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FOURIER COEFFICIENTS OF CONTINUOUS FUNCTIONS AND A CLASS OF MULTIPLIERS

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PLAN

Key-words : Multiplier - Interpolation inequalities - Fourier coefficients.

INTRODUCTION

The question as to whether every square-summable non-negative sequence $\{a_n\}_{n \in \mathbb{Z}}$ could be majorized by the sequence of moduli of Fourier coefficients of a continuous function f was posed by Sidon in 1932 (see [12]) and was answered in the affirmative only 45 years later by de Leeuw, Katznelson and Kahane [3] (in fact they proved that, in addition, the estimate $\|f\|_{C(\mathbb{T})} \leqslant \text{const}\,(\Sigma |a_n|^2)^{1/2}$ can always be ensured). This result was then refined in the author's paper [7]. It was proved in [7], among other things, that if $a_n = 0$ for $n < 0$ then the function f in question can be chosen in such a way that its Fourier coefficients corresponding to the negative integers vanish. This suggests the following definition.

Let G be a compact abelian group with dual (discrete) group Γ. Suppose that X is a Banach space of measurable functions on G such that $X \subset L^2(G)$ (a set theoretic inclusion) and the natural imbedding of X to $L^2(G)$ is continuous. A subset E of Γ is said to be massive for X (or X-massive) if there is a constant K such that for every numerical family $\{a_\gamma\}_{\gamma \in E}$ in $l^2(E)$ there is a function f in X with

$$|\hat{f}(\gamma)| \geqslant |a_\gamma| \quad \text{for} \quad \gamma \in E; \qquad \hat{f}(\gamma) = 0 \quad \text{for} \quad \gamma \notin E;$$
$$\|f\|_X \leqslant K\Bigl(\sum_{\gamma \in E} |a_\gamma|^2\Bigr)^{1/2}.$$

So, the result of the author [7] mentioned above means that the set of non-negative integers $\mathbb{Z}_+$ is massive for the space $C(\mathbb{T})$ ($\mathbb{T}$ is the group of the circle, $\mathbb{T} = \{z \in \mathbb{C} : |z|\}$). In fact, in [7] a stronger result was established, namely, that this set is massive for a smaller space U consisting of all functions f such that the two series $\sum_{n \geqslant 0} \hat{f}(n) z^n$, $\sum_{n<0} \hat{f}(n) z^n$ converge uniformly on $\mathbb{T}$. The norm in U is defined as follows :

$$\|f\|_U = \sup\left\{\left|\sum_{k \leqslant n \leqslant l} \hat{f}(n)\zeta^n\right| : \ k, l \in \mathbb{Z}, k \leqslant l; \zeta \in \mathbb{T}\right\}.$$

It is well-known that every set is massive for L^p, $2 \leqslant p < \infty$ (see [16], Ch. 5, Theorem 8.6). On the other hand, not every set is massive for $C(\mathbb{T})$ (for example, no infinite Sidon set is $C(\mathbb{T})$-massive; in particular, the geometric progression $\{2^n\}_{n\geqslant 0}$ is not $C(\mathbb{T})$-massive). Thus it is natural to try to describe the sets massive for $C(\mathbb{T})$ or smaller spaces (e.g., for U) or at least, to begin with, to give as many examples of such sets as possible. For the first time I heard of this general problem several years ago from Prof. J.-P. Kahane.

The starting point for the present work was the following concrete question: set $E_0 = \bigcup_{k\geqslant 1} [2^{2k}, 2^{2k+1}]$; is E_0 massive?

The contents of [7] suggests that the question of massiveness of a given set E for $C(\mathbb{T})$ is connected with the behaviour on the space $L^1(\mathbb{T})$ of the corresponding multiplier M_E,

$$(M_E f)(z) \overset{\text{def}}{=} \sum_{n\in E} \hat{f}(n)z^n \qquad \text{or} \qquad (M_E f)\hat{\ } = \hat{f}\mathbb{1}_E,$$

$\mathbb{1}_E$ standing for the indicator function of E. For E to be $C(\mathbb{T})$-massive it is sufficient, e.g., that M_E be of weak type (1,1) i.e. satisfy the inequality

$$\operatorname{mes}\{|M_E f| > \lambda\} \leqslant \operatorname{const} \lambda^{-1} \|f\|_{L^1(\mathbb{T})}.$$

Just this consideration was used in [7] to prove that $\mathbb{Z}_+$ is massive. But for the above set E_0 this scheme fails because M_{E_0} is not of weak type (1,1). Incidentally, the absence of weak type implies that M_{E_0} does not act from L^1 even to the space of measurable functions with the topology of convergence in measure (since M_{E_0} commute with translations) — see [9], [11].

Nevertheless E_0 is $C(\mathbb{T})$-massive. The proof is based on the fact that M_{E_0} is still « L^1-regular » to a certain extent, namely, we shall see that it satisfies the « interpolation inequality »

$$\|M_{E_0} f\|_{L^p} \leqslant C \|f\|_{L^1}^{\alpha} \|M_{E_0} f\|_{L^2}^{1-\alpha}, \tag{1}$$

where $1 < p < 2$ and C, α, $0 < \alpha < 1$ depend on p only. It is worth noting that if T is an operator of weak type (1,1) then (1) with T substitued for M_{E_0} is automatically true; moreover, the same holds if T acts continuously from L^1 to L^r for some r, $0 < r < 1$. But this continuity property is not necessary for the inequality of type (1) — besides M_{E_0}, the multiplier $M_{\mathbb{Z}_+\times\mathbb{Z}_+}$ in the case of the group $\mathbb{T}^2$ can provide an example. In [7] an interpolation inequality for the last mentioned multiplier was established and then used in the proof of the fact that $\mathbb{Z}_+ \times \mathbb{Z}_+$ is massive for $C(\mathbb{T}^2)$.

Now the problem arises to investigate multipliers satisfying inequalities of type (1). This problem proves to be interesting in itself, so it would be unnatural to restrict ourselves only to multipliers generated by indicator functions. In the sequel we denote by M_x the multiplier generated by a bounded function x on $\mathbb{Z}$:

$$(M_x f)^{\wedge} \stackrel{\text{def}}{=} x\hat{f}$$

(this makes sense at least for $f \in L^2(\mathbb{T})$). The function x is called the symbol of the multiplier in question. If $x = \mathbb{1}_E$ with $E \subset \mathbb{Z}$ we still write M_E for M_x.

The paper is divided into two parts. In the first part we describe two classes of multipliers satisfying the inequality of type (1) or a stronger inequality (we shall see that in many cases the L^1-norm of the function f on the right can be replaced by its norm in a wider space, e.g. in U^*). The second part is devoted to applications of these results. For example, it is proved there that the set E_0 is massive for U (and not only for $C(\mathbb{T})$, as already claimed). Moreover, if we fix some points n_k, $n_k \in [2^{2k}, 2^{2k+1}]$ then for every sequence $\{a_n\}$ in $l^2(E_0)$ there is a function f in U such that: $\hat{f}(n_k) = a_{n_k}$; $|\hat{f}(n)| \geqslant |a_n|$ *for* $n \in E_0$; $\hat{f}(n) = 0$ for $n \notin E_0$; $\|f\|_U \leqslant C\left(\sum_{n \in E_0} |a_n|^2\right)^{1/2}$.

It is also worth mentioning that in the second part of the paper we give examples of sets massive for $C(\mathbb{T})$ but not for U. One of them is $\mathbb{Z}_- \cup \{2^k\}_{k \geqslant 1}$, where $\mathbb{Z}_- \stackrel{\text{def}}{=} \mathbb{Z} \backslash \mathbb{Z}_+$. It is easy to see that this set is not massive even for the space of such functions f in $C(\mathbb{T})$ that $\sum_{n \geqslant 0} \hat{f}(n) z^n$ is the Fourier series of a continuous function. There is another natural class intermediate between $C(\mathbb{T})$ and U, namely, the space U_{symm} of functions representable as uniform limits of symmetric partial sums of their Fourier series. Examples of sets massive for $C(\mathbb{T})$ but not for U_{symm} will be presented as well. Note that the previous results stating that the class of sequences of Fourier coefficients is « close » to l^2 (beginning with the material cited in the introduction of [7] and finishing by the result mentioned in the preceding paragraph) essentially satisfy the principle « all that is true for $C(\mathbb{T})$ can be done for U_{symm} also ».

Now we introduce some more notation (some have already been introduced in passing) : m is the normalized Lebesgue measure on $\mathbb{T}$; $\|.\|_p$ is the norm in the space $L^p(m)$ or $L^p(X,m)$, where X is a Banach space, $0 < p \leqslant \infty$ (we do not hesitate to use the term « norm » for the functional $f \mapsto \left(\int |f|^p\, dm\right)^{\frac{1}{p}}$ if $p < 1$) ; C_A is the subspace of $C(\mathbb{T})$ consisting of the functions f with $\hat{f}(n) = 0$ *for* $n < 0$; z is the identity mapping of $\mathbb{T}$.

For a Banach space X, $\mathscr{P}(X)$ denotes the class of X-valued trigonometric polynomials, i.e. of functions g of the form

$$g(\zeta) = \sum_{n \in Z} x_n \zeta^n, \qquad \zeta \in \mathbb{T}, \tag{2}$$

where $x_n \in X$ and the set $\{n : x_n \neq 0\}$ is finite. Set $\mathscr{P}_A(X) = \{g \in \mathscr{P}(X) :$ the x_n 's in (2) vanish for $n \in \mathbb{Z}_-\}$. $\mathscr{P}_-(X)$ is defined analogously, with $\mathbb{Z}_+$ subsituted for $\mathbb{Z}_-$. (In other words, $\mathscr{P}_A(X)$ and $\mathscr{P}_-(X)$ are the sets of X-valued analytic and anti-analytic polynomials, respectively.) For $0 < p < \infty$ we denote by $H^p(X)$ the closure of $\mathscr{P}_A(X)$ in $L^p(X,m)$ and by $H^p_-(X)$ the closure of $\mathscr{P}_-(X)$ in $L^p(X,m)$. If $X = \mathbb{C}$ we write simply $\mathscr{P}$, $\mathscr{P}_A$, $\mathscr{P}_-$, H^p, H^p_-.

The Riesz projections $\mathbb{P}_+$ and $\mathbb{P}_-$ are defined on $\mathscr{P}(X)$ as follows : if g is given by (2) then

$$\mathbb{P}_+ g = \sum_{n \geqslant 0} x_n z^n, \qquad \mathbb{P}_- g = \sum_{n < 0} x_n z^n.$$

It is well-known (see, e.g., [4], p. 484) that if X is a Hilbert space then $\mathbb{P}_+$ and $\mathbb{P}_-$ can be extended, for $1 < p < \infty$, to continuous projections of $L^p(X)$ onto $H^p(X)$ and $H^p_-(X)$ respectively, their $L^p(X)$-norms being bounded by const $p^2(p-1)^{-1}$. Moreover, $\mathbb{P}_+$ and $\mathbb{P}_-$ are of weak type (1,1) if X is a Hilbert space :

$$m\{\|\mathbb{P}_+ g\|_X > \lambda\} \leqslant \text{const}\ \lambda^{-1}\|g\|_1, \qquad \lambda > 0$$

for $g \in \mathscr{P}(X)$, and the same is true for $\mathbb{P}_-$ (consult the same book [4], p. 486).

Different constants in estimates for the most part will be denoted by C (with or without indices) ; the variations of constants from one estimate to another are not always reflected in notation.

PART I. INEQUALITIES

1. Statements.

1.1. Symbols Vanishing on $\mathbb{Z}_-$. We are interested in inequalities of the type $\|M_x f\|_p \leqslant C\|f\|_1^\alpha \|M_x f\|_q^{1-\alpha}$ and their refinements (here $1<p<q$, $0<\alpha<1$ and x is a bounded function on $\mathbb{Z}$). Note that in the non-trivial case, when M_x is not of weak type $(1,1)$, estimates of this sort are the more interesting, the more « intensively » the function x vanishes (because one should not expect to obtain much information from the knowledge of $\|M_x f\|_q$ if x vanishes « very intensively »). In support of this remark we show that the estimate in question trivially holds if $q = 2$ (the most important case for applications), $\delta \overset{\text{def}}{=} \inf\{|x(n) : n \in \mathbb{Z}\} > 0$ and M_x is bounded in L^p (the latter will be fulfilled in the most of examples considered). Indeed, we have $\|f\|_2 \leqslant \delta^{-1}\|M_x f\|_2$ and thus

$$\|M_x f\|_p \leqslant C\|f\|_p \leqslant C\|f\|_1^\alpha \|f\|_2^{1-\alpha} \leqslant C\delta^{-1}\|f\|_1^\alpha \|M_x f\|_2^{1-\alpha},$$

where $p = \alpha + (1-\alpha)2^{-1}$.

In applications the case when $x^{-1}(0) \supset \mathbb{Z}_-$ often occurs (this holds e.g. for $x=\mathbb{1}_{E_0}$, where E_0 is defined in the introduction). We start with considering this case. First we describe a basic interpolation estimate involving H^p-functions.

A sequence $\{I_k\}_{k\geqslant 1}$ of subintervals of the half-line $(0,\infty)$ ($I_k=[a_k,b_k]$) is called separated if there is $B > 1$ such that the intervals (a_kB^{-1}, b_kB) are mutually disjoint (we also use the term « B-separated sequence » if we want to name the constant B explicitely).

Let $\{I_k\}_{k\geqslant 1}(I_k=[a_k,b_k])$ be a separated sequence of intervals and suppose that $b_k < a_{k+1}$ for all k. Set $J_0 = [0,a_1)$ $J_k = (b_k, a_{k+1})$ for $k \geqslant 1$ and define a function y on $\mathbb{Z}$ by setting $y(n) = 0$ for $n < 0$, $y(n) = t_k$ for $n \in J_k$ and $y(n) = s_k$ for $n \in I_k$.

Theorem 1. — *Suppose that the above sequences $\{s_k\}_{k\geqslant 1}$ and $\{t_k\}_{k\geqslant 0}$ are uniformly bounded and $|s_k| \leqslant$ const $\min\{|t_{k-1}|, |t_k|\}$ with a constant independent of k. Let $r \leqslant 1 < p < q$, $p^{-1} = \theta r^{-1} + (1-\theta)q^{-1}$. Then* *for* $f \in \mathscr{P}_A$ *we have* $\|M_y f\|_p \leqslant C\|f\|_r^{\theta}\|M_y f\|_q^{1-\theta}$ *(C depends only on* r, p, q *and* y*).*

Note that the estimate of Theorem 1 is true for $r > 1$ as well, but this case is not interesting because M_y acts in L^r for $r > 1$ (see the beginning of the next section). Note also that it is very easy to extend the estimate to all f in H^r (in this case $M_y f$ should be understood as the boundary values of the function $\sum_{n\geqslant 0} y(n)c_n z^n$ analytic in the disc $\{|z|<1\}$, c_n being the Taylor coefficients of f; if $M_y f \notin H^s$ we agree that $\|M_y f\|_s = \infty$).

Now we can state an interpolation inequality valid for arbitrary (i.e. not only analytic) functions.

COROLLARY. – *Let* x *be a function on* $\mathbb{Z}$ *with* $x|\mathbb{Z}_- = 0$. *Suppose that for some* $r < 1$ *we have* $\|M_x f\|_r \leqslant C\|f\|_1$ *for all* f *in* $\mathscr{P}$. *If* y *is from Theorem* 1 *then for* $1 < p < q$ *and* $f \in \mathscr{P}$

$$\|M_{xy} f\|_p \leqslant C_1\|f\|_1^{\theta}\|M_{xy} f\|_q^{1-\theta}, \qquad p^{-1} = r^{-1}\theta + q^{-1}(1-\theta).$$

Indeed, if $f \in \mathscr{P}$ then $M_x f \in \mathscr{P}_A$ and by Theorem 1

$$\|M_y M_x f\|_p \leqslant C\|M_x f\|_r^{\theta}\|M_y M_x f\|_q^{1-\theta} \leqslant C_1\|f\|_1^{\theta}\|M_{xy} f\|_q^{1-\theta}. \qquad \square$$

It is clear that if $\|M_x f\|_r \leqslant C|||f|||$, $f \in \mathscr{P}$ for some norm $|||\,.\,|||$ then the inequality of the corollary holds with $\|f\|_1$ on the right replaced by $|||f|||$. We shall use this in Part II to prove that certain sets are massive for some spaces smaller than $C(\mathbb{T})$ (actually a little bit more refined inequality will be used). Now we note that the function $x = \mathbb{1}_{\mathbb{Z}_+}$ does satisfy the hypotheses of Corollary by the classical Kolmogorov theorem. The latter has been strenghtened considerably in [15], namely, it was proved there that $\|M_{\mathbb{Z}_+} f\|_r \leqslant C_r\|f\|_{U^*}$, $0 < r < 1$, where

$$\|f\|_{U^*} \overset{\text{def}}{=} \sup\left\{\left|\int \bar{f} g\, dm\right| : g \in U, \|g\|_U \leqslant 1\right\}.$$

Thus we obtain the following refinement of inequality (1) of the introduction:

$$\|M_{E_0} f\|_p \leqslant \text{const } \|f\|_{U^*}^{\theta}\|M_{E_0} f\|_2^{1-\theta}, \qquad f \in L^2.$$

(To prove this, combine the estimate for $M_{\mathbb{Z}_+}$ from [15] just quoted with Theorem 1 in which it should be taken $I_k = [2^{2k-1}+1, 2^{2k}-1], s_k = 0, t_k = 1$.)

Now we give a generalization of Theorem 1. Roughly speaking, it means that on the intervals I_k we are free to apply again certain multiplier transforms and, moreover, the process can be iterated. This will lead to massive sets of more sophisticated structure.

Define by induction a sequence $\mathscr{A}_n$ of classes of functions on $\mathbb{Z}$. $\mathscr{A}_0$ consists of all functions y as in Theorem 1. We say that the collection of intervals $\{I_k\}$ mentioned in this theorem is adjusted to y. We assume that the end-points of I_k's are in $\mathbb{Z}_+\backslash\{0\}$.

Suppose that we have defined the class $\mathscr{A}_j$ and for every function u in $\mathscr{A}_j$ a collection of intervals $\{I_k\}$ adjusted to it, where $I_k = [a_k, b_k]$, $a_k,\ b_k \in \mathbb{Z}_+\backslash\{0\}$. Fix such u and $\{I_k\}$ and suppose that in each I_k a point c_k and two (finite or empty) systems of intervals $\{\Delta_{kl}\}$ and $\{\Delta'_{kl}\}$ are chosen. Assume further that the end-points of these intervals are integers, $\Delta_{kl} \subset (a_k, c_k]$, $\Delta'_{kl} \subset (c_k, b_k]$ and that for each fixed k each of the « translated » systems $\{\Delta_{kl} - a_k\}$, $\{b_k - \Delta'_{kl}\}$ is B-separated with some B independent of k. Let functions y_k and y'_k be constructed by these « translated » systems according to the same rule as y from Theorem 1 was constructed by I_k's mentioned there. Of course, the numbers $\{t_j\}$ and $\{s_j\}$ used in the construction vary from one system to another. But we assume that all of them as well as the constants in the inequalities $|s_j| \leqslant \mathrm{const} \min\{|t_{j-1}|, |t_j|\}$ are bounded uniformly in k. Define now a function v:

$$v(n) = \begin{cases} u(n), & n \notin \bigcup_n I_k \\ u(n)y_k(n+a_k), & n \in [a_k, c_k] \\ u(n)y'_k(b_k - n), & n \in (c_k, b_k]. \end{cases}$$

The collection of all intervals Δ_{kl}, Δ'_{kl} is said to be adjusted to v. The class $\mathscr{A}_{j+1}$ consists of all functions v so obtained.

THEOREM 1 *bis*. — *If* $u \in \mathscr{A}_j$ *and* $r \leqslant 1 < p < q$ *then* $\|M_u f\|_p \leqslant C\|f\|_r^\alpha \|M_u f\|_q^{1-\alpha}$ *for* $f \in \mathscr{P}_A$. *Here* α, $0 < \alpha < 1$ *depends on* j *and* C *does not depend on* f.

COROLLARY. — *If* x *is as in Corollary to Theorem* 1 *and* $u \in \mathscr{A}_j$ *then* $\|M_{xu} f\|_p \leqslant C\|f\|_1^\alpha \|\mathrm{M}_{xu} f\|_2^{1-\alpha}$ *for all* $f \in \mathscr{P}$. □

The proof of Theorem 1 *bis* presented in this article gives $\alpha = \theta^{j+1}$ where θ is from Theorem 1. It is worth noting that, in general, the estimate $\|M_v f\|_p \leqslant C|||f|||^\beta \|M_v f\|_q^{1-\beta}$ is the stronger, the greater β is, whatever be a (quasi) norm $|||\,.\,|||$. This follows immediately from the fact that to prove this estimate it is sufficient to check it under the additional assumption $|M_v f\|_q = 1$, $|||f||| \leqslant 1$ (we shall use this fact in the sequel several times). Indeed, suppose that under this assumption we have $\|M_v f\|_p \leqslant C|||f|||^{\beta}$. Then the same is true with no restrictions on $|||f|||$, because for $|||f||| > 1$ we have $\|M_v f\|_p \leqslant \|M_v f\|_q = 1 < |||f|||^{\beta}$. Now the homogeneity allows us to get rid of the assumption $\|M_v f\|_q = 1$ as well.

We shall not need to know the best possible α, though in principle this might be of some interest. Note in this connection that the estimate in Corollary to Theorem 1 remains true with θ defined by $p^{-1} = \theta + (1-\theta)q^{-1}$. See Section 3 of Part II for more detail.

1.2. Symbols Bounded away from 0 on $\mathbb{Z}_-$. A bounded function x on $\mathbb{Z}$ is said to satisfy the Hörmander-Mikhlin condition if

$$\text{(1)} \qquad \sup_k 2^k \sum_{2^k \leqslant |n| \leqslant 2^{k+1}} |x(n+1)-x(n)|^2 < \infty .$$

We recall that this condition guarantees that M_x is continuous in L^p for $1 < p < \infty$ and is of weak type (1,1) (see the next section for more information). So we can use as « x » in Corollary to Theorem 1 any bounded function satisfying (1) and vanishing on $\mathbb{Z}_-$. The function $u = xy$ (for which this Corollary gives the interpolation inequality) in interesting cases does not satisfy any longer the Hörmander-Mikhlin condition even if x does. But the Marcinkiewicz condition

$$\text{(2)} \qquad \sup_k \sum_{2^k \leqslant |n| \leqslant 2^{k+1}} |u(n+1)-u(n)| < \infty$$

is evidently fulfilled if x satisfies (1). Recall (see e.g. [14]) that, for any bounded function u, (2) implies that M_u acts in L^p, but in general the weak type (1,1) inequality fails for such multipliers. The interpolation inequality may fail for them either, as the example of $u = \mathbb{1}_{\{2^k : k \geqslant 1\}}$ shows. Indeed, if for this u we had $\|M_u f\|_p \leqslant C\|f\|_1^{\alpha}\|M_u f\|_q^{1-\alpha}$ for some $1 < p < q$ then M_u would act in L^1 (because $\|M_u f\|_r \asymp \left(\sum_{k \geqslant 1} |\hat{f}(2^k)|^2\right)^{1/2}$ for every r, $0 < r < \infty$).

In fact a moment reflection shows that the gap between condition (2) and the condition "$u = xy$ with x satisfying (1) and $y \in \mathscr{A}_0''$ is rather big. It turns out, nevertheless, that an interpolation inequality follows from the Marcinkiewicz condition provided $\inf_{n<0} |u(n)| > 0$.

THEOREM 2. — *Suppose that* $\sup\{|u(n)| : n \in \mathbb{Z}\} < \infty$, $\inf\{|u(n)| : n < 0\} > 0$ *and* u *satisfies* (2). *Then* $\|M_u f\|_p \leqslant C_{p,q,\alpha} \|f\|_1^\alpha \|M_u f\|_q^{1-\alpha}$. *Here* $1 < p < q$, *and* α *can be any positive number strictly less than* θ *defined by* $p^{-1} = \theta + (1-\theta)q^{-1}$.

As it was with Theorem 1, the construction that leads to Theorem 2 can be iterated. Let $\mathscr{B}_0$ be the class of all functions u satisfying the hypotheses of Theorem 2. Every separated sequence of intervals in $\mathbb{Z}_+$ is said to be ajusted to every function in $\mathscr{B}_0$. If for some n the class $\mathscr{B}_n$ has already been defined, take u in $\mathscr{B}_n$ and a sequence $\{I_l\}, I_l = [a_l, b_l]$ adjusted to u, and suppose that $\inf\{|u(n)| : n \notin \bigcup_l I_l\} > 0$. Fix a point $c_l \in [a_l, b_l]$ and consider for each l two functions on $\mathbb{Z}$, y_l and y'_l, such that

$$\sup_{l,n} |y_l(n)| < \infty, \qquad \sup_{l,k} \sum_{2^k \leqslant |n| \leqslant 2^{k+1}} |y_l(n+1) - y_l(n)| < \infty$$

and the same with y' in place of y. Then consider the function v,

$$v(t) = \begin{cases} u(t), & t \notin \bigcup_l I_l \\ u(t) y_l(t - a_l), & t \notin [a_l, c_l] \\ u(t) y'_l(t - b_l), & t \in (c_l, b_l]. \end{cases}$$

The class $\mathscr{B}_{n+1}$ consists of all functions v obtained in this way. By a system of intervals ajusted to v we mean any system $\mathscr{E}$ of intervals such that : (i) for each $I \in \mathscr{E}$ there exists l with either $I \subset [a_l, c_l]$ or $I \subset [c_l, b_l]$, and (ii) for each l the systems $\{I - a_l : I \in \mathscr{E}, I \subset [a_l, c_l]\}$ and $\{-I + b_l : I \in \mathscr{E}, I \subset [c_l, b_l]\}$ are B-separated with some B independent of l.

THEOREM 2 *bis*. — *Let* $u \in \mathscr{B}_n$, $1 < p < q$. *Then for some* C *and* $\beta, 0 < \beta < 1$, *we have the interpolation estimate* $\|M_u f\|_p \leqslant C\|f\|_1^\beta \|M_u f\|_q^{1-\beta}$.

1.3. Commentary. The Plan of the Exposition. In proofs of some theorems stated we repeatedly use the inequality $\|.\|_r \leqslant \|.\|_s$ for $r \leqslant s$. Thus it is unlikely to be possible to transfer all these theorems to the real line in place of $\mathbb{T}$ literally. But of course they can be transferred in some form which is close to literal in simple cases. For example if $E_0 = \bigcup_{k \geqslant 1} [2^{2k}, 2^{2k+1}]$ (but this time the intervals are considered in $\mathbb{R}$)

then for the operator $M_{E_0}, M_{E_0}f \stackrel{\text{def}}{=} (\mathbb{1}_{E_0}\hat{f})^\vee$ we have the estimate

$$\|M_{E_0}f\|_{L^p(\mathbb{R})} \leqslant C_{p,q} \|f\|^{\theta}_{L^1(\mathbb{R})} \|M_{E_0}f\|^{1-\theta}_{L^2(\mathbb{R})}$$

for all f in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ (here $1 < p < q, p^{-1} = \theta + (1-\theta)q^{-1}$). We do not discuss the case of $\mathbb{R}$ any more.

Note also that for the time being the theory considered is essentially one-dimensional: the most natural questions on interpolation inequalities for multipliers in the case of multidimensional tori $\mathbb{T}^n$ are open. It is unknown, for example, whether the inequality

$$\|Mf\|_{L^p(\mathbb{T}^n)} \leqslant C \|f\|^{\alpha}_{L^1(\mathbb{T}^n)} \|Mf\|^{1-\alpha}_{L^2(\mathbb{T}^n)}$$

is true for $M = M_{(\mathbb{Z}_+)^n}$ and $n \geqslant 3$. See [8] for a discussion of this problem. For $n = 2$ this inequality was proved by the author [7]. See also [1] for some refinements of this inequality still pertaining to the case $n = 2$.

Proofs of Theorems 1 and 1 *bis* on the one hand and 2 and 2 *bis* on the other are based on different ideas and are presented in Sections 4 and 5 respectively. To shorten calculations we prove completely only Theorems 1 and 2. The other two will be verified under many simplifying assumptions, but it will be quite clear how one can get rid of them.

In Section 2 more or less standard facts about multipliers are stated in the form convenient to us. Nothing except these facts is needed to prove Theorems 2 and 2 *bis*. To the contrary, Theorems 1 and 1 *bis* are finer. To prove them we shall need an extra inequality discussed in Section 3. The idea of its proof was used (for similar purposes) by the author in [6] and [7] and then by Bourgain [1].

Note also that Theorems 1 and 1 *bis* are the most interesting for applications. Theorems 2 and 2 *bis* are used rather as sources of counterexamples and should be considered as certain complements to the first two.

2. Information on multipliers.

2.1. The Hörmander-Mikhlin Condition and H^r-Multipliers. Suppose that a function $x : \mathbb{Z} \to \mathbb{C}$ satisfies $|x(n)| \leqslant A, n \in \mathbb{Z}$ and also the Hörmander-Mikhlin condition

$$\text{(1)} \qquad \sup_k 2^k \sum_{2^k \leqslant |n| \leqslant 2^{k+1}} |x(n+1) - x(n)|^2 = C < \infty .$$

Then for the function $K, K(\theta) = \sum_n x(n)e^{in\theta}$ (assume e.g. that the set $\{n : x(n) \neq 0\}$ is finite to avoid technical difficulties with the definition of K) we have

$$\text{(2)} \qquad \int_{2|\theta| \leqslant |t| < \pi} |K(t-\theta) - K(t)|\, dt \leqslant \text{const},$$

the constant depending on C and A only. (The proof of the counterpart of this fact for $\mathbb{R}$ in place of $\mathbb{T}$ can be found, e.g., in [4], p. 210-214; for $\mathbb{T}$ essentially the same argument works).

It is the last estimate for K that allows one to prove the properties of M_x with x satisfying (1) listed in the beginning of subsection 1.2 (consult [4], Section II.5). We have in addition

$$\|M_x f\|_1 \leqslant C\|f\|_1 \qquad \text{for} \qquad f \in H^1$$

with C depending only on A and « const » in (2) (see e.g. [2], p. 581, or [4]).

Moreover, suppose that $\{x_l\}$ is a sequence of functions on $\mathbb{Z}$ that satisfy (1) uniformly in l and assume $\sup_{n,l} |x_l(n)| < \infty$. Define an operator T on $L^p(l^2,m)$ (this space consists of sequences of functions $\{g_l\}$ with $\int (\Sigma|g_l|^2)^{p/2} dm < \infty$) by the formula $T(\{g_l\}) = \{M_{x_l} g_l\}$. Then, for $1 < p < \infty$, T is continuous: $\|Tg\|_p \leqslant C_p \|g\|_p, g \in L^p(l^2,m)$ (see e.g. Theorem 3.11 in Section V.3 of [4]).

In [2] it was proved that for a bounded function x satisfying (1) the operator M_x is a multiplier of H^r for $r > 2/3$. To prove Theorems 1 and 1 *bis* we shall need a result of the same kind for arbitrary $r > 0$. If r is small, (1) should be replaced by a condition requiring greater « smoothness » of the symbol of the multiplier. For us the following result will suffice.

PROPOSITION. — *Let g be a function in $C^\infty(0,\infty)$ satisfying $|(d/dt)^\beta g(t)| \leqslant C_\beta t^{-\beta}$ for all non-negative integers β. Define a function x on $\mathbb{Z}$ by $x(n) = 0$ for $n \leqslant 0$, $x(n) = g(n)$ for $n > 0$. Then the multiplier M_x maps H^r to H^r for every r, $0 < r \leqslant 1$, and L^p to L^p for $1 < p < \infty$. Moreover, the norm of M_x for each particular r or p depends only on constants C_β.*

Sketch of the proof. — We prove only the H^r-part of the proposition by reducing it to the counterpart for the real line that can be found in the monograph [4] (Section III.7, Theorem 7.30). Suppose for simplicity that g has compact support (this is sufficient for our purposes but in fact leads to no loss of generality) and extend g by 0 to the negative half-axis. Let F be the inverse Fourier transform of g and $f(t) = \sum_{n \in \mathbb{Z}} x(n)e^{int}$. Then f is the periodization of F: $f(t) = \sum_{l \in \mathbb{Z}} F(t + 2\pi l)$ (the series converges rapidly).

M_x is the operator of convolution with f, so it is sufficient to prove that for every r-atom a on the unit circle the $L^r(\mathbb{T})$-norm of $a * f$ is bounded by some constant depending only on a certain number of C_β's. (Consult the same monograph [4] for characterizations of H^r-spaces in terms of atoms). (It is probably worth noting that some specific features of the situation we consider make it particularly simple. The above uniform L^r-estimate trivially implies the uniform estimate of functions $a * f$ in the space H^r as well, because $a * f \in H^2$ for every r-atom a in view of the fact that $x(n) = 0$ for $n < 0$).

Now we can write

$$4\pi^2 \|a*f\|^r_{L^r} \leqslant \sum_{l \in \mathbb{Z}} \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} a(u)F(t-u+2\pi l)\, du \right|^r dt = \int_{\mathbb{R}} \left| \int_{-\pi}^{\pi} a(u)F(t-u)\, du \right|^r dt.$$

If we extend a by 0 to the complement of $[-\pi,\pi]$ we obtain an r-atom for the real line and the last expression is nothing but $\|a*F\|^r_{L^r(\mathbb{R})}$. This is bounded by a constant depending on a finite number of C_β's by the result quoted in the beginning of this proof. □

The next lemma easily follows from the Proposition.

LEMMA 1. — *Let $A > 1$. There exist functions φ_j, $j \geqslant 0$ on $\mathbb{Z}$ with the following properties :*

(*a*) $\varphi_j \geqslant 0$, $\sum_j \varphi_j = \mathbb{1}_{\mathbb{Z}_+ \setminus \{0\}}$, $\varphi_j(n) = 0$ *for* $n \notin [A^{j-1}, A^{j+1}]$.

(*b*) *If* $u = \sum \varepsilon_j \varphi_j$, $|\varepsilon_j| \leqslant 1$ *then for every fixed r the operators* M_u *are bounded uniformly in all collections* $\{\varepsilon_j\}$ *as operators from* H^r *to* H^r *for* $r \leqslant 1$ *and from* L^r *to* L^r *for* $r > 1$.

Proof. — Let α be a function in $C^\infty(\mathbb{R})$ *such that* $\alpha(s) = 1$ for $s \geqslant A^{-1/4}$, $\alpha(s) = 0$ for $s \leqslant A^{-1/2}$ and $0 \leqslant \alpha \leqslant 1$ everywhere. Define ψ by $\psi(s) = \alpha(s)$ for $s \leqslant 1$ and $\psi(s) = 1 - \alpha(sA^{-1})$ for $s \geqslant 1$. Clearly $\psi \in C^\infty(\mathbb{R})$ and if we set $\psi_j(s) = \psi(sA^{-j})$ then $\sum_{j \geqslant 0} \psi_j(s)$ is 1 for $s \geqslant 1$ and 0 for $s \leqslant 0$. If $|\varepsilon_j| \leqslant 1$ then the function $g = \sum \varepsilon_j \psi_j$ satisfies the hypotheses of the proposition with some constants C_β independent of the collection $\{\varepsilon_j\}$. So we can set $\varphi_j = \psi_j | \mathbb{Z}$. □

COROLLARY. — *With the above functions* φ_j *we have*

$$\left\| \left(\sum_j |M_{\varphi_j} f|^2 \right)^{1/2} \right\|_r \leqslant C_r \|f\|_r$$

for all f in H^r *if* $r \leqslant 1$ *and for all f in* L^r *if* $r > 1$.

Proof. — Let r_j be the Rademacher functions. We get from Lemma 1 that

$$\|\sum r_j(t) M_{\varphi_j} f\|_r^r \leqslant C_r^r \|f\|_r^r$$

uniformly in $t \in [0,1]$. Now integrate this in t over $[0,1]$ and apply the Khintchine inequality. □

2.2. The Littlewood-Paley Decomposition. This is the name of the following statement.

Let $\{a_n\}_{n \in \mathbb{Z}}$ be a strictly increasing family of integers such that

$$\sup_k \operatorname{card} \{n : a_n \in [2^k, 2^{k+1}] \cup [-2^{k+1}, -2^k]\} = C < \infty. \tag{3}$$

Set $I_n = [a_n, a_{n+1})$. Then for $1 < p < \infty$ we have the two-sided estimate

$$c_p \|f\|_p \leqslant \left\| \left(\sum_n |M_{I_n} f|^2 \right)^{1/2} \right\|_p \leqslant C_p \|f\|_p, \ f \in L^p(\mathbb{T}) \tag{4}$$

(c_p and C_p do not depend on f).

The standard argument leading to this theorem (that can be found e.g. in [14], Chapter IV, Section 5) after iterations gives the analogous result for more general decompositions of $\mathbb{Z}$ into intervals. (Cf. also [13] for the case of $\mathbb{R}$.) We give the precise statement.

Define by induction a class $\mathscr{D}$ of decompositions of $\mathbb{Z}$ into intervals. First we attribute to $\mathscr{D}$ all decompositions generated by sequences satisfying (3), as above. Next, if $D \in \mathscr{D}$ and $D = \{[a_s, b_s)\}$, we attribute to $\mathscr{D}$ all decompositions that can be obtained by further subdivision of intervals $[a_s, b_s)$ by some points $a_s = a_{1,s} < a_{2,s} < \cdots < a_{u,s} < b_{v,s} < \cdots < b_{1,s} = b_s$ such that the sequences $\{a_{j,s} - a_s\}_{1 \leqslant j \leqslant u}$ and $\{b_s - b_{j,s}\}_{1 \leqslant j \leqslant v}$ satisfy (3) with C independent of s (u and v can depend on s).

Generalized Littlewood-Paley Theorem. Estimate (4) is true for every decomposition of class $\mathscr{D}$.

As to the proof we note only that in the argument leading to the classical Littlewood-Paley decomposition the Hörmander-Mikhlin theorem is ordinarily used. To obtain the generalization just stated by iterating this argument the variant of this theorem for $L^p(l^2, m)$ mentioned in the preceding subsection should be employed.

2.3. The Marcinkiewicz Multiplier Theorem. This theorem has already been mentioned in Section 1. The Marcinkiewicz condition (inequality (2) in Section 1) is weaker than that of Hörmander and Mikhlin but does not guarantee the weak type (1.1) estimate for the multiplier in question. On H^1 we have, however, a substitute for this estimate: if u is a bounded function satisfying (2) of Section 1 and $w_k = \mathbb{1}_{[2^k, 2^{k+1})}$, $k \geqslant 0$ then for $f \in H^1$ we have

$$m\{(\sum |M_{uw_k} f|^2)^{1/2} > \lambda\} \leqslant \text{const } \lambda^{-1} \|f\|_1 .$$

Note that this statement is unlikely to have multidimensional counterparts.

We shall need a similar fact for l^2-valued functions as well as a variant of the Marcinkiewicz theorem for them. Here is the statement.

Complement to the Marcinkiewicz Theorem. Let $\{v_s\}$ be a sequence of functions on $\mathbb{Z}$ satisfying

$$\sup_{s,n} |v_s(n)| < \infty , \qquad \sup_{s,k} \sum_{2^k \leqslant |n| \leqslant 2^{k+1}} |v_s(n+1) - v_s(n)| < \infty .$$

(*a*) If $1 < p < \infty$ and $(\sum |f_s|^2)^{1/2} \in L^p$ then

$$\|(\sum |M_{v_s} f_s|^2)^{1/2}\|_p \leqslant \text{const } \|(\Sigma |f_s|^2)^{1/2}\|_p .$$

(*b*) Let $A > 1$, $w_k = \mathbb{1}_{\mathbb{Z} \cap [A^k, A^{k+1})}\ (k \geqslant 0)$. If $f_s \in H^1$ for all s then

$$m\left\{\left(\sum_{k,s} |M_{w_k v_s} f_s|^2\right)^{1/2} > \lambda\right\} \leqslant \text{const } \lambda^{-1} \left\|\left(\sum_s |f_s|^2\right)^{1/2}\right\|_1 .$$

Proof. — We begin with (*b*) restricting ourselves to the scalar valued case. (I was not able to find a reference even for this case, though the proof is standard.) So in place of the sequence $\{v_s\}$ we have only one function v.

For each k let X_k be the continuous function on $\mathbb{R}$ equal to 1 on $[A^k, A^{k+1})$, to zero on $(-\infty, A^{k-1}) \cup (A^{k+2}, +\infty)$ and linear on each of the intervals $[A^{k-1}, A^k]$ and $[A^{k+1}, A^{k+2}]$. Set $x_k = X_k | \mathbb{Z}$. Then the functions $\sum \varepsilon_k x_k$ with $|\varepsilon_k| \leqslant 1$ satisfy the Hörmander-Mikhlin condition uniformly in all collections $\{\varepsilon_k\}$. Thus we have for $f \in H^1$:

$$\|(\sum |M_{x_k} f|^2)^{1/2}\|_1 \leqslant \text{const } \|f\|_1 . \tag{5}$$

Set $g_k = M_{x_k} f$, $a_k = \min\{n \in \mathbb{Z} : n \geqslant A^k\}$. Then

$$M_{w_k v} f = v(a_k)(z^{a_k} \mathbb{P}_+(\bar{z}^{a_k} g_k) - z^{a_{k+1}} \mathbb{P}_+(\bar{z}^{a_{k+1}} g_k)) + \sum_{a_k < n \leqslant a_{k+1}} (z^n \mathbb{P}_+(\bar{z}^n g_k) - z^{a_{k+1}} \mathbb{P}_+(\bar{z}^{a_{k+1}} g_k))(v(n) - v(n-1)) .$$

Denote the term written before the summation sign by h_k. Since v is bounded, we obtain by the weak type (1,1) inequality for $\mathbb{P}_+$ on $L^2(l^2)$:

$$m\left\{\left(\sum_k |h_k|^2\right)^{1/2} > \lambda\right\} \leqslant C\lambda^{-1} \left\|\left(\sum_k |g_k|^2\right)^{1/2}\right\|_1 .$$

Thus by (5) all we need will be proved if we establish the estimate

$$m\left\{\left(\sum_k \left(\sum_{a_k < n \leqslant a_{k+1}} |v(n) - v(n-1)|\, |\mathbb{P}_+(\bar{z}^n g_k)|\right)^2\right)^{1/2} > \lambda\right\} \leqslant C\lambda^{-1} \left\|\left(\sum_k |g_k|^2\right)^{1/2}\right\|_1 \tag{6}$$

and the analogous estimate with $\mathbb{P}_+(\bar{z}^n g_k)$ replaced by $\mathbb{P}_+(\bar{z}^{a_{k+1}} g_k)$ (the proof of the latter is the same as for (6) and will be omitted).

Set $d_n = |v(n)-v(n-1)|^{1/2}$, $\varphi_{k,n} = d_n \bar{z}^n g_k$ (for each k the index n satisfies $a_k < n \leqslant a_{k+1}$). Then

$$m\left\{\left(\sum_{k,n} |\mathbb{P}_+\varphi_{k,n}|^2\right)^{1/2} > \lambda\right\} \leqslant C\lambda^{-1}\left\|\left(\sum_{k,n} |\varphi_{k,n}|^2\right)^{1/2}\right\|_1 .$$

This implies (6) because, on the one hand, for a fixed k we have by the Hölder inequality and the Marcinkiewicz condition for v:

$$\sum_{a_k<n\leqslant a_{k+1}} |v(n)-v(n-1)|\,|\mathbb{P}_+(\bar{z}^n g_k)| =$$

$$\sum_{a_k<n\leqslant a_{k+1}} |v(n)-v(n-1)|^{1/2}\,|\mathbb{P}_+(\varphi_{k,n})| \leqslant \text{const}\left(\sum_{a_k<n\leqslant a_{k+1}} |\mathbb{P}_+(\varphi_{k,n})|^2\right)^{1/2},$$

and on the other hand

$$\sum_{k,n} |\varphi_{k,n}|^2 \leqslant \left(\sup_k \sum_{a_k<n\leqslant a_{k+1}} |d_n|^2\right)\sum_k |g_k|^2 \leqslant \text{const}\sum_k |g_k|^2,$$

again by the Marcinkiewicz condition.

The l^2-valued case of (b) can be proved by a routine repetition of essentially the same argument. Hint: Write first $\|(\sum |f_s|^2)^{1/2}\|_1 \asymp \int_0^1 \|\sum r_s(t) f_s\|_1\, dt$, the r_s being the Rademacher functions. For almost every t the function $\sum r_s(t) f_s$ is in H^1 and so we can apply to it the multiplier with symbol $\sum r_k(\tau)x_k$. This leads to

$$\left\|\left(\sum_{s,k} |M_{x_k} f_s|^2\right)^{1/2}\right\|_1 \leqslant \text{const}\left\|\left(\sum_s |f_s|^2\right)^{1/2}\right\|_1$$

in place of (5). And so on.

To prove (a) one should repeat with minor changes the standard proof of the Marcinkiewicz theorem (see e.g., [14], Chapter IV, Section 6 for the latter in the case of $\mathbb{R}$ instead of $\mathbb{T}$). □

3. Interpolation inequalities for vector-valued spaces L^r/H^r.

3.1. The Statement. Commentary. Let X be a Banach space. Denote by $|||\cdot|||_t$, $0 < t < \infty$ the norm in the quotient space $L^t(X)/H^t_-(X)$. (The definition of $H^t_-(X)$ and $H^t(X)$ was given in the Introduction. We also recall that $\mathcal{P}(X)$ denotes the space of X-valued trigonometric polynomials).

THEOREM 3. — *Let* $0 < r < p < q < \infty$, $p^{-1} = \theta r^{-1} + (1-\theta)q^{-1}$. *Suppose also that the number* s *defined by* $s^{-1} = p^{-1} - q^{-1}$ *is strictly greater than* 1. *Then*

$$|||g|||_p \leqslant C_{p,q,r} |||g|||_r^{\theta} |||g|||_q^{1-\theta}$$

for every g *in* $\mathscr{P}(X)$.

Remark. — The proof presented below gives $C_{p,q,r} \leqslant C2^{1/p}(r+1)r^{-1}s^3(s-1)^{-1}$ for some numerical constant C.

COROLLARY. — *Suppose in addition that* $p > 1$ *(in this case automatically* $s > 1$*) and* X *is a Hilbert space. Then for* $g \in \mathscr{P}(X)$ *we have*

$$\|\mathbb{P}_+ g\|_p \leqslant C'_{p,q,r} \|g\|_r^{\theta} \|\mathbb{P}_+ g\|_q^{1-\theta}.$$

Proof of the Corollary. — If h generates the same class as g in the quotient space $L^p(X)/H^p_-(X)$ then $\mathbb{P}_+ g = \mathbb{P}_+ h$. Thus $\|\mathbb{P}_+ g\|_p \leqslant Cp^2(p-1)^{-1}|||g|||_p$. On the other hand it is evident that $|||g|||_r \leqslant \|g\|_r$ and $|||g|||_q \leqslant \|\mathbb{P}_+ g\|_q$. □

Let us comment the Corollary. It has just been used that for $t > 1$ we have $|||g|||_t \leqslant \|\mathbb{P}_+ g\|_t \leqslant C(t)|||g|||_t$ for all g, and hence for $r > 1$ the estimate reduces (to within the constant) to the trivial estimate $\|\mathbb{P}_+ g\|_p \leqslant \|\mathbb{P}_+ g\|_r^{\theta} \|\mathbb{P}_+ g\|_q^{1-\theta}$. If $r = 1$, the situation is slightly more complicated, but still the inequality in question can easily be derived from the weak type $(1,1)$ estimate for $\mathbb{P}_+$. But for $r < 1$ no kind of such argument will work. Thus the conclusion of the Corollary for $r < 1$ can be considered as a certain substitute for the L^r-regularity of $\mathbb{P}_+$.

The main ingredient of the proof of Theorem 3 given below (a « trick » with multiplication by an appropriate outer function) has already been used by the author in [6] (Lemma 1) and [7] (Section 4) for $X = \mathbb{C}$, but sharp estimates (with the exponent θ) were not presented there. It [1] (proposition 4.1) the same trick was employed to prove the counterpart of the Corollary for the Lorentz space $L^{1,\infty}$ (again for $X = \mathbb{C}$):

$$\|\mathbb{P}_+ g\|_p \leqslant C_{p,q} \|g\|_{1,\infty}^{\theta} \|\mathbb{P}_+ g\|_q^{1-\theta}, \tag{1}$$

where $1 < p < q$, $p^{-1} = \theta + (1-\theta)q^{-1}$, $\|g\|_{1,\infty} \overset{\text{def}}{=} \sup_{\lambda>0} \lambda m\{|g|>\lambda\}$.

In applications $r < 1$, $p > 1$ will always hold.

We emphasize that in Theorem 3 no hypotheses on $L^p(X)$-regularity of $\mathbb{P}_+$ are involved. We have formulated Theorem 3 and its corollary for vector-valued functions because this will really be needed in the sequel. But it should be noted that the vector-valued situation presents here no complications compared with the scalar case.

Note also that we can do all of this with H^t substituted for H^t_- and $\mathbb{P}_-$ for $\mathbb{P}_+$.

3.2. The Proof of Theorem 3. Recall that $\mathscr{P}_-(X)$ is the set of X-valued antianalytic polynomials. It is clear that for every $\varepsilon>0$ there are g_1, g_2 in $\mathscr{P}(X)$ such that $g - g_1 \in \mathscr{P}_-(X)$, $g - g_2 \in \mathscr{P}_-(X)$ and $\|g_1\|_r \leqslant (1+\varepsilon)\,|||g|||_r$, $\|g_2\|_q \leqslant (1+\varepsilon)\,|||g|||_q$. Set $f = g_1 - g_2$ (so $f \in \mathscr{P}_-(X)$). Define a function α on $\mathbb{T}$ by $\alpha(\zeta) = \lambda\|g_1(\zeta)\|_X^{-1}$ if $\|g_1(\zeta)\|_X > \lambda$ and $\alpha(\zeta) = 1$ if $\|g_1(z)\|_X \leqslant \lambda$. Let τ be the outer function for which $|\tau| = \alpha$ a.e.: $\tau = \exp(\log\alpha + iH(\log\alpha))$, where H is the harmonic conjugation operator. It is clear that $\bar{\tau}f \in H^t_-(X)$ for all t. We show that with λ appropriately chosen the function $g_2 + \bar{\tau}f$ is just the representative of the class generated by g whose L^p-norm admits the estimate we want, to within ε. We have

$$\|g_2+\bar{\tau}f\|_p \leqslant C(\|\bar{\tau}(g_2+f)\|_p + \|(1-\bar{\tau})g_2\|_p) = C(\|\bar{\tau}g_1\|_p+\|(1-\bar{\tau})g_2\|_p),$$

where $C = 1$ if $p \geqslant 1$ and $C = 2^{1/p}$ if $p < 1$. By the definition of τ, $|\tau| \leqslant 1$ and $\|\bar{\tau}(\zeta)g_1(\zeta)\|_X \leqslant \lambda$ a.e. Thus $\|\bar{\tau}g_1\|_p \leqslant (\lambda^{p-r}\|g_1\|_r^r)^{1/p}$. The term $\|(1-\bar{\tau})g_2\|_p$ will be estimated by using the Hölder inequality:

$$\|(1-\bar{\tau})g_2\|_p \leqslant \|g_2\|_q\left(\int|1-\tau|^s\,dm\right)^{1/s}.$$

Let us estimate the integral on the right. Let $e = \{\zeta\in\mathbb{T} : \|g_1(\zeta)\|_X > \lambda\}$. Then $|1-\tau| \leqslant 2$ on e and on $\mathbb{T}\backslash e$ we have

$$|1 - \tau| = |1 - \exp(iH(\log\alpha))| \leqslant \text{const}\,|H(\log\alpha)|.$$

So

$$\left(\int|1-\tau|^s\,dm\right)^{1/s} \leqslant C(m(e) + \int_{\mathbb{T}}|H(\log\alpha)|^s\,dm)^{1/s} \leqslant$$
$$C(\lambda^{-r}\|g_1\|_r^r)^{1/s} + C_s\left(\int_e|\log\alpha|^s\,dm\right)^{1/s}.$$

Consider the distribution function γ, $\gamma(t) = m\{\|g_1\|_X > t\}$. Clearly $\gamma(t) \leqslant t^{-r}\|g_1\|_r^r$. Now

$$\left(\int_e |\log \alpha|^s \, dm\right)^{1/s} = \left(\int_{\|g_1\|_X > \lambda} (\log \|\lambda^{-1} g_1\|_x)^s \, dm\right)^{1/s} =$$

$$\left(\int_\lambda^\infty s\left(\log \frac{t}{\lambda}\right)^{s-1} \gamma(t) \frac{dt}{t}\right)^{1/s} \leqslant C(\lambda^{-r}\|g_1\|_r^r \int_1^\infty (\log \sigma)^{s-1} \sigma^{-r-1} d\sigma)^{1/s} = C_{r,s}(\lambda^{-r}\|g_1\|_r^r)^{1/s}.$$

Thus finally $\left(\int |1-\tau|^s dm\right)^{1/s} \leqslant C_{r,s}(\lambda^{-r}\|g_1\|_r^r)^{1/s}$ and

$$\|g_2 + \bar{\tau} f\|_p \leqslant C_{p,q,r}\left(\lambda^{1-\frac{r}{p}} \|g_1\|_r^{\frac{r}{p}} + \lambda^{-\frac{r}{s}} \|g_1\|_r^{\frac{r}{s}} \|g_2\|_q\right).$$

Now it suffices to put $\lambda = \|g_1\|_r^{-q^{-1}/(r^{-1}-q^{-1})} \|g_2\|_q^{-r^{-1}/(r^{-1}-q^{-1})}$. □

4. Symbols vanishing on $\mathbb{Z}_-$.

4.1. The Proof of Theorem 1. Let $f \in \mathcal{P}_A$. Choose a constant $A > 1$ so close to 1 that each interval $[A^{j-1}, A^{j+1}]$ ($j \geqslant 0$) meets at most one of the intervals I_k from the definition of the function y before the statement of Theorem 1. (Since the sequence $\{I_k\}$ is separated, such a choice is possible.) Let $\{\varphi_j\}_{j \geqslant 0}$ be the sequence of functions given by Lemma 1 (subsection 2.1) for this A. By the Corollary to that lemma

$$\left\|\left(\sum_j |M_{\varphi_j} f|^2\right)^{1/2}\right\|_r \leqslant C\|f\|_r. \tag{1}$$

Consider three sets of values of j:

$$G_1 = \{j:\ \varphi_j \mathbb{1}_{I_k} = \varphi_j \quad \text{for some} \quad k\}$$

$$G_2 = \{j:\ \varphi_j \mathbb{1}_{I_k} = 0 \quad \text{for all} \quad k\}$$

$$G_3 = \mathbb{Z}_+ \setminus (G_1 \cup G_2).$$

If $j \in G_3$ then there is a unique $k = k(j)$ with $\varphi_j(a_{k(j)}) \neq 0$ or $\varphi_j(b_{k(j)}) \neq 0$ (it is possible that the both inequalities hold; recall that $I_k = [a_k, b_k]$). Set $u_j = \mathbb{1}_{(-\infty, a_k)} \varphi_j$, $v_j = \mathbb{1}_{[a_k, b_k]} \varphi_j$, $w_j = \mathbb{1}_{(b_k, +\infty)} \varphi_j$ with $k = k(j)$.

Fix an integer N and a numerical sequence $\{\alpha_j\}$ and consider the sequence $\{\alpha_j \bar{z}^{b_{k(j)}+1} M_{\varphi_j} f\}_{j \in G_3, j \leqslant N}$. This sequence is an l^2-valued trigonometric polynomial. We apply to it the Corollary to Theorem 3. Since $\mathbb{P}_+(\bar{z}^{b_{k(j)}+1}M_{\varphi_j}f) = \bar{z}^{b_{k(j)}+1}M_{w_j}f$, we get

$$\left(\int\left(\sum_{j\in G_3}|\alpha_j M_{w_j}f|^2\right)^{p/2}dm\right)^{1/p}\leqslant$$
$$C\left(\int\left(\sum_{j\in G_3}|\alpha_j M_{\varphi_j}f|^2\right)^{r/2}dm\right)^{\theta/r}\left(\int\left(\sum_{j\in G_3}|\alpha_j M_{w_j}f|^2\right)^{q/2}dm\right)^{(1-\theta)/q},$$

where $p^{-1} = \theta r^{-1} + (1-\theta)q^{-1}$. (We have extended the summation to all $j\in G_3$ because the estimate is uniform in N). In virtue of (1) the first factor on the right is majorized by $C\|f\|_r^\theta$, provided $\sup_j|\alpha_j| < \infty$. If in addition $|\alpha_j| \leqslant \text{const}\,|t_{k(j)+1}|$ for all j (t_k's are from the statement of Theorem 1) then it is easy to see that the second factor on the right is majorized by $C\|M_yf\|_q^{1-\theta}$. (Indeed, by Corollary to Lemma 1, $\left\|\left(\sum_{j\in G_3}|M_{\varphi_j y}f|^2\right)^{1/2}\right\|_q \leqslant C\|M_yf\|_q$. Using the continuity of $\mathbb{P}_+$ in $L^q(l^2)$ we get from this that $\left\|\left(\sum_{j\in G_3}|t_{k(j)+1}M_{w_j}f|^2\right)^{1/2}\right\|_q \leqslant C\|M_yf\|_q$). So if the α_k's are subjected to the both conditions then

$$\left\|\left(\sum_{j\in G_3}|\alpha_j M_{w_j}f|^2\right)^{1/2}\right\|_p \leqslant C\|f\|_r^\theta\|M_yf\|_q^{1-\theta}. \tag{2}$$

If, moreover, $|\alpha_j| \leqslant \text{const}\,|t_{k(j)}|$ for all $j\in G_3$ then we get in the same way (by using the variant of Corollary to Theorem 3 with $\mathbb{P}_+$ replaced by $\mathbb{P}_-$):

$$\left\|\left(\sum_{j\in G_3}|\alpha_j M_{u_j}f|^2\right)^{1/2}\right\|_p \leqslant C\|f\|_r^\theta\|M_yf\|_q^{1-\theta}. \tag{3}$$

From the Hölder inequality and (1) we obtain

$$\left\|\left(\sum_{j\in G_3}|s_{k(j)}M_{\varphi_j}f|^2\right)^{1/2}\right\|_p \leqslant C\|f\|_r^\theta\left\|\left(\sum_{j\in G_3}|s_{k(j)}M_{\varphi_j}f|^2\right)^{1/2}\right\|_p^{1-\theta},$$

and again the second factor on the right is majorized by $\|M_yf\|^{1-\theta}$, in virtue of the continuity of the Riesz projection in $L^q(l^2)$ and the inequality $|s_k| \leqslant \text{const}\min(|t_k|, |t_{k+1}|)$ from the hypotheses of Theorem 1. Combining this inequality with (2) and (3) in which we put $\alpha_j = s_{k(j)}$ we find

$$\left\|\left(\sum_{j\in G_3}|s_{k(j)}M_{v_j}f|^2\right)^{1/2}\right\|_p \leqslant C\|f\|_r^\theta\|M_yf\|_q^{1-\theta}.$$

Set now $\alpha_j = t_{k(j)}$ in (3) and $\alpha_j = t_{k(j)+1}$ in (2). Then (3) and (2) together with the last inequality give

$$\left\| \left(\sum_{j \in G_3} |M_{y\varphi_j} f|^2 \right)^{1/2} \right\|_p \leqslant C \|f\|_r^\theta \|M_y f\|_q^{1-\theta}.$$

The analogous estimate with G_1 or G_2 in place of G_3 is also true and is much simpler: if $j \in G_1 \cup G_2$ then y is constant on the support of φ_j and so the desired estimate follows from the Hölder inequality and (1). Thus

$$\left\| \left(\sum_{j \geqslant 0} |M_{\varphi_j} M_y f|^2 \right)^{1/2} \right\|_p \leqslant C \|f\|_r^\theta \|M_y f\|_q^{1-\theta}.$$

Since for every $n \in \mathbb{Z}_+$ the relation $\varphi_j(n) \neq 0$ can hold for at most two adjacent values of j, we can apply the Littlewood-Paley theorem to the sums over odd and even j's separately to obtain that the term on the left majorizes $\|M_y f\|_p$. □

4.2. Concerning the Proof of Theorem 1 bis. We restrict ourselves to symbols of class $\mathscr{A}_1$ (the further advance is made by repeating the same procedure). Let y and $\{I_k\}$ be as in Theorem 1 and let intervals Δ_{kl}, $\Delta_{kl} \subset I_k$ and functions y_k be such as described in the definition of $\mathscr{A}_1$ (for simplicity we assume that c_k from this definition coincides with b_k for each k and so there are no intervals Δ_k' and functions y_k'). So we consider the multiplier with symbol α,

$$\alpha(n) = \begin{cases} y(n), & n \notin \bigcup_k I_k \\ y(n) y_k(n + a_k), & n \in I_k. \end{cases}$$

We shall prove the interpolation inequality for functions f in $\mathscr{P}_A$ satisfying $\|f\|_r \leqslant 1$, $\|M_\alpha f\|_q = 1$ (see the end of subsection 1.1 for a reduction of the general case to this).

We keep the notation from the proof of Theorem 1. Some estimates obtained in the course of that proof will be used. Let $E = \bigcup_k I_k$, $\beta = y \mathbb{1}_E$, $\gamma = y - \beta$. We can apply Theorem 1 to γ and obtain

$$(4) \quad \left\| \left(\sum_j |M_{\gamma\varphi_j} f|^2 \right)^{1/2} \right\|_p \leqslant C \|f\|_r^\theta \|M_\gamma f\|_q^{1-\theta} \leqslant C \|f\|_r^\theta,$$

because $\|M_\gamma f\|_q \leqslant \text{const}\, \|M_\alpha f\|_q$ in view of the Marcinkiewicz theorem.

(Note that when considering the classes $\mathscr{A}_n$ with $n > 1$ one should use at the same point the L^q-continuity of a multiplier whose symbol does not satisfy the Marcinkiewicz condition. This continuity property can be derived from Generalized Littlewood-Paley theorem and part (*a*) of the Complement to the Marcinkiewicz Theorem stated in Section 2). Since $|s_k| \leqslant \text{const} \min\{|t_k|, |t_{k+1}|\}$, it follows from (4) that

$$\left\|\left(\sum_{j \in G_3} |s_{k(j)} M_{u_j+w_j} f|^2\right)^{1/2}\right\|_p \leqslant C\|f\|_r^\theta .$$

From this and (1) we derive that

$$\left\|\left(\sum_{j \in G_3} |s_{k(j)} M_{v_j} f|^2\right)^{1/2}\right\|_r \leqslant 2^{1/r}\left(\left\|\left(\sum_{j \in G_3} |M_{\varphi_j} f|^2\right)^{1/2}\right\|_r + \right.$$

$$\left.\left\|\left(\sum_{j \in G_3} |s_{k(j)} M_{u_j+w_j} f|^2\right)^{1/2}\right\|_r\right) \leqslant C(\|f\|_r + \|f\|_r^\theta) \leqslant C_1\|f\|_r^\theta ,$$

since $\|f\|_r \leqslant 1$ and $0 < \theta < 1$. It follows directly from (1) that $\left\|\left(\sum_{j \in G_1} |s_{k(j)} M_{\varphi_j} f|^2\right)^{1/2}\right\|_r \leqslant C\|f\|_r$ (here $k(j)$ denotes the unique number k with $\varphi_j \mathbb{1}_{I_k} = \varphi_j$). Together with the preceding estimate this yields

$$\left\|\left(\sum_j |M_{\beta\varphi_j} f|^2\right)^{1/2}\right\|_r \leqslant C\|f\|_r^\theta .$$

Thus we have « extracted » the part of the symbol y which is subject to changes during the passage from y to α. Now we repeat the proof of Theorem 1 with small variations. The preceding estimate and the Khintchine inequality imply

$$\int_0^1 \int_{\mathbb{T}} \left|\sum_{j \in G_1 \cup G_3} r_j(t) \bar{z}^{d_j} M_{\beta\varphi_j} f\right|^r dm\, dt \leqslant C\|f\|_r^{\theta r} ,$$

where d_j is the left end of the unique interval I_k that intersects the support of φ_j. Let $\{\psi_m\}$ be the functions given by Lemma 1 with parameter A so close to 1 that for every fixed m and k the translated system of intervals $\{\Delta_{kl} - a_k\}$ contains at most one intersecting the support of ψ_m. (Recall that all these systems of intervals are B-separated with the same B). Applying the multiplier with symbol $\sum_m r_m(\tau)\psi_m$ to the function $\sum_{j \in G_1 \cup G_3} r_j(t) \bar{z}^{d_j} M_{\beta\varphi_j} f$ we find by Lemma 1, the counterpart of the Khintchine inequality for the system $\{r_j(t)r_m(\tau)\}$ and the preceding estimate that

$$\left\|\left(\sum_{j,m}|M_{\beta\xi_{j,m}}f|^2\right)^{1/2}\right\|_r \leqslant C\|f\|_r^{\theta}, \tag{5}$$

where $\xi_{j,m}(n) = \varphi_j(n)\psi_m(n-d_j)$. This is the counterpart of (1) from the proof of Theorem 1 and we apply the procedure used there. Namely, we single out the pairs (j,m) for which at least one of the end-points of some Δ_{kl} lies in the support of $\xi_{j,m}$. Then we apply appropriately Corollary of Theorem 3 to the sum over such pairs, and so on. The result of all this will be

$$\|M_\delta f\|_p \leqslant C\|f\|_r^{\theta^2},$$

where $\delta = \alpha \mathbb{1}_{\cup I_k}$. ($\theta^2$ appears because, in contradistinction to (1), the exponent θ is already present in (5) on the right. It should be noted that at the end of the argument we use the Generalized Littlewood-Paley theorem in place of the ordinary one used in the proof of Theorem 1). Since (4) means that $\|M_\gamma f\|_p \leqslant C\|f\|_r^{\theta}$, and $\gamma + \delta = \alpha$ we finally obtain $\|M_\alpha f\|_p \leqslant C\|f\|_r^{\theta^2}$. □

5. Symbols, bounded away from 0 on $\mathbb{Z}_-$.

5.1. The Proof of Theorem 2. We assume without loss of generality that $\|f\|_1 \leqslant 1$, $\|M_u f\|_q = 1$. The last equality implies that $\|\mathbb{P}_- f\|_q \leqslant \text{const}$, since the Marcinkiewicz condition for the function $\mathbb{1}_{\mathbb{Z}_-}u^{-1}$ can easily be verified. On the other hand, $m\{|\mathbb{P}_- f| > \lambda\} \leqslant C\lambda^{-1}\|f\|_1$, and so for $1 < r < q$ we have $\|\mathbb{P}_- f\|_r \leqslant C_r\|f\|_1^{\sigma}$, where $r^{-1} = \sigma + (1-\sigma)q^{-1}$ (note that $\sigma \to 1$ as $r \to 1$). Fix some r, $1 < r < q$. Since u satisfies the Marcinkiewicz condition, the last estimate gives

$$\|\mathbb{P}_- M_u f\|_r = \|M_u \mathbb{P}_- f\|_r \leqslant C\|f\|_1^{\sigma}. \tag{1}$$

Besides,

$$\|\mathbb{P}_+ f\|_1 \leqslant \|f\|_1 + \|\mathbb{P}_- f\|_1 \leqslant \|f\|_1 + \|\mathbb{P}_- f\|_r \leqslant C_r\|f\|_1^{\sigma}. \tag{2}$$

Now we apply part (b) of the Complement to the Marcinkiewicz Theorem (see subsection 2.3) in the scalar case. Let $w_k = \mathbb{1}_{\mathbb{Z}\cap[A^k, A^{k+1})}$ ($A > 1$ is fixed). The part (b) just mentioned implies that

$$m\left\{\left(\sum_k |M_{w_k u}\mathbb{P}_+ f|^2\right)^{1/2} > \lambda\right\} \leqslant C\lambda^{-1}\|\mathbb{P}_+ f\|_1. \tag{3}$$

The Littlewood-Paley theorem directly gives the estimate

$$\left\|\left(\sum_k |M_{w_k u}\mathbb{P}_+ f|^2\right)^{1/2}\right\|_q \leqslant C\|M_u\mathbb{P}_+ f\|_q \leqslant C\|M_u f\|_q \leqslant C.$$

« Interpolating » between this estimate and (3) we obtain

$$\left\|\left(\sum_k |M_{w_k u}\mathbb{P}_+ f|^2\right)^{1/2}\right\|_p \leqslant C\|\mathbb{P}_+ f\|_1^\theta,$$

where $p^{-1} = \theta + (1-\theta)q^{-1}$. Another application of the Littlewood-Paley Theorem gives now $\|\mathbb{P}_+ M_u f\|_p \leqslant C\|\mathbb{P}_+ f\|_1^\theta$. Taking into account (1) (where we put $r=p$) and (2) (where we do not specify r for the moment) we conclude that

$$\|M_u f\|_p \leqslant \|\mathbb{P}_+ M_u f\|_p + \|\mathbb{P}_- M_u f\|_p \leqslant C(\|f\|_1^{\theta\sigma} + \|f\|_1^\theta) \leqslant 2C\|f\|_1^{\theta\sigma},$$

since $\|f\|_1 \leqslant 1$. Recall now that σ is close to 1 if r is close to 1, and so the exponent $\theta\sigma$ can be chosen arbitrarily close to θ. □

5.2. Concerning the Proof of Theorem 2 *bis*. We restrict ourselves to symbols of class $\mathscr{B}_1$ (as in Theorem 1 *bis*, to advance further the procedure should be repeated). Moreover, we consider for the sake of simplicity only symbols v of the form

$$v(n) = \begin{cases} u(n), & n \notin \bigcup_l I_l \\ u(n)y_l(n-a_l), & n \in I_l. \end{cases}$$

Here u satisfies the hypotheses of Theorem 1, $\{I_l\}$ is a separated family of intervals in $\mathbb{Z}_+$ with $\inf\left\{|u(n)| : n \notin \bigcup_l I_l\right\} > 0$ and $\{y_l\}$ is a uniformly bounded sequence of functions on $\mathbb{Z}$ that satisfy the Marcinkiewicz condition uniformly in l. Again we assume that $\|M_v f\|_q = 1$, $\|f\|_1 \leqslant 1$.

Let $\mathrm{E} = \bigcup_l I_l$, $\varphi = u\mathbb{1}_{\mathbb{Z}\setminus E}$, $\psi = u - \varphi$. The function φ satisfies the hypotheses of Theorem 2, and thus

$$\|M_\varphi f\|_p \leqslant C\|f\|_1^\alpha \|M_\varphi f\|_q^{1-\alpha}.$$

The function $\mathbb{1}_{\mathbb{Z}\setminus E}\varphi^{-1}$ satisfies the Marcinkiewicz condition, so $\|M_{\mathbb{Z}\setminus E} f\|_p \leqslant C\|M_\varphi f\|_p$. Besides (again by the Marcinkiewicz theorem) $\|M_\varphi f\|_q \leqslant C\|M_v f\|_q \leqslant C$. So

(4) $\qquad \|M_\varphi f\|_p \leqslant C\|f\|_1^\alpha \qquad$ and $\qquad \|M_{\mathbb{Z}\setminus E} f\|_q \leqslant c\|f\|_1^\alpha.$

(Again, for classes $\mathscr{B}_n$ with $n>1$ the Marcinkiewicz theorem is not applicable at this point. The L^t-continuity of multipliers required for $n>1$ can be derived from the Generalized Littlewood-Paley Theorem and part (*a*) of the Complement to the Marcinkiewicz Theorem stated in Section 2).

It follows from (4) that

$$\|M_E f\|_1 \leqslant c(\|f\|_1 + \|f\|_1^\alpha) \leqslant c_1 \|f\|_1^\alpha .$$

Choose now a sequence $\{\varphi_l\}$ of functions on $\mathbb{Z}$ such that $\varphi_l | I_l \equiv 1$, $\varphi_l | I_k = 0$ for $k \neq l$ and all functions $\sum \varepsilon_l \varphi_l$, $|\varepsilon_l| \leqslant 1$ are uniformly bounded and uniformly satisfy the Hörmander-Mikhlin condition. (Such a sequence exists in view of the fact that the intervals $\{I_l\}$ are separated. It can be constructed by using the same idea as in the beginning of the proof of the Complement to the Marcinkiewicz theorem in subsection 2.3.) Since $M_E f \in H^1$ we obtain from a result quoted in subsection 2.1 and the Khintchine inequality that

$$\left\| \left(\sum_l |M_{I_l} f|^2 \right)^{1/2} \right\|_1 \leqslant C \|M_E f\|_1 \leqslant C \|f\|_1^\alpha .$$

Note that nothing will be changed if we replace $M_{I_l} f$ in the leftmost term by $\bar{z}^{a_l} M_{I_l} f$ (a_l is the left end of I_l). We do this and then apply part (*b*) of the Complement to the Marcinkiewicz theorem (this time in the vector-valued case, with the sequence of symbols $\{\mathbb{1}_{[0, b_l - a_l]} u((\cdot) + a_l) y_l(\cdot)\}$). We get

$$m\left\{ \left(\sum_{k,l} |M_{w_{kl}} M_v M_{I_l} f|^2 \right)^{1/2} > \lambda \right\} \leqslant C\lambda^{-1} \|f\|_1^\alpha , \tag{5}$$

where $w_{kl}(n) = w_k(n - a_l)$, $w_k = \mathbb{1}_{\mathbb{Z} \cap [A^k, A^{k+1})}$ and $A > 1$ is fixed in advance.

The further manipulations are analogous to those in the proof of Theorem 2 after estimate (3). Namely, the L^2-norm of the « quadratic function » in the left-hand term of (5) is majorized by $\|M_E M_v f\|_q$ which is less than or equal to $C\|M_v f\|_q = C$. Then we « interpolate » between this estimate and (5), and so on. At last we arrive at the inequality $\|M_\psi f\|_p \leqslant C \|f\|_1^\beta$ for some β, $0 < \beta < 1$. Combining this with the first estimate of (4) we complete the proof. □

PART II. APPLICATIONS

1. General results.

1.1. The Statement. Our aim is to prove a general theorem connecting interpolation inequalities and massiveness of some sets. Let X be a Banach space of functions on the circle continuously embedded into $L^{\infty}(\mathbb{T})$ (in applications X will be continuously embedded even into $C(\mathbb{T})$). Denote by X_2 the closure of X in $L^2(\mathbb{T})$ and by P the orthogonal projection of $L^2(\mathbb{T})$ onto X_2. If $f \in L^2$ we set

$$\|f\|_{X^*} = \sup \left\{ \left| \int g\bar{f}\, dm \right| : g \in X,\ \|g\|_X \leqslant 1 \right\}.$$

We postulate the following « axiom »:

A1. There exist C, α, $0 < \alpha < 1$ and p, $1 < p < 2$ such that for all f in X_2

$$\|f\|_p \leqslant C\|f\|_{X^*}^{\alpha}\|f\|_2^{1-\alpha}.$$

An equivalent form of the same condition:

$$\|Pg\|_p \leqslant C\|g\|_{X^*}^{\alpha}\|Pg\|_2^{1-\alpha}$$

for all g in $L^2(\mathbb{T})$. Indeed, it is evident that $\|Pg\|_{X^*} = \|g\|_{X^*}$, $g \in L^2$.

THEOREM 4. — *Suppose that X satisfies $A1$. Let $\{\varphi_n\}$ and $\{\psi_n\}$ be two orthonormal systems lying in X_2 and orthogonal to each other. Suppose that $\{\varphi_n\}$ is uniformly bounded in L^{∞} and on the linear span of $\{\psi_n\}$ the norms of the spaces L^1 and L^2 are equivalent. Let $\Sigma|a_n|^2 \leqslant 1$, $\Sigma|b_n|^2 \leqslant 1$. Then there is a function f in X such that $\left|\int f\bar{\varphi}_n\, dm\right| \geqslant |a_n|$, $\int f\bar{\psi}_n\, dm = b_n$ for all n and $\|f\|_X$ does not exceed some constant independent of $\{a_n\}$ and $\{b_n\}$.*

1.2. Another « Axiom ». To prove Theorem 4 we follow more or less closely the scheme presented in [7]. This scheme should, however, be modified to give the « exact interpolation » by Fourier coefficients with respect to $\{\psi_n\}$, as stated. To make the appropriate modification it is convenient to introduce another « axiom ».

A2. There exist C, α, $0 < \alpha < 1$ and p, $1 < p < 2$ such that for every f in X_2 there is a function h in L^2 with $Ph = f$ and

$$\|h\|_p \leqslant C\|f\|_{X^*}^{\alpha}\|f\|_2^{1-\alpha}.$$

It is clear that A2 is implied by A1. In fact A2 is strictly weaker than A1 but we do not dwell on presenting a counterexample.

LEMMA 2. — *Suppose that X satisfies $A1$ and let G be a subspace of X_2 on which the norms $\|.\|_1$ and $\|.\|_2$ are equivalent. Set $Z = \{h \in X : \int h\bar{g}\,dm = 0$ for all $g \in G\} = X \cap G^{\perp}$. ($Z$ is a closed subspace of X.) Then Z satisfies $A2$.*

Proof. — For every g in $L^2(\mathbb{T})$ denote by F_g the functional on X defined by $F_g(x) = \int x\bar{g}\,dm$. Note first that for $g \in G$

$$(1) \qquad c_1\|g\|_2 \leqslant \|F_g\|_{X^*} \leqslant c_2\|g\|_2 .$$

Indeed, the second inequality is valid because X is continuously embedded into $L^2(\mathbb{T})$. To prove the first one write down $\|g\|_p \leqslant C\|F_g\|_{X^*}^{\alpha}\|g\|_2^{1-\alpha}$ and note that $\|g\|_p \asymp \|g\|_2$ for g in G.

Thus the subspace $\{F_g : g \in G\}$ of X^* is norm-closed and reflexive. Hence it is closed in the topology $\sigma(X^*,X)$ as well (because its ball is weakly compact, thus $\sigma(X^*,X)$ compact and thus $\sigma(X^*,X)$ closed). Consequently this space coincides with its bipolar that is nothing but the annihilator of Z. So we have : every functional F on X vanishing on Z is of the form F_g for some $g \in G$.

Now let $f \in Z_2$. By the Hahn-Banach theorem there is a functional F on X with $F|Z = F_f|Z$ and $\|F\|_{X^*} = \|f\|_{Z^*}$. $F - F_f$ vanishes on Z and so $F - F_f = F_g$ for some $g \in G$. Set $h = f + g$. Then, since X satisfies A1,

$$\|h\|_p \leqslant C\|h\|_{X^*}^{\alpha}\|h\|_2^{1-\alpha} = C\|f\|_{Z^*}^{\alpha}\|h\|_2^{1-\alpha}.$$

To complete the proof note that $\|h\|_2 \leqslant \|f\|_2 + \|g\|_2 \leqslant \|f\|_2 + C\|F_g\|_{X^*} \leqslant \|f\|_2 + C(\|F_f\|_{X^*} + \|F_h\|_{X^*}) = \|f\|_2 + C(\|F_f\|_{X^*} + \|f\|_{Z^*}) \leqslant$ const $\|f\|_2$. $\square$

1.3. The condition $\mathcal{D}_\omega$ and the Proof of Theorem 4. Let $\Phi = \{\varphi_n\}$ be an orthonormal system lying in X_2. Let ω be a strictly decreasing positive function on $\mathbb{R}_+$ with $\lim_{t\to\infty} \omega(t) = 0$. We say that the pair (X,Φ) satisfies the condition $\mathcal{D}_\omega$ if for every t, $t \geqslant 0$ and every sequence $\{c_n\}$ with $\Sigma|c_n|^2 = 1$ there are numbers ε_n, $\varepsilon_n = \pm 1$ such that the function F defined by $F = \Sigma c_n \varepsilon_n \varphi_n$ can be represented in the form $F = G + H$ with $G \in X$, $H \in L^2$ and $\|G\|_X \leqslant t$, $\|H\|_2 \leqslant \omega(t)$.

Theorem on Massiveness (see [7], Section 2). Suppose that (X,Φ) satisfies $\mathcal{D}_\omega$ where $\omega(t) = O(t^{-\beta})$ as $t \to \infty$, for some positive β. Then there is a constant A such that for every sequence $\{c_n\}$ with $\Sigma|c_n|^2 = 1$ there exists a function f in X such that $\left|\int f\bar{\varphi}_n \, dm\right| \geqslant |c_n|$ for all n and $\|f\|_X \leqslant A$.

Lemma 3. — *Suppose that X satisfies A2 and Φ is an orthogonal system uniformly bounded in L^∞ and lying in X_2. Then (X,Φ) satisfies $\mathcal{D}_\omega$ for some ω decreasing near infinity as a power function.*

In [7] a similar statement was given (Theorem 3 there), but it involved an axiom of type A1 in place of A2.

Proof. — Let α and p be from A2, $q^{-1}+p^{-1}=1$. If $\sum|c_n|^2=1$ then the Khintchine inequality implies that for some $\varepsilon_n = \pm 1$ we have $\|F\|_q \leqslant$ const for $F = \Sigma\varepsilon_n c_n \varphi_n$, where the constant depends on Φ and q only. We show that this F can be represented as $F = G + H$ with all the properties we need.

The existence of the above decomposition with $\|G\|_X \leqslant t$, $\|H\|_2 \leqslant \delta$ is equivalent to the relation $F \in B_1 + B_2$ where B_1 is the ball of radius t in X and B_2 is the ball of radius δ in X_2 (both centered at 0). The set $B_1 + B_2$ is convex and has nonempty interior in X_2. Thus by the separation theorem applied in this space the relation $F \in B_1 + B_2$ will be established if we prove the following claim.

Let $f \in X_2$ and suppose that

$$\sup\left\{\left|\int (b_1+b_2)\bar{f}\, dm\right| : b_1 \in B_1,\ b_2 \in B_2\right\} \leqslant 1\,; \tag{2}$$

then $\left|\int F\bar{f}\, dm\right| \leqslant 1/2$.

But (2) implies that $\|f\|_{X^*} \leqslant t^{-1}$ and $\|f\|_2 \leqslant \delta^{-1}$ (to see this take first the supremum only over the pairs of the form $(b_1,0)$ and then only over the pairs of the form $(0,b_2)$). Finding for f the function h provided by axiom A2 and taking into account that $\int F\bar{f}\,dm = \int F\bar{h}\,dm$ we obtain

$$\left|\int F\bar{f}\,dm\right| \leqslant \|F\|_q\|h\|_p \leqslant \text{const}\ \|f\|_{X^*}^{\alpha}\|f\|_2^{1-\alpha} \leqslant \text{const}\ t^{-\alpha}\delta^{\alpha-1}.$$

If $\delta = dt^{-\frac{\alpha}{1-\alpha}}$ then $\left|\int F\bar{f}\,dm\right| \leqslant \text{const}\, d^{-(1-\alpha)} < 1/2$ for d large enough. Thus we can take $\omega(t) = dt^{-\frac{\alpha}{1-\alpha}}$ with such a d. □

Proof of Theorem 4. – Note first that the operator T defined by $Tf = \left\{\int f\,\bar{\psi}_n\,dm\right\}$ maps X onto l^2. Indeed, this claim is equivalent to the estimate $\|T^*c\|_{X^*} \geqslant \text{const}\,\|c\|_{l^2}$, $c = \{c_n\} \in l^2$, or (which is the same) $\|\sum \bar{c}_n\psi_n\|_{X^*} \geqslant \text{const}\,(\sum|c_n|^2)^{1/2}$. The latter has already been proved (see inequality (1) in the preceding subsection).

Now suppose that $\{a_n\}$ and $\{b_n\}$ with $\Sigma|a_n|^2 \leqslant 1$, $\Sigma|b_n|^2 \leqslant 1$ are given. Since T is onto, there is $h \in X$ with $\int h\bar{\psi}_n\,dm = b_n$ for all n and $\|h\|_X \leqslant$ const. Consider the space Z, $Z = \left\{x \in X : \int x\bar{\psi}_n = 0 \text{ for all } n\right\}$. This space satisfies A2 (by Lemma 2). Hence Lemma 3 and the Theorem on Massiveness imply that there is $g \in Z$ with $\|g\|_X \leqslant$ const and $\left|\int g\bar{\varphi}_n\,dm\right| \geqslant |a_n| + \left|\int h\bar{\varphi}_n\,dm\right|$ for all n. The function $f = g + h$ is the one we need. □

Note that in this section we did not aim at full generality. For example, everything could happen on an abstract measure space in place of the circle. We could also demand only that the system $\{\varphi_n\}$ in Theorem 4 be uniformly L^s-bounded for some $s > 2$.

2. Examples and counterexamples.

2.1. Massive Subsets of $\mathbb{Z}_+$. Let Y be a Banach space of measurable functions on $\mathbb{T}$ continuously embedded into $C(\mathbb{T})$ and satisfying the following conditions:

(i) The set $\mathscr{P}_A$ is dense in Y (in particular Y lies as a set in the disc-algebra C_A).

(ii) $\overline{\lim\limits_{n\to\infty}} \|z^n\|_Y^{1/n} \leqslant 1$.

These conditions guarantee that every functional F on Y is uniquely determined by its Fourier coefficients $F(z^n)$, $n \geqslant 0$, and, for each F, the series $\sum\limits_{n\geqslant 0} F(z^n)\zeta^n$ converges for $|\zeta| < 1$ and defines an analytic function in the disc, denoted by $\mathscr{K}F$. Consider one more « axiom », A0 (compare with [5]).

A0. There is r, $0 < r < 1$ such that for every $F \in Y^*$ the function $\mathscr{K}F$ lies in H^r and satisfies the estimate $\|\mathscr{K}F\|_r \leqslant c\|F\|_{Y^*}$ (c being independent of F).

It was proved in Vinogradov's paper [15] that the space U_A, $U_A \overset{\text{def}}{=} \{f \in U : \hat{f}(n)=0 \text{ for } n<0\}$ satisfies A0 with every r, $0 < r < 1$. In his Doctor Thesis Vinogradov constructed many examples of spaces smaller than U_A and also satisfying A0. For example, consider a finite collection of operators $T_1, \ldots, T_n$, each T_j being either multiplication by a function in U_A, or composition with a conformal automorphism of the disc, or the Toeplitz operator with an antianalytic symbol. Then the space $\{f \in U_A : T_j f \in U_A \text{ for } 1\leqslant j\leqslant n\}$ satisfies A0. We do not go into further discussion. We only mention that among these spaces there are some for which A0 is fulfilled only with a very small r.

LEMMA 4. – *Let $E \subset \mathbb{Z}_+$ and suppose that $\mathbb{1}_E \in \mathscr{A}_j$ for some j (the classes $\mathscr{A}_j$ were defined before Theorem 1). Suppose that Y satisfies A0 and set $X = \{f \in Y : \hat{f}(n)=0$ for $n \notin E\}$. Then X satisfies A1.*

Proof. – As earlier, let X_2 be the closure of X in $L^2(m)$ and let $f \in X_2$. By the Hahn-Banach theorem there is a functional F on Y such that $F(x) = \int x\bar{f}\, dm$ for $x \in X$ and $\|F\|_{Y^*} = \|f\|_{X^*}$. By A0 the function Φ, $\Phi = \mathscr{K}F$ lies in H^r and $\|\Phi\|_r \leqslant c\|F\|_{Y^*} \leqslant c\|f\|_{X^*}$. By

Theorem 1 *bis* we have for $1 < p < 2$

$$\|M_E\Phi\|_p \leqslant C\|\Phi\|_r^{\alpha}\|M_E\Phi\|_2^{1-\alpha} \leqslant C'\|f\|_{X^*}^{\alpha}\|M_E\Phi\|_2^{1-\alpha}.$$

Now it suffices to note that $F(z^n) = \overline{\hat{f}(n)}$ for $n \in E$ and thus $M_E\Phi(z) = \overline{f(\bar{z})}$. □

COROLLARY. – *Under the hypotheses of Lemma* 4 *the set* E *is* Y-*massive*.

This follows directly from Theorem 4. □

In this corollary we used only a part of Theorem 4 that concerned the system $\{\varphi_n\}$. Now an application will be given that involves this theorem in full generality. Recall that a set F, $F \subset \mathbb{Z}$ is said to be of type $\Lambda_p (1<p<\infty)$ if on the linear span of the set $\{z^n\}_{n \in F}$ the norms of $L^p(\mathbb{T})$ and $L^1(\mathbb{T})$ are equivalent. A classical example (for all p at once) is any Hadamard lacunary set, in particular any geometric progression. Consult [10] for some other examples.

To make the statement more transparent we take $j = 0$, $Y = U_A$ in Lemma 4.

THEOREM 5. – *Let* $E = \mathbb{Z}_+ \setminus \bigcup_{k \geqslant 1} I_k$, *where* $\{I_k\}$ *is a separated sequence of intervals in* $\mathbb{Z}_+$. *Suppose that* F *is a* Λ_2-*subset of* E. *Let* $\sum_{n \in E} |a_n|^2 = 1$. *Then there is a function* g *in* U_A *with* $\hat{g}(n) = 0$ *for* $n \notin E$, $|\hat{g}(n)| \geqslant |a_n|$ *for* $n \in E$ *and* $\hat{g}(n) = a_n$ *for* $n \in F$, *and, moreover, whose norm in* U_A *does not exceed some constant independent of the sequence* $\{a_n\}$. □

Taking in Lemma 4, $j > 0$ we obtain massive sets of more sophisticated structure. For example Theorem 5 remains true if we enlarge E by $\bigcup_{k \in K} \left(I_k \setminus \left(\bigcup_l \Delta_{kl} \right) \right)$ for a certain infinite set $K \subset \{1,2,3,\ldots\}$. Here for each fixed k, $\{\Delta_{kl}\}$ is a B-separated sequence appropriately disposed in I_k, B being independent of k. We refer the reader to Part I, Subsection 1.1 for precise information on how the intervals Δ_{kl} should be disposed for the indicator function of E so enlarged to be in $\mathscr{A}_1$. Then we can repeat the same procedure with some of the Δ_{kl}, etc.

2.2. Massive sets containing $\mathbb{Z}_-$.

LEMMA 5. – *If* $\mathbb{1}_E \in \mathscr{B}_j$ *for some* j *then the space* C_E, $C_E \overset{\text{def}}{=} \{f \in C(\mathbb{T}) : \hat{f}(n) = 0$ *for* $n \notin E\}$ *satisfies* $A1$.

We refer the reader to Part I, subsection 1.2 for the definition of $\mathscr{B}_j$. Note that $\mathbb{1}_E \in \mathscr{B}_j$ implies $E \supset \mathbb{Z}_-$.

Proof. – Let $f \in (C_E)_2 = L^2_E \overset{\text{def}}{=} \{f \in L^2 : \hat{f}(n) = 0$ for $n \notin E\}$. There is a mesure μ such that $\int x\bar{f}dm = \int x\,d\bar{\mu}$ for $x \in C_E$ and $\|\mu\| = \|f\|_{(C_E)^*}$. The spectrum of the measure $\mu - fm$ lies in $\mathbb{Z}\backslash E$ and hence in $\mathbb{Z}_+$, so by the F. and M. Riesz theorem (see e.g. [16], Chapter VII, Theorem 8.1) $\mu = hm$ for some function h. Applying Theorem 2 *bis* we obtain $\|f\|_p = \|M_E h\|_p \leqslant C\|h\|_1^\alpha \|M_E h\|_2^{1-\alpha} \leqslant C\|f\|^\alpha_{(C_E)_*}\|f\|_2^{1-\alpha}$ for $1 < p < 2$. □

Instead of the reference to the F. and M. Riesz theorem we could write the above inequality for the convolution of μ with a Fejér kernel.

The next result is an immediate consequence of Lemma 5 and Theorem 4.

THEOREM 6. – *If* $\mathbb{1}_E \in \mathscr{B}_j$ *then* E *is massive for* $C(\mathbb{T})$.

(Here it is also possible to involve Λ_2-sets, as in Theorem 5.)

One cannot hope to replace $C(\mathbb{T})$ by a much smaller space (as U) in Theorem 6. We discuss two examples in the cases $j = 0$ and $j = 1$.

The relation $\mathbb{1}_E \in \mathscr{B}_0$ means that $E \supset \mathbb{Z}_-$ and $E \cap \mathbb{Z}_+$ is the union of mutually disjoint intervals (note that a one-point set is also an interval) so that for each n the number of their end-points in $[2^n, 2^{n-1}]$ does not exceed some constant independent of n. In particular, the set $\mathbb{Z}_- \cup \{2^n\}_{n \geqslant 1} \overset{\text{def}}{=} F$ satisfies $\mathbb{1}_F \in \mathscr{B}_0$.

Consider the space X, $X = \{f \in C(\mathbb{T}) : \mathbb{P}_+ f \in C(\mathbb{T})\}$, supplied with the norm $\|f\|_X = \|f\|_\infty + \|\mathbb{P}_+ f\|_\infty$. It is clear that F is not X-massive (because $\sum |\hat{f}(2^n)| < \infty$ for every function f in X with $\hat{f}(k) = 0$ for $k \notin F$). Thus F is not massive for U as well.

There is one more rather « popular » space (besides X) lying between $C(\mathbb{T})$ and U. It consists of the functions f whose symmetric partial

Fourier sums $S_n f$, $S_n f \overset{\text{def}}{=} \sum_{|k| \leqslant n} \hat{f}(k) z^k$ converge to f uniformly. The author was not able to find out whether or not the above set F was massive for this space. But sets non-massive for it with the indicator functions in $\mathscr{B}_1$ can easily be found. We state a more general result. Namely, fix a strictly increasing sequence of integers $\{n_k\}$ and set $Y = \{f : \|f - S_{n_k} f\|_\infty \to 0 \text{ as } k \to \infty\}$. The norm in Y is defined by $\|f\|_Y = \sup_k \|S_{n_k} f\|_\infty$.

PROPOSITION. – *There is a set E non-massive for Y with $\mathbb{1}_E \in \mathscr{B}_1$.*

Proof. – Without loss of generality we can assume that $n_{k+1}/n_k \geqslant 2$ (for Y becomes wider if we replace $\{n_k\}$ by a subsequence). Define a set E by the following conditions : $\mathrm{E} \supset \mathbb{Z}_-$, $\mathrm{E} \supset [n_{2j}, n_{2j+1}]$ for $j \geqslant 1$, $\mathrm{E} \supset [0, n_1]$; $\mathrm{E} \cap (n_{2j-1}, n_{2j}) = \{n_{2j-1} + 2^l\}_{0 \leqslant l \leqslant s_j}$, where $s_j \to \infty$ but $n_{2j-1} + 2^{s_j} \leqslant 2n_{2j-1}$. Clearly $\mathbb{1}_E \in \mathscr{B}_1$. Let f be a function in Y with spectrum in E. Set $g = S_{n_{2j}} f - \mathrm{S}_{n_{2j-1}} f$. Then the spectrum of f lies in the union of two intervals, $[-n_{2j}, -n_{2j-1}]$ and $[n_{2j-1}, 2n_{2j-1}]$, and its intersection with the second interval is contained in the translated geometric progression $\{n_{2j-1} + 2^l\}_{0 \leqslant l \leqslant s_j}$. The convolution of g with the product of an appropriate de la Vallée Poussin kernel by an appropriate power of z gives the function h with $\hat{h}(k) = \hat{g}(k)$ for $k \in [n_{2j-1}, 2n_{2j-1}]$ and $\hat{h}(k) = 0$ for $k \notin [n_{2j-1}, 2n_{2j-1}]$. Besides, $\|h\|_\infty \leqslant C\|f\|_Y$ with C independent of j. Hence for each j

$$\sum_{0 \leqslant l \leqslant s_j} |\hat{f}(n_{2j-1} + 2^l)| \leqslant C\|f\|_Y .$$

Consequently, E is not massive for Y. (It should be noted thas this statement would still be true even if we had not imposed any restriction on the norm of f with $|\hat{f}(n)| \geqslant |a_n|$ in the definition of massive sets).

□

3. Concluding remarks.

3.1. On the Exponent θ in the Corollary to Theorem 1. Let Y be a space satisfying conditions (i) and (ii) from the beginning of Section 2. If x is a bounded function on $\mathbb{Z}$, $x|\mathbb{Z}_- = 0$, we set for F in Y^*

$$M_x F(\zeta) = \sum_{n \geqslant 0} x_n F(z^n) \zeta^n$$

(this is an analytic function in the disc; for $x = \mathbb{1}_{\mathbb{Z}_+}$ we obtain $\mathscr{H}F$ from Section 2).

As in the proof of Lemma 4 one obtains with the help of Theorem 1 : if for some r, $0 < r < 1$ we have $\|M_x F\|_r \leqslant \text{const}\,\|F\|_{Y^*}$ for all F in Y^* then for every y in $\mathscr{A}_0$

$$(1) \qquad \|M_{xy}F\|_p \leqslant C\|F\|_{Y^*}^{\theta}\|M_{xy}F\|_q^{1-\theta}, \qquad F \in Y^*,$$

where $p^{-1} = r^{-1}\theta + q^{-1}(1-\theta)$, $1 < p < q$.

PROPOSITION. — *If the space Y is translation invariant then under the hypotheses just made the estimate* (1) *is true with θ defined by* $p^{-1} = \theta + (1-\theta)q^{-1}$.

Sketch of the proof. — Let $\{\varphi_j\}$ be the functions from the proof of Theorem 1. By (1) in Section 4 of Part I we have

$$\left\|\left(\sum_j |M_{x\varphi_j}F|^2\right)^{1/2}\right\|_r \leqslant C\|M_x F\|_r \leqslant C\|F\|_{Y^*}, \qquad F \in Y^*.$$

Since Y is translation invariant, this inequality implies the stronger one (see [9], Lemma 1) :

$$m\left\{\left(\sum_j |M_{x\varphi_j}F|^2\right)^{1/2} < \lambda\right\} \leqslant C'\lambda^{-1}\|F\|_{Y^*}, \qquad \lambda > 0, \qquad F \in Y^*.$$

Then one should repeat the argument from the proof of Theorem 1 using this inequality in place of (1) in Section 4 of Part 1. (Note also that one should use estimate (1) from Section 3 of Part 1 in place of the Corollary to Theorem 3.) $\square$

3.2. Interpolation Inequalities and L^p-Continuity $(p>1)$. All the time we considered the interpolation inequality $\|M_x f\|_p \leqslant C\|f\|_1^{\alpha}\|M_x f\|_q^{1-\alpha}$ as a regularity property of M_x on $L^1(\mathbb{T})$, and, in particular, as a certain substitute for the weak type (1, 1) inequality. It should be kept in mind, however, that this substitute is not always quite good.

Example. — There is a set E, $E \subset \mathbb{Z}$ such that M_E is discontinuous in L^s for $1 \leqslant s < 4/3$ but the inequality $\|M_E f\|_p \leqslant C_p\|f\|_1^{\alpha}\|M_E f\|_2^{1-\alpha}$, $\alpha = \alpha(p)$ is valid for all p, $1 < p < 2$.

(Were M_E of weak type (1, 1), m_E would be L^s-continuous for all s, $1 < s < \infty$.)

For E we can take $\mathbb{Z}\backslash F$ where F is a set of type Λ_4 but not of type $\Lambda_{4+\varepsilon}$ for any $\varepsilon > 0$ (cf. [10] for a construction). M_E is discontinuous in L^s for $s < 4/3$ (for otherwise M_F would be bounded in L^s and hence F would be of type $\Lambda_{s'}$).

As to the interpolation inequality, it is sufficient to check it for $4/3 \leqslant p < 2$. For such p the operator M_F, and hence M_E, acts in $L^p(\mathbb{T})$, and so $\|M_E f\|_p \leqslant C\|f\|_p \leqslant C\|f\|_1^{\alpha}\|f\|_2^{1-\alpha}$ for an appropriate $\alpha = \alpha(p)$. Assuming that $\|M_E f\|_2 = 1$, $\|f\|_1 \leqslant 1$ we find (by using the fact that F is of type Λ_4): $\|M_F f\|_2 \leqslant C\|M_F f\|_1 \leqslant C(\|f\|_1 + \|M_E f\|_2) \leqslant 2C$, whence $\|f\|_2 \leqslant 1 + 2C$. Thus $\|M_E f\|_p \leqslant \text{const}\,\|f\|_1^{\alpha}$. $\square$

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