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BANACH SPACES WHICH ARE M -IDEALS IN THEIR BIDUAL HAVE PROPERTY (u)

by G. GODEFROY and D. LI

1. Introduction.

A Banach X is an M -ideal in its bidual if the relation

$$\|y+t\| = \|y\| + \|t\|$$

holds for every $y \in X^*$ and every $t \in X^\perp \subseteq X^{***}$. The spaces $c_0(I) - I$ any set-equipped with their canonical norm belong to this class, which also contains e.g. certain spaces $K(E, F)$ of compact operators between reflexive spaces (see [11]) and certain spaces of the form $C(G)/C_\Lambda(G)$ where G is an abelian compact group and Λ is a subset of the discrete dual group (see [5]). This class has been carefully investigated, in particular by A. Lima and by the « West-Berlin school », since the notion of M -ideal was introduced by Alfsen and Effros in 1972 [1].

We will show in this paper that these spaces somehow « look like » c_0 ; more precisely, that they share the property (u) with this latter space. This solves affirmatively a question that was pending for several years, and provides improvements of some results of [6] and [10].

Our proof uses non-linear arguments. The key lemma is actually a special case of a fundamental lemma ([1], lemma 1.4.) of the original article of Alfsen and Effros.

Notation. — The closed unit ball of a Banach space Z is denoted by Z_1 , and its dual by Z^* . The topology defined on Z^* by the pointwise convergence on Z is denoted by ω^* . The canonical injection from a Banach space X into its bidual X^{**} is denoted by i . A sequence (x_m) in X is said to be a weakly unconditionally convergent series —

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in short, w.u.c. series — if for every $y \in X^*$,

$$\sum_{m=0}^{\infty} |y(x_m)| < \infty.$$

If (x_m) is a w.u.c. series, then clearly the sequence

$$s_k = \sum_{m=0}^k x_m \quad (k \geq 1)$$

is weakly Cauchy and thus it converges in (X^{**}, w^*) ; we note $\Sigma^* x_m = \lim_{k \rightarrow \infty} (s_k)$ in (X^{**}, ω^*) . A Banach space X has the property (u)

([14]; see [12], p. 32) if every $z \in X^{**}$ which is in the sequential closure of X in (X^{**}, ω^*) may be written

$$z = \Sigma^* x_m$$

for some w.u.c. series (x_m) in X .

If $\tau: Z_1^* \rightarrow \mathbf{R}$ is a real-valued function defined on a dual unit ball Z_1^* , we denote by $\hat{\tau}$ the smallest concave ω^* -u.s.c. function which is greater than τ on Z_1^* . The function $\hat{\tau}$ is the infimum of the affine continuous functions on (Z_1^*, ω^*) which maximize τ on Z_1^* . The reader should consult [2] for a presentation of the basic facts about M -ideals. Similar ideas to those we use in this work are to be found e.g. in [15].

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2. The main result.

We will now prove:

THEOREM 1. — *Let X be a Banach space which is an M -ideal in its bidual. Then X has the property (u).*

Proof. — If i denotes the canonical injection from X into X^{**} , i^{**} is an isometric injection from X^{**} into X^{****} which is distinct from

the canonical injection if X is not reflexive. We will use a simplified notation that we now define: i^* denotes the canonical projection from X^{***} onto X^* of kernel $i(X)^\perp = X^\perp$, and then of course $i^{**}(z) = z \circ i^*$.

From now on, we assume that X is a real Banach space. By ([2], p. 22), we can do so without loss of generality. The notation $\mathbf{1}_{X_1^\perp}$ denotes the characteristic function of the subset $X_1^\perp$ of the unit ball X_1^{***} of X^{***} . Thus for any $z \in X^{**}$, $(\mathbf{1}_{X_1^\perp} \cdot z \vee 0)$ denotes the supremum of 0 and of the pointwise product of z and $\mathbf{1}_{X_1^\perp}$.

With this notation, we have the following crucial lemma.

LEMMA 2. — *If X is an M -ideal in its bidual X^{**} , then for every $z \in X^{**}$ and every $t \in X_1^{***}$ one has*

$$\langle z - i^{**}(z), t \rangle = [\widehat{\mathbf{1}_{X_1^\perp} \cdot z \vee 0}](t) - [\widehat{\mathbf{1}_{X_1^\perp} \cdot z \vee 0}](-t).$$

This lemma is actually a special case of ([1], lemma I.4). For sake of completeness, we give a simplified proof of this special case.

Proof. — If Ψ is a function from X_1^{***} to $\mathbf{R}^+$, we define

$$\mathfrak{G}^-(\Psi) = \{(t, \lambda) \in X_1^{***} \times \mathbf{R}^+ \mid 0 \leq \lambda \leq \Psi(t)\}$$

we let $\tau = (\mathbf{1}_{X_1^\perp} \cdot z \vee 0)$, and

$$B = \{(t, z(t)) \in X_1^\perp \times \mathbf{R}^+ \mid z(t) \geq 0\}.$$

We clearly have

$$(1) \quad \mathfrak{G}^-(\hat{\tau}) = \overline{\text{conv}}^* (\mathfrak{G}^-(\tau)).$$

On the other hand,

$$(2) \quad X_1^{***} = \text{conv}(X_1^* \cup X_1^\perp) \text{ since } X \text{ is an } M\text{-ideal in } X^{**}$$

$$(3) \quad \text{if } 0 \leq \lambda \leq z(t), (t, \lambda) \in \text{conv} \{(t, 0); (t, z(t))\}.$$

From (1), (2) and (3) follows

$$\begin{aligned} \mathfrak{G}^-(\hat{\tau}) &= \overline{\text{conv}}^* (\mathfrak{G}^-(\tau)) \\ &= \text{conv} \{(X_1^{***} \times \{0\}) \cup B\} \\ &= \text{conv} \{(X_1^* \times \{0\}) \cup (X_1^\perp \times \{0\}) \cup B\} \end{aligned}$$

since by w^* -compactness we don't need to take w^* -closures.

For every $t \in X_1^{***}$, $(t, \hat{\tau}(t)) \in \mathfrak{G}^-(\hat{\tau})$, hence we may write $(t, \hat{\tau}(t)) = \alpha_1(t_1, 0) + \alpha_2(t_2, 0) + \alpha_3(t_3, z(t_3))$ with :

$$\begin{cases} t_1 \in X^* \\ t_2 \in X_1^\perp \\ t_3 \in X_1^\perp; \quad z(t_3) \geq 0 \\ \alpha_1, \alpha_2, \alpha_3 \geq 0, \quad \alpha_1 + \alpha_2 + \alpha_3 = 1. \end{cases}$$

Since $t = \alpha_1 t_1 + (\alpha_2 t_2 + \alpha_3 t_3)$ is the unique decomposition of t on the direct sum $X^* \oplus X^\perp$ one has

$$t - i^*(t) = \alpha_2 t_2 + \alpha_3 t_3.$$

Since $z(t_3) \geq 0$, one has $\tau(t_3) = z(t_3)$ and $\tau(-t_3) = 0$. Since $\hat{\tau}$ is concave, one has

$$\begin{aligned} \hat{\tau}(t) &= \alpha_3 z(t_3) \\ &\geq \sum_{i=1}^3 \alpha_i \hat{\tau}(t_i) \\ &\geq \sum_{i=1}^3 \alpha_i \tau(t_i) \\ &= \alpha_2 \tau(t_2) + \alpha_3 \tau(t_3) \\ &= \alpha_2 \tau(t_2) + \alpha_3 z(t_3) \end{aligned}$$

hence $\alpha_2 \tau(t_2) \leq 0$; if $\alpha_2 = 0$ we may take $t_2 = 0$ as well; if $\alpha_2 > 0$ this implies $\tau(t_2) \leq 0$, hence $z(t_2) \leq 0$. In both cases, we have $z(-t_2) \geq 0$ and thus $z(-t_2) = \tau(-t_2)$.

Again by concavity of $\hat{\tau}$, one has

$$\begin{aligned} \hat{\tau}(-t) &\geq \sum_{i=1}^3 \alpha_i \hat{\tau}(-t_i) \\ &\geq \sum_{i=1}^3 \alpha_i \tau(-t_i) \\ &= \alpha_2 \tau(-t_2) + \alpha_3 \tau(-t_3) \\ &= \alpha_2 z(-t_2) \end{aligned}$$

hence

$$-\hat{\tau}(-t) \leq \alpha_2 z(t_2)$$

and therefore

$$\hat{\tau}(t) - \hat{\tau}(-t) \leq \alpha_3 z(t_3) + \alpha_2 z(t_2)$$

and

$$\begin{aligned}\alpha_3 z(t_3) + \alpha_2 z(t_2) &= \langle z, \alpha_2 t_2 + \alpha_3 t_3 \rangle \\ &= \langle z, t - i^*(t) \rangle \\ &= \langle z - i^{**}(z), t \rangle.\end{aligned}$$

Now the functions $\Phi(t) = \hat{\tau}(t) - \hat{\tau}(-t)$ and $(z - i^{**}(z))$ are both odd functions on X_1^{***} and they satisfy $\Phi \leq z - i^{**}(z)$; hence necessarily $\Phi = z - i^{**}(z)$ on X_1^{***} . $\square$

We now come back to the proof of theorem 1. By lemma 2, for every $z \in X^{**}$, we can write

$$\forall t \in X_1^{***}, \quad \langle i^{**}(z), t \rangle = (z(t) - \hat{\tau}(t)) + \hat{\tau}(-t)$$

hence if we let

$$\begin{aligned}h_1(t) &= z(t) - \hat{\tau}(t) \\ h_2(t) &= -\hat{\tau}(-t)\end{aligned}$$

we have $i^{**}(z) = h_1 - h_2$ and h_1, h_2 are both l.s.c. on (X_1^{***}, w^*) .

We need now a topological argument for going down to (X_1^*, w^*) .

LEMMA 3 (Saint-Raymond). — *Let K be a compact topological space and $S: K \rightarrow K'$ be a continuous surjection. Let f be a function from K' to $\mathbf{R}$ which is such that $(f \circ S)$ is the difference of two l.s.c. functions on K . Then f is the difference of two l.s.c. functions on K' .*

Proof. — Write $f \circ S = g_1 - g_2$ where g_1, g_2 are l.s.c. on K ; we define for $y \in K'$

$$\begin{aligned}\tilde{g}_1(y) &= \inf \{g_1(t) \mid S(t) = y\} \\ \tilde{g}_2(y) &= \inf \{g_2(t) \mid S(t) = y\}\end{aligned}$$

the functions $\tilde{g}_i (i=1,2)$ are l.s.c. on K' . Indeed, pick $\alpha < \tilde{g}_i(y)$; this means

$$(1) \quad \forall t \in S^{-1}(y), \quad g_i(t) > \alpha.$$

Since g_i is l.s.c. and $S^{-1}(y)$ is compact, (1) implies that there exists $\varepsilon > 0$ and an open neighbourhood V of $S^{-1}(y)$ such that

$$(2) \quad \forall t \in V, \quad g_i(t) > \alpha + \varepsilon.$$

Again by compactness, there exists a neighbourhood W of y such

that $S^{-1}(W) \subseteq V$; by (2) and the definition of $\tilde{g}_i$, this implies

$$\forall y' \in W, \quad \tilde{g}_i(y') \geq \alpha + \varepsilon > \alpha$$

and thus $\tilde{g}_i$ is l.s.c.

We show now that $f = \tilde{g}_1 - \tilde{g}_2$; for every $y \in K'$ and $t \in S^{-1}(y)$, one has

$$\begin{aligned} \tilde{g}_1(y) &\leq g_1(t) = f \circ S(t) + g_2(t) \\ &= f(y) + g_2(t) \end{aligned}$$

hence by definition of $\tilde{g}_2$

$$\tilde{g}_1(y) \leq f(y) + \tilde{g}_2(y).$$

On the other hand,

$$\begin{aligned} f(y) + \tilde{g}_2(y) &\leq f(y) + g_2(t) \\ &= f \circ S(t) + g_2(t) \\ &= g_1(t) \end{aligned}$$

and thus by definition of $\tilde{g}_1$,

$$f(y) + \tilde{g}_2(y) \leq \tilde{g}_1(y)$$

and this concludes the proof of lemma 3. □

Let us now conclude the proof of the theorem. Since

$$i^{**}(z) = z \circ i^* = h_1 - h_2$$

with h_1 and h_2 l.s.c. on (X_1^{***}, w^*) , we may apply lemma 3 with $f = z$, $S = i^*$ and $K' = (X_1^*, w^*)$; this lemma provides us with the l.s.c. functions $\tilde{h}_1$ and $\tilde{h}_2$ on (X_1^*, w^*) such that $z = \tilde{h}_1 - \tilde{h}_2$.

If now $z = \lim_{n \rightarrow \infty} x_n$ in (X^{**}, w^*) , where (x_n) is a sequence in X , we let

$$Y = \overline{\text{span}} \{x_n | n \geq 1\}$$

and we call Q the canonical quotient map from X^* onto Y^* ; since $z \in Y^{\perp\perp} = Q^*(Y^{**})$, there is $z' \in Y^{**}$ such that $z = z' \circ Q$; again by lemma 3, there exist two l.s.c. functions $\tilde{h}_1$ and $\tilde{h}_2$ on (Y_1^*, w^*) such that

$$z' = \tilde{h}_1 - \tilde{h}_2.$$

But since Y is separable, the w^* -topology on Y_1^* is defined by a metric d , and then classically the sequences $f_n^i (i=1,2)$ defined for $y \in Y_1^*$ and $n \geq 1$ by

$$f_n^i(y) = \inf \{ \tilde{h}_i(y') + nd(y, y') \mid y' \in Y_1^* \}$$

are increasing sequences of continuous functions on (Y_1^*, w^*) which converge pointwise to $\tilde{h}_i$. Now the sequence $u_n (n \geq 0)$ of continuous functions on (Y_1^*, w^*) defined by

$$\begin{aligned} u_0 &= f_1^1 - f_1^2 \\ u_n &= f_{n+1}^1 + f_n^2 - f_n^1 - f_{n+1}^2 \quad (n \geq 1) \end{aligned}$$

satisfies

$$\sum_{n=0}^{\infty} |u_n(y)| < \infty, \quad \forall y \in Y_1^*$$

and

$$\sum_{n=0}^{\infty} u_n(y) = z'(y), \quad \forall y \in Y_1^*.$$

But we still have

$$z'(y) = \lim_{n \rightarrow \infty} x_n(y), \quad \forall y \in Y_1^*$$

in this situation, a classical lemma of Pelczynski [14] (see [12], p. 32), which relies on a convex combination argument, shows that there is a sequence $(c_n)_{n \geq 0}$ in Y with

$$\sum_{n=0}^{\infty} |c_n(y)| < \infty, \quad \forall y \in Y_1^*$$

and

$$\sum_{n=0}^{\infty} c_n(y) = z'(y), \quad \forall y \in Y_1^*$$

and since $z = Q^*(z')$ and $c_n = Q^*(c_n)$, this shows that

$$z = \Sigma^* c_n$$

and (c_n) is a w.u.c. series in X . □

Before mentioning a few applications of our result, we would like to mention that the proof provides an explicit expression of $z \in X^{**}$

as a difference of two l.s.c. functions on (X_1^*, w^*) ; indeed, if we define for $y \in X_1^*$

$$v(y) = \inf \{z(t) - [\widehat{1_{X_1^*} \cdot z} \vee 0](t) \mid t \in X_1^{***}, i^*(t) = y\}$$

then the functions v and $(v - z)$ are both l.s.c. on (X_1^*, w^*) .

3. Applications.

We gather in this section a few consequences of our result.

3.1. P. Saab and the first-named author showed in ([6], Theorem 1) that if X is an M -ideal in its bidual then X has the property (V) of Pelczynski; the proof uses «pseudo-balls» ([3]) and the local reflexivity principle. Since such an X does not contain $\ell^1(N)$, our result is an improvement of ([6], Theorem 1), and of course also of the fact ([10]) that non-reflexive M -ideals in their bidual contain $c_0(N)$.

Another result of [6] is a structural result (Corollary 6) for certain spaces E such that $K(E)$ is an M -ideal in $L(E)$. The proof uses Banach algebras techniques that require to work with complex Banach spaces. This is not needed any more, and our result together with the proofs of ([6], Theorem 4 and Corollary 6) implies for instance the

PROPOSITION 4. — *Let E be a separable reflexive space with A.P. such that $K(E)$ is an M -ideal in $L(E)$. Then E is complemented in a reflexive space with an unconditional finite dimensional decomposition.*

There are some similarities between the techniques of [6] and of the present work; the main difference is that instead of using l.s.c. affine functions on a non-symmetric convex set — namely, the state space of a Banach algebra — we employ l.s.c. convex functions on a symmetric convex set — namely, a dual unit ball.

3.2. A Banach space Y is said to have to property (X) [7] if the following holds: $z \in Y^{**}$ belongs to Y if and only if for every w.u.c. series (y_n) in Y^* ,

$$z(\Sigma_* y_n) = \Sigma z(y_n)$$

where $(\Sigma_* y_n)$ denotes the limit of the sequence $\left\{s_k = \sum_{n=1}^k y_n \mid k \geq 1\right\}$ in (Y^*, w^*) . This condition roughly means that an abstract Radon-

Nikodym theorem is available for deciding which elements of Y^{**} belong to Y . Property (X) is equivalent to saying that $Y < \ell^1(N)$ for Edgar's ordering of Banach spaces [4]. For more details about this property, the reader may consult the recent survey [8].

Let us recall now the following easy

Claim. — If X is separable, does not contain $\ell^1(N)$ and has the property (u), then X^* has the property (X).

Proof of the claim. — We must show that every $t \in X^{***}$ such that $t(\Sigma_* z_n) = \Sigma t(z_n)$ for every w.u.c. series in X^{**} belongs to X^* . We can write $t = y + t'$ with $y \in X^*$ and $t' \in X^\perp$; since $y(\Sigma_* z_n) = \Sigma y(z_n)$ by w^* -continuity of y , we also have $t'(\Sigma_* z_n) = \Sigma t'(z_n)$.

Since X is separable, does not contain $\ell^1(N)$ and has (u), every $z \in X^{**}$ can be written $z = \Sigma^* x_n = \Sigma_* i(x_n)$ for some w.u.c. series in X ; but since $t'(x_n) = 0$ for every n , this implies $t'(z) = 0$, hence $t' = 0$ and $t = y \in X^*$.

Now this claim, together with theorem 1, shows:

PROPOSITION 5. — If a separable Banach space X is an M -ideal in its bidual, then X^* has the property (X).

By ([8], Theorem VII.8) this implies the following

COROLLARY 6. — Let X be a separable Banach space X which is an M -ideal in its bidual, and let Y be an arbitrary Banach space. Let $T: X^{**} \rightarrow Y^*$ be a bounded linear operator. The following are equivalent:

- (1) there is an operator $T_0 = Y \rightarrow X^*$ such that $T_0^* = T$,
- (2) $\ker(T)$ and $T(X_1^{**})$ are w^* -closed,
- (3) T is $(w^* - w^*)$ -Borel,
- (4) T is $(w^* - w^*)$ -strongly Baire measurable.

Let us conclude this work with a few natural questions.

Question 3.4. — Does there exist a separable Asplund space with property (u) which is not isomorphic to an M -ideal in its bidual? It looks reasonable to believe that this question has a positive answer; a candidate example is the space $K(L^p)$ ($1 < p < \infty$, $p \neq 2$) which has (u) ([12], Th. 3) but is not M -ideal of its bidual for its canonical norm [11].

Let us also mention that a separable $\mathcal{L}^\infty$ -space which is isomorphic to an M -ideal in its bidual is in fact isomorphic to $c_0(N)$ [9]. We do not know whether any isomorphic predual of $\ell^1(N)$ which has property (u) is isomorphic to $c_0(N)$.

Question 3.5. — A reformulation of Proposition 5 is that if Y is a separable space such that there exists a projection $\pi : Y^{**} \rightarrow Y$ with :

$$(a) \|z\| = \|\pi(z)\| + \|z - \pi(z)\|, \quad \forall z \in Y^{**}$$

(b) $(\ker \pi)$ w^* -closed,

then Y has the property (X). It is not known whether the assumption (b) can be removed, or whether (a) alone implies the weaker property (V^*) (see [14]), or at least that Y contains a complemented copy of $\ell^1(N)$ if it is not reflexive.

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