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MITSURU IKAWA

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DECAY OF SOLUTIONS OF THE WAVE EQUATION IN THE EXTERIOR OF SEVERAL CONVEX BODIES

by Mitsuru IKAWA

1. Introduction.

Let $\mathcal{O}$ be an open bounded set in $\mathbf{R}^3$ with smooth boundary Γ. We set $\Omega = \mathbf{R}^3 - \mathcal{O}$. Suppose that Ω is connected. Consider the following acoustic problem

$$(1.1)\quad \begin{cases} \square u(x,t) = \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0 & \text{in } \Omega \times (-\infty,\infty) \\ u(x,t) = 0 & \text{on } \Gamma \times (-\infty,\infty) \\ u(x,0) = f_1(x) & \\ \dfrac{\partial u}{\partial t}(x,0) = f_2(x) & \end{cases}$$

where $\Delta = \sum_{j=1}^{3} \partial^2/\partial x_j^2$. We define the local energy of u in $\Omega(R)$ at time t by

$$E(u,R;t) = \frac{1}{2}\int_{\Omega(R)} \left\{ |\nabla u(x,t)|^2 + \left|\frac{\partial u}{\partial t}(x,t)\right|^2 \right\} dx,$$

$$\Omega(R) = \Omega \cap \{x\,;\ |x| < R\}.$$

Concerning the uniform decay of local energy, Morawetz, Ralston and Strauss [MRS] and Melrose [Me1] showed that, when $\mathcal{O}$ is non-

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trapping (1), the exponential decay of the type

$$E(u,R\,;\ t) \leqslant C_R e^{-\alpha|t|}\, E(u,R\,;\ 0) \tag{1.2}$$

for all

$$f = \{f_1, f_2\} \in (C_0^\infty(\Omega(R)))^2$$

holds, where α is a positive constant independent of R. On the other hand Ralston [R] proved that the exponential decay of the type (1.2) does not hold for trapping obstacles.

On the uniform decay of the local energy for trapping obstacles, the author considered in [I1] an example of trapping obstacles $\mathcal{O}$ which consists of two disjoint strictly convex bodies, and we showed that the exponential decay of the form

$$E(u,R\,;\ t) \leqslant C_R e^{-\alpha t}\left\{\|f_1\|^2_{H^7(\Omega)} + \|f_2\|^2_{H^6(\Omega)}\right\} \tag{1.3}$$

for all

$$f = \{f_1, f_2\} \in (C_0^\infty(\Omega(R)))^2$$

holds. The purpose of the present paper is to extend the result in [I1] to the case that $\mathcal{O}$ consists of several disjoint strictly convex bodies, namely

$$\mathcal{O} = \bigcup_{j=1}^{J} \mathcal{O}_j,$$

where $\mathcal{O}_j$, $j = 1, 2, \ldots, J$, are disjoint bounded open sets in $\mathbf{R}^3$ such that $\Gamma_j = \partial\mathcal{O}_j$ are smooth and the Gaussian curvature of Γ_j is strictly positive at every point of Γ_j.

As a result of the former studies [LP1, R, Me1, I1] we can say that the behavior of solution to (1.1) is in close connection with the properties of the broken rays of the geometric optics in Ω, and especially with the periodic rays in the case of trapping obstacles. For a periodic ray γ in Ω we denote by d_γ the length of γ, and by β_γ and β'_γ the eigenvalues of the Poincaré map of γ with the absolute values less than 1. On the configuration of $\mathcal{O}_j$ we assume the following :

(H.1) For all $\{j_1, j_2, j_3\} \in \{1, 2, \ldots, J\}^3$ such that $j_l \neq j_{l'}$, if $l \neq l'$, the convex hull of $\overline{\mathcal{O}}_{j_1}$ and $\overline{\mathcal{O}}_{j_2}$ has no intersection with $\overline{\mathcal{O}}_{j_3}$.

(1) For the definition, see, for example, [MeS].

(H.2) There exists $\alpha > 0$ such that

$$\Sigma\lambda_\gamma d_\gamma e^{\alpha d_\gamma} < \infty,$$

where the summation is taken over all the primitive periodic rays γ in Ω, and

$$\lambda_\gamma = |\beta_\gamma\beta'_\gamma|^{1/2}.$$

The main result is

THEOREM 1. – *Suppose that* (H.1) *and* (H.2) *are satisfied. Then we have an exponential decay of local energy of the type*

$$E(u,R;t) \leqslant C_R e^{-at}(\|f_1\|^2_{H^3(\Omega)} + \|f_2\|^2_{H^2(\Omega)})$$

for all

$$\{f_1,f_2\} \in (C_0^\infty(\Omega(\mathrm{R}))^2,$$

where a is a positive constant independent of R.

Remark. – Consider the case that all $\mathcal{O}_j$ are balls with radius r. Then the condition

$$\mathrm{dis}(\mathcal{O}_j,\mathcal{O}_l) \geqslant rJ \qquad \text{for all} \qquad j \neq l$$

implies (H.2).

In the case of $J = 2$, since there is only one primitive periodic ray in Ω, not only the exponential decay of local energy but also the distribution of the scattering matrix is studied well [G, I2, I3]. On the other hand, when $J \geqslant 3$ the geometry of Ω is more complicated. Namely, under the hypothesis (H.1) there are infinitely many primitive periodic rays in Ω, which makes difficult to extract the asymptotic behavior of solutions as $t \to \infty$ in a simple form. Therefore we can only show the non-existence of poles of the scattering matrix in a certain strip, which implies the exponential decay of solutions (see the next section).

As for the hypothesis in Theorem 1, we may say that (H.1) is not essential for the exponential decay of local energy. Probably we can show the same decay without (H.1) at the prise of certain technical complications. But (H.2) is used essentially in the proof of Theorem 1. We do not know at present whether the exponential decay of the type (1.3) holds without (H.2).

The author has obtained the result of the present paper during his stay in l'Institut Fourier, and the results was announced in [I5]. The author would like to express his sincere gratitude to Prof. Y. Colin de Verdière and Prof. L. Guillopé for stimulating discussions.

2. Reduction of the problem.

As considered in [LP1], the decay of local energy is closely related to the spectral property of Δ in Ω. Theorem 1 is derived easily from the fact that the resolvent of Δ can be continued holomorphically into a strip $\{\mu; -a<\operatorname{Re}\mu\leqslant 0\}$ $(a>0)$. More precisely, consider the boundary value problem with parameter $\mu\in\mathbf{C}$

$$(2.1)\qquad \begin{cases}(\mu^2-\Delta)u = 0 & \text{in } \Omega\\ u = g & \text{on } \Gamma,\end{cases}$$

where $g\in C^\infty(\Gamma)$. For $\operatorname{Re}\mu > 0$, (2.1) has a unique solution in $H^2(\Omega)$. Write the solution u as

$$u = U(\mu)g.$$

Then $U(\mu)$ can be regarded as an $\mathscr{L}(C^\infty(\Gamma),\ C^\infty(\bar{\Omega}))$-valued holomorphic function in $\operatorname{Re}\mu > 0$. Recall that $U(\mu)$ can be extended to a function holomorphic in $\operatorname{Re}\mu \geqslant 0$ and meromorphic in the whole complex plane $\mathbf{C}$ (see, for example, [LP1], [Mi]).

In this case we have

THEOREM 2.1. – *Suppose that* (H.1) *and* (H.2) *are satisfied. Set*

$$F(\alpha) = \Sigma_\gamma \lambda_\gamma d_\gamma e^{\alpha d_\gamma}(1-\lambda_\gamma e^{\alpha d_\gamma})^{-1},$$
$$a_0 = \sup\{\alpha; F(\beta)<\infty, \textit{ for all } \beta<\alpha\}.$$

Then, for any $\varepsilon > 0$, $U(\mu)$ *is holomorphic in*

$$D_\varepsilon = \{\mu; \operatorname{Re}\mu > -(a_0-\varepsilon), |\mu|\geqslant C_\varepsilon\}$$

and we have

$$\sup_{x\in\Omega(R)} |(U(\mu)g)(x)| \leqslant C_{R,\varepsilon}(\|g\|_{H^2(\Gamma)} + |\mu|^2\|g\|_{L^2(\Gamma)}),$$

for all $\mu\in D_\varepsilon$.

Remark 2.2. – Since there exists $\beta > 0$ such that $\lambda_\gamma < e^{-\beta d_\gamma}$ for all γ, we have $|1-\lambda_\gamma e^{\beta d_\gamma}|^{-1} \leqslant C$ for all γ. Therefore a_0 is necessarily positive under the assumption (H.2).

Remark 2.3. — Note that the relationship between the poles of the scattering matrix $\mathscr{S}(z)$ and those of $U(\mu)$, that is, z is a pole of $\mathscr{S}$ if and only if $\mu = iz$ is that of U ([LP1, Theorem 5.1 of Chapter V]). Thus, there exists $\alpha > 0$ such that $\{z; \operatorname{Im} z < \alpha\}$ does not contain pole of $\mathscr{S}(z)$ (cf. the conjecture on the distribution of the poles of the scattering matrix in [LP1, page 158]).

Since the derivation of Theorem 1 from Theorem 2 is the same as in Section 2 of [I1] we omit the proof. Hereafter we shall use the notation $|\cdot|_p(\omega)$ as in [I1, I2], which stands for the norm of $\mathscr{B}^p(\omega)$.

In order to show Theorem 2.1 the following proposition is essential.

PROPOSITION 2.4. — *Let m be an oscillatory data on Γ_j of the form*

$$m(x;k) = e^{ik\psi(x)}g(x)$$

satisfying Condition A of Definition 4.2. *We fix an positive integer N arbitrarily. Then there is a function $w(x,\mu;k)$ such that*

(i) $w(\cdot,\mu;k)$ *is* $C^\infty(\bar{\Omega})$*-valued holomorphic function in* $D = \{\mu; \operatorname{Re}\mu > -a_0\}$,

(ii) $w(\cdot,\mu;k) \in L^2(\Gamma)$ *for* $\operatorname{Re}\mu > 0$,

(iii) $(\mu^2 - \Delta)w(x,\mu;k) = 0$ *in* Ω *for all* $\mu \in D$,

(iv) $|w(\cdot,\mu;k)|_p(\Omega(R)) \leqslant C_{p,\varepsilon}k^p(|\psi|_{N2+p}(\Gamma_j)+1)|g|_{N2+p}(\Gamma_j)$ *for all* $\mu \in D_\varepsilon$,

(v) $|w(\cdot,\mu;k) - m(\cdot,k)|_p(\Gamma_j) \leqslant C_{p,\varepsilon}(|\psi|_{N2+p}(\Gamma_j)+1)|g|_{N2+p}(\Gamma_j)k^{-N+p}$ *for* $\mu \in D_\varepsilon$ *such that* $|\mu + ik| < a_0 + 1$.

By the same argument as in [I3] we can derive Theorem 2.1 from Proposition 2.4. Therefore the rest of the paper will be devoted to the proof of Proposition 2.2.

3. On the behavior of phase functions and broken rays.

From now on we suppose that $\mathcal{O} = \bigcup_{j=1}^{N} \mathcal{O}_j$ satisfies (H.1). As a fundamental preparation to investigate the behavior of solutions to the problem (1.1) we consider the behaviors of broken rays in Ω and sequences of phase functions.

Let ρ be a positive constant such that

$$\bar{\mathcal{O}} \subset \{x; |x| < \rho\},$$

and set

$$d_{j,l} = \operatorname{dis}(\mathcal{O}_j, \mathcal{O}_l), \qquad d_{\max} = \max_{j \neq l} d_{j,l}, \qquad d_{\min} = \min_{j \neq l} d_{j,l}.$$

For $x \in \Gamma$, $n(x)$ denotes the unit outer normal of Γ at x, and we set

$$\Sigma_x^+ = \{\xi \in \mathbf{R}^3; |\xi| = 1, n(x) \cdot \xi \geqslant 0\},$$

and

$$\Sigma^+\Gamma = \{(x,\xi); x \in \Gamma, \xi \in \Sigma_x^+\}.$$

We denote by $\mathscr{X}(x,\xi)$ the broken ray according to the law of geometric optics starting from $x \in \Gamma$ in the direction $\xi \in \Sigma_x^+$, by $X_1(x,\xi)$, $X_2(x,\xi), \ldots$, the points of reflection of the broken ray and by $\Xi_l(x,\xi)$ the direction of ray reflected at $X_l(x,\xi)$. More precisely, if

$$\{x + \tau\xi; \tau > 0\} \cap \Gamma = \varnothing,$$

we set $L_0(x,\xi) = \{x + \tau\xi; \tau \geqslant 0\}$. If $\{x + \tau\xi; \tau > 0\} \cap \Gamma \neq \varnothing$, we set

$$\begin{aligned}
\tau_0(x,\xi) &= \inf\{\tau; \tau > 0, x + \tau\xi \in \Gamma\}, \\
L_0(x,\xi) &= \{x + \tau\xi; 0 \leqslant t \leqslant \tau_0(x,\xi)\}, \\
X_1(x,\xi) &= x + \tau_0(x,\xi)\xi, \\
\Xi_1(x,\xi) &= \xi - 2(n(X_1(x,\xi)),\xi)n(X_1(x,\xi)).
\end{aligned}$$

When $\{X_1 + \tau\Xi_1; \tau > 0\} \cap \Gamma = \varnothing$, $L_1(x,\xi) = \{X_1 + \tau\Xi_1; \tau \geqslant 0\}$. Otherwise we set

$$\begin{aligned}
\tau_1(x,\xi) &= \inf\{\tau; \tau > 0, X_1 + \tau\Xi_1 \in \Gamma\}, \\
L_1(x,\xi) &= \{X_1 + \tau\Xi_1; 0 \leqslant \tau \leqslant \tau_1\}, \\
X_2(x,\xi) &= X_1 + \tau_1\Xi_1, \\
\Xi_2(x,\xi) &= \Xi_1 - 2(n(X_2),\Xi_1)n(X_2).
\end{aligned}$$

Thus we define successively $\tau_l(x,\xi)$, $X_l(x,\xi)$, $\Xi_l(x,\xi)$, $L_l(x,\xi)$ until $\{X_l + \tau\Xi_l; \tau > 0\} \cap \Gamma = \varnothing$. If there exists l_0 such that for $\tau_l(x,\xi)$, $X_l(x,\xi)$, $\Xi_l(x,\xi)$ are defined for $l \leqslant l_0$ and $\{X_{l_0} + \tau\Xi_{l_0}; \tau > 0\} \cap \Gamma = \varnothing$ then we set

$$\mathscr{X}(x,\xi) = \bigcup_{l=0}^{l_0} L_l(x,\xi),$$

$$^{\#}\mathscr{X}(x,\xi) = l_0.$$

Otherwise

$$\mathscr{X}(x,\xi) = \bigcup_{l=0}^{\infty} L_l(x,\xi),$$

$$^{\#}\mathscr{X}(x,\xi) = \infty .$$

We denote by $\mathrm{Od}\,\mathscr{X}(x,\xi)$ a sequence $(j_l)_{l=0}^{\#\mathscr{X}(x,\xi)}$ such that

$$X_l(x,\xi) \in \Gamma_{j_l} \qquad \text{for all} \qquad 0 \leqslant l \leqslant {}^{\#}\mathscr{X}(x,\xi),$$

and call it the order of reflection of $\mathscr{X}(x,\xi)$. For an integer $q \leqslant {}^{\#}\mathscr{X}(x,\xi)$ we set

$$\mathrm{Od}_q\mathscr{X}(x,\xi) = (j_0, j_1, j_2, \ldots, j_q).$$

We denote by $X(\tau;x,\xi)$ the representation of $\mathscr{X}(x,\xi)$ by the length τ of the ray from x to the point X on the broken ray.

From the assumption (H.1) we have

LEMMA 3.1. – *There exists $\delta_1 > 0$ and $d_0 > 0$ with the following properties : Let $(x,\xi) \in \Sigma^+\Gamma$. If*

$$-\,n(X_1(x,\xi))\cdot\xi \leqslant \delta_1, \tag{3.1}$$

the reflected ray does not pass the d_0 neighborhood of $\mathcal{O}$, that is, $L_1(x,\xi)$ is a half line and

$$L_1(x,\xi) \cap \{y; \mathrm{dis}\,(\mathcal{O}-\mathcal{O}_{j_1}, y) \leqslant d_0\} = \varnothing .$$

Proof. – Let $x \in \Gamma_j$, $X_1 \in \Gamma_{j_1}$. Suppose that

$$n(X_1)\cdot\xi = 0, \tag{3.2}$$

and that $L_1 \cap (\mathcal{O}-\mathcal{O}_{j_1}) \neq \varnothing$. If $X_2 \in \Gamma_{j_2}$, evidently $j_2 \neq j$. Note that (3.2) implies $\Xi_1 = \xi$. Namely, X_1 is on a segment xX_2, which means that

$$(\text{convex hull of } \overline{\mathcal{O}}_j \text{ and } \overline{\mathcal{O}}_{j_2}) \cap \overline{\mathcal{O}}_{j_1} \ni X_1 .$$

This contradicts (H.1). Thus it is shown that $L_1 \cap (\mathcal{O}-\mathcal{O}_{j_1}) = \varnothing$ holds provided (3.2). Since X_1 and Ξ_1 are continuous in x and ξ on condition that X_1 exists, the assertion of Lemma follows from the compactness of $\Sigma^+\Gamma$. Q.E.D.

We set

$$\Gamma_{p,(j)} = \{x \in \Gamma_p;\ -n(x)\cdot(x-y)/|x-y| \geqslant \delta_1 \text{ for all } y \in \Gamma_j\}.$$

A real valued smooth function defined in an open set in $\mathbf{R}^3$ which satisfies $|\nabla\varphi| = 1$ is called *phase function*, and a surface

$$\mathscr{C}_\varphi(x) = \{y; \varphi(y) = \varphi(x)\}$$

is called the wave front of φ passing x. Note that $-\nabla\varphi(y)$ is the unit normal of $\mathscr{C}_\varphi(x)$ at y.

DEFINITION 3.2. – *We say that a phase function φ defined in $\mathscr{U}$ satisfies Condition P on Γ_j when*

(i) *the principal curvatures of the wave front with respect to $-\nabla\varphi$ are non-negative at every point in $\mathscr{U}$,*

(ii) $$\{y + \tau\nabla\varphi(y); \tau \geqslant 0, y \in \mathscr{U} \cap \Gamma_j\} \supset \bigcup_{l \neq j} \overline{\mathscr{O}}_l.$$

Let φ be a phase function satisfying Condition P on Γ_j. We define φ_p for $p \neq j$ by the following way: for $x = X_1(y, \nabla\varphi) + \tau\Xi_1(y, \nabla\varphi)$ such that $X_1(y, \xi) \in \Gamma_p$, $\varphi_p(x) = \varphi(X_1(y, \xi)) + \tau$. We set

$$\mathscr{U}_p(\nabla\varphi) = \bigcup_{X_1(x,\xi) \in \Gamma_{p,(j)}} \{X_1(x, \nabla\varphi) + \tau\Xi_1(x, \nabla\varphi); \tau \geqslant 0\}.$$

Concerning the principal curvatures of $\mathscr{C}_{\varphi_p}$ we have

LEMMA 3.3 (Section 4 of [I1]). – *Let $x \in \Gamma_{p,(j)}$. Then all the principal cuvatures of $\mathscr{C}_{\varphi_p}(x) \geqslant 2K(x)$, where $K(x)$ denotes the minimum of the principal curvatures of Γ_p at x.*

Now the following lemma is obvious from the definition of φ_p.

LEMMA 3.4. – *It holds that*

$$|\nabla\varphi_p| = 1 \quad \text{in } \mathscr{U}_p,$$
$$\varphi_p = \varphi \quad \text{on } \Gamma_{p,(j)}.$$

The above two lemmas imply

LEMMA 3.5. – *If φ satisfies Condition P on Γ_j, then for every $p \neq j$ φ_p satisfies Condition P on Γ_p.*

We denote the correspondance from φ to φ_p as

$$\varphi_p = \Phi_j^p \varphi.$$

Since $\Phi_j^p\varphi$ satisfies Condition P on Γ_p for $p \neq j$ by Lemma 3.4, we can repeat this procedure. To express successive applications of Φ_j^p's, we introduce notations. Set

$$I^{(n)} = \{\mathbf{j} = (j_0, j_1, \ldots, j_n)\,;\, j_l \in \{1, \ldots, J\} \quad \text{for} \quad 0 \leqslant l \leqslant n,$$
$$j_l \neq j_{l+1} \quad \text{for} \quad 0 \leqslant l \leqslant n-1\},$$
$$I_j^{(n)} = \{\mathbf{j} = (j_0, j_1, \ldots, j_n) \in I^{(n)}; j_0 = j\},$$

and

$$I = \bigcup_{n=0}^{\infty} I^{(n)}, \qquad I_j = \bigcup_{n=0}^{\infty} I_j^{(n)}.$$

For $\mathbf{j} \in I^{(n)}$ $|\mathbf{j}|$ stands for $n + 1$.

For each $\mathbf{j} = (j_0, j_1, \ldots, j_n) \in I_j$ we define a phase function $\varphi_{\mathbf{j}}$ inductively by

$$\varphi_{\mathbf{j}} = \Phi_{j_{n-1}}^{j_n} \varphi_{\mathbf{j}'}, \quad \mathbf{j}' = (j_0, j_1, \ldots, j_{n-1})$$

and we regard $\varphi_{\mathbf{j}}$ as a function defined in

$$\mathscr{U}_{\mathbf{j}}(\nabla\varphi) = \bigcup_{X_n(x, \nabla\varphi) \in \Gamma_{j_n, (j_{n-1})}} \{X_n(x, \nabla\varphi) + \tau\Xi_n(x, \nabla\varphi); \tau \geqslant 0\}.$$

We use sometimes the notation

$$\Phi_{\mathbf{j}} = \Phi_{j_{n-1}}^{j_n} \cdot \Phi_{j_{n-2}}^{j_{n-1}} \ldots \Phi_{j_0}^{j_1}.$$

Set

$$\mathscr{V}(\nabla\varphi) = \bigcup_{y \in \mathscr{U} \cap \Gamma_j} L_0(y, \nabla\varphi).$$

Define a mapping $\Psi_j(\nabla\varphi)$ from $\mathscr{V}(\nabla\varphi)$ into Γ_j by

$$\mathscr{V}(\nabla\varphi) \ni x = y + \tau\nabla\varphi(y) \to \Psi_j(\nabla\varphi)x = y \in \Gamma_j.$$

Let $\mathbf{j} = (j_0, j_1, \ldots, j_n) \in I_j^{(n)}$. Define $X^{-l}(x, \nabla\varphi_{\mathbf{j}})$ for $x \in \mathscr{V}(\nabla\varphi_{\mathbf{j}})$ and $0 \leqslant l \leqslant |\mathbf{j}|$ by

$$X^{-l}(x, \nabla\varphi_{\mathbf{j}}) = \Psi_{j_{n-l+1}}(\nabla\varphi_{(j, j_1, \ldots, j_{n-l+1})}) \ldots \Psi_{j_n}(\nabla\varphi_{(j, j_1, \ldots, j_n)})x.$$

Now we consider the behavior of broken rays which stay in $\Omega(\rho)$. First note the following apparent fact :

LEMMA 3.6. — *Let φ be a smooth phase function defined in an open set $\mathscr{U}$. Suppose that the both principal curvatures of $\mathscr{C}_\varphi(x_0)$ are greater* *than* $\varkappa \geqslant 0$ *at every point of* $\mathscr{C}_\varphi(x_0)$. *Then for any* x, $y \in \mathscr{C}_\varphi(x_0)$ *and* $\tau \geqslant 0$, *we have*

$$\mathrm{dis}(x+\tau\nabla\varphi(x), \quad y + \tau\nabla\varphi(y)) \geqslant (1+\tau\varkappa)\ \mathrm{dis}(x,y).$$

Next we consider the reflection on the boundary.

LEMMA 3.7. — *Let* φ *be a phase function sastisfying Condition P on* Γ_j, *and let* x *and* y *be points on* Γ_j *such that* $x_1 = X_1(x,\nabla\varphi)$ *and* $y_1 = X_1(y,\nabla\varphi)$ *are together on* $\Gamma_{p,(j)}$. *Suppose that* $\varphi(x_1) \leqslant \varphi(y_1)$. *Denote by* x_1^i *the point on the half line* $\{x+\tau\nabla\varphi(x,\nabla\varphi);\tau\geqslant 0\}$ *such* that $\varphi(x_1^i) = \varphi(y_1)$, *and by* x_1^r *the point on* $L_1(x,\xi)$ *such that* $\varphi_p(x_1^r) = \varphi(y_1)$. *Then we have*

$$\mathrm{dis}\,(x_1^i, y_1) \leqslant \mathrm{dis}\,(x_1^r, y_1). \tag{3.3}$$

Proof. — We set $D^- = \{z; n(x_1)\cdot(z-x_1)\leqslant 0\}$. The law of reflection of the geometric optics means

$$\nabla\varphi_p(x_1) = \nabla\varphi(x_1) - 2(n(x_1)\cdot\nabla\varphi(x_1))n(x_1),$$

which implies that

$$\mathrm{dis}\,(x_1+\tau\nabla\varphi(x_1^i),z)\leqslant \mathrm{dis}\,(x_1+\tau\nabla\varphi(x_1^r),z) \tag{3.4}$$

for $\tau \geqslant 0$, $z \in D^-$.

From the convexity of $\mathscr{O}_p$ we have

$$\mathscr{O}_p \subset D^-,$$

from which it follows $y_1 \in D^-$. Thus setting $z=y_1$, $\tau=\varphi(y_1) - \varphi(x_1)$ in (3.4) we have (3.3). Q.E.D.

PROPOSITION 3.8. — *Let* φ *be a phase function satisfying Condition P on* Γ_j. *Suppose that*

$$x,\ y \in \{z \in \Gamma_j; n(z)\cdot\nabla\varphi(z)\geqslant\delta_1\}. \tag{3.5}$$

If

$$\mathrm{Od}_q\mathscr{X}(x,\nabla\varphi) = \mathrm{Od}_q\mathscr{X}(y,\nabla\varphi),$$

we have

$$|x-y| \leqslant C\alpha^q \quad (0<\alpha<1) \tag{3.6}$$

where α *and* C *are independent of* φ, j *and* q.

Proof. – Set $x_0 = x$, $y_0 = y$, and $x_p = X_p(x,\nabla\varphi)$, $y_p = X_p(y,\nabla\varphi)$ for $p = 1,2,,\ldots,q$. Let φ_{j}, $\mathbf{j} \in I_j$ be phase functions defined successively according to the process mentioned after Lemma 3.5. We set

$$\mathrm{Od}\,\mathscr{X}(x,\nabla\varphi) = (j,j_1,j_2,\ldots).$$

For each $0 \leqslant p \leqslant q$ we set

$$w_p = x_p,\ z_p = y_p \quad \text{if} \quad \varphi_{(j,j_1,\ldots,j_{p-1})}(x_p) \leqslant \varphi_{(j,j_1,\ldots,j_{p-1})}(y_p)$$

and $w_p = y_p$, $z_p = x_p$ if not. In order to apply Lemma 3.7 to a pair of $\varphi_{(j,j_1,\ldots,j_{p-1})}$ and x_p, y_p we denote by w_p^i and w_p^r the points corresponding to x^i and x^r in Lemma 3.7. Then the assumption (3.5) and the positivity of the principal curvatures of the wave front of φ imply

$$|x-y| \leqslant \delta^{-1}|w_0^r - z_0|. \tag{3.7}$$

Evidently, $|\varphi(w_1^i)-\varphi(w_0^r)| = |\varphi(z_1)-\varphi(z_0)| \geqslant \mathrm{dis}\,(\mathscr{O}_j,\mathscr{O}_{j_1}) \geqslant d_{\min}$. Applying Lemma 3.6 to φ we have

$$|z_1 - w_1^i| \geqslant |z_0 - w_0^r|. \tag{3.8}$$

Then from Lemma 3.7 it follows that

$$|z_1 - w_1^r| \geqslant |z_1 - w_1^i|. \tag{3.9}$$

Since the principal curvatures of $\varphi_{(j,j_1)}$ are greater than $2K$ on $\Gamma_{j_1,(j)}$, the application of Lemma 3.5 gives

$$|z_2 - w_2^i| \geqslant (1+2d_{\min}K)|z_1 - w_1^r|. \tag{3.10}$$

Next applying Lemma 3.7, we have

$$|z_2 - w_2^r| \geqslant |z_2 - w_2^i|. \tag{3.11}$$

Thus from (3.7) $\sim$ (3.11) we have

$$|z_2 - w_2^r| \geqslant (1+2d_{\min}K)\delta|x-y|.$$

Repeating this argument we have for any $p \leqslant q$

$$|z_p - w_p^r| \geqslant (1+2d_{\min}K)^{p-1}|z_0 - w_0^r|.$$

Obviously $|z_p - w_p| \leqslant \rho$, $|w_p - w_p^r| \leqslant \rho$. Thus it must holds that

$$(1+2d_{\min}K)^{p-1}\,\delta|x-y| \leqslant 2\rho \qquad \text{for all} \qquad p \leqslant q,$$

from which it follows that

$$|x-y| \leqslant 2\rho\,\delta^{-1}(1+2d_{\min}K)^{-(q-1)}.$$

Thus we have (3.6). Q.E.D.

Next we consider the behavior of $\mathscr{X}(x,\nabla\varphi)$ and $\mathscr{X}(y,\nabla\tilde{\varphi})$ for two different phase functions φ and $\tilde{\varphi}$. First we prepare the following :

Lemma 3.9. — *Let φ and φ be phase functions satisfying Condition P on Γ_j. Suppose that the principal curvatures of the wave front of φ are greater than $\varkappa > 0$ on $\Gamma_j \cap \mathscr{U}$. Then we have for all $p \neq j$*

$$\sup_{x\in\Gamma_{p,(j)}} |\nabla\varphi_p(x)-\nabla\tilde{\varphi}_p(x)| \leqslant \alpha_\varkappa \sup_{x\in\Gamma_j\cap\mathscr{U}} |\nabla\varphi(x)-\nabla\tilde{\varphi}(x)| \tag{3.12}$$

where $0 < \alpha_\varkappa < 1$ and independent of j and p.

Proof. — Suppose that

$$z = y + \tau\nabla\varphi(y) = w + \eta\nabla\tilde{\varphi}(w) \tag{3.13}$$

where $z \in \Gamma_p$, y, $w \in \Gamma_j$ and τ, η are positive numbers. If $y = w$ we have immediately $\nabla\varphi_p(z) = \nabla\tilde{\varphi}_p(z)$. Suppose that $y \neq w$. Denote by π the plane on which z, y and w lie. The intersection of π and Γ_j is a smooth curve, which we represent as $x = y(\sigma)$ by σ the length of curve from y to x. Suppose that $w = y(s)(s>0)$. We introduce a coordinate (y_1, y_2) in π such that (0,0) corresponds to y and y_2-axis is the direction $\nabla\varphi(y)$ and w lies in $y_1 > 0$. Denote as

$$y(\sigma) = (y_1(\sigma), y_2(\sigma)),$$

and set

$$i(\sigma) = \nabla\varphi(y(\sigma)), \qquad j(\sigma) = \nabla\tilde{\varphi}(y(\sigma)).$$

Note that the strict convexity of Γ_j implies

$$\frac{dy_1}{d\sigma}(\sigma) \geqslant c > 0 \qquad \text{for all} \qquad \sigma \in (0,s).$$

Indeed, since at the point such that $dy_1/d\sigma = 0$ $dy/d\sigma$ is parallel to $i(0)$, it is impossible to hold (3.13). Denote by $i_1(\sigma)$ and $j_1(\sigma)$ the y_1-component of $i(\sigma)$ and $j(\sigma)$ respectively. Since the principal curvatures of wave front of φ is greater than $\varkappa$, we have

$$\frac{di_1}{d\sigma}(\sigma) \geqslant \varkappa\frac{dy_1}{d\sigma}(\sigma) \qquad \text{for all} \qquad 0 \leqslant \sigma \leqslant s.$$

Set

$$\int_0^1 \frac{di_1}{d\sigma}(\theta\sigma)\,d\theta = I(\sigma), \qquad \int_0^1 \frac{dy_1}{d\sigma}(\theta\sigma)\,d\theta = Y(\sigma).$$

Then $I(\sigma) \geqslant \varkappa Y(\sigma)$. Comparing the y_1-component of the both sides of (3.13) we have

$$y_1(s) + \eta j_1(s) = y_1(0) + \tau i_1(0).$$

Taking account of $i_1(0) = 0$ we have

$$y_1(s) - y_1(0) = -\eta(i_1(s) - i_1(0)) - \eta(j_1(s) - i_1(s)).$$

By using the above notation we can write the above relation as

$$sY(s) = -\eta s I(s) + \eta(j_1(s) - i_1(s)),$$

from which we have

$$s = -(Y(s) + \eta I(s))^{-1}\eta(j_1(s) - i_1(s)).$$

Now

$$\begin{aligned} j_1(s) - i_1(0) &= j_1(s) - i_1(s) + i_1(s) - i_1(0) \\ &= sI(s) + j_1(s) - i_1(s) = Y(s)(Y(s) + \eta I(s))^{-1}(j_1(s) - i_1(s)), \end{aligned}$$

i.e.,

$$j_1(s) - i_1(s) = (1 + \eta Y(s)^{-1} I(s))(j_1(s) - i_1(0)).$$

Thus we have

$$|j_1(s) - i_1(s)| \geqslant (1 + d_{j,p}\varkappa)|j_1(s) - i_1(0)|.$$

By using $|j(s)| = |i(s)| = 1$ we have

$$|j(s) - i(s)| \geqslant (1 + d_{j,p}\varkappa)|j(s) - i(0)|,$$

which is nothing but $|\nabla\varphi_p(z) - \nabla\tilde{\varphi}_p(z)| \leqslant (1 + d_{j,p}\varkappa)^{-1}|\nabla\varphi(w) - \nabla\tilde{\varphi}(w)|$. Since z is arbitrary on $\Gamma_{p,(j)}$ we have (3.12) from the above inequality. Q.E.D.

COROLLARY 3.10. — *Let φ and $\tilde{\varphi}$ be phase functions satisfying Condition P on Γ_j. Set*

$$\varphi_{\mathbf{j}} = \Phi_{\mathbf{j}}\varphi, \qquad \tilde{\varphi}_{\mathbf{j}} = \Phi_{\mathbf{j}}\tilde{\varphi}.$$

There exists a constant $0 < \alpha < 1$, which is independent of φ and $\tilde{\varphi}$, such that

$$|\nabla\varphi_{\mathbf{j}} - \nabla\tilde{\varphi}_{\mathbf{j}}|(\Gamma_{\mathbf{j}}) \leqslant \alpha^{|\mathbf{j}|-1}|\nabla\varphi - \nabla\tilde{\varphi}|(\Gamma_j).$$

Proof. – By Lemma 3.4 we see that the principal curvatures of the wave front of $\varphi_{\mathbf{j}}$ and $\tilde{\varphi}_{\mathbf{j}} \geqslant 2K$ if $|\mathbf{j}| \geqslant 1$. Then applying Lemma 3.9 we have

$$|\nabla\varphi_{(j,j_1,\ldots,j_n)} - \nabla\tilde{\varphi}_{(j,j_1,\ldots,j_n)}|(\Gamma_{j_n,(j_{n-1})}) \leqslant$$
$$\alpha_K|\nabla\varphi_{(j,j_1,\ldots,j_{n-1})} - \nabla\tilde{\varphi}_{(j,j_1,\ldots,j_{n-1})}|(\Gamma_{j_{n-1},(j_{n-2})}) \leqslant$$
$$\alpha_K^{n-1}|\nabla\varphi_{j_1} - \nabla\tilde{\varphi}_{j_1}|(\Gamma_{j_1,(j)}) \leqslant \alpha_K^{n-1}|\nabla\varphi - \nabla\tilde{\varphi}|(\Gamma_j). \quad \text{Q.E.D.}$$

By using the argument in Section 5 of [I1] we can derive the convergence of derivatives of $\nabla\varphi$ and $\nabla\tilde{\varphi}$.

PROPOSITION 3.11. – *It holds that*

$$|\nabla\varphi_{\mathbf{j}} - \nabla\tilde{\varphi}_{\mathbf{j}}|_p(\Gamma_{\mathbf{j}}) \leqslant C_p\alpha^{|\mathbf{j}|-1}|\nabla\varphi - \nabla\tilde{\varphi}|_p(\Gamma_j), \qquad p = 1, 2, \ldots.$$

With the aid of Proposition 3.11 we can prove the following proposition by the same procedure as in Section 4 of [I3].

PROPOSITION 3.12. – *It holds that*

$$|X^{-l}(x,\nabla\varphi_{\mathbf{j}}) - X^{-l}(x,\nabla\tilde{\varphi}_{\mathbf{j}})|_p(\Gamma_{\mathbf{j}}) \leqslant C_p\alpha^{|\mathbf{j}|-1}.$$

Now we turn to consideration of the periodic rays in Ω. Let γ be a periodic ray in Ω. Take one of the reflecting points x_0 of γ, and trace the ray starting from x_0. Suppose that we pass the reflecting points $x_1, x_2, \ldots, x_n$ one after another, and go back to x_0 from x_n. Namely,

$$\gamma = \bigcup_{l=0}^{n} \overline{x_l x_{l+1}} \quad (x_{n+1} = x_0). \tag{3.12}$$

Suppose that $x_l \in \Gamma_{j_l}$, and set $\mathbf{j} = (j_0, j_1, \ldots, j_n) \in I^{(n)}$. For a periodic ray γ we set

$$\mathscr{I}(\gamma) = \{\mathbf{i} = (i_0, i_1, \ldots, i_n) \in I^{(n)};\ {}^{\exists}y_l \in \Gamma_{i_l} \text{ such that}$$
$$\gamma = \bigcup_{l=0}^{n} \overline{y_l y_{l+1}}\ (y_{n+1} = y_0)\}.$$

Obviously, if $(j_0, j_1, \ldots, j_n) \in \mathscr{I}(\gamma)$, $\mathscr{T}\mathbf{j} = (j_1, j_2, \ldots, j_n, j_0) \in \mathscr{I}(\gamma)$ and $\mathscr{R}\mathbf{j} = (j_n, j_{n-1}, \ldots, j_0) \in \mathscr{I}(\gamma)$. Thus, if $\mathbf{j} \in \mathscr{I}(\gamma)$, then we have

$$\mathscr{I}(\gamma) = \{\mathscr{T}^l\mathbf{j}, \mathscr{R}\mathscr{T}^l\mathbf{j}; l = 0, 1, \ldots, n\}. \tag{3.13}$$

For any finite sequence $\mathbf{j} = (j_0, j_1, \ldots, j_n) \in I^{(n)}$ such that $j_0 \neq j_n$ there exists a periodic ray γ such that $\mathbf{j} \in \mathscr{I}(\gamma)$. Indeed, consider

$$\min\left\{\sum_{l=0}^{n} |y_l - y_{l+1}|; y_l \in \Gamma_{j_l}, l=0,1,\ldots,n, \text{ and } y_{n+1} = y_0\right\}.$$

Evidently it exists because $\sum_{l=0}^{n} |y_l - y_{l+1}|$ is continuous in $(y_0, y_1, \ldots, y_n) \in \Gamma_{j_0} \times \Gamma_{j_1} \times \cdots \times \Gamma_{j_n}$ and $\Gamma_{j_0} \times \Gamma_{j_1} \times \cdots \times \Gamma_{j_n}$ is compact. If $(x_0, x_1, \ldots, x_n) \in \Gamma_{j_0} \times \Gamma_{j_1} \times \cdots \times \Gamma_{j_n}$ is an n-tuple of points which gives the minimum, for a broken line in Ω

$$\gamma = \bigcup_{l=0}^{n} \overline{x_l x_{l+1}}$$

it is easy to check that at each x_l γ verifies the law of reflection of the geometric optics. Thus, γ is a periodic ray in Ω.

Now we show that

$$\mathscr{I}(\gamma) = \mathscr{I}(\tilde{\gamma}) \tag{3.14}$$

implies $\gamma = \tilde{\gamma}$. From (3.14) there exist sequences of reflection points $(x_0, x_1, \ldots, x_n)$ of γ and $(\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n)$ of $\tilde{\gamma}$ such that x_l, $\tilde{x}_l \in \Gamma_{j_l}$, $l = 0, 1, \ldots, n$. Set $\xi = (x_1 - x_0)/|x_1 - x_0|$ and $\tilde{\xi} = (\tilde{x}_1 - \tilde{x}_0)/|\tilde{x}_1 - \tilde{x}_0|$.

Choose phase functions φ and $\tilde{\varphi}$ satisfying Condition P on Γ_{j_0} such that $\nabla\varphi(x_0) = \xi$ and $\nabla\tilde{\varphi}(\tilde{x}_0) = \tilde{\xi}$. Since γ and $\tilde{\gamma}$ are periodic, we have

$$X_{n+1}(x_0, \nabla\varphi) = x_0, \qquad X_{n+1}(\tilde{x}_0, \nabla\tilde{\varphi}) = \tilde{x}_0,$$
$$\Xi_{n+1}(x_0, \nabla\varphi) = \xi, \qquad \Xi_{n+1}(\tilde{x}_0, \nabla\tilde{\varphi}) = \tilde{\xi}.$$

Therefore it follows that for any integer $r \geqslant 1$

$$X_{r(n+1)}(x_0, \nabla\varphi) = x_0, \qquad X_{r(n+1)}(\tilde{x}_0, \nabla\tilde{\varphi}) = \tilde{x}_0.$$

Then we can write the above relations as

$$x_0 = X^{-s(n+1)}(x_0, \nabla\varphi_{rj}), \qquad \tilde{x}_0 = X^{-s(n+1)}(\tilde{x}_0, \nabla\tilde{\varphi}_{rj}) \quad (s \leqslant r).$$

Then we have

$$|x_0 - \tilde{x}_0| \leqslant |X^{-s(n+1)}(x_0, \nabla\varphi_{rj}) - X^{-s(n+1)}(\tilde{x}_0, \nabla\tilde{\varphi}_{rj})| +$$
$$|X^{-s(n+1)}(\tilde{x}_0, \nabla\varphi_{rj}) - X^{-s(n+1)}(\tilde{x}_0, \nabla\tilde{\varphi}_{rj})| = I + II.$$

The application of Proposition 3.8 gives

$$|I| \leqslant C\alpha^{s(n+1)}.$$

On the other hand, from Proposition 3.12 we have

$$|II| \leqslant C\alpha^{(r-s)(n+1)}.$$

Thus we have for any $1 \leqslant s \leqslant r$

$$|x_0 - \tilde{x}_0| \leqslant C(\alpha^{s(n+1)} + \alpha^{(r-s)(n+1)}),$$

which implies $x_0 = \tilde{x}_0$. By the same argument, for other reflecting points we have $x_l = \tilde{x}_l$. Hence $\gamma = \tilde{\gamma}$. Thus we have

THEOREM 3.13. – *For any* $\mathbf{j} = (j_0, \ldots, j_n) \in I^{(n)}$ *such that* $j_0 \neq j_n$ *there exists uniquely a periodic ray* γ *in* Ω *such that*

$$\mathbf{j} \in \mathcal{I}(\gamma).$$

4. Asymptotic solutions in the exterior of one convex body.

In this section we fix $j \in \{1,2,\ldots,J\}$ arbitrarily. Let a be an oscillatory function on $\Gamma_j \times \mathbf{R}$ of the form

$$a(x,t;k) = e^{ik(\psi(x)-t)} f(x,t;k) \tag{4.1}$$

where $k \geqslant 1$, $\psi \in C^\infty(\Gamma_j)$ and $f \in C^\infty(\Gamma_j \times (0,\infty))$.

DEFINITION 4.1. – *We say that a boundary data a of the form* (4.1) *satisfies Condition B on* Γ_j *if*

(i) *there exists a phase function* φ *satisfying Condition P in* Γ_j *such that*

$$\varphi = \psi \quad \text{on} \quad \bigcup_{t,k} \operatorname{supp} f(.,t;k),$$

(ii) $|f(.,.;k)|_p(\Gamma_j \times \mathbf{R}) \leqslant C_p$ *for all* $k \geqslant 1, p = 0,1,\ldots.$

DEFINITION 4.2 – *We say that a boundary data on* Γ_j *of the form*

$$m(x;k) = e^{ik\psi(x)} g(x;k) \tag{4.2}$$

satisfies Condition A on Γ_j *if*

(i) *there exists a phase function* φ *satisfying Condition P in* Γ_j *such that*

$$\varphi = \psi \qquad on \quad \bigcup_{t,k} \operatorname{supp} g(\cdot;k),$$

(ii) $|g(.;k)|_p(\Gamma_j) \leqslant C_p$ *for all* $k \geqslant 1, p = 0,1, \ldots.$

We set

$$\Omega_j = \mathbf{R}^3 - \bar{\mathcal{O}}_j, \qquad j = 1,2, \ldots, J.$$

First we consider asymptotic solutions of the problem

$$\text{(4.3)} \qquad \begin{cases} \Box\, u = 0 & \text{in} \quad \Omega_j \times \mathbf{R} \\ \quad u = a & \text{on} \quad \Gamma_j \times \mathbf{R} \\ \operatorname{supp} u \subset \Omega_j \times (0,\infty). \end{cases}$$

For m satisfying Condition B we can construct an asymptotic solution u of the form

$$\text{(4.4)} \qquad u(x,t;k) = e^{ik(\varphi(x)-t)} \sum_{l=0}^{N} v_l(x,t;k)\,(ik)^{-l}.$$

Indeed, as is known (see, for example, [KSL], [I1]), when $v_l, l = 0,1,2, \ldots, N$, satisfy

$$\begin{cases} Tv_0 = 0 & \text{in} \quad \Omega_j \times \mathbf{R} \\ v_0 = f(x,t;k) & \text{on} \quad \Gamma_j \times \mathbf{R}, \end{cases}$$

and for $l \geqslant 1$

$$\begin{cases} Tv_l = \Box v_{l-1} & \text{in} \quad \Omega_j \times \mathbf{R} \\ v_l = 0 & \text{on} \quad \Gamma_j \times \mathbf{R} \end{cases}$$

where

$$T = 2\frac{\partial}{\partial t} + 2\nabla\varphi\cdot\nabla + \Delta\varphi,$$

it holds that

$$\text{(4.5)} \qquad \begin{cases} \Box u = e^{ik(\varphi-t)}(ik)^N \Box v_N & \text{in} \quad \Omega_j \times \mathbf{R} \\ \quad u = a & \text{on} \quad \Gamma_j \times \mathbf{R}. \end{cases}$$

Moreover we have

LEMMA 4.3. — *It holds that*

$$\text{(4.6)} \quad \operatorname{supp} v_l(\cdot,\cdot;k) \subset \{(y+\tau\nabla\varphi(y), t+\tau); \tau \geqslant 0, (y,t) \in \operatorname{supp} f(\cdot,\cdot;k)\},$$

$$\text{(4.7)} \quad |v_l|_p(\Omega_j(R)\times \mathbf{R}) \leqslant C_p|\nabla\varphi|_{p+2l}(\Gamma_j)|f|_{p+2l}(\Gamma_j\times\mathbf{R}).$$

By means of the proof of Lemma 3.3 of [I4] we have

LEMMA 4.5. — *Let m be an oscillatory boundary data of the form (4.2) satisfying Condition A on Γ_j and let $h(t) \in C^\infty(0,\infty)$. If we set*

$$\text{(4.8)} \quad \begin{aligned} f(x,t;k) &= g(x;k)h(t), \\ a(x,t;k) &= e^{ik(\psi(x)-t)} f(x,t;k), \end{aligned}$$

then a satisfies Condition B on Γ_j and v_l constructed for a can be expressed as

$$\text{(4.9)} \quad v_l(x,t;k) = \sum_{q=0}^{2l} g_{l,q}(x;k)h^{(q)}(t-(\varphi(x)-\varphi(X^{-1}(x,\nabla\varphi))),$$

$$|g_{l,q}|_p(\Omega_j(R)) \leqslant C_p|\nabla\varphi|_{p+2l}(\Gamma_j)|g|_{p+2l}(\Gamma_j).$$

Especially

$$\begin{aligned} g_{0,0}(x;k) &= \Lambda_\varphi(x)g(X^{-1}(x,\nabla\varphi);k), \\ \Lambda_\varphi(x) &= \{G_\varphi(x)/G_\varphi(X^{-1}(x,\nabla\varphi))\}^{1/2} \end{aligned}$$

where $G_\varphi(x)$ denotes the Gaussian curvature of $\mathscr{C}_\varphi(x)$ at x.

Take a function $b(x,t;k) \in C^\infty(\mathbf{R}^3\times\mathbf{R})$ with the following properties: b is equal to the right hand side of (4.5) in $\Omega_j \times \mathbf{R}$, and

$$\operatorname{supp} b \cap \{\mathcal{O}\times\mathbf{R}\} \subset \{(x,t); \operatorname{dis}((x,t),\operatorname{supp} f(\cdot,\cdot;k)) < 1/2\},$$

$$\text{(4.10)} \quad |b|_p(\mathcal{O},\mathbf{R}) \leqslant C_p k^{-N+p}|\nabla\varphi|_{p+2N}(\Gamma_j)|f|_{p+2N}(\Gamma_j\times\mathbf{R}).$$

Let $z(x,t;k)$ be the solution of

$$\text{(4.11)} \quad \begin{cases} \Box z = -b & \text{in } \mathbf{R}^3\times\mathbf{R} \\ \operatorname{supp} z \subset \mathbf{R}^3 \times \{t \geqslant 0\}. \end{cases}$$

Then it follows from (4.10) that

$$\text{(4.12)} \quad |z|_p(\Omega_j(R)\times\mathbf{R}) \leqslant C_{p,R}k^{-N+p+2}|\nabla\varphi|_{p+2N}(\Gamma_j)|f|_{p+2N}(\Gamma_j\times\mathbf{R}).$$

We set

$$w = u + z.$$

Then from the choice of b we have

$$\text{(4.13)} \quad \Box w = 0 \quad \text{in } \Omega_j \times \mathbf{R}.$$

Suppose that

$$\operatorname{supp} f(\cdot,\cdot;k) \subset \Gamma_j \times (T, T+1).$$

Then we have

$$\operatorname{supp}\{w(\cdot, T+1;k), w_t(\cdot, T+1;k)\} \subset \{x;\ \operatorname{dis}(x,\mathcal{O}) \leqslant 3/2\}.$$

If we choose $\rho_j > 0$ as $\mathcal{O}_j \subset \{x; |x| < \rho_j\}$, the Huygens principle implies

$$(4.14)\quad \operatorname{supp} w(\cdot,\cdot;k) \subset \{(x,t); t \geqslant T-1,\ t-(T-1)-(\rho_j+1) \leqslant |x| \leqslant t-(T-1)+(\rho_j+1)\}.$$

Let f be a boundary data of the form (4.8). We consider the Laplace transform in t variable of the asymptotic solution w constructed in the above, that is,

$$\hat{w}(x,\mu;k) = \int_{-\infty}^{\infty} e^{-\mu t} w(x,t;k)\, dt.$$

By setting

$$s(x,\mu;k) = \hat{w}(x,\mu;k)/\hat{h}(\mu+ik),$$

we have

PROPOSITION 4.4. — *For a boundary data*

$$m(x;k) = e^{ik\psi(x)} g(x;k)$$

satisfying Condition A, there exists a function $s(x,\mu;k)$ *such that*

(i) *for each* $k \in \mathbf{R}$ $s(\cdot,\mu;k)$ *is a* $C^\infty(\bar{\Omega})$*-valued entire function,*

(ii) $$s(\cdot,\mu;k) \in L^2(\Omega_j) \quad \text{if} \quad \operatorname{Re}\mu > 0,$$

(iii) $$(\mu^2-\Delta)s(x,\mu;k) = 0 \quad \text{in} \quad \Omega_j,$$

(iv) $$s(x,\mu;k) = \sum_{l=0}^{N} \Big(\sum_{p=0}^{2l} g_{l,p}(x;k)(ik+\mu)^p\Big)(ik)^{-l} \times e^{-(\mu+ik)(\varphi(x)-\psi(X^{-1}(x,\nabla\varphi))} + r(x,\mu;k),$$

where r *satisfies*

$$(4.15)\quad |r(\cdot,\mu;k)|_p(\Omega_j(R)) \leqslant C_{R,p,x} e^{-\operatorname{Re}\mu(R+\rho_j+1)} k^{-N+p+2} |\nabla\varphi|_{p+2N}(\Gamma_j)|g|_{p+2N}(\Gamma_j)$$

for $-\alpha \leqslant \operatorname{Re}\mu \leqslant 1$, $|\operatorname{Im}(\mu+ik)| < 1$,

(v) $$s(x,\mu;k) = m(x,k) + r(x,u;k) \quad \text{on} \quad \Gamma_j.$$

Proof. — The estimate (4.14) of the support of w implies (i), and (iii) follows immediately from (4.14). (iv) and (v) are evident from the properties of u except for the estimate (4.15). If we choose $h(t)$ so that

$$|\hat{h}(\mu)| \geqslant 1 \quad \text{for} \quad |\mathrm{Im}\,\mu| \leqslant 1, \quad -\alpha \leqslant \mathrm{Re}\,\mu \leqslant 1,$$

then (4.15) follows from (4.12) since

$$r(x,\mu;k) = \hat{z}(x,\mu;k)/\hat{h}(ik+\mu).$$

From now on, we shall use the following definition for the brevity of statement.

DEFINITION 4.5. — *Let* ω *be an open set in* $\mathbf{R}^3$ *and let* $\mathscr{D}$ *be a domain in* $\mathbf{C}$. *We say that a function* $s(x,\mu;k)$ *satisfies Condition S in* $(\omega,\mathscr{D})$ *when*

(i) *for each* $k \in \mathbf{R}$ $s(\cdot,\mu;k)$ *is a* $C^\infty(\bar{\omega})$*-valued holomorphic function in* $\mathscr{D}$,

(ii) $$s(\cdot,\mu;k) \in L^2(\omega) \quad \textit{if} \quad \mathrm{Re}\,\mu > 0,$$

(iii) $$(\mu^2-\Delta)s(x,\mu;k) = 0 \quad \textit{in } \omega \textit{ for all} \quad \mu \in \mathscr{D}.$$

We denote the solution s constructed in Proposition 4.4 for m by

$$s(\cdot,\mu;k) = S_j(\mu)m(\cdot;k).$$

Thus S_j may be regarded as a mapping from the set of boundary data satisfying Condition A on Γ_j into the set of functions satisfying Condition S in $(\Omega_j, \mathbf{C})$.

Let $\chi \in C^\infty(\mathbf{R})$ be a function such that

$$\chi(t) = \begin{cases} 1 & \text{for} \quad t \leqslant -\delta_1 \\ 0 & \text{for} \quad t \geqslant -\delta_1/2, \end{cases}$$

where δ_1 is the constant in Lemma 3.1. As to the asymptotic solution $s = S_j m$ we shall use the following notations:

$$B_j^l(\mu)m = S_j(\mu)m|_{\Gamma_l},$$

$$B_j^l(\mu;q)m = \chi\left(\frac{\partial\varphi}{\partial n}\right)e^{ik\varphi}\sum_{p=0}^{2q} g_{q,p}(x;k)(ik+\mu)^p e^{-(\mu+ik)(\varphi(x)-\varphi(X^{-1}(x,\nabla\varphi))}|_{\Gamma_l},$$

$$\tilde{B}_j^l(\mu;q)m = \left(1-\chi\left(\frac{\partial\varphi}{\partial n}\right)\right)e^{ik\varphi}\sum_{p=0}^{2q} g_{q,p}(x;k)(ik+\mu)^p \times$$
$$e^{-(\mu+ik)(\varphi(x)-\varphi(X^{-1}(x,\nabla\varphi))}|_{\Gamma_l},$$

$$A_j^l(\varphi)g(x) = \chi\left(\frac{\partial\varphi}{\partial n}\right)\Lambda_\varphi(x)g(X^{-1}(x,\nabla\varphi);k)|_{\Gamma_l}.$$

It follows from Proposition 4.5 that the dependency of $B_j^l(\mu)$, $B_j^l(\mu\,;\,q)$, $\tilde{B}_j^l(\mu;q)$ on μ is holomorphic.

DEFINITION 4.6. – *We say that a boundary data*

$$\tilde{m}(x;k) = e^{ik\psi(x)}\tilde{g}(x;k)$$

satisfies Condition E on Γ_j *when*

$$|\tilde{g}(\cdot,k)|_p(\Gamma_j) \leqslant C_p \qquad \text{for all} \qquad k\,(p=0,1,\cdots)$$

and

$$\{x+\tau\xi(x);\tau\geqslant 0,\ x\in\bigcup_k \operatorname{supp} g(\cdot,k)\} \cap (d_0\text{-neighborhood of } \mathcal{O}-\mathcal{O}_j)=\varnothing,$$

where $\xi(x)$ *is a vector in* $\mathbf{R}^3$ *such that* $|\xi(x)|=1$, $n(x)\cdot\xi(x)\geqslant 0$ *and*

$$\xi(x) - (\xi(x)\cdot n(x))n(x) = \mathrm{grad}_\Gamma\psi.$$

We have the following Proposition by means of Proposition 7.5 of [I1].

PROPOSITION 4.7. – *For a boundary data* $\tilde{m}$ *satisfying Condition E on* Γ_j *we have a function* $\tilde{s}(x,\mu;k)$ *such that*

(i) $\tilde{s}$ *satisfies Condition S in* $(\Omega_j,\mathbf{C})$,

(ii) $|\tilde{s}(\cdot,\mu;k)|_p(\Omega_j(R)) \leqslant C_{R,p}e^{-\mathrm{Re}\mu(R+\rho_j+1)}k^p|\nabla\varphi|_{p+2N}(\Gamma_j)|g|_{p+2N}(\Gamma_j)$,

(iii) $|\tilde{s}(\cdot,\mu;k)-\tilde{m}(\cdot,k)|_p(\Gamma_j) \leqslant$

$$C_{R,p}e^{-\mathrm{Re}\mu(R+\rho_j+1)}k^{-N+p}|\nabla\varphi|_{p+2N}(\Gamma_j)|g|_{p+2N}(\Gamma_j),$$

(iv) *for* $l\neq j$

$$|\tilde{s}(\cdot,\mu;k)|_p(\Gamma_l) \leqslant C_{R,p}e^{-\mathrm{Re}\mu(R+\rho_j+1)}k^{-N+p}|\nabla\varphi|_{p+2N}(\Gamma_j)|g|_{p+2N}(\Gamma_j).$$

We denote by $S_j(\mu)$ the mapping from a boundary data $\tilde{m}$ satisfying Condition E on Γ_j to $\tilde{s}$ a function satisfying Condition S in $(\Gamma_j,\mathbf{C})$ that is constructed in Proposition 4.7, that is,

$$\tilde{s}(\cdot,\mu;k) = \tilde{S}_j(\mu)\tilde{m}(\cdot,k).$$

5. Construction of asymptotic solutions in Ω.

In this section we shall construct a first approximation of the solution to the problem

$$(5.1)\qquad \begin{cases}(\mu^2-\Delta)u = 0 & \text{in} \quad \Omega \\ u = m & \text{on} \quad \Gamma,\end{cases}$$

where m is a boundary data of the form (4.2) satisfying Condition A on Γ_j.

Define $m_{\mathbf{j}}$ and $s_{\mathbf{j}}$, $\tilde{s}_{\mathbf{j}}$ for $\mathbf{j} = (j,j_1,\cdots j_n) \in I_j$ by

$$m_{\mathbf{j}}(x,\mu;k) = B_{\mathbf{j}}(\mu)m = B^{j_n}_{j_{n-1}}(\mu,0)\cdot B^{j_{n-1}}_{j_{n-2}}(\mu,0)\cdot\cdots\cdot B^{j_1}_{j}(\mu,0)m\,;$$

$$s_{\mathbf{j}}(x,\mu;k) = S_{j_n}(\mu)m_{\mathbf{j}},\ \tilde{s}_{\mathbf{j}}(x,\mu;k) = \sum_{l\neq j_n}\tilde{S}_l(\mu)\tilde{B}^l_{j_n}(\mu,0)m_{\mathbf{j}}.$$

Set

$$(5.2)\qquad w^{(0)}(x,\mu;k) = \sum_{\mathbf{j}\in I_j}(-1)^{|\mathbf{j}|-1}\{s_{\mathbf{j}}(x,\mu\,;k) + \tilde{s}_{\mathbf{j}}(x,\mu;k)\}.$$

In order to investigate the convergence of (5.2), first we shall make a decomposition of I_j. For $\mathbf{i} = (i_0,i_1,\cdots,i_m)\in I^{(m)}$ and $\mathbf{j} = (j_0,j_1,\cdots j_n)\in I^{(n)}$ such that $i_n \neq j_0$ we denote by $(\mathbf{i},\mathbf{j})$ an element in $I^{(m+n+1)}$ defined by

$$(\mathbf{i},\mathbf{j}) = (i_0,i_1,\cdots,i_m,j_0,j_1,\cdots,j_n).$$

Let $\mathbf{i} = (i_0,i_1,\cdots,i_m)\in I^{(m)}$ such that $i_0\neq i_m$, and let $r\in N$. We denote by $r\mathbf{i}$ an element in $I^{(r(m+1)-1)}$ defined by

$$r\mathbf{i} = (\overbrace{\mathbf{i},\mathbf{i},\cdots,\mathbf{i}}^{r}).$$

We say that $\mathbf{i} = (i_0,i_1,\cdots,i_m)\in I^{(m)}$ is primitive when $i_0\neq i_m$ and there are no $\mathbf{j} = (j_0,j_1,\cdots j_n)\in I^{(n)}$ and $r\geqslant 2$ such that $\mathbf{i} = r\mathbf{j}$. Denote by $I(\mathscr{P})$ the set of all the primitive elements in I.

Set for $\mathbf{i} = (i_0,i_1,\cdots,i_m)\in I(\mathscr{P})$

$$P'\mathbf{i} = \{(r\mathbf{i},i_0,i_1,\cdots,i_s)\,;r\geqslant 1, 0\leqslant s\leqslant m-1\},$$

$$P\mathbf{i} = P'\mathbf{i}\cup\{\mathbf{i}\}.$$

LEMMA 5.1. – *For each $1 \leqslant j \leqslant J$ there exist $\mathscr{I}_j$ and $\mathscr{I}'_j$ subsets of $I(\mathscr{P})$ such that $P\mathbf{j}$, $P'\mathbf{j}'$ ($\mathbf{j} \in \mathscr{I}_j, \mathbf{j}' \in \mathscr{I}'_j$) are disjoint with one another and*

$$I_j = \Big(\bigcup_{\mathbf{j} \in \mathscr{I}_j} P\mathbf{j}\Big) \cup \Big(\bigcup_{\mathbf{j}' \in \mathscr{I}'_j} P'\mathbf{j}'\Big). \tag{5.3}$$

Proof. – It is sufficient to prove for $j = 1$. Now we prove the following assertion by the induction in n: for $n \geqslant 1$ there are $\mathscr{I}_1^{(l)}, \mathscr{I}_1^{(l)'} \subset I(\mathscr{P}) \cap I_1^{(l)}$ such that

$$P\mathbf{j},\ P'\mathbf{j}',\ \mathbf{j} \in \mathscr{I}_1^{(l)},\ \mathbf{j}' \in \mathscr{I}_1^{(l)'}\ l = 1, 2, \cdots, n \text{ are disjoint} \tag{5.4}$$

and

$$I_1^{(n)} \subset \bigcup_{l=1}^{n} \Big\{\Big(\bigcup_{\mathbf{j} \in \mathscr{I}_1^{(l)}} P\mathbf{j}\Big) \cup \Big(\bigcup_{\mathbf{j}' \in \mathscr{I}_{1,}^{(l)}} P'\mathbf{j}'\Big)\Big\}. \tag{5.5}$$

It is evident that this assertion implies the statement of Lemma.

We set
$$\mathbf{j}_p^{(1)} = (1,p), \qquad 2 \leqslant p \leqslant J.$$

Evidently $\mathbf{j}_p^{(1)} \in I(\mathscr{P}) \cap I_1^{(1)}$, and

$$I_1^{(1)} \subset \bigcup_{p=2}^{J} P\mathbf{j}_p^{(1)}.$$

Thus in the case of $n = 1$, (5.4) and (5.5) hold by choosing $\mathscr{I}_1^{(1)} = \{(1,p); 2 \leqslant p \leqslant J\}$, $\mathscr{I}_1^{(1)'} = \varnothing$.

Suppose that (5.4) and (5.5) hold for $n = m$. Let $\mathbf{j} = (1, j_1, \cdots, j_{m+1})$ be an element in $I_1^{(m+1)}$ such that

$$\mathbf{j} \notin \bigcup_{l=1}^{m} \Big\{\Big(\bigcup_{\mathbf{j}' \in \mathscr{I}_1^{(l)}} P\mathbf{j}\Big) \cup \Big(\bigcup_{\mathbf{j}' \in \mathscr{I}_1^{(l)}} P'\mathbf{j}'\Big)\Big\}. \tag{5.6}$$

If we set $\mathbf{j}' = (1, j_1, \ldots, j_m)$ we have from (5.5) for $n = m$

$$\mathbf{j}' \in \bigcup_{l=1}^{m} \Big\{\Big(\bigcup_{\mathbf{j} \in \mathscr{I}_1^{(l)}} P\mathbf{j}\Big) \cup \Big(\bigcup_{\mathbf{j}' \in \mathscr{I}_1^{(l)}} P'\mathbf{j}'\Big)\Big\},$$

namely we can write $\mathbf{j}'$ as

$$\mathbf{j}' = (r\mathbf{i}, 1, i_1, \ldots, i_s), \qquad \mathbf{i} = (1, i_1, i_2, \ldots, i_p) \in \mathscr{I}_1^{(p)} \cup \mathscr{I}_1^{(p)'} \tag{5.7}$$
$$r \geqslant 1, \qquad 1 \leqslant s \leqslant p - 1,$$

or

(5.8) $\mathbf{j}' = r\mathbf{i}, \qquad \mathbf{i} = (1, i_1, i_2, \ldots, i_p) \in \mathscr{I}_1^{(p)} \cup \mathscr{I}_1^{(p)\prime}, \qquad r \geqslant 1.$

In case of $j_{m+1} = 1$, it happens only the case (5.7). Indeed, in case of (5.8) we have $\mathbf{j} \in P'\mathbf{i} \subset P\mathbf{i}$. This contradicts (5.6). In case of (5.7) evidently $\mathbf{j}' \in I(\mathscr{P})$ and $\mathbf{j} \in P'\mathbf{j}'$. Consider the case of $j_{m+1} \neq 1$. Now for the both cases of (5.7) and (5.8) we have $\mathbf{j} \in I(\mathscr{P})$. Setting

$$\mathscr{I}_1^{(m+1)} = \{\mathbf{j} = (1, j_1, j_2, \ldots, j_{m+1})\,;\ \text{satisfying (5.6) and } j_{m+1} \neq 1\},$$
$$\mathscr{I}_1^{(m+1)\prime} = \{\mathbf{j} = (1, j_1, j_2, \ldots, j_{m+1})\,;\ \text{satisfying (5.6) and } j_{m+1} = 1\}.$$

We have for $\mathbf{j}$ satisfying (5.6)

$$\mathbf{j} \in \Big(\bigcup_{\mathbf{i} \in \mathscr{I}_1^{(m+1)}} P\mathbf{i}\Big) \cup \Big(\bigcup_{\mathbf{i}' \in \mathscr{I}_1^{(m+1)}} P'\mathbf{i}'\Big).$$

It is evident that the disjointness of (5.4) holds for $n = m + 1$. Thus the assertion is proved for $n = m + 1$. Q.E.D.

To show the convergence of (5.2) we have to express $m_{\mathbf{j}}$ and $s_{\mathbf{j}}$ more explicitely. Let $\mathbf{j} = (j, j_1, \ldots, j_n) \in I_j^{(n)}$. It is easy to show the following by the induction:

(5.9) $$m_{\mathbf{j}}(x, \mu; k) = e^{ik\varphi_{\mathbf{j}}(x)} A_{\mathbf{j}}(\varphi) g(x) e^{-(\mu + ik)(\varphi_{\mathbf{j}}(x) - \psi(X^{-|\mathbf{j}|}(x, \nabla\varphi_{\mathbf{j}})))}$$

where we define $A_{\mathbf{j}}$ by

$$A_{\mathbf{j}}(\varphi)g(x) = A_{j_{n-1}}^{j_n}(\varphi_{(j, j_1, \ldots, j_{n-1})}) \cdot \mathrm{A}_{j_{n-2}}^{j_{n-1}}(\varphi_{(j, j_1, \ldots, j_{n-2})}) \cdots$$
$$\mathrm{A}_{j_1}^{j_2}(\varphi_{(j, j_1)}) \cdot \mathrm{A}_j^{j_1}(\varphi) g(x).$$

Indeed, for $\mathbf{j} = (j, j_1)$ from the definition of $B_j^{j_1}(\mu; 0)$

$$m_{(j, j_1)}(x, \mu : k) = e^{ik\varphi(x)} A_j^{j_1}(\varphi) g(x; k) e^{-(\mu + ik)(\varphi(x) - \psi(X^{-1}(x, \nabla\varphi)))}.$$

By using the fact that

$$\varphi_{(j, j_1)} = \varphi \qquad \text{on} \quad \Gamma_{j_1, (j)},$$
$$\varphi = \psi \qquad \text{on} \quad \Gamma_j,$$

we see that (5.9) is valid for $n = 1$. Suppose that (5.9) is valid for $n = l$. Let $\mathbf{j} = (j, j_1, \ldots, j_{l+1})$, $\mathbf{j}' = (j, j_1, \ldots, j_l)$. Since (5.9) holds for

$\mathbf{j}'$, we have

$$m_{\mathbf{j}'}(x,\mu;k) = e^{ik\varphi_{\mathbf{j}'}(x)}A_{\mathbf{j}'}(\varphi)g(x)e^{-(\mu+ik)(\varphi_{\mathbf{j}'}(x)-\psi(X^{-|\mathbf{j}'|}(x,\nabla\varphi_{\mathbf{j}'})))}.$$

Now from the definition we have

$$m_{\mathbf{j}}(x,\mu;k) = B_{j_l}^{j_{l+1}}(\mu;0)m_{\mathbf{j}'}(x,\mu;k) =$$

$$e^{ik\varphi_{\mathbf{j}'}(x)}A_{j_l}^{j_{l+1}}(\varphi_{\mathbf{j}'})A_{\mathbf{j}'}(\varphi)g(x)e^{-(\mu+ik)(\varphi_{\mathbf{j}'}(x)-\psi(X^{-|\mathbf{j}'|}(x,\nabla\varphi_{\mathbf{j}'})))}$$

Note that we have on $\Gamma_{j_{l+1}}$

$$\varphi_{\mathbf{j}} = \varphi_{\mathbf{j}'}, \qquad X^{-|\mathbf{j}'|}(x,\nabla\varphi_{\mathbf{j}'}) = X^{-|\mathbf{j}|}(x,\nabla\varphi_{\mathbf{j}}).$$

Evidently

$$A_{j_l}^{j_{l+1}}(\varphi_{\mathbf{j}'})A_{\mathbf{j}'}(\varphi) = A_{\mathbf{j}}(\varphi).$$

Hence (5.9) holds for any $\mathbf{j} \in I_j^{(l+1)}$. Thus we have shown that (5.9) is valid for any $\mathbf{j} \in I_j$.

Note that Lemma 3.1 implies

$$\frac{\partial\varphi_{(j,j_1,\ldots,j_{n-l})}}{\partial n}(X^{-l}(x,\nabla\varphi_{\mathbf{j}})) \geqslant \delta_1 \qquad \text{for any} \qquad x \in \mathscr{U}_{\mathbf{j}}(\nabla\varphi).$$

Thus from the definition of $A_{\mathbf{j}}$ we have

$$(5.10) \qquad A_{\mathbf{j}}(\varphi)g(x) = \chi\left(\frac{\partial\varphi_{\mathbf{j}}}{\partial n}\right)\Lambda_{\varphi,\mathbf{j}}(x)g(X^{-n}(x,\nabla\varphi_{\mathbf{j}})),$$

$$\Lambda_{\varphi,\mathbf{j}}(x) = \Lambda_{\varphi_{(j,j_1,\ldots,j_n)}}(x)\Lambda_{\varphi_{(j,j_1,\ldots,j_{n-1})}}(X^{-1}(x,\nabla\varphi_{\mathbf{j}}))\ldots\Lambda_{\varphi}(X^{-(n-1)}(x,\nabla\varphi_{\mathbf{j}})).$$

LEMMA 5.2. — *Let $\mathbf{i} = (j,i_1,\ldots,i_n) \in \mathscr{I}_j$, and let γ be a periodic ray in Ω such that $\mathbf{i} \in \mathscr{I}(\gamma)$. Suppose that*

$$(5.11) \qquad \gamma = \bigcup_{l=0}^{n} \overline{x_l x_{l+1}}, \qquad x_l \in \Gamma_{i_l}, \qquad x_{n+1} = x_0.$$

Then there exist phase functions $\varphi_{\mathbf{i},l}^{\infty}$, $l = 0,2,\ldots,n$, such that

(i) $\varphi_{\mathbf{i},l}^{\infty}$ *satisfies Condition P on* Γ_{j_l},

(ii) $\varphi_{\mathbf{i},l}^{\infty}(x_l) = 0$,

(iii) $\Phi_{i_l}^{i_{l+1}}\varphi_{\mathbf{i},l}^{\infty} = \varphi_{\mathbf{i},l+1}^{\infty} + d_{(\gamma)i_l}, 0 \leqslant l \leqslant n-1, \Phi_{i_n}^{j}\varphi_{\mathbf{i},n}^{\infty} = \varphi_{\mathbf{i},0}^{\infty} + d_{(\gamma)i_n}$,

where $d_{(\gamma)i_l} = |x_{l+1} - x_l|$.

Proof. — Take a phase function ψ satisfying Condition P on Γ_j such that

$$\psi(x_0) = 0, \qquad \nabla\psi(x_0) = (x_1 - x_0)/|x_1 - x_0|.$$

For $r \geqslant 0$, $0 \leqslant l \leqslant n$, we set

$$\psi_{ri,l} = \Phi_{(ri,l)}\psi.$$

Evidently we have for all $r \geqslant 0$, $0 \leqslant l \leqslant n$

$$(\nabla\psi_{ri,l})(x_l) = (x_{l+1} - x_l)/|x_{l+1} - x_l|,$$

$$\text{(5.12)} \quad \begin{cases} \psi_{ri,l}(x_l) = rd_\gamma + |x_1 - x_0| + \cdots + |x_l - x_{l-1}| \quad (n \geqslant l \geqslant 1), \\ \psi_{(r+1)i,l}(x_l) = \psi_{ri,l}(x_l) + d_\gamma . \end{cases}$$

Applying Corollary 3.10 to ψ and $\Phi_i\psi$, and we have

$$|\nabla\psi_{ri,l} - \nabla\psi_{(r+1)i,l}|_p(\Gamma_{i_l}) \leqslant C_p\alpha^{rn+s},$$

which implies the existence of a smooth vector $\eta_{i,l}$ such that

$$\text{(5.13)} \qquad |\nabla\psi_{ri,l} - \eta_{i,l}|_p(\Gamma_{i_l}) \leqslant C_p\alpha^{rn+l}.$$

Then from (5.12) and (5.13) it follows that

$$\varphi_{i,l}^{\infty}(x) = \lim_{r\to\infty} (\psi_{ri,l}(x) - (rd_\gamma + d_{(\gamma)i_0} + \cdots + d_{(\gamma)i_l}))$$

exists and

$$|\varphi_{i,l}^{\infty} - (\psi_{ri,l} - (rd_\gamma + d_{(\gamma)i_0} + \cdots + d_{(\gamma)i_l}))|_p(\Gamma_{i_l}) \leqslant C_p\alpha^{nr+l}.$$

By using (5.12) we have (ii) from the above estimate.

By Lemma 3.9 we have for $0 \leqslant l \leqslant n - 1$

$$|\Phi_{i_l}^{i_{l+1}}(\varphi_{i,l}^{\infty} + rd_\gamma + d_{(\gamma)i_0} + \cdots + d_{(\gamma)i_l}) - \Phi_{i_l}^{i_{l+1}}\psi_{ri,l}|_p(\Gamma_{i_l}) \leqslant C_p\alpha^{nr+l},$$

$$\Phi_{i_l}^{i_{l+1}}(\varphi_{i,l}^{\infty} + rd_\gamma + d_{(\gamma)i_0} + \cdots + d_{(\gamma)i_l}) = \Phi_{i_l}^{i_{l+1}}\varphi_{i,l}^{\infty} + rd_\gamma + d_{(\gamma)i_0} + \cdots + d_{(\gamma)i_l}.$$

On the other hand

$$\Phi_{i_l}^{i_{l+1}}\psi_{ri,l} - (rd_\gamma + d_{(\gamma)i_0} + \cdots + d_{(\gamma)i_l}) =$$
$$\psi_{ri,l+1} - (rd_\gamma + d_{(\gamma)i_0} + \cdots + d_{(\gamma)i_l}) \to \varphi_{i,l+1}^{\infty} - d_{(\gamma)i_{l+1}}.$$

Thus we have (iii). Q.E.D.

LEMMA 5.3. – *We have*

$$(5.14)\qquad |\nabla\varphi_{(r\mathbf{i},i_0,\dots,i_l)}-\nabla\varphi^{\infty}_{\mathbf{i},l}|_p(\Gamma_{i_l}) \leqslant C_p\alpha^{nr+l}.$$

Proof. – Apply Lemma 3.9 to $\varphi_{\mathbf{j}}$ and $\psi_{\mathbf{j}} = \Phi_{\mathbf{j}}\psi$, and we have

$$|\nabla\varphi_{\mathbf{j}}-\nabla\psi_{\mathbf{j}}|_p(\Gamma_{i_l}) \leqslant C_p\alpha^{|\mathbf{j}|}.$$

On the other hand, since $\eta_{\mathbf{i},l}$ in (5.13) is equal to $\nabla\varphi^{\infty}_{\mathbf{i},l}$ we have

$$|\nabla\psi_{(r\mathbf{i},i_0,\dots,i_n)}-\nabla\varphi^{\infty}_{\mathbf{i},l}|_p(\Gamma_i) \leqslant C_p\alpha^{nr+l}.$$

Combining the above estimates we have (5.14). Q.E.D.

LEMMA 5.4. – *There exists uniquely a point $x^{\infty}_{\mathbf{i}}$ in Γ_j such that*

$$(5.15)\qquad |X_{rn+l}(x^{\infty}_{\mathbf{i}},\nabla\varphi)-x_l| \leqslant C\alpha^{nr+l} \quad \textit{for all}\quad r\geqslant 0,\ 0\leqslant l\leqslant |\mathbf{i}|,$$

where C is a constant independent of $\mathbf{i}$.

Proof. – Set

$$x^{rn+l}_{\mathbf{i}} = X^{-rn}(x_l,\nabla\varphi_{(r\mathbf{i},i_0,\dots,i_l)}).$$

Evidently we have

$$x^{(r+r')n+l}_{\mathbf{i}} = X^{-(r+r')n}(x_l,\nabla\varphi_{((r+r')\mathbf{i},i_0,\dots,i_l)}) =$$
$$X^{-rn}(X^{-r'n}(x_l,\nabla\varphi_{((r+r')\mathbf{i},i_0,\dots,i_l)}),\nabla\varphi_{r\mathbf{i}}).$$

Since $X^{-r'n}(x_l,\nabla\varphi_{((r+r')\mathbf{i},i_0,\dots,i_l)})\in\Gamma_{i_l}$ an application of Proposition 3.8 gives

$$|x^{(r+r')n+l}_{\mathbf{i}}-x^{rn+l}_{\mathbf{i}}| \leqslant C\alpha^{nr+l}.$$

Thus $\lim_{r\to\infty} x^{rn+l}_{\mathbf{i}}$ exists. Denoting the limit point by $x^{\infty}_{\mathbf{i}}$ we have

$$(5.16)\qquad |x^{rn+l}_{\mathbf{i}}-x^{\infty}_{\mathbf{i}}| \leqslant C\alpha^{nr+l}.$$

Note that

$$X^{-r'n}(x_l,\nabla\varphi^{\infty}_{\mathbf{i},l}) = x_l,\qquad \text{for}\qquad r'\geqslant 0,\ \ 0\leqslant l\leqslant n-1.$$

Then by using (5.14) we have

$$|X^{-r'n}(x_l,\nabla\varphi_{((r+r')\mathbf{i},i_0,\dots,i_l)})-x_l| \leqslant C\alpha^{nr+l}.$$

Since $X^{-r'n}(x_l,\nabla\varphi_{((r+r')\mathbf{i},i_0,\dots,i_l)}) = X_{rn+l}(x^{(r+r')n+l}_{\mathbf{i}},\nabla\varphi)$ we have

$$|X_{rn+l}(x^{(r+r'),n+l}_{\mathbf{i}},\nabla\varphi)-x_l| \leqslant C\alpha^{nr+l}.$$

Then letting $r'\to\infty$ we have (5.15). Q.E.D.

By following the argument in Section 4 of [I2] we can derive from Lemmas 5.3 and 5.4 the following

PROPOSITION 5.5. — *Let* $\mathbf{j} = (r\mathbf{i}, i_0, \ldots, i_l) \in P\mathbf{i}$. *Then we have for* $s \leqslant |\mathbf{j}|/2$

$$|X^{-s}(\cdot, \nabla\varphi_{\mathrm{j}}) - X^{-s}(\cdot, \nabla\varphi^{\infty}_{\mathrm{i},l})|_p(\Gamma_{\mathrm{j}}) \leqslant C_p\alpha^{|\mathrm{j}|/2},$$

$$|X^{-(|\mathrm{j}|-s+1)}(\cdot, \nabla\varphi_{\mathrm{j}}) - X_s(x^{\infty}_{\mathrm{i}}, \nabla\varphi)|_p(\Gamma_{\mathrm{j}}) \leqslant C_p\alpha^{|\mathrm{j}|/2}.$$

With the aid of Proposition 5.5 we have the following as in Section 7 of [I2].

PROPOSITION 5.6. — *There exists a constant* $d_{\varphi,\mathrm{i}}$ *such that*

$$|\varphi_{\mathrm{j}} - (\varphi^{\infty}_{\mathrm{i},l} + rd_{\gamma} + d_{\varphi,\mathrm{i}} + d_{(\gamma)i_0} + d_{(\gamma)i_1} + \cdots + d_{(\gamma)i_l})|_p(\Gamma_{\mathrm{j}}) \leqslant C_p\alpha^{|\mathrm{j}|}$$

holds for all $\mathbf{j} = (r\mathbf{i}, i_0, i_1, \ldots, i_n) \in P\mathbf{i}$, *and the set* $\{d_{\varphi,\mathrm{i}}\}_{\mathrm{i} \in \mathscr{I}_j \cup \mathscr{I}'_j}$ *is bounded in* $\mathbf{R}$.

Now we look for an asymptotic formula for $\Lambda_{\varphi,\mathrm{j}}(x)$. Set for $0 \leqslant l \leqslant n$

$$\lambda_{\mathrm{i},l} = \Lambda_{\varphi^{\infty}_{\mathrm{i},l}}(x_{l+1}).$$

Note that

$$\pi^{n}_{l=0}\lambda_{\mathrm{i},l} = \lambda_{\gamma}$$

(see, for example [BGR, Section 2]).

By employing the agument of Section 5 of [I2] we have

PROPOSITION 5.7. — *It holds that for all* $\mathbf{j} \in P\mathbf{i}$

$$|\Lambda_{\varphi,\mathrm{j}}(\cdot) - (\lambda_{\mathrm{i}})^{\mathrm{j}} a_{\mathrm{i},l}(\cdot) b_{\mathrm{i}}|_p(\Gamma_{\mathrm{j}}) \leqslant C_p(\lambda_{\mathrm{i}})^{\mathrm{j}}\alpha^{|\mathrm{j}|}. \tag{5.17}$$

Here (i) *we denote by* $(\lambda_{\mathrm{i}})^{\mathrm{j}}$

$$(\lambda_{\mathrm{i}})^{\mathrm{j}} = \lambda^{r}_{\gamma}\lambda_{\mathrm{i},0} \cdots \lambda_{\mathrm{i},l} \qquad \textit{for} \qquad \mathbf{j} = (r\mathbf{i}, i_0, i_1, \ldots, i_l),$$

(ii) $a_{\mathrm{i},l}(x)$, $l = 0, 1, \ldots, n$ *are smooth functions in* $\mathscr{U}_{i_l}(\nabla\varphi^{\infty}_{\mathrm{i},l})$ *such that* $|a_{\mathrm{i},l}|_p(\mathscr{U}_{i_l}(\nabla\varphi^{\infty}_{\mathrm{i},l})) \leqslant C_p$, *where* C_p *is independent of* $\mathbf{i}$ *and* l,

(iii) b_{i} *is a positive constant depending on* φ, *and we have*

$$|b_{\mathrm{i}}| \leqslant C \qquad \textit{for all} \qquad \mathbf{i}.$$

Set

$$m_{\mathbf{i},l}^{\infty}(x,\mu;k) = e^{ik\varphi_{\mathbf{i},l}^{\infty}(x)} a_{\mathbf{i},l}(x).$$

With the aids of Propositions 5.6, 5.7 and (5.16) we have

$$(5.18) \quad |m_{\mathbf{j}} - g(x_{\mathbf{i}}^{\infty};k)b_{\mathbf{i}}e^{-\mu d_{\varphi,\mathbf{i}}}(\lambda_{\mathbf{i}}e^{-\mu d_{\mathbf{i}}})^{\mathbf{j}} m_{\mathbf{i},l}^{\infty}|_p(\Gamma_{\mathbf{j}}) \leqslant C_p k^p \cdot k\alpha^{|\mathbf{j}|}(\lambda_{\mathbf{i}}e^{-\mathrm{Re}\,\mu d_{\mathbf{i}}})^{\mathbf{j}} |g|_{p+1}(\Gamma_j)$$

for $\mathbf{j} = (r\mathbf{i}, i_0, i_1, \ldots, i_l)$ where we use a notation

$$(\lambda_{\mathbf{i}}e^{-\mu d_{\mathbf{i}}})^{\mathbf{j}} = (\lambda_\gamma e^{-\mu d_\gamma})^r \lambda_{\mathbf{i},0} e^{-\mu d(\gamma)i_0} \ldots \lambda_{\mathbf{i},l} e^{-\mu d(\gamma)i_l}.$$

By setting

$$s_{\mathbf{i},l}^{\infty}(x,\mu\,;k) = S_{i_l}(\mu) m_{\mathbf{i},l}^{\infty},$$

$$t_{\mathbf{j}}(x,\mu;k) = s_{\mathbf{j}} - g(x_{\mathbf{i}}^{\infty};k)b_{\mathbf{i}}e^{-\mu d_{\varphi,\mathbf{i}}}(\lambda_{\mathbf{i}}e^{-\mu d_{\mathbf{i}}})^{\mathbf{j}} s_{\mathbf{i},l}^{\infty}(x,\mu;k),$$

from (5.18) we have

$$(5.19) \quad |t_{\mathbf{j}}|_p(\Omega(R)) \leqslant C_{p,R} k^p \alpha^{|\mathbf{j}|} e^{-\mathrm{Re}\mu\, d_{\gamma,\mathbf{i}}} (\lambda_{\mathbf{i}}e^{-\mathrm{Re}\,\mu d_{\mathbf{i}}})^{\mathbf{j}} |g|_{p+1}(\Gamma_j).$$

Thus

$$\sum_{\mathbf{j}\in P\mathbf{i}} s_{\mathbf{j}} = \sum_{l=0}^{n} \sum_{r=1}^{\infty} s_{(r\mathbf{i}, i_0, i_1, \ldots, i_l)} = g(x_{\mathbf{i}}^{\infty};k)b_{\mathbf{i}}e^{-\mu d_{\varphi,\mathbf{i}}} \sum_{l=0}^{n} (\lambda_{\mathbf{i}}e^{-\mu d_{\mathbf{i}}})^{(i_0, i_1, \ldots, i_l)} s_{\mathbf{i},l}^{\infty} \times \sum_{r=1}^{\infty} \lambda_\gamma^r e^{-r\mu d_\gamma} + \sum_{\mathbf{i}\in P\mathbf{i}} t_{\mathbf{j}} = g(x_{\mathbf{i}}^{\infty};k)b_{\mathbf{i}}e^{-\mu d_{\varphi,\mathbf{i}}} I + II.$$

For $\mathbf{i} = (i_0, i_1, \ldots, i_p)$, $\mathbf{h} = (h_0, h_1, \ldots, h_q) \in I$, with the notation $\mathbf{h} \leqslant \mathbf{i}$ we signify that $q \leqslant p$ and $h_l = i_l$ for all $l \leqslant q$. Then I is expressed as

$$I = \lambda_\gamma e^{-\mu d_\gamma}(1 - \lambda_\gamma e^{-\mu d_\gamma})^{-1} \sum_{\mathbf{i}'\leqslant\mathbf{i}} (\lambda_{\mathbf{i}}e^{-\mu d_{\mathbf{i}}})^{\mathbf{i}'} s_{\mathbf{i},l}^{\infty}.$$

In order to estimate I and II we prepare the following

LEMMA 5.8. — *Suppose that*

$$\beta < a_0.$$

Then we have

$$\sup_{\mathbf{i}} \sum_{\mathbf{i}'\leqslant\mathbf{i}} (\lambda_{\mathbf{i}}e^{\beta d_{\mathbf{i}}})^{\mathbf{i}'} \leqslant CF(\beta),$$

where the supremum is taken over $\mathbf{i} \in \mathscr{I}_j \cup \mathscr{I}'_j$.

Proof. – First we admit that an inequality

$$(5.20) \qquad (\lambda_{\mathrm{i}} e^{\beta d_{\mathrm{i}}})^{(i_0, i_1, \ldots, i_l)} \leqslant C \lambda_{\tilde{\gamma}} e^{\beta d_{\tilde{\gamma}}}$$

holds where $\tilde{\gamma}$ is a periodic ray in Ω such that $(i_0, i_1, \ldots, i_l) \in \mathscr{I}(\tilde{\gamma})$ and C is a constant independent of γ and $(i_0, i_1, \ldots, i_l)$. Obviously the assertion of Lemma follows from (5.20).

Set

$$\tilde{\gamma} = \bigcup_{s=0}^{l} \overline{y_s y_{s+1}}, \qquad y_{l+1} = y_0, \qquad y_s \in \Gamma_{i_s}$$

and $\varphi_{\tilde{\gamma},s}^{\infty}$ be phase functions in Lemma 5.2 for

$$\tilde{\mathrm{i}} = (i_0, i_1, \ldots, i_l) \in \mathscr{I}(\tilde{\gamma}).$$

Then from Proposition 3.11 we have

$$|\nabla \varphi_{\gamma,s}^{\infty} - \nabla \varphi_{\tilde{\gamma},s}^{\infty}|_p(\Gamma_{i_s}) \leqslant C_p \alpha^s.$$

Then it follows that

$$|y_s - x_s| \leqslant \alpha^{\min(s, l-s)}.$$

Therefore from the above two estimates we have

$$|\lambda_{\mathrm{i},s}/\lambda_{\tilde{\mathrm{i}},s} - 1| \leqslant C\alpha^{\min(s,l-s)},$$
$$|d_{\gamma,i_s}/d_{\tilde{\gamma},i_s} - 1| \leqslant C\alpha^{\min(s,l-s)}.$$

Substituting these estimates we have (5.20). Q.E.D.

Now from the form of I we have

$$|I| \leqslant CF(-\mathrm{Re}\,\mu)\lambda_\gamma e^{-\mathrm{Re}\,\mu d_\gamma}(1 - \lambda_\gamma e^{-\mathrm{Re}\,\mu d_\gamma})^{-1}.$$

On the other hand from (5.19) we have

$$|II| \leqslant \sum_{\mathrm{j} \in Pi} |t_{\mathrm{j}}|_p(\Omega(R)) \leqslant \Sigma C_p k^p \alpha^{|\mathrm{j}|} (\lambda_{\mathrm{i}} e^{-\mathrm{Re}\,\mu d_{\mathrm{i}}})^{\mathrm{j}} |g|_{p+1}(\Gamma_j) =$$
$$C_p k^p \sum_{r=1}^{\infty} \lambda_\gamma^r e^{-\mathrm{Re}\,\mu r d_\gamma} \alpha^{r|\mathrm{i}|} \sum_{\mathrm{i}' \leqslant \mathrm{i}} (\lambda_{\mathrm{i}} e^{-\mathrm{Re}\,\mu d_{\mathrm{i}}})^{\mathrm{i}'} \alpha^{|\mathrm{i}'|} |g|_{p+1}(\Gamma_j) \leqslant$$
$$C_p F(-\mathrm{Re}\,\mu - \beta_0)\lambda_\gamma e^{(-\mathrm{Re}\,\mu - \beta_0)d_\gamma} |g|_{p+1}(\Gamma_j),$$

where $\beta_0 = -\log \alpha / d_{\max}$.

Set

$$s_{P\mathrm{i}} = \sum_{\mathrm{j} \in P\mathrm{i}} (-1)^{|\mathrm{j}|-1}(s_{\mathrm{j}} \mp s_{\mathrm{j}}).$$

Concerning $s_{P\mathrm{i}}|_\Gamma - m$, note that s_j and $\tilde{s}_\mathrm{j}$ are so chosen that the coefficient of k_0 of its expansion in k^{-1} vanishes. Then summing up the above estimates we have

PROPOSITION 5.9. — *Let* $\mathbf{i} \in \mathscr{I}_j \cup \mathscr{I}'_j$, *and let* γ *be a periodic ray in* Ω *such that* $\mathbf{i} \in \mathscr{I}(\gamma)$. *Set*

$$D_\gamma = \{\mu; \mathrm{Re}\ \mu > (\log \lambda_\gamma)/d_\gamma\}.$$

Then a series $s_{P\mathrm{i}}$ *converges absolutely in* D_γ *and has an estimate*

$$(5.21)\quad |s_{P\mathrm{i}}|_p(\Omega(R)) \leqslant C_{p,R} M_{2N+p} k^p F(-\mathrm{Re}\ \mu) \times \lambda_\gamma e^{-\mathrm{Re}\ \mu d_\gamma}(1 - \lambda_\gamma e^{-\mathrm{Re}\ \mu d_\gamma})^{-1},$$

where

$$M_l = (|\nabla\psi|_{l+2}(\Gamma_j) + 1)|g|_l(\Gamma_j),$$

and $C_{p,R}$ *is a constant independent of* ψ, g *and* $\mathbf{i}$. *Moreover* $s_{P\mathrm{i}}$ *satisfies*

(i) *Condition S in* (Ω, D_γ),

(ii) $s_{P\mathrm{i}}|_\Gamma - m = \sum_{\mathrm{j} \in P\mathrm{i}} \{\sum_{h=1}^{N} k^{-h} m_{\mathrm{j},h}(x,\mu;k) + \tilde{m}_\mathrm{j}(x,\mu;k)\}$,

where

$$(5.22)\quad m_{\mathrm{j},h}(x,\mu;k) = e^{ik\varphi_\mathrm{j}(x)} g_{\mathrm{j},h}(x,\mu;k),$$

$$(5.23)\quad |g_{\mathrm{j},h}|_p(\Gamma_\mathrm{j}) \leqslant C_{p,h} M_{2h+p} (\lambda_\mathrm{i} e^{-\mathrm{Re}\ \mu d_\mathrm{i}})^\mathrm{j}$$

$$(5.24)\quad |\tilde{m}_\mathrm{j}|_p(\Gamma_\mathrm{j}) \leqslant c_p k^{-N+p} M_{2N+p} (\lambda_\mathrm{i} e^{-\mathrm{Re}\ \mu d_\mathrm{i}})^\mathrm{j}.$$

Now we turn to consideration of the convergence of $w^{(0)}$ of (5.2). First we remark that we have from (3.13)

$$\#\{\mathbf{j}; \mathbf{j} \in \mathscr{I}(\gamma)\} = 2|\mathbf{j}|.$$

On the other hand since $|\mathbf{i}| \leqslant d_\gamma/d_{\min}$, we have

$$(5.25)\quad \#\{\mathbf{j}; \mathbf{j} \in \mathscr{I}(\gamma)\} \leqslant C d_\gamma.$$

From Lemma 5.1 we have

$$\sum_{\mathrm{j} \in Ij} * = \sum_{\mathrm{i} \in \mathscr{I}_j} (\sum_{\mathrm{j} \in P\mathrm{i}} *) + \sum_{\mathrm{i} \in \mathscr{I}'_\mathrm{j}} \left(\sum_{\mathrm{j} \in P'\mathrm{i}} *\right) =$$

$$\sum_{\gamma} : \text{primitive periodic ray} \left\{ \sum_{\mathrm{i} \in \mathscr{I}_j \cap \mathscr{I}(\gamma)} \left(\sum_{\mathrm{j} \in P\mathrm{i}} *\right) + \sum_{\mathrm{i} \in \mathscr{I}'_\mathrm{j} \cap \mathscr{I}(\gamma)} \left(\sum_{\mathrm{j} \in P'\mathrm{i}} *\right)\right\}.$$

Thus for $\mu \in \bigcap_{\gamma} \mathscr{D}_{\gamma}$ we have from Proposition 5.9

$$\sum_{j \in I_j} |s_j + \tilde{s}_j|_p(\Omega(R)) \leqslant C_{P,R} M_{2N+p} F(-\operatorname{Re}\mu) k^p \times$$

$$\sum_{\gamma}^{\#} \{\mathbf{i}; \mathbf{i} \in \mathscr{I}(\gamma) \cap (\mathscr{I}_j \cup \mathscr{I}'_j)\} \lambda_\gamma e^{-\operatorname{Re}\mu d_\gamma} (1 - \lambda_\gamma e^{-\operatorname{Re}\mu d_\gamma})^{-1}.$$

Thus we have

PROPOSITION 5.10. — *The function* $w^{(0)}$ *defined by* (5.2) *satisfies Condition S in* (Ω, D), *and it is represented on the boundary* Γ *as*

$$w^{(0)} - m = \sum_{j \in I_j} \left\{ \sum_{h=1}^{N} k^{-h} m_{j,h} + k^{-N} \tilde{m}_j \right\},$$

where $m_{j,h}$ *and* $\tilde{m}_j$ *have the properties* (5.22)-(5.24).

6. Proof of Proposition 2.2.

In the previous section we have constructed a first approximation of the solution of (5.1). To arrive Proposition 2.2 it suffices to repeat the preceding argument.

Since $m_{j,h}$ is a boundary data satisfying Condition A we can apply the construction procedure in Proposition 5.10 to each $m_{j,h}$. Denote the corresponding function by $w^{(0)}_{j,h}$. From Proposition 5.10 we have for each $\mathbf{j} = (j_0, j_1, \cdots, j_s) \in I_j$ and $h \geqslant 1$

$$w^{(0)}_{j,h} - m_{j,h} = \sum_{j' \in I_{j_n}} \left\{ \sum_{h'=1}^{N} h^{-h'} m_{j,h,j',h'} + k^{-N} \tilde{m}_{j,j'} \right\}$$

where $m_{j,h,j',h'}$ and $\tilde{m}_{j,h,j'}$ satisfy

$$m_{j,h,j',h'}(x, \mu; k) = e^{ik\varphi_{j,j'}(x)} g_{j,h,j',h'}(x, \mu; k),$$

$$|g_{j,h,j',h'}|_p(\Gamma_{j'}) \leqslant C_p M_{2N+p} (\lambda_i e^{-\operatorname{Re}\mu d_i})^j (\lambda_{i'} e^{-\operatorname{Re}\mu d_{i'}})^{j'},$$

$$|\tilde{m}_{j,h,j'}|_p(\Gamma_{j'}) \leqslant C_p k^{-N+p} M_{2N+2h+p} (\lambda_i e^{-\operatorname{Re}\mu d_i})^j (\lambda_{i'} e^{-\operatorname{Re}\mu d_{i'}})^{j'},$$

for $\mathbf{j}' \in P\mathbf{i}'$.

By setting

$$w^{(1)} = -\sum_{h=1}^{N} k^{-h} \sum_{j \in I} w^{(0)}_{j,h'},$$

we have that $w^{(0)} + w^{(1)}$ satisfies Condition S in (Ω,D) and

$$|w^{(0)}+w^{(1)}-m|_p(\Omega(R)) \leqslant C_{p,R}M_{4N+p}k^{-2+p}(F(-\mathrm{Re}\,\mu))^3.$$

Repeating this procedure we get

$$w = w^{(0)} + w^{(1)} + \cdots + w^{(N)}$$

which satisfies Condition S in (Ω,D) and

$$|w-m|_p(\Gamma) \leqslant C_pM_{(2N)^2+p}k^{-N+p}(F(-\mathrm{Re}\,\mu))^{N+2}.$$

Thus we proved Proposition 2.2.

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Mitsuru IKAWA,
Departement of Mathematics
Osaka University
Toyonaka 560 (Japan).