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ON RIESZ PRODUCT MEASURES ; MUTUAL ABSOLUTE CONTINUITY AND SINGULARITY

by S. J. KILMER and S. SAEKI

In his 1973 paper [9], J. Peyrière gives a very simple criterion for two Riesz product measures to be mutually singular. Although the result is stated only for the circle group, both it and its proof extend *mutatis mutandis* to the case of the general compact abelian group. He also gives several sufficient conditions for one Riesz product measure to be absolutely continuous with respect to another. However, all of these conditions involve some strong lacunarity constraints on the underlying dissociate set. A little later, in 1974 G. Brown and W. Moran [1] obtained, among other things, a sufficient condition for absolute continuity which is independent of the underlying dissociate set. G. Ritter's 1978 paper [11] contains an improved version of their condition.

The present paper gives further criteria for determining mutual absolute continuity and singularity of Riesz product measures. The first section consists of some basic definitions and results about weak convergence of measures. For the reader's convenience, we have also included a proof of Peyrière's theorem on mutual singularity. In the second section we give some sufficient conditions for one Riesz product measure to be absolutely continuous with respect to another. One of our results (Corollary 1.2) contains the Brown-Moran-Ritter theorem mentioned above as a special case.

In the final section we shall introduce two families (μ_ω) and (ν_ω) of « random » Riesz product measures, where ω runs through the countably infinite product of the circle group. We shall establish the dichotomy

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that either ν_ω is absolutely continuous with respect to $\mu_\omega (\nu_\omega \ll \mu_\omega)$ almost surely, or μ_ω and ν_ω are mutually singular $(\mu_\omega \perp \nu_\omega)$ almost surely. Our Theorem 2 gives an explicit criterion for determining which one of these alternatives occurs. Finally we shall give two applications of this probabilistic result to Riesz product measures of a certain type on the circle group.

We are greatly influenced by the well-known paper [6] of S. Kakutani on infinite product measures and Peyrière's pioneering paper [9] on the subject. We shall use their ideas freely without any further explicit references.

1. Basic definitions and results.

Throughout the paper, let G be a nondiscrete LCA group with dual Γ, and let $M(G)$ be the convolution algebra of all regular complex Borel measures on G (cf. C. C. Graham and O. C. McGehee [2], E. Hewitt and K. A. Ross [3], W. Rudin [13], and J. L. Taylor [14]). As usual, we define the Fourier transform of $\mu \in M(G)$ by

$$\hat{\mu}(\gamma) = \int \bar{\gamma}\, d\mu \qquad \text{for all} \qquad \gamma \in \Gamma .$$

Let $C(G)$ denote the space of all bounded continuous complex-valued functions on G.

In order to introduce Riesz product measures which are more general than the usual Riesz product measures, we need the following result which is (essentially) well-known in the field of probability theory ; see, e.g., Chap. IV of M. Loève [7].

PROPOSITION A. — *Let (μ_α) be a norm-bounded net in $M(G)$, and let ϕ be a bounded Haar measurable function on Γ such that*

(i) *$\hat{\mu}_\alpha \to \phi$ weak-$*$ in $L^\infty(\Gamma)$.*

Then (μ_α) converges to some $\mu \in M(G)$ in the weak-$*$ topology of $M(G)$, and $\hat{\mu} = \phi$ locally (Haar) a.e. on Γ. If, in addition,

(ii) $\lim \|\mu_\alpha\| = \|\mu\|$,

then $\lim \mu_\alpha = \mu$ *weakly, i.e.,*

(iii) $\lim \displaystyle\int f\, d\mu_\alpha = \int f\, d\mu$ *for all $f \in C(G)$.*

Proof. – For $\psi \in L^1(\Gamma)$, the « inverse » Fourier transform of ψ is defined by

$$\psi^{\vee}(x) = \int \psi(\gamma)\gamma(x)\, d\gamma \qquad \text{for all} \qquad x \in G,$$

where $d\gamma$ denotes Haar measure on Γ. Thus $[L^1(\Gamma)]^{\vee}$ is a uniformly dense subalgebra of $C_0(G)$.

Now $\psi \in L^1(\Gamma)$ implies

$$\text{(1)} \quad \lim \int \psi^{\vee}\, d\mu_\alpha = \lim \int \hat{\mu}_\alpha(\gamma^{-1})\psi(\gamma)\, d\gamma = \int \phi(\gamma^{-1})\psi(\gamma)\, d\gamma$$

by Fubini's theorem and (i). Moreover, the net (μ_α) is norm-bounded by the hypotheses, and $[L^1(\Gamma)]^{\vee}$ is uniformly dense in $C_0(G)$. Therefore (1) ensures that (μ_α) converges weak-$*$ to some $\mu \in M(G)$ and that $\hat{\mu} = \phi$ locally a.e. on Γ.

Now suppose that (ii) obtains. Then the net (μ_α) is uniformly tight in the following sense : given $\varepsilon > 0$, there exists a compact subset K of G such that

$$\text{(2)} \qquad \max\{|\mu_\alpha|(G\backslash K), |\mu|(G\backslash K)\} < \varepsilon$$

eventually. In fact, choose $g \in C_c(G)$ so that

$$\text{(3)} \qquad |g| \leqslant 1 \text{ on } G \qquad \text{and} \qquad \int g\, d\mu > \|\mu\| - \varepsilon .$$

Since $\mu_\alpha \to \mu$ weak-$*$, it follows from (ii) and (3) that

$$\text{(4)} \qquad \left|\int g\, d\mu_\alpha\right| > \|\mu_\alpha\| - \varepsilon$$

eventually. Define K to be the closed support of g. Then

$$|\mu|(K) \geqslant \left|\int g\, d\mu\right| > \|\mu\| - \varepsilon$$

by (3), and so $|\mu|(G\backslash K) < \varepsilon$. Similarly (4) assures that $|\mu_\alpha|(G\backslash K) < \varepsilon$ eventually. These two inequalities establish (2).

Finally, let $f \in C(G)$. Given $\varepsilon > 0$, select a compact subset K of G as in (2). Also select $h \in C_c(G)$ such that $0 \leqslant h \leqslant 1$ on G and $h = 1$

on K. By (2), we then have

$$\left|\int f\,d(\mu_\alpha-\mu)\right| \leqslant \left|\int (1-h)f\,d(\mu_\alpha-\mu)\right| + \left|\int hf\,d(\mu_\alpha-\mu)\right| \leqslant \|f\|_u.2\varepsilon + \left|\int hf\,d(\mu_\alpha-\mu)\right|$$

eventually. Since $hf \in C_c(G)$ and $\mu_\alpha \to \mu$ weak-$*$, it follows that

$$\left|\int f\,d(\mu_\alpha-\mu)\right| < (2\|f\|_u+1)\varepsilon$$

eventually. Since $\varepsilon > 0$ was arbitrary, this establishes (iii).

Remark (I). – The above proposition may be used to remove an awkward condition in Lemmas 1 and 2 of L. Pigno and S. Saeki [10]. That is, the set $(\Lambda+S)\cup T$ there may be replaced by Γ. This subtle point is overlooked in the proof of Theorem A.7.1 of Graham and McGehee [2].

Now let Θ be a subset of Γ. We denote by $W(\Theta)$ the set of all elements γ of Γ of the form

$$\text{(W)} \qquad \gamma = \theta_1^{\varepsilon_1}\theta_2^{\varepsilon_2}\ldots\theta_n^{\varepsilon_n},$$

where the θ_k are distinct elements of Θ, $\varepsilon_k = 1$ or -1 if $\theta_k^2 \neq 1$, and $\varepsilon_k = 1$ if $\theta_k^2 = 1$. (For $n=0$, we interpret such a product to be 1.) Following E. Hewitt and H. S. Zuckerman [5], we call Θ a *dissociate set* if each element of $W(\Theta)$ has a unique representation of the form (W) except for the order of the factors. Two elements of $W(\Theta)$ will be said to be *nonoverlapping* if their representations of the form (W) have no θ_k in common.

Now let Θ be a dissociate subset of Γ, and let a be a complex-valued function on Θ. For each finite subset Φ of Θ, define

$$P(\Phi,a) = \Pi\,\{1+\mathrm{Re}\,[a(\theta)\theta] : \theta\in\Phi\},$$

so this is a trigonometric polynomial on G. Since Θ is dissociate, it is easy to show that there exists a unique function $\hat\rho = \hat\rho(\Theta,a)$ on Γ such that

$$\text{(R.1)} \qquad \hat\rho = 0 \quad \text{on} \quad \Gamma\setminus W(\Theta);$$

and

$$\text{(R.2)} \qquad P(\Phi,a) = \sum\{\hat{\rho}(\gamma)\gamma : \gamma \in W(\Phi)\}$$

for each finite subset Φ of Θ. If G is compact and a is bounded by 1, then $\hat{\rho}$ is the Fourier transform of a probability measure on G, called a Riesz product measure [5]. For accounts and applications of these measures, we refer the reader to Graham and McGehee [2], Hewitt and Ross [4], and J. M. López and K. A. Ross [8].

PROPOSITION B. – *Let (Θ,a,σ) be a triple consisting of a dissociate subset Θ of Γ, a complex function a on Θ bounded by 1, and a probability measure $\sigma \in M(G)$. Let $\hat{\rho} = \hat{\rho}(\Theta,a)$ be as above, and let $S(\hat{\sigma}) = \{\gamma \in \Gamma : \hat{\sigma}(\gamma) \neq 0\}$. Suppose that*

(*a*) *$\gamma^{-1}\gamma' \notin S(\hat{\sigma})$ whenever γ, $\gamma' \in W(\Theta)$ are distinct, and that*

(*b*) *the family $\{\gamma S(\hat{\sigma}) : \gamma \in W(\Theta)\}$ of subsets of Γ is locally finite. Then there exists a unique probability measure $\mu \in M(G)$ such that*

(*c*) $\hat{\mu}(\chi) = \sum\{\hat{\rho}(\gamma)\hat{\sigma}(\gamma^{-1}\chi) : \gamma \in W(\Theta)\}$, $\forall \chi \in \Gamma$.
Moreover, $\hat{\mu} = \hat{\rho}$ on $W(\Theta)$ and

(*d*) $\lim \displaystyle\int f.P(\Phi,a)\, d\sigma = \int f\, d\mu$, $\forall f \in C(G)$,

where the finite subsets of Θ are directed by set-inclusion.

Proof. – Let $\chi \in \Gamma$ be given. If $\gamma \in \Gamma$ and $\hat{\rho}(\gamma)\hat{\sigma}(\gamma^{-1}\chi) \neq 0$, then $\gamma \in W(\Theta)$ and $\chi \in \gamma S(\hat{\sigma})$ by the definitions of $\hat{\rho}$ and $S(\hat{\sigma})$. It follows from (*b*) and (*a*) that there are at most finitely many such γ. Therefore the right-hand side of (*c*) is (essentially) a finite sum. Let $\phi(\chi)$ denote this finite sum, so that $\phi = \hat{\rho}$ on $W(\Theta)$ by (*a*).

Now let Φ be any finite subset of Θ. Since a is bounded by 1, the trigonometric polynomial $P(\Phi,a)$ is nonnegative. Moreover, $1 \in W(\Phi)$, $\hat{\rho}(1) = 1$ and σ is a probability measure. It follows from the definition of $P(\Phi,a)$ and (*a*) that

$$\text{(1)} \qquad \|P(\Phi,a)\sigma\| = \int P(\Phi,a)\, d\sigma = 1.$$

Next suppose that V is an open subset of Γ such that V meets only finitely many $\gamma S(\hat{\sigma})$ with $\gamma \in W(\Theta)$. Then Θ contains a finite subset Φ_0 such that

$$\text{(2)} \qquad \gamma \in W(\Theta) \quad \text{and} \quad V \cap [\gamma S(\hat{\sigma})] \neq \varnothing \Rightarrow \gamma \in W(\Phi_0).$$

From (2) and the definition of ϕ, we infer that

$$(3) \qquad \phi(\chi) = \sum\{\hat{\rho}(\gamma)\hat{\sigma}(\gamma^{-1}\chi) : \gamma \in W(\Phi_0)\}, \qquad \forall \chi \in V.$$

If Φ is a finite subset of Θ containing Φ_0, then the definition of $\hat{\rho}$ and (2) ensure that

$$[P(\Phi,a)\sigma]^{\wedge}(\chi) = \sum\{\hat{\rho}(\gamma)\hat{\sigma}(\gamma^{-1}\chi) : \gamma \in W(\Phi)\} = \sum\{\hat{\rho}(\gamma)\hat{\sigma}(\gamma^{-1}\chi) : \gamma \in W(\Phi_0)\}$$

for all $\chi \in V$. Accordingly we have proved that

$$(4) \qquad [P(\Phi,a)\sigma]^{\wedge} = \phi \text{ on } V$$

whenever Φ is a finite set such that $\Phi_0 \subset \Phi \subset \Theta$.

By (*b*), each element of Γ has a neighborhood V with the above property. Therefore $\phi \in C(\Gamma)$ by (1) and (4), and the functions $[P(\Phi,a)\sigma]^{\wedge}$ converge weak-* to $\phi \in L^{\infty}(\Gamma)$ again by (1) and (4). It follows from Proposition A that the probability measures $P(\Phi,a)\sigma$ converge weak-* to some $\mu \in M(G)$ and that $\hat{\mu} = \phi$ on Γ (notice that both $\hat{\mu}$ and ϕ are continuous). Plainly μ is a nonnegative measure. Moreover, $\hat{\mu} = \phi = \hat{\rho}$ on $W(\Theta)$ by (3) and (*a*). In particular, $\hat{\mu}(1) = 1$, and so μ is a probability measure. Therefore (*d*) follows from Proposition A and (1). The uniqueness of μ is obvious and the proof is complete.

DEFINITIONS. — Let (Θ,a,σ) and μ be as in Proposition B. Then we call (Θ,a,σ) a *dissociate triple* and μ the *Riesz product measure* based on (Θ,a,σ). Also we write $\mu = R(\Theta,a,\sigma)$. If G is a compact group and σ is its Haar measure of norm 1, we shall merely write $\mu = R(\Theta,a)$.

Remark (II). — Our definition of a Riesz product measure is slightly more general than the corresponding definition in [2 ; p. 219]. Also note the resemblance between Proposition B and Lemma 1 of [10].

The following result is essentially due to Peyrière [9].

THE PEYRIÈRE THEOREM. — *Let $\mu = R(\Theta,a,\sigma)$ and $\nu = R(\Psi,b,\tau)$ be two Riesz product measures on G. Suppose that $a(\theta)$ and $b(\theta)$ are real for all $\theta \in \Theta \cap \Psi$ with $\theta^2 = 1$, and that*

$$\text{(i)} \qquad \Sigma\{|a(\theta)-b(\theta)|^2 : \theta \in \Theta\cap\Psi\} = \infty .$$

Then μ and ν are mutually singular.

Proof. – We modify Peyrière's proof as follows.

Given $\varepsilon > 0$, there exist two trigonometric polynomials f, g on G such that $g - f = 1$ identically and such that

$$\text{(1)} \qquad \max\left\{\int |f|^2\, d\mu, \int |g|^2\, d\nu\right\} < \varepsilon.$$

To see this, we use (i) to obtain finitely many distinct elements $\theta_1, \ldots, \theta_n \in \Theta \cap \Psi$ such that $\sum_1^n |a(\theta_k) - b(\theta_k)|^2 > 4/\varepsilon$. Define $\alpha_k = a(\theta_k)/2$ if $\theta_k^2 \neq 1$ and $\alpha_k = \mathrm{Re}[a(\theta_k)] = a(\theta_k)$ if $\theta_k^2 = 1$. Similarly define β_k for $k = 1, 2, \ldots, n$. Then $\sum_1^n |\alpha_k - \beta_k|^2 > 1/\varepsilon$ and so there exist $c_1, \ldots, c_n \in \mathbf{C}$ such that

$$\text{(2)} \qquad \sum_1^n |c_k|^2 < \varepsilon \quad \text{and} \quad \sum_1^n c_k(\alpha_k - \beta_k) = 1.$$

Now define

$$f = \sum_1^n c_k(\bar{\theta}_k - \alpha_k) \quad \text{and} \quad g = \sum_1^n c_k(\bar{\theta}_k - \beta_k).$$

Then $g - f = 1$ by (2). Moreover, the definition of μ ensures that

$$\int |f|^2\, d\mu = \sum_j \sum_k c_j \bar{c}_k \int (\bar{\theta}_j \theta_k - \bar{\theta}_j \bar{\alpha}_k - \alpha_j \theta_k + \alpha_j \bar{\alpha}_k)\, d\mu =$$

$$\sum_{j \neq k} c_j \bar{c}_k (\alpha_j \bar{\alpha}_k - \alpha_j \bar{\alpha}_k - \alpha_j \bar{\alpha}_k + \alpha_j \bar{\alpha}_k) + \sum_1^n |c_k|^2 (1 - |\alpha_k|^2 - |\alpha_k|^2 + |\alpha_k|^2) =$$

$$0 + \sum_1^n |c_k|^2 (1 - |\alpha_k|^2) < \varepsilon \text{ by (2)}.$$

Similarly

$$\int |g|^2\, d\nu \leqslant \sum_1^n |c_k|^2 < \varepsilon,$$

which confirms (1).

It follows that there exist two sequences (f_n) and (g_n) of trigonometric polynomials on G such that

$$\text{(3)} \qquad g_n - f_n = 1 \quad \text{on} \quad G \quad \text{for all} \quad n,$$

and

$$\text{(4)} \qquad \sum_1^\infty \left[\int |f_n|^2\, d\mu + \int |g_n|^2\, d\nu\right] < \infty.$$

By (4), $f_n \to 0$ $\mu - a.e.$ and $g_n \to 0$ $\nu - a.e.$ Therefore μ and ν are concentrated on the sets $\{\lim_n f_n = 0\}$ and $\{\lim_n g_n = 0\}$, respectively. By (3), these sets are disjoint. Consequently μ and ν are mutually singular, as desired.

Remark (III). – Brown and Moran [1] give a « stronger » version of Peyrière's singularity criterion, namely, the version with $\Theta \cap \Psi$ replaced by $\Theta \cup \Psi$. However, this is false in general; their proof contains two errors. The following counterexample was sent to Moran in January 1976 by the second author of the present paper.

Suppose that G is compact and that $\Theta = \{\theta_n\}_0^\infty$ is any dissociate set in Γ such that θ_0 has order 2. Then

$$\Theta' = \{\theta_0, \theta_0\theta_1, \theta_0\theta_2, \dots\}$$

is a dissociate set, as is easily seen. Choose and fix any sequence (a_n) in $(-1,1)$ so that $\sum_0^\infty a_n^2 = \infty$. Let $\mu = R(\Theta, a)$ and $\mu' = R(\Theta', a')$, where $a(\theta_0) = a'(\theta_0) = a_0$ and $a(\theta_n) = a'(\theta_0\theta_n) = a_n$ for all $n \geqslant 1$. We claim that μ *and* μ' *are neither mutually singular nor absolutely continuous with each other.* In particular, Brown and Moran's claim is false.

In order to confirm our claim, first define

$$P_n = (1 + a_0\theta_0)\prod_{k=1}^{n}\{1 + \mathrm{Re}[a_k\theta_k]\},$$

$$P'_n = (1 + a_0\theta_0)\prod_{k=1}^{n}\{1 + \mathrm{Re}[a_k\theta_0\theta_k]\},$$

and

$$Q_n = (1 + a_0\theta_0)\prod_{k=1}^{n}\{1 - \mathrm{Re}[a_k\theta_k]\}$$

for all $n = 1, 2, \dots$ Thus $\mu = \lim P_n\sigma$ and $\mu' = \lim P'_n\sigma$, where (and below) σ is the normalized Haar measure of G and the limits are taken with respect to the weak-$*$ topology of $M(G)$. Let $\nu = \lim Q_n\sigma$, which is another Riesz product measure. Since $\sum_1^\infty |a_n - (-a_n)|^2 = \infty$ by hypothesis, it follows from Peyrière's theorem that μ and ν are mutually singular.

Now notice that $\theta_0 = \pm 1$ on G. Since μ is a probability measure and $\mu(\{\theta_0 = 1\}) - \mu(\{\theta_0 = -1\}) = \hat{\mu}(\theta_0) = a_0 \in (-1,1)$, it follows that $\mu(\{\theta_0 = k\}) \neq 0$ for $k = 1$ and -1. Similarly $\mu'(\{\theta_0 = -1\})$ is nonzero.

In addition we have

$$(1+\theta_0)\mu' = \lim (1+\theta_0)P'_n\sigma = \lim (1+\theta_0)P_n\sigma = (1+\theta_0)\mu .$$

Hence $\mu' = \mu \neq 0$ on the closen subgroup $\{\theta_0=1\}$ of G. Similarly

$$(1-\theta_0)\mu' = \lim (1-\theta_0)P'_n\sigma = \lim (1-\theta_0)Q_n\sigma = (1-\theta_0)\nu .$$

Since μ and ν are mutually singular, it follows that μ and μ' are mutually singular (and both nonzero) on the coset $\{\theta_0=-1\}$. The disparate behavior on the two cosets confirms our claim.

Notice also that if we choose $a_0 = 1$ in the above example, then $\mu = \mu'$ throughout G although $\sum\{|a(\theta)|^2 : \theta \in \Theta\backslash\Theta'\} = \infty$.

2. Criteria for absolute continuity.

Throughout this section, we shall choose and fix two dissociate triples (Θ,a,σ) and (Ψ,b,σ) such that $|a(\theta)| < 1$ for all $\theta\in\Theta$. Let

$$\mu = R(\Theta,a,\sigma) \qquad \text{and} \qquad \nu = R(\Psi,b,\sigma)$$

be the corresponding Riesz product measures on G. Define $a = 0$ off Θ and $b = 0$ off Ψ. For $p > 0$, we write $\nu\in L^p(\mu)$ to mean that $\nu \ll \mu$ and $d\nu/d\mu \in L^p(\mu)$.

THEOREM 1. — *Let $1 \leqslant p < \infty$ be given. Suppose that either* (i) *$p \leqslant 2$ and the series*

$$(*)\qquad \sum\{|a(\gamma)-b(\gamma)|^p/(1-|a(\gamma)|)^{p-1} : \gamma\in\Theta\cup\Psi\}$$

converges, or (ii) *$p > 2$ and the series in* $(*)$ *converges for both p and 2 (in place of p). Then $\nu\in L^r(\mu)$ for $r = p^2/(2p-1)$. If, in addition, $\Theta = \Psi$, then $\nu\in L^p(\mu)$.*

To prove this, we need three lemmas. The first two of them are well-known and have no direct relationship with Riesz products. The third lemma is essentially a list of notation that we need later.

LEMMA 1.1. — *For each real positive number p, there exists a finite constant c_p such that*

(i) $(1+t)^p \leqslant 1 + pt + c_p(|t|^p+t^2),\ \forall t \geqslant -1$.

In case $0 < p \leqslant 2$, such a c_p can be chosen to satisfy

(ii) $(1+t)^p \leqslant 1 + pt + c_p|t|^p,\ \forall t \geqslant -1$.

Proof. – As is well-known, the binomial expansion

$$(1+t)^p = 1 + pt + t^2 \sum_{k=2}^{\infty} \binom{p}{k} t^{k-2}$$

converges absolutely for $t \in [-1,1]$. Define $c_p = 2^p + \sum_2^{\infty} |\binom{p}{k}|$. Then (i) is obvious since $(1+t)^p \leqslant 2^p t^p$ if $t \geqslant 1$. To check (ii), it will suffice to note that $0 < p \leqslant 2$ and $|t| \leqslant 1$ imply $t^2 \leqslant |t|^p$.

LEMMA 1.2. – *Let μ' and ν' be two measures in $M^+(G)$ and let $1 < p \leqslant \infty$. If there exists a norm-bounded net (f_α) in $L^p(\mu')$ such that $f_\alpha \mu' \to \nu'$ in the weak-$*$ topology of $M(G)$, then $\nu' \in L^p(\mu')$.*

The proof is left to the reader.

LEMMA 1.3. – *Suppose that $\Theta = \Psi$ and that $\{\theta \in \Theta : a(\theta) \neq b(\theta)\}$ is countably infinite. Let $\theta_1, \theta_2, \ldots$ be any enumeration of the distinct elements of this countable set, let $\Theta_n = \Theta \setminus \{\theta_1, \ldots, \theta_n\}$ and $\mu(n) = R(\Theta_n, a|\Theta_n, \sigma)$ for $n \geqslant 1$. Writing $a_k = a(\theta_k)$ and $b_k = b(\theta_k)$, define*

$$A_k = 1 + \mathrm{Re}[a_k\theta_k], \qquad P_n = \prod_1^n A_k;$$

$$B_k = 1 + \mathrm{Re}[b_k\theta_k], \qquad Q_n = \prod_1^n B_k;$$

and

$$a(n) = \begin{cases} b & \text{on} \quad \Theta\setminus\Theta_n, \\ a & \text{on} \quad \Theta_n \end{cases}$$

for k, $n \geqslant 1$. Then we have :

(i) $\mu = P_n\mu(n)$ *and*

$$(Q_n/P_n)\mu = Q_n\mu(n) = R(\Theta, a(n), \sigma);$$

(ii) *If c_k, $d_k \in \mathbf{C}$ for $k = 1, 2, \ldots, n$, then*

$$\int \prod_1^n (c_k + \mathrm{Re}[d_k\theta_k])\, d\mu(n) = \prod_1^n c_k.$$

(iii) $(Q_n/P_n)\mu \to \nu$ *weakly as $n \to \infty$.*

Proof. — We direct the finite subsets of Θ_n by set-inclusion. Thus $\lim P(\Theta,a)\sigma = \mu(n)$ weakly by Proposition B. Multiplying both sides by P_n, we obtain $\mu = P_n\mu(n)$ again by Proposition B. Notice that $|a| < 1$ on Θ by one of our basic hypotheses. So $P_n > 0$ on G and we may therefore take the quotient Q_n/P_n. The remainder of (i) is obvious.

For (ii), it will suffice to note that $\mu(n) = R(\Theta,a'(n),\sigma)$, where $a'(n) = 0$ on $\Theta\backslash\Theta_n$ and $a'(n) = a$ on Θ_n.

For (iii), we have to repeat some arguments used in the proof of Proposition B. Let V be an open subset of Γ as there. So there exists a finite subset Φ_0 of Θ such that

$$(1) \qquad \hat{v}(\chi) = \sum\{\hat{v}(\gamma)\hat{\sigma}(\gamma^{-1}\chi) : \gamma \in W(\Phi_0)\}, \qquad \forall\chi \in V.$$

Plainly this set Φ_0 can be chosen independently of v, so (1) holds with v replaced by the Riesz product measures $(Q_n/P_n)\mu = R(\Theta,a(n),\sigma)$. Thus

$$(2) \quad [(Q_n/P_n)\mu]^\wedge(\chi) = \sum\{[(Q_n/P_n)\mu]^\wedge(\gamma)\hat{\sigma}(\gamma^{-1}\chi) : \gamma \in W(\Phi_0)\}, \quad \forall\chi \in V$$

for all $n \geqslant 1$. Moreover, it is easy to show that

$$(3) \qquad \lim_n [(Q_n/P_n)\mu]^\wedge(\gamma) = \hat{v}(\gamma), \qquad \forall\gamma \in W(\Theta).$$

Since $W(\Phi_0)$ is a finite set, it follows from (1)-(3) that $\lim_n[(Q_n/P_n)\mu]^\wedge = \hat{v}$ uniformly on V. Therefore, upon repeating arguments similar to those used in the last paragraph of the proof of Proposition B, we conclude that $(Q_n/P_n)\mu \to v$ weakly as $n \to \infty$, which establishes (iii).

Proof of theorem 1. — First suppose that $\Theta = \Psi$. Since the series in $(*)$ converges by hypothesis, $\{\theta \in \Theta : a(\theta) \neq b(\theta)\}$ is at most countable. If this set is finite, Lemma 1.3 (i) and its proof show that $v = (Q/P)\mu$ for some nonnegative trigonometric polynomials P, Q on G with $\inf P > 0$. So there is no loss of generality in assuming that the set in question is countably infinite. In what follows, we shall preserve all the notations in Lemma 1.3.

Now suppose that $p = 1$, so $\sum_1^\infty |a_k - b_k| < \infty$ by (i). For each natural number n, we have

$$|P_n - Q_n| = |\prod_1^n A_k - \prod_1^n B_k| \leqslant |A_1 - B_1|(\prod_2^n A_k) + B_1|A_2 - B_2|\left(\prod_3^n A_k\right) + \cdots + \left(\prod_1^{n-1} B_k\right)|A_n - B_n| \leqslant |a_1 - b_1|g_1 + \cdots + |a_n - b_n|g_n,$$

where

$$g_j = g_{j,n} = (B_1 \ldots B_{j-1})(A_{j+1} \ldots A_n)$$

for all $j = 1, 2, \ldots, n$. It follows from Lemma 1.3 (i) and (ii) that

$$(1) \quad ||\mu - (Q_n/P_n)\mu|| = ||(P_n - Q_n)\mu(n)|| \leqslant \sum_1^n |a_j - b_j| . ||g_i\mu(n)|| = \sum_1^n |a_j - b_j| \leqslant \sum_1^\infty |a_j - b_j| .$$

Since $\mu - (Q_n/P_n)\mu \to \mu - \nu$ weakly by Lemma 1.3 (iii), we infer from (1) that $||\mu - \nu|| \leqslant \sum_1^\infty |a_j - b_j|$. Moreover, $(Q_n/P_n)\mu$ is the Riesz product measure $R(\Theta, a(n), \sigma)$ by Lemma 1.3 (i). So the last inequality applied to $(Q_n/P_n)\mu$ in place of μ yields $||(Q_n/P_n)\mu - \nu|| \leqslant \sum_{n+1}^\infty |a_j - b_j|$ for all n. Since $\sum_1^\infty |a_j - b_j| < \infty$, we conclude that $(Q_n/P_n)\mu \to \nu$ in the norm of $M(G)$, and so $\nu \ll \mu$.

Next suppose that $1 < p \leqslant 2$. Let $d_k = b_k - a_k$ and $D_k = B_k - A_k = \mathrm{Re}[d_k\theta_k]$ for all k. Since $A_k \geqslant 1 - |a_k| > 0$ while $B_k \geqslant 0$ on G, it follows from Lemma 1.3 and Lemma 1.1 (ii) that

$$(2) \quad |Q_n/P_n|^p P_n = \prod_{k=1}^n A_k(1 + D_k/A_k)^p \leqslant \prod_{k=1}^n A_k(1 + pD_k/A_k + c_p|D_k|^p/A_k^p) = \prod_{k=1}^n (A_k + pD_k + c_p|D_k|^p/A_k^{p-1}) \leqslant \prod_{k=1}^n (1 + \mathrm{Re}[a'_k\theta_k] + c_p|d_k|^p/[1 - |a_k|]^{p-1}),$$

where $a'_k = a_k + pd_k$ for all k. Therefore Lemma 1.3 (i) and (ii) ensure that

$$(3) \quad \int |Q_n/P_n|^p d\mu = \int |Q_n/P_n|^p P_n d\mu(n) \leqslant \prod_{k=1}^n \{1 + c_p|d_k|^p/[1 - |a_k|]^{p-1}\}$$

for all $n \geqslant 1$. Since the series in $(*)$ converges, (3) shows that $(Q_n/P_n)_1^\infty$ is a norm-bounded sequence in $L^p(\mu)$. Moreover, $(Q_n/P_n)\mu \to \nu$ weakly (hence weak-$*$) by Lemma 1.3 (iii). As $p > 1$, it follows from Lemma 1.2 that $\nu \in L^p(\mu)$.

In case $p > 2$, we use Lemma 1.1 (i) in place of Lemma 1.1 (ii). Thus a calculation similar to the one in (2) shows that $|Q_n/P_n|^p P_n$ is less than or equal to

$$\prod_1^n \left(1 + \mathrm{Re}\,[a_k'\theta_k] + \frac{c_p|d_k|^p}{(1-|a_k|)^{p-1}} + \frac{c_p|d_k|^2}{1-|a_k|}\right); \tag{4}$$

hence

$$\int |Q_n/P_n|^p d\mu \leqslant \prod_1^n \left(1 + \frac{c_p|d_k|^p}{(1-|a_k|)^{p-1}} + \frac{c_p|d_k|^2}{1-|a_k|}\right) \tag{5}$$

for all n, again by Lemma 1.3. Via arguments like those at the end of the preceding paragraph, we conclude that $\nu \in L^p(\mu)$.

In case $\Theta \neq \Psi$, define $a' = a$ on $\Theta \cap \Psi$ and $a' = 0$ on $\Theta \setminus \Psi$. Then the hypotheses of the present theorem are fulfilled with (Ψ, b) replaced by (Θ, a'). Letting $\mu' = R(\Theta, a', \sigma)$, we therefore have $\mu' \in L^p(\mu)$ by the result for the case $\Theta = \Psi$. Similarly we have $\nu \in L^p(\mu')$ by the same result. (Notice that $\mu' = R(\Psi, a', \sigma)$, where $a' = 0$ on $\Psi \setminus \Theta$.) Finally pick $f \in L_+^p(\mu')$ and $g \in L_+^p(\mu)$ so that $\nu = fu'$ and $\mu' = g\mu$. Then $\nu = fg\mu$ and $fg \in L^r(\mu)$ with $r = p^2/(2p-1)$, as is easily seen from Hölder's inequality. This completes the proof.

Corollary 1.1. — *Suppose that $|a-b|/(1-|a|)$ is bounded on $\Theta \cup \Psi$ and that*

$$\sum \left\{ \frac{|a(\gamma)-b(\gamma)|^q}{(1-|a(\gamma)|)^{q-1}} : \gamma \in \Theta \cup \Psi \quad \text{and} \quad a(\gamma) \neq b(\gamma) \right\} < \infty$$

for some real number $q \leqslant 2$. Then $\nu \in L^p(\mu)$ for all real positive numbers p.

Proof. — Choose a finite positive number C so that $|a-b|/(1-|a|) \leqslant C$ on $\Theta \cup \Psi$. Since $q \leqslant 2$, $p \leqslant 2$ implies

$$|a-b|^p/(1-|a|)^{p-1} = (|a-b|/(1-|a|))^{p-q} . |a-b|^q/(1-|a|)^{q-1} \leqslant C^{p-q}|a-b|^q/(1-|a|)^{q-1}$$

whenever $a \neq b$. It follows from the hypotheses that the series in Theorem 1 converges for all real $p \geqslant 2$. Therefore $\nu \in L^r(\mu)$ for all $r = p^2/(2p-1)$ with $p \geqslant 2$. Since μ is a probability measure, this ensures that $\nu \in L^p(\mu)$ for all real positive p, as desired.

Remark (IV). – Suppose that $a \in \ell^2(\Theta)$ [and $|a| < 1$ on Θ]. Then, since $\sigma = R(\Theta,0,\sigma)$, it is a direct consequence of Corollary 1.1 that $\sigma \in L^p(\mu)$ for all real $p > 0$. By the same corollary with μ and $\nu = \sigma$ interchanged, we also have $\mu \in L^p(\sigma)$ for all such p.

The following result for $p = 2$ was first proved by Brown and Moran [1] in a slightly weaker form and was later improved by Ritter [11] to the present form (both for usual Riesz product measures).

COROLLARY 1.2. – *If*

$$\text{(i)} \qquad \sum \left\{ \frac{|a(\gamma)-b(\gamma)|^p}{(2-|a(\gamma)+b(\gamma)|)^{p-1}} : \gamma \in \Theta \cup \Psi \right\} < \infty$$

for some $1 \leqslant p \leqslant 2$, *then* $\nu \ll \mu$.

Proof. – After arguing as in the last paragraph of the proof of Theorem 1, we may and do assume that $\Theta = \Psi$.

Now let $c = (a+b)/2$. Then $|c| < 1$ on Θ and

$$\frac{|c-b|^p}{(1-|c|)^{p-1}} = \frac{1}{2} \frac{|a-b|^p}{(2-|a+b|)^{p-1}}.$$

Therefore Theorem 1 combined with (i) ensures that $\nu \ll \mu_c$, where $\mu_c = R(\Theta,c,\sigma)$.

Next let $T = \{|a-c| \leqslant (1-|c|)/2\}$, let

$$d = \begin{cases} a & \text{on } T \\ c & \text{on } \Theta \backslash T, \end{cases}$$

and define $\mu_d = R(\Theta,d,\sigma)$. Then

$$1 - |a| = 1 - |c+(a-c)| \geqslant 1 - |c| - |a-c| \geqslant (1-|c|)/2$$

on T and $|d| < 1$ on Θ. Thus

$$\sum_{\Theta} \frac{|d(\theta)-c(\theta)|^p}{(1-|d(\theta)|)^{p-1}} = \sum_{T} \frac{|a(\theta)-c(\theta)|^p}{(1-|a(\theta)|)^{p-1}} \leqslant 2^{-1} \sum_{\Theta} \frac{|a(\theta)-b(\theta)|^p}{(1-|c(\theta)|)^{p-1}} < \infty$$

by (i); hence $\mu_c \ll \mu_d$ by Theorem 1. Furthermore, letting $U = \Theta\backslash T$, we have

$$\sum_{\Theta} |a(\theta)-d(\theta)| = \sum_{U} |a(\theta)-c(\theta)| \leqslant$$

$$2^{p-1} \sum_{U} |a(\theta)-c(\theta)| \left(\frac{|a(\theta)-c(\theta)|}{(1-|c(\theta)|)}\right)^{p-1} \leqslant 2^{p-2} \sum_{\Theta} \frac{|a(\theta)-b(\theta)|^p}{(2-|a(\theta)+b(\theta)|)^{p-1}} < \infty .$$

Therefore $\mu_d \ll \mu$ by Theorem 1 with $p = 1$. Combining all these, we conclude that $\nu \ll \mu$.

Remark (V). – Suppose that all the elements of Θ have odd orders, that these orders are uniformly bounded, and that $0 \leqslant a \leqslant 1$ on Θ. Then $u = u(\Theta,a) > 0$, where

$$u = \inf\{1+\mathrm{Re}\,[a(\theta)\theta(x)] : \theta \in \Theta \quad \text{and} \quad x \in G\}.$$

In the proof of Theorem 1 (for the case $p > 1$), we have used the estimate $A_k \geqslant 1 - |a_k|$. In the present case, we may instead use the better estimate $A_k \geqslant u$. Therefore a momentary glance at the proofs of Theorem 1 and Corollary 1.1 shows that the condition

$$\sum \{|a(\gamma)-b(\gamma)|^2 : \ \gamma \in \Theta \cup \Psi\} < \infty$$

is strong enough to ensure that $\nu \in L^p(\mu)$ for all real $p > 0$. (A weaker version of this result is proved in Brown and Moran [1] by using Kakutani's theorem [6]; also Ritter [11 : (4.6)] gives a similar but weaker result than ours.) On the other hand, if the last series diverges with $\Theta \cup \Psi$ replaced by $\Theta \cap \Psi$, then μ and ν are mutually singular by Peyrière's theorem.

This observation indicates that the sufficient conditions in Theorem 1 can be improved in some special cases. However, the above example might be too special. Most of the results in the next section provide better information in this direction.

3. Random Riesz product measures.

Throughout this section, we fix two dissociate triples (Θ,a,σ) and (Θ,b,σ) on G, where $\Theta = \{\theta_k\}_1^\infty$ is countably infinite. As before, we shall assume that $|a| < 1$ on Θ. Let Ω denote the product of countably (infinitely) many replicas of the circle group. The space Ω will always be equipped with its Haar measure of norm 1. For each $\omega = (\omega_k)_1^\infty \in \Omega$, write $a\omega = (a_k\omega_k)$ and $b\omega = (b_k\omega_k)$, where $a_k = a(\theta_k)$ and $b_k = b(\theta_k)$ for all $k \geqslant 1$.

THEOREM 2. – *Let* $\mu_\omega = R(\Theta, a\omega, \sigma)$ *and* $\nu_\omega = R(\Theta, b\omega, \sigma)$ *for all* $\omega \in \Omega$. *Then we have either*

(i) *almost surely* $\nu_\omega \ll \mu_\omega$, *or*

(ii) *almost surely* $\nu_\omega \perp \mu_\omega$.

Moreover (i) *holds if and only if*

$$\sum_{k=1}^{\infty} |a_k - b_k|^2 \left\{ \frac{\cos^2 (s_k - t_k)}{(2 - |a_k + b_k|)^{1/2}} + 1 \right\} < \infty ,$$

where the s_k *and* t_k *are real numbers such that* $(a_k + b_k) \exp(-is_k) \geqslant 0$ *and* $(b_k - a_k) \exp(-it_k) \geqslant 0$ *for all* $k \geqslant 1$.

To prove this, we need three lemmas. The first two of them are implicit in Kakutani [6]. The inequalities in the proof of the first lemma are used in Brown and Moran [1] and in Ritter [11].

LEMMA 2.1. – *Let* ρ *be a nonnegative measure, and let* $(f_n)_1^\infty$ *be a sequence in* $L_+^1(\rho)$. *Suppose that*

(*a*) $\int f_n \, d\rho \leqslant 1$ *for all* $n \geqslant 1$, *and*

(*b*) $\sum_1^\infty \left[1 - \int (f_n f_{n+1})^{1/2} d\rho \right]^{1/2} < \infty$.

Then $(f_n)_1^\infty$ *converges in the* L^1*-norm*.

Proof. – Schwarz' inequality and two applications of (*a*) yield

$$\|f_m - f_n\|_1 \leqslant \|f_m^{1/2} + f_n^{1/2}\|_2 . \|f_m^{1/2} - f_n^{1/2}\|_2 \leqslant 2\|f_m^{1/2} - f_n^{1/2}\|_2 \leqslant 2.2^{1/2}\left[1 - \int (f_m f_n)^{1/2} d\rho \right]^{1/2}$$

for all m, n. Hence $\sum_1^\infty \|f_n - f_{n+1}\|_1 < \infty$ by (*b*). Thus the desired conclusion follows from the completeness of $L^1(\rho)$.

LEMMA 2.2. — *Let ρ, τ be two nonnegative finite measures on a measurable space X. Then the following properties are equivalent :*

(*a*) $\rho \perp \tau$.

(*b*) *Given $\varepsilon > 0$, there exists a nonnegative measurable function f on X such that $f > 0$, τ — a.e. and such that*

$$\left(\int f\,d\rho\right)\left(\int f^{-1}\,d\tau\right) < \varepsilon .$$

Proof. — Suppose that (*a*) obtains. Then the Hahn decomposition theorem provides two disjoint measurable sets A, B such that $X = A \cup B$, ρ is concentrated on A and τ is concentrated on B. Given $\varepsilon > 0$, define $f = \varepsilon\xi_A + \varepsilon^{-1}\xi_B$, where ξ denotes the characteristic function of a set. Thus

$$\left(\int f\,d\rho\right)\left(\int f^{-1}\,d\tau\right) = \varepsilon\rho(A).\varepsilon\tau(B).$$

Since $\rho(X)$ and $\tau(X)$ are finite, this establishes (*b*).

Conversely suppose (*b*) obtains. Let $\tau' = \rho \wedge \tau$, i.e., let τ' be the largest measure such that $\tau' \leqslant \rho$ and $\tau' \leqslant \tau$. Given $\varepsilon > 0$, let f be the function furnished by (*b*). Since $f > 0$, τ — *a.e.*, we also have $f > 0$, τ' — *a.e.* Therefore Schwarz' inequality yields

$$\tau'(X) = \int f^{1/2}.f^{-1/2}d\tau' \leqslant \left(\int f\,d\tau'\right)^{1/2}\left(\int f^{-1}\,d\tau'\right)^{1/2} \leqslant$$
$$\left(\int f\,d\rho\right)^{1/2}\left(\int f^{-1}\,d\tau\right)^{1/2} < \varepsilon^{1/2}$$

by (*b*). Since $\varepsilon > 0$ was arbitrary, this proves that $\tau' = 0$, i.e., that $\rho \perp \tau$.

LEMMA 2.3. — *Let $\alpha, \beta \in \mathbf{C}$ be such that $|\alpha| < 1$ and $|\beta| \leqslant 1$. Write*

$$I(\alpha,\beta) = \frac{1}{2\pi}\int_0^{2\pi} \{(1+\mathrm{Re}[\alpha e^{iu}])(1+\mathrm{Re}[\beta e^{iu}])\}^{1/2}\,du,$$

$$\gamma = 2^{-1}(\beta+\alpha) = |\gamma|e^{is} \quad \textit{and} \quad \delta = 2^{-1}(\beta-\alpha) = |\delta|e^{it},$$

where s and t are real. Then

(i) $I(\alpha,\beta) \geqslant 1 - |\delta|^2\left\{\dfrac{\cos^2(s-t)}{(1-|\gamma|)^{1/2}} + \sin^2(s-t)\right\}$,

(ii) $I(\alpha,\beta) \leqslant 1 - \dfrac{|\delta|^2}{8}\left\{\dfrac{\cos^2(s-t)}{(1-|\gamma|)^{1/2}} + \sin^2(s-t)\right\}$, *and*

(iii) $I(z\alpha, z\beta) = I(\alpha,\beta)$ *for all* $z \in \mathbf{C}$ *with* $|z| = 1$.

Proof. – If $|z| = 1$, then

$$(1)\quad \{(1+\mathrm{Re}[\alpha z])(1+\mathrm{Re}[\beta z])\}^{1/2} = \{(1+\mathrm{Re}[\gamma z]-\mathrm{Re}[\delta z])(1+\mathrm{Re}[\gamma z]+\mathrm{Re}[\delta z])\}^{1/2} = (1+\mathrm{Re}[\gamma z])\left\{1-\left(\frac{\mathrm{Re}[\delta z]}{1+\mathrm{Re}[\gamma z]}\right)^2\right\}^{1/2}.$$

Since $(1-v^2)^{1/2} \geqslant 1 - v^2$ for all $v \in [-1,1]$, it follows from (1) that

$$(2)\quad I(\alpha,\beta) \geqslant \frac{1}{2\pi}\int_0^{2\pi}\left\{1+\mathrm{Re}[\gamma e^{iu}]-\frac{(\mathrm{Re}[\delta e^{iu}])^2}{1+\mathrm{Re}[\gamma e^{iu}]}\right\}du =$$

$$1 - \frac{|\delta|^2}{2\pi}\int_0^{2\pi}\frac{\cos^2(t+u)}{1+|\gamma|\cos(s+u)}du = 1 - \frac{|\delta|^2}{2\pi}\int_0^{2\pi}\frac{\cos^2(t-s+u)}{1+|\gamma|\cos u}du.$$

To evaluate the last integral, we first expand the numerator

$$\cos^2(t-s+u) = \{\cos(s-t)\cos u + \sin(s-t)\sin u\}^2,$$

and then use the following elementary formulas :

$$(3)\quad \int_0^{2\pi}\frac{\cos u\ \sin u\ du}{1+|\gamma|\cos u} = 0,$$

$$(4)\quad \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos^2 u\ du}{1+|\gamma|\cos u} = \frac{1}{(1-|\gamma|^2)^{1/2}+1-|\gamma|^2},$$

and

$$(5)\quad \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin^2 u\ du}{1+|\gamma|\cos u} = \frac{1}{(1-|\gamma|^2)^{1/2}+1}.$$

These formulas are easily derived using either $v = \tan(u/2)$ or Cauchy's Residue Theorem. From (2)-(5), we infer that

$$I(\alpha,\beta) \geqslant 1 - |\delta|^2\left\{\frac{\cos^2(s-t)}{(1-|\gamma|^2)^{1/2}+1-|\gamma|^2}+\frac{\sin^2(s-t)}{(1-|\gamma|^2)^{1/2}+1}\right\} \geqslant 1 - |\delta|^2\left\{\frac{\cos^2(s-t)}{(1-|\gamma|)^{1/2}}+\sin^2(s-t)\right\},$$

which establishes (i).

For (ii), notice that $(1-v^2)^{1/2} \leqslant 1 - v^2/2$ for all $v \in [-1,1]$. Therefore (1) yields

$$\{(1+\mathrm{Re}[\alpha z])(1+\mathrm{Re}[\beta z])\}^{1/2} \leqslant 1 + \mathrm{Re}[\gamma z] - 2^{-1}\frac{(\mathrm{Re}[\delta z])^2}{1 + \mathrm{Re}[\gamma z]}.$$

So a similar estimate as above establishes (ii).

Part (iii) is obvious.

Proof of theorem 2. – For $\omega = (\omega_k) \in \Omega$, $x \in G$ and k, $n \geqslant 1$, define

$$A_k(\omega) = A_k(\omega,x) = 1 + \mathrm{Re}[a_k\omega_k\theta_k(x)],$$

and

$$P_n(\omega) = P_n(\omega,x) = \prod_1^n A_k(\omega,x).$$

Define $B_k(\omega)$ and $Q_n(\omega)$ similarly with a_k replaced by b_k. Therefore

$$\mu_\omega = R(\Theta,a\omega,\sigma) = \lim_n P_n(\omega)\sigma,$$

and

$$\nu_\omega = R(\Theta,b\omega,\sigma) = \lim_n Q_n(\omega)\sigma$$

weakly by Proposition B. Write $H_0 = 1$,

$$(1) \qquad H_n(\omega) = Q_n(\omega)/P_n(\omega) = \prod_1^n (B_k(\omega)/A_k(\omega)), \qquad \forall n \geqslant 1,$$

and $E(f(\omega))$ for the expectation of $f \in L^1(\Omega)$. Thus

$$(2) \qquad E(A_k(\omega)) = E(B_k(\omega)) = 1, \qquad \forall k \geqslant 1.$$

Now let $m \leqslant n \leqslant r$ be nonnegative integers. Then

$$[H_m(\omega)H_n(\omega)]^{1/2}P_r(\omega) = \left(\prod_1^m B_k(\omega)\right)\left(\prod_{m+1}^n A_k(\omega)B_k(\omega)\right)^{1/2}\left(\prod_{n+1}^r A_k(\omega)\right)$$

by (1). Since the projections $\omega \to \omega_k$ are stochastically independent, it follows from (2) and Lemma 2.3 that

$$(3) \qquad E([H_m(\omega)H_n(\omega)]^{1/2}P_r(\omega)) = \prod_{m+1}^n E([A_k(\omega)B_k(\omega)]^{1/2}) = \prod_{m+1}^n I(\theta_k(x)a_k, \theta_k(x)b_k) = \prod_{m+1}^n I(a_k, b_k)$$

for all $x \in G$. On the other hand,

$$(4) \qquad \int_G H_n(\omega) P_r(\omega)\, d\sigma = \int_G Q_n(\omega)\Big(\prod_{n+1}^{r} A_n(\omega)\Big)\, d\sigma = 1$$

by (1) and condition (*a*) of Proposition B. Therefore Schwarz' inequality applied to the measure $P_r(\omega)\sigma$ yields

$$(5) \qquad \int_G [H_m(\omega) H_n(\omega)]^{1/2} P_r(\omega)\, d\sigma \leqslant 1, \qquad \forall \omega \in \Omega.$$

Since $P_r(\omega)\sigma \to \mu_\omega$ weakly, we also have

$$(6) \quad \int_G [H_m(\omega) H_n(\omega)]^{1/2} d\mu_\omega = \lim_r \int_G [H_m(\omega) H_n(\omega)]^{1/2} P_r(\omega)\, d\sigma$$

for each $\omega \in \Omega$. In particular, the left-hand side of (6) is Borel measurable *qua* function of ω. It follows from (5) and (6) that

$$\begin{aligned}(7) \quad E&\left(\int_G [H_m(\omega) H_n(\omega)]^{1/2} d\mu_\omega\right) \\ &= \lim_r E\left(\int_G [H_m(\omega) H_n(\omega)]^{1/2} P_r(\omega)\, d\sigma\right) \text{ by Lebesgue} \\ &= \lim_r \int_G E([H_m(\omega) H_n(\omega)]^{1/2} P_r(\omega))\, d\sigma \text{ by Fubini} \\ &= \int_G \prod_{m+1}^{n} I(a_k, b_k)\, d\sigma \text{ by (3)} \\ &= \prod_{m+1}^{n} I(a_k, b_k)\end{aligned}$$

whenever $m \leqslant n$.

Now assume that the series in Theorem 2 converges, so

$$\sum_{k=1}^{\infty} |d_k|^2 \left(\frac{\cos^2 (s_k - t_k)}{(1-|c_k|)^{1/2}} + 1\right) < \infty,$$

where $c_k = (b_k + a_k)/2 = |c_k| \exp(is_k)$ and $d_k = (b_k - a_k)/2 = |d_k| \exp(it_k)$ for all k. Then we can choose natural numbers $N_1 < N_2 < \cdots$ so that

$$(8) \qquad \prod_{m+1}^{n} \left[1 - |d_k|^2 \left(\frac{\cos^2 (s_k - t_k)}{(1-|c_k|)^{1/2}} + 1\right)\right] > 1 - 4^{-j}$$

for all $n \geqslant m \geqslant N_j$ and all $j = 1, 2, \ldots$ It follows from (7), Lemma 2.3 and (8) that

$$E\left(\int_G [H_m(\omega) H_n(\omega)]^{1/2} d\mu_\omega\right) > 1 - 4^{-j}$$

whenever $n \geqslant m \geqslant N_j$. Taking (5) and (6) into account, we therefore infer from Schwarz' inequality and $E(1) = 1$ that

$$E\left(\left[1 - \int_G [H_m(\omega) H_n(\omega)]^{1/2} d\mu_\omega\right]^{1/2}\right) \leqslant \left[E\left(1 - \int_G [H_m(\omega) H_n(\omega)]^{1/2} d\mu_\omega\right)\right]^{1/2} < 2^{-j}$$

for all $n \geqslant m \geqslant N_j$. Hence

$$E\left(\sum_{j=1}^{\infty}\left[1 - \int_G [H_{N_j}(\omega) H_{N_{j+1}}(\omega)]^{1/2} d\mu_\omega\right]^{1/2}\right) < 1\,;$$

in particular,

$$(9) \qquad \sum_{j=1}^{\infty}\left[1 - \int_G [H_{N_j}(\omega) H_{N_{j+1}}(\omega)]^{1/2} d\mu_\omega\right]^{1/2} < \infty \text{ almost surely.}$$

Moreover, we have

$$(10) \qquad \int_G H_n(\omega)\, d\mu_\omega = \lim_r \int_G H_n(\omega) P_r(\omega)\, d\sigma = 1 \qquad \forall n \geqslant 1$$

by (4). It follows from (9), (10) and Lemma 2.1 that almost surely $(H_{N_j}(\omega))_{j=1}^{\infty}$ converges in the norm of $L^1(\mu_\omega)$. Recalling that $H_n(\omega) = Q_n(\omega)/P_n(\omega)$ for all n, we therefore conclude from Lemma 1.3 that almost surely $\nu_\omega \in L^1(\mu_\omega)$.

Conversely, suppose that the series in Theorem 2 diverges. Then, since $\sin^2 t + \cos^2 t = 1$, it is obvious that

$$(11) \qquad \sum_{k=1}^{\infty} |d_k|^2 \left(\frac{\cos^2(s_k - t_k)}{(1 - |c_k|)^{1/2}} + \sin^2(s_k - t_k)\right) = \infty\,.$$

Now (7) with $m = 0$ and Lemma 2.3 yield

$$E\left(\int_G H_n(\omega)^{1/2} d\mu_\omega\right) = \prod_1^n I(a_k, b_k) \leqslant \prod_1^n\left[1 - \frac{|d_k|^2}{8}\left(\frac{\cos^2(s_k - t_k)}{(1 - |c_k|)^{1/2}} + \sin^2(s_k - t_k)\right)\right]$$

for all n. From this and (11) it follows that

$$\lim_n E\left(\int_G H_n(\omega)^{1/2} d\mu_\omega\right) = 0\,;$$

hence

$$E\left(\liminf_n \int_G H_n(\omega)^{1/2} d\mu_\omega\right) = 0$$

by Fatou's lemma ; hence

$$\liminf_n \int_G H_n(\omega)^{1/2} d\mu_\omega = 0 \quad \text{almost surely.} \tag{12}$$

Moreover, $\nu_\omega = Q_n(\omega)\nu_\omega(n)$ by Lemma 1.3, and so

$$H_n(\omega) = Q_n(\omega)/P_n(\omega) > 0, \qquad \nu_\omega - a.e. \tag{13}$$

for all $n \geqslant 1$ and $\omega \in \Omega$. Furthermore, Schwarz' inequality yields

$$\int_G H_n(\omega)^{-1/2} d\nu_\omega = \int_G [P_n(\omega)/Q_n(\omega)]^{1/2} Q_n(\omega)\, d\nu_\omega(n) \leqslant \left(\int_G P_n(\omega)\, d\nu_\omega(n) \int_G Q_n(\omega)\, d\nu_\omega(n)\right)^{1/2} = 1 \tag{14}$$

for all $\omega \in \Omega$. By (12)-(14) we may apply Lemma 2.2 to conclude that almost surely μ_ω and ν_ω are mutually singular. This completes the proof of Theorem 2.

Remark (VI). — Ritter [12 : (4.2)] observes that, in the case of usual (generalized) Riesz product measures μ, ν, the condition which naturally corresponds to (12) in the above proof implies the mutual singularity of μ, ν.

THEOREM 3. — *Write* $b_k - a_k = |b_k - a_k| \exp(it_k)$ *and* $a_k = |a_k| \exp(iu_k)$, *where the* t_k *and* u_k *are real for all* $k \geqslant 1$. *Suppose that* $1 < p < \infty$ *and that :*

(i) *in case* $p < 3/2$, $\sum_1^\infty |a_k - b_k|^p < \infty$;

(ii) *in case* $p = 3/2$,

$$\sum_1^\infty |a_k - b_k|^{3/2}\{|\cos(t_k - u_k)|^{3/2} \log((1 - |a_k|)^{-1}) + 1\} < \infty\,;$$

(iii) *in case* $3/2 < p \leqslant 2$,

$$\sum_{1}^{\infty} |a_k - b_k|^p \{|\cos(t_k - u_k)|^p / (1 - |a_k|)^{p-3/2} + 1\} < \infty ;$$

(iv) *in case* $p > 2$, $\sum_{1}^{\infty} |a_k - b_k)|^p / (1 - |a_k|)^{p-3/2} < \infty$ *and*

$$\sum_{1}^{\infty} |a_k - b_k|^2 \{|\cos(t_k - u_k)|^2 / (1 - |a_k|)^{1/2} + 1\} < \infty .$$

Then $\nu_\omega \in L^p(\mu_\omega)$ *almost surely.*

We believe that the conditions in (i)-(iii) are best possible, although we have not attempted to prove it. On the other hand, the first condition in case (iv) can be slightly relaxed in the case $2 < p \leqslant 3$. This will become clear from our proof of Theorem 3 and the lemma given below. If the reader feels that the above result is too messy, he is strongly encouraged to give a direct proof that, e.g., the condition in (ii) implies the convergence of the series in Theorem 2.

LEMMA 3.1. – *Let* $1 < p < \infty$ *and let*

$$I_p(\alpha, \delta) = \int_0^{2\pi} |\cos(t + \delta)|^p / (1 - \alpha \cos t)^{p-1} \, dt$$

for $\alpha \in [0, 1)$ *and real* δ. *Then there exists a finite constant* M_p, *depending only on p and independent of* α *and* δ, *such that :*

(i) *if* $p < 3/2$, *then* $I_p(\alpha, \delta) \leqslant M_p$;

(ii) *if* $p = 3/2$, *then*

$$I_p(\alpha, \delta) \leqslant M_p \{|\cos \delta|^p \log((1 - \alpha)^{-1}) + 1\};$$

(iii) *if* $3/2 < p < 3$, *then*

$$I_p(\alpha, \delta) \leqslant M_p \{|\cos \delta|^p / (1 - \alpha)^{p-3/2} + 1\};$$

(iv) *if* $p = 3$, *then*

$$I_3(\alpha, \delta) \leqslant M_3 \{|\cos \delta|^3 / (1 - \alpha)^{3/2} + |\sin \delta|^3 \log((1 - \alpha)^{-1}) + 1\};$$

(v) *if* $p > 3$, *then*

$$I_p(\alpha, \delta) \leqslant M_p / (1 - \alpha)^{p-3/2}.$$

Proof. — In this proof, the symbols K_1, K_2, ..., will denote finite positive constants which depend only on p and are independent of α and δ.

If $p < 3/2$, then

$$I_p(\alpha,\delta) \leqslant 4\int_0^{\pi/2} 1/(1-\cos t)^{p-1}\,dt \leqslant 4\int_0^2 1/(t^2/4)^{p-1}\,dt = 4^p\int_0^2 t^{-2(p-1)}\,dt = K_1 < \infty$$

since $2(p-1) < 1$. This establishes (i).

So assume that $p \geqslant 3/2$. If $\alpha \leqslant 1/2$, then the integrand in the definition of $I_p(\alpha,\delta)$ is less than or equal to $1/(1-\alpha)^{p-1} \leqslant 2^{p-1}$, so $I_p(\alpha,\delta) \leqslant 2\pi . 2^{p-1}$. Therefore we may also assume that $\alpha > 1/2$.

Now define

$$C_p(\alpha) = \int_0^{2\pi} |\cos t|^p/(1-\alpha\cos t)^{p-1}\,dt \tag{1}$$

for $\alpha \in (1/2, 1)$. Then we have

$$C_p(\alpha) \leqslant 4\int_0^{\pi/2} (\cos t)/(1-\alpha\cos t)^{p-1}\,dt \leqslant 4\int_0^{\pi/2} (\cos t/2)/(1-\alpha+2\alpha\sin^2(t/2))^{p-1}\,dt \leqslant 8\int_0^1 1/(1-\alpha+2\alpha u^2)^{p-1}\,du \leqslant 8\int_0^1 (1-\alpha+u^2)^{-(p-1)}\,du$$

since $2\alpha > 1$. In particular,

$$C_{3/2}(\alpha) \leqslant 8\int_0^1 (1-\alpha+u^2)^{-1/2}du = 8\,[\log\,(u+(1-\alpha+u^2)^{1/2}]_0^1 < 8\,[2+\log((1-\alpha)^{-1})]. \tag{2}$$

In case $p > 3/2$, set $A = (1-\alpha)^{1/2}$. Then

$$C_p(\alpha)/8 \leqslant \left(\int_0^A + \int_A^1\right)(1-\alpha+u^2)^{-(p-1)}\,du < A(1-\alpha)^{-(p-1)} + \int_A^1 u^{-(2p-2)}\,du < A(1-\alpha)^{1-p} + A^{3-2p}/(2p-3);$$

hence

$$C_p(\alpha) \leqslant K_2/(1-\alpha)^{p-3/2} \quad \text{for} \quad p > 3/2. \tag{3}$$

Now we estimate

$$S_p(\alpha) = \int_0^{2\pi} |\sin t|^p/(1-\alpha\cos t)^{p-1}\, dt \tag{4}$$

for $\alpha \in (1/2, 1)$. Note that

$$S_p(\alpha) \leqslant 4\int_0^{\pi/2} |\sin t|^p/(1-\alpha\cos t)^{p-1}\, dt \leqslant$$
$$4.2^p \int_0^{\pi/2} |\sin(t/2)|^p/(1-\alpha+\sin^2 t/2))^{p-1}\, dt.$$

Letting $u = \sin^2(t/2)$ and $v = 1 - \alpha + u$, we therefore have

$$S_p(\alpha) \leqslant 8.2^p\int_0^1 u^{(p-1)/2}/(1-\alpha+u)^{p-1}\, du =$$
$$8.2^p\int_{1-\alpha}^{2-\alpha} (v-1+\alpha)^{(p-1)/2}/v^{p-1}\, dv \leqslant 8.2^p\int_{1-\alpha}^{2} v^{-(p-1)/2} dv.$$

Hence

$$S_p(\alpha) \leqslant 8.2^p\int_0^2 v^{-(p-1)/2} dv = K_3 \quad \text{for} \quad p < 3, \tag{5}$$

$$S_3(\alpha) \leqslant 64\int_{1-\alpha}^2 v^{-1}\, dv \leqslant 64\,[1+\log((1-\alpha)^{-1})], \tag{6}$$

and

$$S_p(\alpha) \leqslant 8.2^p\int_{1-\alpha}^2 v^{-(p-1)/2} dv \leqslant K_4/(1-\alpha)^{p-3/2} \quad \text{for} \quad p > 3. \tag{7}$$

Finally, note that

$$|\cos(t+\delta)|^p \leqslant 2^p(|\cos\delta\cos t|^p + |\sin\delta\sin t|^p)$$

for all real t and δ. Hence

$$I_p(\alpha,\delta) \leqslant 2^p\{|\cos\delta|^p C_p(\alpha) + |\sin\delta|^p S_p(\alpha)\} \tag{8}$$

by (1) and (4). The desired results for $p \geqslant 3/2$ are obtained by combining (2) and (3) with (5)-(8). We leave the details to the reader.

Proof of theorem 3. – We shall preserve all the notations in the proof of Theorem 2 except for $d_k = b_k - a_k = |d_k| \exp(it_k)$. However, we shall omit ω in $A_k(\omega)$, $B_k(\omega)$, etc. Write $D_k = B_k - A_k = \mathrm{Re}\,[d_k\omega_k\theta_k]$ for each k.

Let c_p be the positive finite constant furnished by Lemma 1.1, and let n, r be two natural numbers with $n < r$. Assuming $p \leqslant 2$, we then have

$$|Q_n/P_n|^p P_r = \left[\prod_{k=1}^{n} (1+D_k/A_k)\right]^p P_r \leqslant$$

$$\left[\prod_{k=1}^{n} (1+pD_k/A_k+c_p|D_k/A_k|^p)\right]\prod_{k=1}^{r} A_k =$$

$$\left[\prod_{k=1}^{n} (A_k+pD_k+c_p|D_k|^p/A_k^{p-1})\right]\prod_{n+1}^{r} A_k.$$

Recall that $E(A_k) = E(B_k) = 1$ for all k and that the ω_k are stochastically independent. It follows from the above inequality that

$$E(|Q_n/P_n|^p P_r) \leqslant \prod_{k=1}^{n} (1+c_pE(|D_k|^p/A_k^{p-1})). \tag{1}$$

Now fix $x \in G$ and choose a real number $v_k = v_k(x)$ so that $\theta_k(x) = \exp(iv_k)$. Then

$$E(|D_k|^p/A_k^{p-1}) = \frac{|d_k|^p}{2\pi}\int_0^{2\pi} \frac{|\cos(t+t_k+v_k)|^p\,dt}{(1+|a_k|\cos(t+u_k+v_k))^{p-1}} =$$

$$\frac{|d_k|^p}{2\pi}\int_0^{2\pi} \frac{|\cos(t+\delta_k)|^p\,dt}{(1-|a_k|\cos t))^{p-1}} = \frac{|d_k|^p}{2\pi} I_p(|a_k|, \delta_k),$$

where $\delta_k = t_k - u_k$ for $k \geqslant 1$. Assuming further that $p < 3/2$, we therefore have

$$E(|D_k|^p/A_k^{p-1}) \leqslant M_p|d_k|^p \quad \text{for all} \quad k \geqslant 1, \tag{2}$$

where M_p is the positive finite constant furnished by Lemma 3.1 (i). Hence

$$E(|Q_n/P_n|^pP_r) \leqslant \Pi_{k=1}^n (1+c_pM_p|d_k|^p) \leqslant \tag{3}$$

$$\exp\left(c_pM_p \sum_1^{\infty} |d_k|^p\right) = C, \text{ say,}$$

for all $r > n$ by (1) and (2). Since $P_r\sigma \to \mu_\omega$ weakly by Proposition B, it follows from Fatou's lemma, Fubini's theorem, and (3) that

$$E\left(\int_G |Q_n/P_n|^p \, d\mu_\omega\right) = E\left(\lim_r \int_G |Q_n/P_n|^p P_r \, d\sigma\right) \leqslant$$

$$\liminf_r E\left(\int_G |Q_n/P_n|^p P_r \, d\sigma\right) = \liminf_r \int_G E(|Q_n/P_n|^p P_r) \, d\sigma \leqslant C$$

for all n. Therefore one more application of Fatou's lemma yields

$$E\left(\liminf_n \int_G |Q_n/P_n|^p \, d\mu_\omega\right) \leqslant C. \tag{4}$$

Notice that $C < \infty$ since $\sum_1^\infty |d_k|^p < \infty$ by the present case assumption. Hence (4) implies that

$$\liminf_n \int |Q_n/P_n|^p \, d\mu_\omega < \infty \quad \text{almost surely.} \tag{5}$$

From (5), Lemmas 1.2 and 1.3 (iii), we conclude that almost surely $\nu_\omega \in L^p(\mu_\omega)$, provided that $1 < p < 3/2$ and $\sum_1^\infty |d_k|^p < \infty$.

The proofs for the other cases will now be clear to the reader. We omit the details.

Finally, we give two applications of Theorem 2 to Riesz product measures on the circle group $\mathbf{T}$, where the basic probability measure σ is chosen to be the normalized Lebesgue measure on $\mathbf{T}$. As usual, we shall regard $\mathbf{T}$ as $\mathbf{R}$ (mod 2π) and identify the dual of $\mathbf{T}$ with the additive group $\mathbf{Z}$ of all integers.

THEOREM 4. — *Let (m_k) and (n_k) be two sequences of natural numbers such that*

$$2(m_1 n_1 + \cdots + m_k n_k) < n_{k+1} \quad \textit{for all } k \geqslant 1. \tag{i}$$

Also let (a_k) and (b_k) be two sequences in $\{z \in \mathbf{C} : |z| \leqslant 1\}$. Define $\mu = R(\{n_k\}, (a_k))$ and $\nu = R(\{n_k\}, (b_k))$. Choose $s_k, t_k \in \mathbf{R}$ so that

$$(a_k + b_k) \exp(-is_k) \geqslant 0 \qquad \textit{and} \qquad (b_k - a_k) \exp(-it_k) \geqslant 0$$

fo all $k \geqslant 1$.

(*a*) *If* $|a_k| < m_k/(1+m_k)$ *and* $|b_k| \leqslant m_k/(1+m_k)$ *for all* k, *and if*

$$\sum_{k=1}^{\infty} |a_k - b_k|^2 \left\{ \frac{\cos^2 (s_k - t_k)}{[2 - |a_k + b_k|(1+m_k)/m_k]^{1/2}} + 1 \right\} < \infty ,$$

then $\nu \ll \mu$.

(*b*) *On the other hand, if*

$$\sum_{k=1}^{\infty} |a_k - b_k|^2 \left\{ \frac{\cos^2 (s_k - t_k)}{[2 - |a_k + b_k| m_k/(1+m_k)]^{1/2}} + 1 \right\} = \infty ,$$

then $\nu \perp \mu$.

Proof. — First of all, note that the n_k form a dissociate set in $\mathbf{Z}$ by (i), and so the definitions of μ and ν make sense.

Next recall that the m th Fejér kernel on $\mathbf{T}$ is the function K_m defined by

$$K_m(t) = \sum_{-m}^{m} [1 - |\ell|/(1+m)] \exp (i\ell t) \quad \text{for} \quad t \in \mathbf{R}.$$

Thus K_m is a nonnegative trigonometric polynomial on $\mathbf{T}$ with $\int K_m \, d\sigma = 1$ (cf. [15]).

Now let $\omega = (\omega_k) \in \Omega$ be given. Writing $\omega_k = \exp (iv_k)$ with real v_k, define $R_k(t) = R_k(\omega, t) = K_{m_k}(v_k + n_k t)$ for $k \geqslant 1$. Thus R_k is non negative and

$$R_k(t) = \sum_{-m_k}^{m_k} [1 - |\ell|/(1+m_k)] \, \omega_k^{\ell} \exp (i\ell n_k t).$$

By using (i), it is easy to show that the sequence $(R_1 R_2 \ldots R_r \sigma)_{r=1}^{\infty}$ converges weak-$*$ to a probability measure $\lambda_\omega \in M(\mathbf{T})$. Moreover,

$$\hat{\lambda}_\omega(n) = \prod_{k=1}^{r} [1 - |\ell_k|/(1+m_k)] \, \omega_k^{\ell_k} \tag{1}$$

if $n \in \mathbf{Z}$ has the form $n = \ell_1 n_1 + \cdots + \ell_r n_r$, where $r \in \mathbf{N}$ and $\ell_k \in \{0, \pm 1, \ldots, \pm m_k\}$ for $k = 1, 2, \ldots, r$; and $\hat{\lambda}_\omega(n) = 0$ if n does not have the above form (cf. [10] and [11]).

Now suppose that the hypotheses in part (*a*) hold. Let $a_k' = a_k(1+m_k)/m_k$ and $b_k' = b_k(1+m_k)/m_k$ for all $k \geqslant 1$. Consider the

Riesz product measures

$$\mu'_\omega = R(\{n_k\}, (a'_k\omega_k)) \qquad \text{and} \qquad \nu'_\omega = R(\{n_k\}, (b'_k\omega_k))$$

for $\omega \in \Omega$. By (1), it is easy to show that $\mu = \mu'_\omega * \lambda_{\bar\omega}$ and $\nu = \nu'_\omega * \lambda_{\bar\omega}$ for all $\omega \in \Omega$, where $\bar\omega = (\bar\omega_k)$. On the other hand, the series in (*a*) converges, and so Theorem 2 ensures that $\nu'_\omega \ll \mu'_\omega$ almost surely. In particular, there exists $\omega \in \Omega$ such that $\nu'_\omega \ll \mu'_\omega$. Since all of μ'_ω, ν'_ω and $\lambda_{\bar\omega}$ are positive measures, it follows that $\nu = \nu'_\omega * \lambda_{\bar\omega} \ll \mu'_\omega * \lambda_{\bar\omega} = \mu$.

Finally, suppose that the series in part (*b*) diverges. Define $\mu''_\omega = R(\{n_k\}, (a''_k\omega_k))$ and $\nu''_\omega = R(\{n_k\}, (b''_k\omega_k))$, where $a''_k = a_k m_k/(1+m_k)$ and $b''_k = b_k m_k/(1+m_k)$ for all k. Then we have $\mu''_\omega = \mu * \lambda_\omega$ and $\nu''_\omega = \nu * \lambda_\omega$ for all $\omega \in \Omega$. Moreover $\mu''_\omega \perp \nu''_\omega$ almost surely by Theorem 2 and the present hypothesis. Consequently we obtain $\mu \perp \nu$, which completes the proof.

COROLLARY 4.1 (*Notation as in Theorem* 4.) — *Suppose that*

(i) $2(m_1n_1 + \cdots + m_kn_k) < n_{k+1}$ *for all* $k \geqslant 1$,

(ii) $\sum_{k=1}^{\infty} (|a_k| + |b_k|)/m_k < \infty$, *and*

(iii) $\max(|a_k|, |b_k|) < m_k/(1+m_k)$ *for all* $k \geqslant 1$.

Then either $\nu \ll \mu$ *or* $\nu \perp \mu$. *Moreover* $\nu \ll \mu$ *obtains if and only if*

$$(*) \qquad \sum_{k=1}^{\infty} |a_k - b_k|^2 \left\{ \frac{\cos^2(s_k - t_k)}{(2 - |a_k + b_k|)^{1/2}} + 1 \right\} < \infty .$$

Proof. — We shall preserve all the notations in the proof of Theorem 4. Define $\mu_\omega = R(\{n_k\}, (a_k\omega_k))$ and $\nu_\omega = R(\{n_k\}, (b_k\omega_k))$ for $\omega \in \Omega$.

We claim that for each $\omega \in \Omega$, the three measures μ_ω, μ'_ω, $\mu''_\omega[\nu_\omega$, ν'_ω, $\nu''_\omega]$ are mutually equivalent, i.e., mutually absolutely continuous. In fact, $\omega \in \Omega$ implies

$$\sum_1^\infty |a_k\omega_k - a'_k\omega_k| = \sum_1^\infty |a_k|/m_k < \infty$$

by (ii). Also $|a_k\omega_k| \leqslant |a'_k\omega_k| = |a_k|(1+m_k)/m_k < 1$ for all k by (iii). Thus two applications of Theorem 1 with $p = 1$ ensure that μ_ω and μ'_ω are mutually equivalent. Similarly μ_ω and μ''_ω are mutually equivalent. The same argument with the *a*'s replaced by the *b*'s shows that ν_ω, ν'_ω, ν''_ω, are mutually equivalent.

Now suppose that the series in $(*)$ converges. Then almost surely μ_ω and ν_ω are mutually equivalent by Theorem 2 and (iii). It follows from the above claim that almost surely μ'_ω and ν'_ω are mutually equivalent. Hence the series in part (*a*) of Theorem 4 converges by Theorem 2. Therefore $\nu \ll \mu$ by Theorem 4 and (iii).

Finally suppose that the series in $(*)$ diverges. Then almost surely μ_ω and ν_ω are mutually singular by Theorem 2 and (iii). Therefore, almost surely μ''_ω and ν''_ω are mutually singular by the above claim. Hence the series in part (*b*) of Theorem 4 diverges by Theorem 2, and so μ and ν are mutually singular by Theorem 4. This completes the proof.

Remarks (VII). – Suppose in Theorem 2 that Θ is an independent set consisting of elements of infinite order and that $\hat{\sigma} = 0$ on $G_p(\Theta)\setminus\{1\}$, where $G_p(\Theta)$ denotes the subgroup of Γ generated by Θ. Then the « almost sure » statements in Theorem 2 may be replaced by the corresponding « sure » statements. The proof of this depends on the fact that Θ is stochastically independent with respect to σ and on the property that $\hat{\sigma}(\theta^n) = 0$ for all $\theta \in \Theta$ and all nonzero $n \in \mathbf{Z}$ (under the present hypotheses). The same comment applies to Theorem 3 as well. We omit the details.

(VIII) Let (n_k) be a sequence of natural numbers. As is well-known, if $n_k \to \infty$ « very rapidly », then the exponential functions $t \to \exp(in_k t)$ behave as if they were stochastically independent (with respect to the normalized Lebesgue measure on the circle group). This gives us an intuitive explanation of why the conclusions of Corollary 4.1 hold. However, the important point of Corollary 4.1 is that, given two sequences (a_k), (b_k) in $\{z \in C : |z| < 1\}$, it provides an explicit sufficient condition on the « speed » of $n_k \to \infty$ in order that the corresponding Riesz product measures be either mutually absolutely continuous or mutually singular.

(IX) Part of Theorem 1 is based on lectures about usual Riesz product measures given by the second author at Tokyo Metropolitan University in 1975 or 1976. Most of the results in sections 2 and 3 (for usual Riesz product measures) are contained in the first author's Ph. D. dissertation at Kansas State University (Summer, 1986), which was written under the direction of the second author.

(X) *Added on May* 22, *1987*. The referee has kindly pointed out to us that Peyrière's paper [16] deals with Riesz product measures on the real line. He has also suggested that the restriction $\max(a_k, b_k) < m_k/(1+m_k)$ in Theorem 4 might be slightly relaxed by using appropriate kernels other than the Fejér kernel.

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