients. Let $S_t' : \overset{1,1}{\mathcal{E}'} \to \overset{1,0}{\mathcal{D}'}$ be the transpose of $S_t : \overset{0,1}{\mathcal{D}} \to \overset{0,0}{\mathcal{E}}$. Then $S_t : \overset{1,1}{\mathcal{E}'} \to \overset{1,0}{\mathcal{D}'}$ depends holomorphically on t. By the hypo-ellipticity of S_t', S_t' maps $\overset{1,1}{\mathcal{D}}$ into $\overset{1,1}{\mathcal{E}}$ and $S_t : \overset{0,1}{\mathcal{D}} \to \overset{1,0}{\mathcal{E}}$ is continuous. As we proved for S_t, we prove, using the hypo-ellipticity of the system $\left\{S_t', \frac{\partial}{\partial \bar{t}^1}, \dots \frac{\partial^t}{\partial \bar{t}^m}\right\}$ that $S_t : \overset{1,1}{\mathcal{D}} \to \overset{1,0}{\mathcal{E}}$ depends holomorphically on t. By taking transposes we obtain $S_t : \overset{0,1}{\mathcal{E}'} \to \overset{0,0}{\mathcal{D}'}$ depending holomorphically on t and coinciding on $\overset{0,1}{\mathcal{D}}$ with the S_t originally given.

This proves Proposition 2.

6. A result on the prolongation of holomorphic functions.

Proposition 3. — *Let* M_0 *be a relatively compact subdomain of* M *with analytic boundary. Then there exists a neighbourhood* U_2 *of* t_0 *with the following property: if* $f(z)$ *is a function which is holomorphic for the structure* $\mathcal{G}(t_1)$ *in* $\overline{M}_1$, $t_1 \in U_2$, *then there exists a function* $F(z, t)$ *in* $M_0 \times U_2$ *which is holomorphic for the structure* $\mathcal{G}(M_0 \times U_2)$ *such that* $F(z, t_1) = f(z)$.

Proof. — Let $S_t : \overset{0,1}{H_\alpha}(M_0) \to H_{1,\alpha}(M_0)$ be right inverses for T_t depending holomorphically on $t \in U_2$. Now $f \in H_{1,\alpha}$. Define

$$F(z, t) = f - S_t T_t f, \quad (t \in U_2).$$

We then have

$$T_t(f - S_t T_t f) = T_t f - T_t S_t T_t f = T_t f - T_t f = 0;$$

since $T_{t_1} f = 0$, $F(z, t_1) = f(z)$. $F(z, t)$ satisfies the system of differential equations

$$\begin{cases} T_t F(z, t) = 0, \\ \dfrac{\partial F}{\partial \bar{t}^i} = 0, \qquad i = 1, \ldots, m. \end{cases}$$

Hence $F(z, t)$ is holomorphic for the structure $\mathcal{S}(M_0 \times U_2)$.

7. Proof of Theorem 1.

We now proceed to prove Theorem 1. Let M_0 be a relatively compact sub-domain of M with analytic boundary such that $\overline{M}_1 \subset M_0$. Let $O_1, \ldots, O_i, \ldots, O_k$ be a finite number of coordinate discs for the structure $\mathcal{S}(t_0)$ with $\overline{O}_i \subset M_0$ and $\cup_i O_i \supset \overline{M}_1$. Let z^i be the coordinate function in O_i. Then $\tilde{\mu} = \mu_i \, d\bar{z}^i \otimes \dfrac{\partial}{\partial z^i}$. Let V_i, $i = 1, \ldots, k$, be neighbourhoods of t_0 such that functions $\zeta_i(z^i, t)$, $z^i \in O_i$, $t \in V_i$ can be defined satisfying conditions i), ii), iii) of Proposition 1. (By an obvious abuse of notation we use the letter z^i to denote a point on the Riemann surface and as well its image by the coordinate function z^i.) Let U_3 and U_2 be neighbourhoods of t_0 given in Proposition 2 and 3. Let V and V′ relatively compact neghbourhoods of t_0 such that $\overline{V} \subset V'$ and $\overline{V'} \subset \bigcap_{i=1,\ldots,k} V_i \cap U_2 \cap U_3$. Let U be a Stein neighbourhood of t_0 contained in V.

We first show that holomorphic functions on $\mathcal{S}(M_1 \times U)$ separate points. Since $(t^1, \ldots, t^m)$ are holomorphic functions on $\mathcal{S}(U \times M_1)$ we have only to consider the case when the points are on the same fibre. Let then (z_1, t_1), (z_2, t_1) be two points $t_1 \in U$, $z_1, z_2 \in M_1$, $z_1 \neq z_2$. Now there exists a function $f(z)$ holomorphic in $\overline{M}_0$ for $\mathcal{S}(t_1)$ with $f(z_1) \neq f(z_2)$. [This is shown, for example, by taking an open set slightly larger than M_0 and using the fact every open Riemann surface is a Stein manifold.] By Proposition 3 there exists a function $F(z, t)$ holomorphic for the structure $\mathcal{S}(M_0 \times U)$ such that $F(z_1, t_1) = f(z)$. Hence $F(z_1, t_1) \neq F(z_2, t_1)$.

Next let (z_1, t_1), $z_1 \in M_1$, $t_1 \in U$ be a point in $M_1 \times U$. We shall show that there exist $(m + 1)$ functions in $M_1 \times U$ which are holomorphic on $\mathcal{S}(M_1 \times U)$ and which form a local coordinate system at (z_1, t_1). Let $f(z)$ be a function holomorphic for $\mathcal{S}(t_1)$ in $\overline{M}_0$ which forms a local coordinate system at z_1 in M_0. Let $F(z, t)$ be an extension of $f(z)$ to $M_0 \times U$ as a function holomorphic for the structure $\mathcal{S}(M_0 \times U)$. Suppose $z_1 \in O_i$. With respect to the coordinate system $(z^i, t^1, \ldots, t^m)$ the Jacobian of $(F, t^1, .., t^m)$ at (z_1, t_1) is

$$\left|\left[\frac{\partial}{\partial z^i} F(z, t)\right]_{(z_1, t_1)}\right|^2 (1 - |\mu_i(z, t)|^2)$$

or

$$\left|\frac{\partial f(z)}{\partial z^i}\right|^2 (1 - |\mu_i(z, t)|^2).$$

But $\left(\frac{\partial f}{\partial z^i}\right)_{z^i = z_1} \neq 0$; for if it were zero, then $\frac{\partial f}{\partial \bar{z}^i} = \mu_i \frac{\partial f}{\partial z^i}$ would be zero so that the Jacobian of f at z_1 would be zero. Thus the Jacobian of $(F, t^1, .., t^m)$ at (z_1, t_1) is different from zero.

Finally we show that given infinite discrete set of points $\{z_n, t_n\}$, $z_n \in M_1$, $t_n \in U$ there exists a holomorphic function Ψ on $\mathcal{S}(M_1 \times U)$ such that the sequence $\{\Psi(z_n, t_n)\}$ is not bounded. Now either the sequence $\{t_n\}$ contains an infinite discrete subset or the sequence $\{z_n\}$ contains an infinite discrete subset. If $\{t_n\}$ contains an infinite discrete subset we can find, since U is Stein, a holomorphic function $F(t)$ in U such that $\{F(t_n)\}$ is not bounded. Then $\Psi(z, t) = F(t)$ is holomorphic on $\mathcal{S}(M_1 \times U)$ and $\Psi(z_n, t_n)$ is not bounded. If $\{z_n\}$ contains an infinite discrete subsequence $\{z_k\}$ let $z_0 \in \overline{M}_1 \subset M_0$ be an adherent point of $\{z_k\}$, $z_0 \notin M_1$. Suppose $z_0 \in O_i$. Consider the currents of degree 0, $F(t) = 1/\zeta_i(z^i, t) - \zeta_i(z_0, t)$ in O_i. Since $F(t)$ is a holomorphic function with values $\overset{0}{\mathcal{D}}'(O_i)$ and $T_t \colon \overset{0}{\mathcal{D}}'(O_i) \to \overset{0,1}{\mathcal{D}}'(O_i)$ depends homorphically on t, $T_t F(t)$ is a holomorphic function t with values in $\overset{0,1}{\mathcal{D}}'(O_i)$. But the supports of $T_t F(t)$ are at z_0 and hence $T_t F(t)$ is a holomorphic function with values in $\overset{0,1}{\mathcal{E}}'(M_0)$. Let S_t be the operators given by Proposition 2, in U_3. Let $\Psi(t) = S_t(T_t F(t))$. Then

$$\begin{cases} T_t \Psi(t) = (T_t F(t)), \\ \dfrac{\partial \Psi(t)}{\partial \bar{t}^i} = 0, \quad i = 1, \ldots, m. \end{cases}$$

Since $z_0 \notin M_1$, $\Psi(t)$ defines a function in $M_1 \times U$ satisfying in $M_1 \times U$

$$\begin{cases} T_t\Psi(z, t) = 0, \\ \dfrac{\partial}{\partial \bar{t}^i}\Psi(z, t) = 0, \quad i = 1, \ldots, m; \end{cases}$$

that is $\Psi(z, t)$ is holomorphic on $\mathcal{G}(M_1 \times U)$. It remains to show that $\{\Psi(z_k, t_k)\}$ is not bounded. Let O'_i be a relatively compact neighbourhood of z_0 such that $\overline{O'_i} \subset O_i$. We may suppose that all z_k belong to O'_i. On $O_i \times V'$ the currents $G(z, t) = F_t(z) - \Psi_t(z)$ satisfies the system of differential equations

$$\begin{cases} T_tG(z, t) = 0 \\ \dfrac{\partial}{\partial \bar{t}^i}G(z, t) = 0, \quad i = 1, \ldots, m. \end{cases}$$

Hence $G(z, t)$ is a C^∞ function in $O_i \times V'$ and is hence bounded on $O'_i \times U$. For $z \in M_1 \times O_i$, $t \in U$

$$\Psi(z, t) = F(z, t) - G(z, t).$$

Hence

$$\Psi(z_k, t_k) = F(z_k t_k) - G(z_k, t_k) = \frac{1}{\zeta(z_k, t_k) - \zeta(z_0, t_k)} - G(z_k, t_k)$$

So

$$|\Psi(z_k, t_k) + G(z_k, t_k)| \geqslant K_2^{-1}/|z_k - z_0|.$$

by Proposition 1. Since $G(z_k, t_k)$ is bounded and z_0 is adherent to z_k, it follows that $\Psi(z_k, t_k)$ is not bounded. This completes the proof of Theorem 1.

8. Proof of Theorem 2.

The proof is essentially same as the proof of Theorem 5. 1 in Kodaira-Spencer [2], once we have Theorem 1. Still we give the complete proof since some changes are required in our case. It is sufficient to prove the theorem without the requirement that Ω be Stein. For, once we have a Φ with Ω an open subset of $M \times \mathbf{C}^m$ we could restrict Φ to $\mathcal{G}(M_1 \times U)$ where U is a sufficiently small Stein neighbourhood of t_0 and obtain Theorem 2 (since $\mathcal{G}(M_1 \times U)$ is Stein by Theorem 1).

Thus it is enough to show that there exist a neighbourhood U' of t_0, an analytic homeomorphism $\Phi: \mathcal{S}(M_1 \times U') \to \Omega$ where Ω is an open submanifold of $M \times \mathbf{C}^m$, and an analytic homeomorphism $\varphi: U' \to \pi(\Omega)$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{S}(U' \times M_1) & \xrightarrow{\Phi} & \Omega \\ \downarrow{\scriptstyle \pi_1} & & \downarrow \\ U' & \xrightarrow{\varphi} & \pi(\Omega) \end{array}$$

We make the following inductive assumption:

A_{p-1}: If the dimension of U_0 is $(p-1)$ and M_1, is any relatively compact subdomain of an open Riemann surface M, then there exists a neighbourhood U of t_0 and a holomorphic map h: $\mathcal{S}(M_1 \times U)$ into M which maps each fibre biholomorphically into M.

Now, assumig A_{p-1} we prove A_p.

Let M_0 be a relatively compact subdomain of M such that $\overline{M}_1 \subset M_0$. Let W be a sufficiently small Stein neighbourhood of t_0 in $\mathbf{C}^p$, with $\overline{W}$ compact, $\overline{W} \subset U_0$. Then $\mathcal{S}(M_0 \times W)$ is a Stein manifold. If $\mathcal{F}$ denotes the holomorphic tangent bundle along the fibres, then by Theorem B on Stein manifolds $H^1(\mathcal{S}(M_0 \times W), \mathcal{F}) = 0$. From the exact sequence

$$H^0(\mathcal{S}(M_0 \times W), \Pi) \to H^0(W, T) \to H^1(\mathcal{S}(M_0 \times W), \mathcal{F})$$

(Π denotes the sheaf of germs of holomorphic vector fields which are projectable, T denotes the sheaf of germs of holomorphic vector fields on W), we see that the vector field $\frac{\partial}{\partial t^p}$ can be lifted into a holomorphic vector field X of $\mathcal{S}(M_0 \times W)$. We may suppose $t_0 = 0$. Let $f(x) = \exp(-\pi^p(x)X)$, $x \in M_1 \times W'$ where $\pi^p(x)$ is defined as follows: if $x = (z, t^1, \ldots, t^p)$, $\pi^p(x) = t^p$. By the complex analytic analogue of Proposition 5. 1 in [2], for a neighbourhood $W' \subset W$, f maps $\mathcal{S}(M_1 \times W')$ holomorphically into $\mathcal{S}(M_0 \times (t^1, \ldots, t^{p-1}, 0))$ mapping the fibre at $(t^1, \ldots, t^p)$ biholomorphically into $\mathcal{S}((t^1, \ldots, t^{p-1}, 0), M_0)$. M_0 being a relatively compact subdomain of M, there exists, by the inductive hypothesis, a holomorphic map g:

$$M_0 \times (t^1, \ldots, t^{p-1}, 0) \to M$$

which maps each fibre biholomorphically into M. Taking the composite $h = g \circ f$ we get a holomorphic map of $\mathcal{S}(U_4 \times M_1)$ where U_4 is a neighbourhood of t_0 in $\mathbf{C}^p$, mapping each fibre biholomorphically into M. This proves A_p.

Once we have proved the assertion A_m, consider the map

$$\Phi : \mathcal{S}(U \times M_1) \to U \times M$$

defined by $(t, z) \to (t, h(t, z))$. Φ is holomorphic and one to one. By a known theorem on holomorphic functions Φ maps $\mathcal{S}(U \times M_1)$ biholomorphically onto an open subset Ω of $U \times M$ and we have the commutative diagram

$$\begin{array}{ccc} S(U \times M_1) & \xrightarrow{\ \Phi\ } & \Omega \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{\text{identity}} & U \end{array}$$

This completes the proof of Theorem 2.

9. Differentiable variations of complex structures. Proof of Theorem 3.

Let $\mathcal{S}(t, M)$ be a differentiable variation of complex structures on an open Riemann surface M, $t \in U_0 \subset R^m$. Let J_t be the almost complex structure tensor corresponding to the structure $\mathcal{S}(t, M)$. On $M \times U_0$ let J denote the tensor along the fibres composed of $\{J_t\}$. If X is a *projectable* vector field on $M \times U_0$ (with respect to the projection $M \times U_0 \to U_0$) we remark that the Lie derivative of J with respect to the vector field X, denoted by [X, J], is defined as a tensor along the fibres.

Let X be a projectable vector field on $M \times U_0$ satisfying the condition $[X, J] = 0$. Let M' (resp U_0') be a relatively compact subdomain of M (resp. U_0'). If exp (sX) denotes the one parameter family of transformations associated with X, exp (sX) is a diffeomorphism of $M' \times U_0'$ into $M \times U_0$ which maps $\mathcal{S}(t, M')$, $t \in U_0'$ biholomorphically into $\mathcal{S}(\exp(s\nu)(t), M)$, where ν denotes the projection of X on U_0. Now referring to the proof of Theorem 2, we see that to prove Theorem 3 it is sufficient to prove

PROPOSITION 4. — *Let $\mathcal{S}(t)$ be a differentiable family of complex structures on an open Riemann surface* M. *Let* M_1 *be a relatively compact subdomain of* M. *Then there exists a neighbourhood* U_1 *of* t_0 *in* R^m *such that every differentiable vector field* (*real*) *on* U_1 *can be lifted into a differentiable vector field* X *on* $M_1 \times U_1$ *satisfying the condition* $[X, J] = 0$, J *denoting the tensor along the fibres composed on the almost complex structure tensors along the fibres.*

Proof of Proposition 4. — Let M_0 be a relatively compact subdomain (of M) with analytic boundary, with $\overline{M}_1 \subset M_0$.

Let Θ_t denote the holomorphic tangent bundle of $\mathcal{S}(t, M)$. Let $\mathcal{F} = U_t\Theta_t$ be the bundle on $M \times U_0$ composed of the holomorphic tangent bundles along the fibres. If U_2 is a spherical neighbourhood of t_0 with $U_2 \subset U_0$ then $\mathcal{F}|M \times U_2$ is differentiably equivalent to the bundle $U_2 \times \Theta_{t_0}$ (Homotopy theorem). It follows that there exist isomorphisms

$$\psi_1(t) : H_{1,\alpha}(M_0, \Theta_t) \rightarrow H_{1,\alpha}(M_0, \Theta_{t_0}),$$
$$\psi_2(t) : \overset{0,1}{H}_\alpha(M_0, \Theta_t) \rightarrow \overset{0,1}{H}_\alpha(M_0, \Theta_{t_0}),$$

depending differentiably on t such that $\psi_1(t_0) = \text{identity}$, $\psi_2(t_0) = \text{identity}$. Let

$$T_t = \psi_2(t)\, d_{\bar{z}}(t)\, \psi_1(t)^{-1} : H_{1,\alpha}(M_0, \Theta_{t_0}) \rightarrow \overset{0,1}{H}_\alpha(M_0, \Theta_{t_0})$$

where $d_{\bar{z}}(t)$ denotes the $d_{\bar{z}}$ operator with respect to the structure $\mathcal{S}(t)$. T_t depends differentiably on t. Since $T_{t_0} = d_{\bar{z}}(t_0)$ admits of a right inverse by Lemma 5, there exist a neighbourhood U_3 of t_0 and operators

$$S_t : \overset{0,1}{H}_\alpha(M_0, \Theta_{t_0}) \rightarrow H_{1,\alpha}(M_0, \Theta_{t_0}), \quad t \in U_3$$

depending differentiably on $t \in U_3$ and such that S_t is a right inverse of T_t (Lemma 2).

Let M_2 be a relatively compact subdomain of M with $\overline{M}_0 \subset M_2$ Let U_4 be a neighbourhood of t_0 such that there exist a finite open covering $O_1, \ldots, O_k$ of M_2 and diffeomorphisms g_i of $O_i \times U_4$ into $\mathbf{C} \times U_4$ which maps $\mathcal{S}(t, O_i)$, $t \in U_4$ biholomorphically into in $\mathbf{C} \times (t)$. [Such a neighbourhood U_4 exists. This follows from the definition of differentiable variation of complex structures in the sense of Kodaira-Spencer. With our definition this follows from the differentiable analogue of Proposition 1]. We denote the coordinate function in $O_i \times U_4$ by (z^i, t).

Let U_1 be a relatively compact neighbourhood of t_0 in R^m such that $\overline{U}_1 \subset U_2 \cap U_3 \cap U_4$. Let $\rho = (\rho_1(t), \ldots, \rho_m(t))$ be a differentiable vector field in U_1. In $O_i \times U_4$ consider the vector field π_i defined by $(0, \rho_1(t), \ldots, \rho_m(t))$ with respect to the coordinate system (z^i, t). We have $[\pi_i, J] = 0$. Put $\theta_{ij} = \pi_i - \pi_j$ in $(O_i \times U_4) \cap (O_j \times U_4)$. Let $\theta'_{ij} = \theta_{ij} - iJ\theta_{ij}$. Then θ'_{ij} are sections of $\mathcal{F}$ over $(O_i \times U_4) \cap (O_j \times U_4)$ whose restriction to each fibre is holomorphic. Evidently there exist differentiable sections $f'_i(z^i, t)$ of $\mathcal{F}$ over $O_i \times U_4$ such that $f'_i - f'_j = \theta'_{ij}$, in $(O_i \times U_4) \cap (O_j \times U_4)$. If we define $\varphi(t) = d_{\bar{z}}(t) f'_i(z^i, t)$, $\varphi(t)$ is a $(0, 1)$ form on $\mathcal{S}(t, M_2)$ with values in Θ_t which depends differentiably on t. Let $\eta(t) = \psi_2(t)\{\varphi(t) \mid \overline{M}_0\}$ [$\psi_2(t)$ is the isomorphism defined earlier.] Then $\eta(t)$ is a differentiable function with values in $\overset{0,1}{H}_\alpha(M_0, \Theta_{t_0})$. For $t \in U_1$, let $h_1(t) = S_t\eta(t)$. Then $h_1(t)$ depends differentiably on t. Let $h_2(t) = \{\psi_1(t)\}^{-1}(h_1(t))$. Then $h_2(t)$ depends differentiably on t and satisfies $d_{\bar{z}}(t) h_2(t) = \varphi(t)$. [It follows easily from differentiability theorem for elliptic differential equations that $h_2(z, t)$ is a differentiable vector field on $M_1 \times U_1$. See proposition 1 in [3]]. Let

$$h(z, t) = \frac{1}{2}\{h_2(z, t) + \bar{h}_2(z, t)\}$$

and

$$f_i(z, t) = \frac{1}{2}\{f'_i(z, t) + \bar{f}'_i(z, t)\}.$$

Define

$$X = \pi_i + h - f_i \qquad \text{in} \qquad (O_i \cap M_1) \times U_1.$$

Then X is globally defined on $M_1 \times U_1$, projects into ρ and satisfies the equation $[X, J] = 0$. This completes the proof of Proposition 4.

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