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INTEGRALS FOR HOLOMORPHIC FOLIATIONS WITH SINGULARITIES HAVING ALL LEAVES COMPACT (*)

by Xavier GOMEZ-MONT

The problem of understanding smooth foliations with all leaves compact has been analysed by Reeb, Epstein and Edward, Millet and Sullivan ([R], [E], [EMS]). In the holomorphic category, these last ones have shown that if the manifold is compact and Kaehler, then the leaf space is Hausdorff, and hence carries a canonical structure of a complex analytic space. Non-singular holomorphic foliations are difficult to find, but singular holomorphic foliations are abundant, at least in projective varieties. The objective of this paper is to give an extension of the theorem of Edwards, Millet and Sullivan to the case of singular holomorphic foliations in projective varieties. Due to the singularities (i.e. the closure of the leaves passing through the same point, or irreducible varieties converging to reducible varieties), the conclusion obtained is in the rational category, instead of the holomorphic one. Our technique is different from the one used in [EMS]: we rely on the Hilbert scheme parametrizing subvarieties of a given projective variety.

To describe the result, let M be a projective integral (i.e. irreducible and reduced) variety over the complex numbers. A holomorphic foliation (with singularities) $\mathcal{F}$ of codimension q in M is a coherent subsheaf of the sheaf of differential 1-forms $\mathcal{F} \subset \Omega_M^1$ such that over a Zariski dense open subset of M it defines a non-singular holomorphic foliation of codimension q . The leaves of $\mathcal{F}$ are the leaves of this non-singular

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foliation. A subvariety of M is quasiprojective if it is of the form $V - W$, with V and W projective subvarieties of M .

In this paper we prove :

THEOREM 3. — *Let $\mathcal{F}$ be a holomorphic foliation (with singularities) of codimension q in the projective integral variety M , and assume that every leaf L of $\mathcal{F}$ is a quasiprojective subvariety of M ; then there is a projective integral variety V of dimension q and a rational map $f: M \rightarrow V$ such that the closure of a general f -fibre is the closure of a leaf of $\mathcal{F}$.*

By pulling back the rational functions on V , one may formulate the result by saying that the set of rational first integrals of the foliation is a function field of transcendence degree q . The variety V is a geometric realization of this function field. We also obtain a generic equidesingularization of the leaves :

COROLLARY 4. — *Let M , $\mathcal{F}$ and V be as in Theorem 3, then there is a quasiprojective smooth variety M' with a smooth, proper and surjective holomorphic map $f': M' \rightarrow V - W$, W a subvariety of V and a holomorphic birational map $\psi: M' \rightarrow M$ such that the image under ψ of an f' -fibre is the closure of an $\mathcal{F}$ -leaf (i.e. $\psi(f'^{-1}(v)) = \bar{L}$).*

The main ingredient in the proof of the theorem consist in showing that there is a unique q dimensional variety in the Hilbert scheme of M parametrizing codimension q integral subvarieties of M which are not contained in the singular locus of $\mathcal{F}$ and which are tangent to $\mathcal{F}$. The Zariski closure of this subvariety is V .

I would like to thank Dominique Cerveau for posing the problem for foliations by curves in projective 3-space, which was the motivation for carrying out this research.

1. The Hilbert scheme.

We will consider projective or quasiprojective varieties with the algebraic Zariski topology. Recall that a subset of a projective variety is constructible if it can be written as a finite disjoint union of locally closed subsets ([H], p. 94).

It has been shown by Grothendieck ([G]) that given a projective variety M , one may introduce into the set $\mathcal{Hilb}(M)$ of subvarieties

of M an algebraic structure converting it into a countable union of projective varieties and there is a proper and flat morphism h

$$(1) \quad \begin{array}{ccc} \mathcal{V}_M & \subset & M \times \mathcal{H}ilb_M \\ h \searrow & & \swarrow p_2 \\ & \mathcal{H}ilb_M & \end{array}$$

such that the fibre over any point in $\mathcal{H}ilb_M$ is the variety it represents, and it is universal for flat and proper families of subvarieties of M . $h: \mathcal{V}_M \rightarrow \mathcal{H}ilb_M$ is called the Hilbert scheme of M .

LEMMA 1. — Let M be a projective variety, A a projective subvariety of M and $f: X \subset M \times T \rightarrow T$ a flat, proper algebraic morphism between quasiprojective varieties parametrizing subvarieties of M . If $X_t = f^{-1}(t)$ denotes the f fibre over t , then the set

$$T' = \{t \in T/X_t \text{ is integral and not contained in } A\}$$

is open in T . The subset

$$T'' = \{t \in T'/\text{Sing}(X_t) \subseteq A\}$$

is constructible in T .

Proof. — The condition of X_t to be integral is open by [G-D] IV.12.2.4.viii, and since the image of $\mathcal{H}ilb(A) \rightarrow \mathcal{H}ilb(M)$ is closed in every connected component of $\mathcal{H}ilb(M)$ ([G] Theorem 3.2), we obtain that T' is open. Applying [G-D] IV.12.1.6.iii to the restriction of f to $X \cap (M - A) \times T'$ we obtain that T'' is constructible.

2. Singular foliations with quasiprojective leaves.

Let M be a projective integral variety over the complex numbers and Ω_M^1 the sheaf of differential 1-forms on M . A holomorphic foliation (with singularities) in M is a coherent subsheaf $\mathcal{F}$ of Ω_M^1 such that $\mathcal{F}$ satisfies the Frobenius integrability condition on a Zariski dense open subset, namely, $d\mathcal{F} \subset \mathcal{F} \wedge \Omega_M^1$ on a Zariski dense open subset, where d is the differential operator on differential forms. The singular set $\text{Sing } \mathcal{F}$ of the foliation $\mathcal{F}$ is the union of the singular points of M and the set of points where $\Omega_M^1/\mathcal{F}$ is not locally free. In $M - \text{Sing } \mathcal{F}$ we have a non-singular holomorphic foliation. The leaves of $\mathcal{F}$ are the leaves of the non-singular holomorphic foliation in $M - \text{Sing } \mathcal{F}$. The

rank of the sheaf $\mathcal{F}$ is called the *codimension of the foliation*. A smooth quasiprojective subvariety Y of M is *tangent to the foliation* if for any irreducible component V of Y , it is not contained in $\text{Sing } \mathcal{F}$ and the image of the sheaf $\mathcal{F}$ under the canonical map $\Omega_M^1|_V \rightarrow \Omega_V^1$ has support on a proper analytic subset.

LEMMA 2. — *Let M'' and T be quasiprojective varieties with M'' non-singular, $\mathcal{F}$ a non-singular holomorphic foliation in M'' and X a closed subvariety of $M'' \times T$ such that the projection to the second factor $g: X \rightarrow T$ is a smooth morphism with connected fibres, then the set of points of T such that X_t is tangent to $\mathcal{F}$ is a Zariski closed subset of T .*

Proof. — Denote by p_j the projections of $M'' \times T$ to the factors and $\omega: \mathcal{F} \rightarrow \Omega_{M''}^1$ the map defining $\mathcal{F}$. Let η be the map between locally free sheaves on X defined by the diagram

$$\begin{array}{ccccc} p_1^* \Omega_{M''}^1|_X & = & \Omega_{p_2}^1|_X & \rightarrow & \Omega_g^1 \rightarrow 0 \\ & & \uparrow & \nearrow & \\ & & p_1^* \omega|_X & \eta & \\ & & p_1^*(\mathcal{F})|_X & & \end{array}$$

where $\Omega_{p_2}^1$ and Ω_g^1 are the sheaves of relative 1-forms (see [H]), p. 176). Let $Z \subset X$ be the Zariski closed set defined by $\eta = 0$. (This is well defined since η is a map of locally free sheaves). X_t is tangent to $\mathcal{F}$ if and only if X_t is contained in Z . The set of points of T such that $Z_t = X_t$ is the set of points where Z_t has dimension $\dim X_t$, which is closed by [Sh], p. 61, applied to the irreducible components of the closure $\bar{Z}$ of Z in $\bar{M} \times T$, where $\bar{M}$ is a projective closure of M'' .

THEOREM 3. — *Let $\mathcal{F}$ be a holomorphic foliation (with singularities) of codimension q in the projective integral variety M , and assume that every leaf L of $\mathcal{F}$ is a quasiprojective subvariety of M ; then there is a projective integral variety V of dimension q and a rational map $f: M \rightarrow V$ such that the closure of a general f -fibre is the closure of a leaf of $\mathcal{F}$.*

Proof. — Let T'' be the constructible set obtained by applying Lemma 1 to those components of the Hilbert scheme of M that parametrize subvarieties of M of codimension q , with $A = \text{Sing } (\mathcal{F})$. Let $\{T_j\}$ be the decomposition of T'' into a disjoint union of a countable number of reduced quasiprojective varieties (this is possible since T'' is

constructible in every connected component of $\mathcal{Hilb}(M)$), and let $\{h'_j: \mathcal{V}_j \rightarrow T_j\}$ be the restriction of the Hilbert scheme (1) of M to T_j . Let S_j be the reduced subvariety of T_j obtained by applying Lemma 2 to the family

$$\mathcal{W}_j = \mathcal{V}_j - A \times T_j \rightarrow T_j$$

of smooth subvarieties of $M\text{-Sing}(\mathcal{F})$.

We claim that there is one and only one irreducible component of one S_j of dimension q . To see this, observe that the projection of $\mathcal{W}_j \cap M \times S_j$ to the first factor is an injective map. This is so, since every fibre of h'_j is a reduced and irreducible variety tangent to $\mathcal{F}$ in $M - \text{Sing } \mathcal{F}$, and hence is completely determined by its set of points. Since the image of this projection is a constructible set, this implies that the dimension of S_j is less than or equal to q . If all of them had dimension less than q then, since there are a countable number of such components, their union would be properly contained in $M - \text{Sing } \mathcal{F}$. But we know that the union is all of $M - \text{Sing } \mathcal{F}$, hence there must be at least one component of $\{S_j\}$ which has dimension q . There can be no more than one such component, since these components have as image in M disjoint constructible sets of total dimension, and by hypothesis M is irreducible.

Let V be the closure (with respect to the Zariski topology) in $\mathcal{Hilb}(M)$ of this component of dimension q , provided with its reduced structure, and let $h_0: \mathcal{V} \rightarrow V$ be the restriction of the Hilbert scheme to V . The projection of $\mathcal{V} \subset M \times V$ to the first factor is surjective, and it is injective over a Zariski dense, hence it is a birational map, with inverse a birational map ϕ .

$$\begin{array}{ccc} & \phi & \\ \mathcal{V} & \xrightarrow{\quad} & M \\ h_0 \searrow & & \swarrow f \\ & V & \end{array}$$

The rational map $f = h_0 \circ \phi: M \rightarrow V$ satisfies the properties required to prove the theorem. $\square$

COROLLARY 4. — *Let M , $\mathcal{F}$ and V be as in Theorem 3, then there is a quasiprojective smooth variety M' with a smooth, proper and surjective*

holomorphic map $f' : M' \rightarrow V - W$, W a subvariety of V and a holomorphic birational map $\psi : M' \rightarrow M$ such that the image under ψ of an f' -fibre is the closure of an $\mathcal{F}$ -leaf (i.e. $\psi(f'^{-1}(v)) = L$).

Proof. — Let $\mathcal{V}' \rightarrow \mathcal{V}$ be a desingularization of $\mathcal{V}$ ([Hi]). The map ψ' from $\mathcal{V}'$ to M obtained by composition

$$\begin{array}{ccccc}
 & & \psi' & & \\
 & & \text{---} & & \\
 \mathcal{V}' & \xrightarrow{\quad} & \mathcal{V} & \xrightarrow{\quad} & M \\
 & \searrow h'' & \downarrow h_0 & \nearrow f & \\
 & & V & &
 \end{array}$$

is a holomorphic birational map. Let $V - W$ be the non-empty Zariski open subset of V where the map h'' is smooth ([H], p. 272) and let $M' = h''^{-1}(V - W)$. Then the restriction f' of h'' and ψ of ψ' to M' satisfy the properties in the statement of the corollary. $\square$

The following lemma gives conditions under which we may apply Theorem 3.

LEMMA 5. — *Let $\mathcal{F}$ be a holomorphic foliation (with singularities) of codimension q in an integral projective variety M such that $\text{Sing}(\mathcal{F})$ has codimension bigger than q . Then the leaves of $\mathcal{F}$ are quasiprojective if and only if they are closed subsets in $M - \text{Sing}(\mathcal{F})$ with the transcendental topology.*

Proof. — A Theorem of Bishop (see [S]) asserts that if the local $2(\dim M - k)$ Hausdorff measure of the closure $\bar{B}$ of a pure codimension k subvariety B of $M - \text{Sing}(\mathcal{F})$ intersected with $\text{Sing}(\mathcal{F})$ is zero, then $\bar{B}$ is an analytic subset of M . The hypothesis on the codimension of $\text{Sing}(\mathcal{F})$ imply that the hypothesis in Bishop's Theorem are satisfied for every closed leaf of $\mathcal{F}$, hence the lemma. $\square$

Remarks. — 1) The flatness condition determines which codimension q subvarieties of M should be considered as leaves of M , even though they are reduced (branching behaviour of the foliation at the leaf), reducible (breaking of the generic leaf into several components; these leaves would give non-Hausdorff points if we consider a naive leaf space) or completely contained in the singular set of $\mathcal{F}$.

2) An example of Suzuki (see [CM], p. 74) gives a holomorphic foliation by curves in the ball B in $\mathbb{C}^2$, singular only at 0 such that

the closure of every leaf is an analytic curve through 0, but there does not exist a germ of a meromorphic function at 0 which is constant along the leaves. There is then a marked difference between the local and the global problems on the existence of meromorphic first integrals in the presence of closed leaves.

3) The existence of a universal object parametrizing complex analytic subspaces of a compact complex manifold was shown by A. Douady ([D]), where the connected components of the parameter space are not necessarily projective, but if M is Kaehler they are compact by a theorem of Fujiki ([F]). Lemmas 1 and 2 hold in the analytic category, and the above argument suffices to prove Corollary 4 for Kaehler manifolds, but the part of the argument of Theorem 3 that uses the algebraic Zariski topology (i.e. the closure of the component of dimension q is a variety of dimension q) does not extend to the analytic category.

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