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ON FUNCTIONS WITH BOUNDED REMAINDER

by **P. HELLEKALEK & G. LARCHER**

0. Introduction.

Let λ denote normalized Haar measure on the one-dimensional torus $\mathbf{R}/\mathbf{Z}$. The following two classes of λ-preserving measurable transformations on $\mathbf{R}/\mathbf{Z}$ are important in ergodic theory as well as in the theory of uniform distribution modulo one.

Let α be an irrational number and $T : \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$, $Tx := \{x + \alpha\}$, $\{\cdot\}$ the fractional part. T is called an "irrational rotation" on $\mathbf{R}/\mathbf{Z}$.

Let $q \geq 2$ be an integer and $T : \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$, $Tx := x - (1 - q^{-k}) + q^{-(k+1)}$, whenever $x \in [1 - q^{-k}, 1 - q^{-(k+1)}[$, $k = 0, 1, \ldots$. T is called a "q-adic von Neumann-Kakutani adding machine transformation" on $\mathbf{R}/\mathbf{Z}$. In the following, T will be called a "q-adic transformation ".

Let $\varphi : [0,1] \to \mathbf{R}$ be a Riemann-integrable function with $\int_0^1 \varphi(t)\, dt = 0$ and let T be either an irrational rotation or a q-adic transformation on $\mathbf{R}/\mathbf{Z}$. Define

$$\varphi_n(x) := \sum_{k=0}^{n-1} \varphi(T^k x) \ ,$$

where $x \in \mathbf{R}/\mathbf{Z}$ and $n \in \mathbb{N}$ (we shall always identify $\mathbf{R}/\mathbf{Z}$ with $[0,1[$).

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The following two questions are of importance in ergodic theory – for the study of skew products – as well as for the study of irregularities in the distribution of sequences in $\mathbf{R}/\mathbf{Z}$:

1. Under which conditions (on φ and x) one has $\sup_n |\varphi_n(x)| < +\infty$?
2. What can be said about limit points of $(\varphi_n(x))_{n\geq 1}$?

The classical example. — Let $\varphi(x) = 1_{[0,\beta[}(x) - \beta$, $0 < \beta \leq 1$. In this now "classical" example, the first question leads to the study of irregularities in the distribution of the sequence $(T^k x)_{k\geq 0}$, $\varphi_n(x)$ being the so-called discrepancy function. For $x = 0$ one gets well-known sequences : in the first case $(\{k\alpha\})_{k\geq 0}$, in the second case the Van-der-Corput-sequence to the base q .

For this example, the first question has been solved completely by elementary and by ergodic methods (for the first type of T see Kesten [8] and Petersen [11], for the second type Faure [2] and Hellekalek [4]). The numbers β with $\sup_n |\varphi_n(0)| < +\infty$, respectively $\sup_n |\varphi_n(x)| < +\infty$, are all known.

The second question is closely related to ergodicity of the skew product (cylinder flow) $T_\varphi : T_\varphi(x,y) = (Tx, y + \varphi(x))$ on the cylinder $\mathbf{R}/\mathbf{Z} \times \mathbf{R}$ (see Oren [10] and Hellekalek [5]). In exactly this context Oren has solved the problem.

In this paper we shall be interested in question 1,2 and ergodicity of the cylinder flow T_φ on $\mathbf{R}/\mathbf{Z} \times \mathbf{R}$ in the case of a q-adic transformation T and $\varphi \in C^1([0,1])$.

1. Results.

Throughout this paper we shall assume $q \geq 2$ to be an integer and T to be a q-adic transformation on $\mathbf{R}/\mathbf{Z}$.

THEOREM 1. — *Let $\varphi \in C^1([0,1])$, let $\int_0^1 \varphi(t)\,dt = 0$ and $\varphi(1) \neq \varphi(0)$. Then every number c such that $|c| \leq |\varphi(1) - \varphi(0)|/2$ is a limit point of the sequence $(\varphi_{q^k}(x))_{k\geq 0}$ for almost all $x \in \mathbf{R}/\mathbf{Z}$, in particular for any x normal to base q .*

THEOREM 2. — *Let $\varphi \in C^1([0,1])$, let $\int_0^1 \varphi(t)\,dt = 0$ and let φ' be Lipschitz continuous on $[0,1]$. Then*

(1) $\varphi(0) = \varphi(1) \Rightarrow \sup_n |\varphi_n(x)| < \infty$ *for all* $x \in \mathbf{R}/\mathbf{Z}$;

(2) $\sup_n |\varphi_n(x)| < \infty$ *for some* $x \in \mathbf{R}/\mathbf{Z} \Rightarrow \varphi(0) = \varphi(1)$;

(3) $\varphi(1) < \varphi(0) \Rightarrow -\infty < \liminf_{n\to\infty} \varphi_n(0)$ *and* $\limsup_{n\to\infty} \varphi_n(0) = +\infty$;

(4) $\varphi(1) > \varphi(0) \Rightarrow -\infty = \liminf_{n\to\infty} \varphi_n(0)$ *and* $\limsup_{n\to\infty} \varphi_n(0) < +\infty$;

(if $\omega(\delta) := \sup\{|\varphi'(x) - \varphi'(y)| : |x-y| < \delta\ ,\ 0 \le x\ ,\ y \le 1\}\ ,\ \delta > 0$, *denotes the modulus of continuity of* φ' *, then* φ' *called Lipschitz-continuous if* $\omega(\delta) \le L \cdot \delta\ ,\ \forall \delta > 0\ ,\ L$ *a positive constant).*

The reader might want to compare theorem 2 *(1)* with theorem 7.8 in [7], and theorem 2 *(3)* and *(4)* with results on the one-sided boundedness of the discrepancy function (see [1]).

THEOREM 3. — *Let* $\varphi \in C^1([0,1])$ *and let* $\int_0^1 \varphi(t)\, dt = 0$. *Then* $\varphi(1) \neq \varphi(0) \Rightarrow \forall x \in \mathbf{R}/\mathbf{Z}$ *normal to base* $q : (\varphi_n(x))_{n\ge 1}$ *is dense in* $\mathbf{R}$.

In particular, if $\varphi(1) \neq \varphi(0)$ *and if* x *is normal to base* q *, then* $\liminf_{n\to\infty} \varphi_n(x) = -\infty$ *and* $\limsup_{n\to\infty} \varphi_n(x) = +\infty$.

The reader might want to compare theorem 3 with corollary C in [10].

THEOREM 4. — *Let* φ *be as in theorem 3 and let* $T_\varphi\ : \mathbf{R}/\mathbf{Z} \times \mathbf{R} \to \mathbf{R}/\mathbf{Z} \times \mathbf{R}\ ,\ T_\varphi(x,y) = (Tx, y + \varphi(x))$. *Then*

(1) $\varphi(1) \neq \varphi(0) \Rightarrow T_\varphi$ *ergodic;*

(2) let φ' *be Lipschitz-continuous on* $[0,1]$. *Then* T_φ *is ergodic if and only if* $\varphi(1) \neq \varphi(0)$.

2. The proofs.

Let $\mathsf{A}(q) = \left\{ \sum_{i=0}^{\infty} z_i q^i\ :\ z_i \in \{0, 1, \ldots, q-1\} \right\}$ denote the compact Abelian group of q-adic integers with the metric

$$\rho(z, z') := q^{-\min\{i : z_i \neq z'_i\}}$$

for $z = \sum_{i=0}^{\infty} z_i q^i \neq z' = \sum_{i=0}^{\infty} z'_i q^i$ and $\rho(z, z) := 0$.

The homeomorphism $S : \mathsf{A}(q) \to \mathsf{A}(q)$, $Sz = z + 1$ ($z \in \mathsf{A}(q)$, $1 := 1 \cdot q^0 + 0 \cdot q^1 + 0 \cdot q^2 + \cdots$) has a unique invariant Borel probability measure on $\mathsf{A}(q)$: the normalized Haar measure. The dynamical system $(\mathsf{A}(q), S)$ is minimal (see [4]).

The map $\Phi : \mathsf{A}(q) \to \mathbf{R}/\mathbf{Z}$, $\Phi\Big(\sum_{i=0}^{\infty} z_i q^i\Big) := \sum_{i=0}^{\infty} z_i q^{-(i+1)} \bmod 1$, is measure preserving, continuous and surjective.

The q-adic representation of an element x of $\mathbf{R}/\mathbf{Z}$, $x = \sum_{i=0}^{\infty} x_i q^{-(i+1)}$ with digits $x_i \in \{0, 1, \ldots, q-1\}$, is unique under the condition $x_i \neq q-1$ for infinitely many i . From now on we shall assume this uniqueness condition to hold for all x . Numbers x with $x_i \neq 0$ for infinitely many i will be called *non-q-adic*. In the following $z = z(x)$ will denote the element

$$z = z(x) := \sum_{i=0}^{\infty} x_i q^i$$

of $\mathsf{A}(q)$ associated with x . One has

$$Tx = \Phi(z + 1)$$

and it is elementary to see :

- $T \circ \Phi(z) = \Phi \circ S(z)$, $\forall z \in \mathsf{A}(q)$
- $x \in [aq^{-k}, (a+1)q^{-k}[$, $0 \leq a < q^k$, $k = 1, 2, \ldots \Rightarrow T^{q^k} x \in [aq^{-k}, (a+1)q^{-k}[$ and therefore $|T^{q^k} x - x| < q^{-k}$.
- T permutes the open elementary q-adic intervals $]aq^{-k}, (a+1)q^{-k}[$, $0 \leq a < q^k$, of length q^{-k} , $k = 1, 2, \ldots$.

PROPOSITION 1. — *Let φ be continuously differentiable on the closed interval $[0,1]$ and let $\int_0^1 \varphi(t)\, dt = 0$. If ω denotes the modulus of continuity of φ' , then for all $k \in \mathbf{N}$ and for all $x \in \mathbf{R}/\mathbf{Z}$*

$$\begin{aligned}
\varphi_{q^k}(x) = {} & (\varphi(1) - \varphi(0))(\rho_k + \sigma_k - 1/2) + \mathcal{O}(\omega(q^{-k})) \\
& + \mathcal{O}(\rho_k \cdot \omega(c(q) \cdot (q^k - z(k))^{-1} \log(q^k - z(k)))) \qquad (1) \\
& + \mathcal{O}(\sigma_k \cdot \omega(c(q) \cdot z(k)^{-1} \log z(k))) ,
\end{aligned}$$

where

$$x = \sum_{i=0}^{\infty} x_i q^{-(i+1)}$$

$$z = z(x) := \sum_{i=0}^{\infty} x_i q^i$$

$$z(k) := \sum_{i=0}^{k-1} x_i q^i \qquad k = 1, 2, \ldots$$
$$\rho_k := (q^k - z(k)) \cdot \Phi(z - z(k))$$
$$\sigma_k := z(k) \cdot \Phi(z - z(k) + q^k)$$

and $c(q)$ is a constant that depends only on q . The $\mathcal{O}$–constants that appear in identity (1) are all bounded from above by a constant that depends only on q and φ .

Proof. — It is easy to prove

$$\varphi_{q^k}(x) = \sum_{i=0}^{q^k-1} \varphi(a_i q^{-k}) + \sum_{i=0}^{q^k-1} \varphi'(a_i q^{-k})(T^i x - a_i q^{-k}) + \mathcal{O}(\omega(q^{-k})) ,$$

where a_i is the uniquely determined integer with $0 \le a_i < q^k$ and $T^i x \in [a_i q^{-k}, (a_i+1)q^{-k}[$. From proposition 1 in [6] it follows that

$$\sum_{i=0}^{q^k-1} \varphi(a_i q^{-k}) = -(\varphi(1) - \varphi(0))/2 + \mathcal{O}(\omega(q^{-k})) .$$

Further

$$T^i x - a_i q^{-k} = \begin{cases} \Phi(z - z(k)) & 0 \le i < q^k - z(k) \\ \Phi(z - z(k) + q^k) & q^k - z(k) \le i < q^k . \end{cases}$$

By theorem 5.4, chapter 2 of [9]

$$(q^k - z(k))^{-1} \sum_{i=0}^{q^k - z(k) - 1} \varphi'(a_i q^{-k}) = \varphi(1) - \varphi(0) + \mathcal{O}(\omega(D_{q^k - z(k)})) ,$$

where $D_{q^k - z(k)}$ denotes the discrepancy of $(a_i q^{-k})_{i=0}^{q^k - z(k) - 1}$. As $a_i q^{-k} = \Phi(z(k) + i)$, this is a string in the Van-der-Corput-sequence to base q , and therefore the following discrepancy estimate holds (see [9] chapter 2, theorem 3.5 for the idea of the proof) :

$$D_{q^k - z(k)} \le c(q)(q^k - z(k))^{-1} \log(q^k - z(k)) , \ k = 1, 2, \ldots ,$$

$c(q)$ a constant that depends only on q .

With the same arguments one proves

$$z(k)^{-1} \sum_{i=q^k - z(k)}^{q^k - 1} \varphi'(a_i q^{-k}) = \varphi(1) - \varphi(0) + \mathcal{O}(\omega(c(q) z(k)^{-1} \log z(k))) .$$

□

COROLLARY 1. — *Let* $n \in \mathbb{N}$, $n = \sum_{i=0}^{s} n_i q^i$, *with* $n_i \in \{0, 1, \ldots, q-1\}$, $0 \le i \le s$, $n_s \neq 0$, *and let* $n(k) := \sum_{i=0}^{k-1} n_i q^i$ *if* $k = 1, \ldots, s+1$, $n(0) := 0$.

If $\sum_{k=0}^{s}{}'$ denotes $\sum_{\substack{k=0 \\ k:n_k \neq 0}}^{s}$ then

$$\varphi_n(x) = \sum_{k=0}^{s}{}' \sum_{\ell=0}^{n_k-1} \sum_{j=0}^{q^k-1} \varphi(T^{n(k)+\ell q^k + j} x) .$$

Let

$$T^{n(k)+\ell q^k} x =: x^{k,\ell} = \sum_{i=0}^{\infty} x_i^{k,\ell} q^{-(i+1)}$$

$$z^{k,\ell} := \sum_{i=0}^{\infty} x_i^{k,\ell} q^i$$

$$z^{k,\ell}(m) := \sum_{i=0}^{m-1} x_i^{k,\ell} q^i \qquad (m = 1, 2, \ldots)$$

$$\rho_{k,\ell} := (q^k - z^{k,\ell}(k)) \cdot \Phi(z^{k,\ell} - z^{k,\ell}(k))$$

$$\sigma_{k,\ell} := z^{k,\ell}(k) \cdot \Phi(z^{k,\ell} - z^{k,\ell}(k) + q^k) .$$

Then proposition 1 implies :

$$\begin{aligned} \varphi_n(x) &= (\varphi(1) - \varphi(0)) \sum_{k=0}^{s}{}' \sum_{\ell=0}^{n_k-1} (\rho_{k,\ell} + \sigma_{k,\ell} - 1/2) \\ &\quad + \mathcal{O}\Big(\sum_{k=0}^{s}{}' n_k \ \omega(q^{-k}) \Big) \qquad (2) \\ &\quad + \mathcal{O}\Big(\sum_{k=0}^{s}{}' \sum_{\ell=0}^{n_k-1} (\rho_{k,\ell} \ \omega(c(q)(q^k - z^{k,\ell}(k))^{-1} \log(q^k - z^{k,\ell}(k))) \\ &\quad + \sigma_{k,\ell} \ \omega(c(q) z^{k,\ell}(k)^{-1} \log z^{k,\ell}(k))) \Big) . \end{aligned}$$

The $\mathcal{O}$–constants in identity (2) are bounded from above by a constant that depends only on q and φ .

Proof of theorem 1. — Let x be normal to base q and let $d = 0, d_0 d_1 d_2 \cdots$ be an arbitrary number in $[0, 1[$. For any index k such that

$x_k < q-1$ we have

$$\rho_k + \sigma_k = (q^k - z(k))\sum_{i\geq k} x_i q^{-i-1} + z(k)\Big(\sum_{i\geq k} x_i q^{-i-1} + q^{-k-1}\Big)$$
$$= \sum_{i\geq 0} x_i q^{-|i-k|-1} .$$

Let $\varepsilon > 0$ be arbitrary. Choose m such that $q^{-m} < \varepsilon$. As x is normal there are infinitely many k such that $x_k < q-1$

$$|\rho_k + \sigma_k - d| = |0, x_k x_{k+1} x_{k+2}\cdots + 0, 0x_{k-1}x_{k-2}\cdots x_0 - d| < q^{-m}$$

(this imposes a condition on the digits $x_k, x_{k\pm 1}, \ldots, x_{k\pm m-1}$)

$$x_{k-m} = q-1 \quad , \quad x_{k-m-1} = 0 .$$

Then

$$z(k) \geq q^{k-m} \quad , \quad q^k - z(k) \geq q^{k-m-1}$$

and, if we choose k sufficiently large,

$$\omega(q^{-k}) < \varepsilon \quad \text{and} \quad \omega(c(q)q^{-k+m+1}\log q^k) < \varepsilon .$$

If we put $c := (\varphi(1) - \varphi(0))(d - 1/2)$, then it follows directly that $|\varphi_{q^k}(x) - c| = \mathcal{O}(\varepsilon)$. □

Proof of theorem 2. — *(1)* : Let $\varphi(1) = \varphi(0)$. It is $\Phi(z - z(k)) < q^{-k}$ and $\Phi(z - z(k) + q^k) < q^{-k}$, $k = 1, 2, \ldots$. Hence for the third term in identity (2) we get the estimate

$$(3) \quad \sum_{k=0}^{s}{}' \sum_{\ell=0}^{n_k-1} (\rho_{k,\ell}\cdots + \cdots \log z^{k,\ell}(k))) \leq 2qLc(q)\sum_{k=0}^{\infty} q^{-k}\log q^k < +\infty .$$

Thus the first part of the theorem is proved.

(2) : Let $\sup_n |\varphi_n(x)| < +\infty$ for some $x \in \mathbb{R}/\mathbb{Z}$ and let $z := z(x)$. The map $\varphi\circ\Phi : \mathbb{A}(q) \to \mathbb{R}$ is continuous and $(\mathbb{A}(q), S)$ is a minimal (topological) dynamical system. We have

$$\sup_n |\varphi_n(x)| = \sup_n |\sum_{k=0}^{n-1} \varphi\circ\Phi(S^k z)| < +\infty .$$

By theorem 14.11 of [3] there is a continuous function $g : \mathbb{A}(q) \to \mathbb{R}$ such that $\varphi\circ\Phi(z) = g(z) - g(Sz)$, $\forall z \in \mathbb{A}(q)$. Hence

$$-(\varphi(1) - \varphi(0))/2 = \lim_{k\to\infty} \varphi_{q^k}(0) = \lim_{k\to\infty} \sum_{i=0}^{q^k-1} \varphi\circ\Phi(S^i 0)$$
$$= \lim_{k\to\infty}(g(0) - g(q^k)) = 0 ;$$

(here we use proposition 1 in [6] to prove the first equality).

(3) : We shall prove $-\infty < \liminf_{n\to\infty} \varphi_n(0)$, then part *(2)* will imply the remaining statement. Because of identity (2) and inequality (3) it is enough to show, for $x = 0$,

$$\Sigma_n := {\sum_{k=0}^{s}}' \sum_{\ell=0}^{n_k-1} (\rho_{k,\ell} + \sigma_{k,\ell} - 1/2) \leq K \ , \ \forall n \in \mathbf{N}$$

with some constant K . If $x = 0$ then $z^{k,\ell} = n(k) + \ell q^k$ and $z^{k,\ell}(k) = n(k)$. Hence $\rho_{k,\ell} = (q^k - n(k))\ell q^{-(k+1)}$ and $\sigma_{k,\ell} = n(k)(\ell+1)q^{-(k+1)}$. Thus

$$\Sigma_n = {\sum_{k=0}^{s}}' n_k((n_k - 1)/(2q) + n(k)q^{-(k+1)} - 1/2) \ .$$

The statement then follows because $(n_k - 1)/(2q) + n(k)q^{-(k+1)} - 1/2 < 0$.

(4) : The idea of the proof is the same as in *(3)*. □

Remark. — In theorem 2 *(1)*, *(3)* and *(4)* one can weaken the condition on the modulus of continuity of φ' to $\omega(\delta) = \mathcal{O}(|\log\delta|^{-1-\varepsilon})$ with some $\varepsilon > 0$.

Proof of theorem 3. — The idea of the proof is as follows. Let $(k_m)_{m\geq 1}$ be a strictly increasing sequence of positive integers. If $n = q^{k_1} + \cdots + q^{k_s}$ then

$$\begin{aligned}\varphi_n(x) = (\varphi(1) - \varphi(0)) \sum_{m=1}^{s} (\rho_{k_m} + \sigma_{k_m} - 1/2) + \mathcal{O}\Big(\sum_{m=1}^{s} \omega(q^{-k_m})\Big) \\ + \mathcal{O}\Big(\sum_{m=1}^{s} \rho_{k_m}\omega(c(q)(q^{k_m} - z^{k_m}(k_m))^{-1} \log(q^{k_m} - z^{k_m}(k_m))\Big) \\ + \sigma_{k_m}\omega(c(q)(z^{k_m}(k_m))^{-1} \log z^{k_m}(k_m)))\end{aligned}$$

with $x = 0, x_0x_1x_2\cdots$, $z = z(x) = \sum_{i=0}^{\infty} x_i q^i$, $z^{k_m} = z + q^{k_1} + \cdots + q^{k_{m-1}}$ and, if $x_{k_m} \leq q-2$,

$$\rho_{k_m} + \sigma_{k_m} = 0 \ , \ \ x_{k_m}x_{k_m+1}\cdots + 0 \ , \ \ 0x_{k_m-1}x_{k_m-2}\cdots x_0 \ .$$

Now, let $d \in \mathbf{R}$, $\varepsilon > 0$ and $x \in [0,1[$ normal to base q be given. We shall prove that there is a positive integer m_0 and a strictly increasing sequence $(k_m)_{m\geq m_0}$ such that

$$|\varphi_n(x) - d| < \varepsilon \ \text{ for all } \ n = q^{k_{m_0}} + \cdots + q^{k_s} \text{ sufficiently large.}$$

Let m_0 be such that $\sum\limits_{m \geq m_0} q^{-m} < \varepsilon$. Let $(a_m)_{m \geq m_0}$ be a sequence in $[0, 1[$ such that

$$d = (\varphi(1) - \varphi(0)) \sum_{m \geq m_0} (a_m - 1/2) .$$

The number x is normal to base q . Hence there are infinitely many $k = k(m)$ such that

1. $x_k \leq q - 2$
2. $x_{k-2m} = 1$
   $x_{k-2m-1} = x_{k+2m} = x_{k+2m+1} = 0$
3. $|\rho_k + \sigma_k - a_m| < q^{-m}(\varphi(1) - \varphi(0))^{-1}$, $\forall m \geq m_0$;

(this condition defines a string of digits $x_{k-2m+1}, \ldots, x_{k+2m-1}$). Hence we may choose a strictly increasing sequence $(k_m)_{m \geq m_0}$ such that these three conditions hold for every k_m and such that

4. $k_m + 2m + 1 < k_{m+1}$
5. $\sum\limits_{m \geq m_0} \omega(q^{-k_m}) < \varepsilon$
6. $\sum\limits_{m \geq m_0} \omega(c(q) q^{-k_m + 2m+1} \log q^{k_m}) < \varepsilon$.

Then if $n = q^{k_{m_0}} + \cdots + q^{k_s}$ $(s \geq m_0)$,

$$|\varphi_n(x) - d| = \mathcal{O}(\varepsilon) ,$$

and therefore the sequence $(\varphi_n(x))_{n \geq 1}$ is dense in $\mathbf{R}$. □

Remark. — Theorem 3 gives an alternative to the proof of theorem 2 *(2)*, this time without a condition on the modulus of continuity of φ' :

If $\sup\limits_n |\varphi_n(x)| < \infty$ for some $x \in [0, 1[$, then this holds for all x by the theorem of Gottschalk and Hedlund. Hence $\varphi(1) = \varphi(0)$, otherwise a contradiction to theorem 3 would arise for any x normal to base q .

Proof of theorem 4.

(1) is proved in the very same way as the theorem of [6].

(2) : Let L_2 stand for $L_2(\mathbf{R}/\mathbf{Z}, \lambda)$. Then $\varphi(1) = \varphi(0)$ implies $\sup\limits_n \|\varphi_n\|_{L_2} < +\infty$. By Lemma 2.2 in [4] there exists an element g of L_2 such that $\varphi = g - g \circ T \pmod{\lambda}$. This implies that $(x, y) \mapsto$

$(Tx, y + \varphi(x) \mod 1)$ is not ergodic on $\mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z}$ and therefore T_φ cannot be ergodic on $\mathbf{R}/\mathbf{Z} \times \mathbf{R}$ (see [5], part. I : remarks). □

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