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ON CONTINUOUS FUNCTIONS WITH NO UNILATERAL DERIVATIVES

by Masayoshi HATA

1. Introduction.

It is known that A. S. Besicovitch in 1925 gave the first example of a continuous function $B(x)$ which has nowhere a unilateral derivative *finite or infinite* by geometrical process. E. D. Pepper [9] has examined this same function $B(x)$, giving a different exposition. The graph of his function is illustrated in Figure 1. Later, A. N. Singh [12, 13] gave the arithmetical definition of $B(x)$ and constructed an infinite class of such non-differentiable functions. On the other hand, A. P. Morse [8] gave an example of a continuous function $f(x)$ satisfying

$$\liminf_{s \rightarrow x \pm} \left| \frac{f(s) - f(x)}{s - x} \right| < \limsup_{s \rightarrow x \pm} \left| \frac{f(s) - f(x)}{s - x} \right| = \infty$$

respectively, for every $x \in (0, 1)$, by arithmetical process.

It seems, however, that their methods are somewhat complicated and inappropriate to the study concerning further properties of such functions. In the present paper we shall develop a simple but powerful method to construct and analyze such singular functions by using certain one-dimensional dynamical systems.

The difficulties of finding such functions may be explained by the fact that the set of functions which have nowhere a unilateral derivative finite or infinite is of only the first category in the space of continuous functions (S. Saks [11]), while the set of functions which have nowhere a finite unilateral derivative is of the second category (S. Banach [1], S. Mazurkiewicz [7] and V. Jarnik [5]).

Key-words : Non-differentiable functions - Knot points - Functional equations.

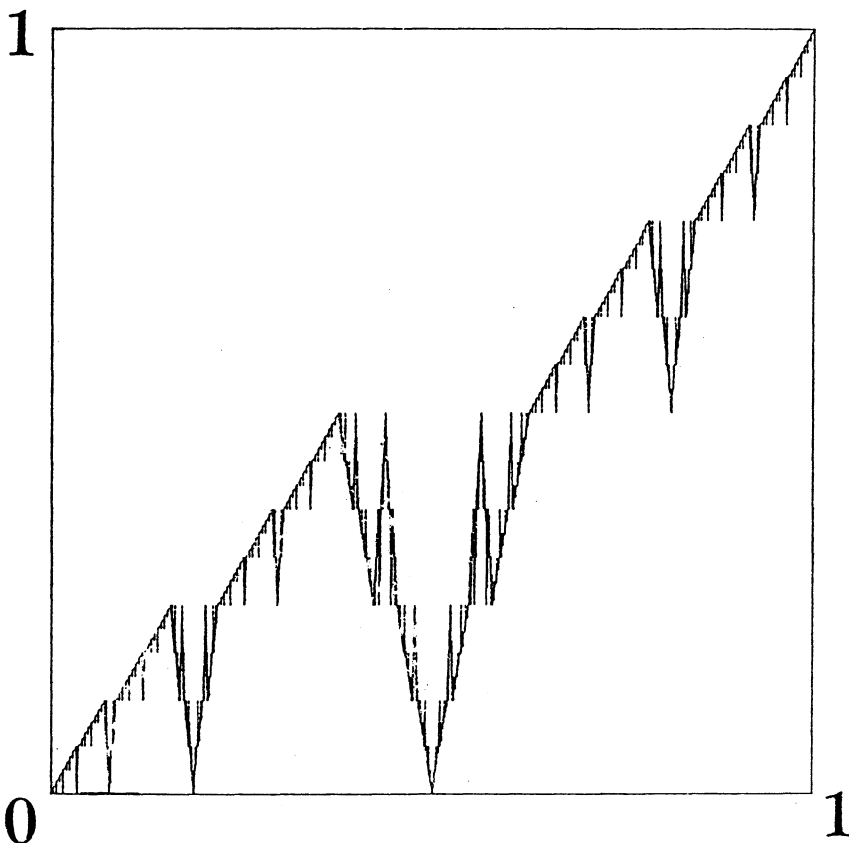


Fig. 1.

2. Main Result.

To state our main theorem, we need some definitions and notations. We denote, as usual, the upper and lower derivatives at x of a real-valued function $f(x)$ on the right by $D^+f(x)$, $D_+f(x)$ respectively. Similarly the upper and lower derivatives, on the left, are denoted by $D^-f(x)$, $D_-f(x)$ respectively. A point x is said to be a knot point of $f(x)$ provided that

$$D^+f(x) = D^-f(x) = \infty \quad \text{and} \quad D_+f(x) = D_-f(x) = -\infty.$$

The set of knot points of $f(x)$ is denoted by $\text{Knot}(f)$. For a measurable

set E , we denote by $|E|$ the Lebesgue measure of E . Our theorem can now be stated as follows :

THEOREM 2.1. — *For any $\alpha \in [0, 1)$ and $\varepsilon \in (0, 1)$, there exists a continuous function $\psi_{\alpha, \varepsilon}(x)$ defined on the unit interval I satisfying the following properties :*

- (1) $\psi_{\alpha, \varepsilon}(x)$ has nowhere a unilateral derivative finite or infinite ;
- (2) $|\text{Knot}(\psi_{\alpha, \varepsilon})| = \alpha$;
- (3) $\psi_{\alpha, \varepsilon}(x)$ satisfies Hölder's condition of order $1 - \varepsilon$.

Remark. — K. M. Garg [3] has shown that the set of knot points of Besicovitch's function is of measure zero. He also showed that, for every continuous function defined on I which has nowhere a unilateral derivative finite or infinite, the set of points at which the upper derivative on one side is $+\infty$, the lower derivative on the other side is $-\infty$, and the other two derivatives are finite and equal has a positive measure in every subinterval of I ; therefore the constant α in our theorem can not be taken to be 1. Note that the set $\text{Knot}(f)$ is of the second category if $f(x)$ is a continuous function which has nowhere a finite or infinite derivative (W. H. Young [14]).

As a corollary, we have immediately

COROLLARY 2.2. — *For any $\alpha \in [0, 2\pi)$ and $\varepsilon \in (0, 1)$, there exists an absolutely convergent cosine Fourier series*

$$\Psi_{\alpha, \varepsilon}(x) = \sum_{n=0}^{\infty} a_{\alpha, \varepsilon, n} \cos nx$$

satisfying the following properties :

- (1) $\Psi_{\alpha, \varepsilon}(x)$ has nowhere a unilateral derivative finite or infinite;
- (2) $|\text{Knot}(\Psi_{\alpha, \varepsilon}|_{[0, 2\pi]})| = \alpha$;
- (3) $\sum_{n=1}^{\infty} |a_{\alpha, \varepsilon, n}|^2 n^{2-\varepsilon} < \infty$.

For the proof of Theorem 2.1, we shall introduce a symbol space in section 3 and certain functional equations in section 4. The fundamental properties of the solution are investigated in sections 5 and 6. We then prove Theorem 2.1 in section 7 using Cantor sets of positive measure.

3. Preliminaries.

We first divide the unit interval I into m subintervals

$$I_1 = [c_0, c_1], I_2 = [c_1, c_2], \dots, I_m = [c_{m-1}, c_m]$$

where $0 = c_0 < c_1 < c_2 < \dots < c_m = 1$, $m \geq 2$ and define the *address* $A(x)$ of a point $x \in I$ by setting $A(x) = j$ for $c_{j-1} \leq x < c_j$, $1 \leq j \leq m$ and $A(c_m) = m$. Let $g_j(x)$ be a strictly monotone, either increasing or decreasing, continuous function defined on the subinterval I_j such that $g_j(I_j) = I$ for $1 \leq j \leq m$. Define the *sign* ε_j to be either $+1$ or -1 according as g_j is monotone increasing or monotone decreasing on I_j . We assume, in addition, that $g_1(x)$ and $g_m(x)$ are monotone increasing; so $\varepsilon_1 = \varepsilon_m = +1$.

Let $\Sigma = \{1, 2, \dots, m\}^{\mathbb{N}}$ be the one-sided symbol space endowed with the metric

$$d(w, z) = \sum_{n=1}^{\infty} 2^{-n} |w_n - z_n| \quad \text{for} \quad w = (w_n), \quad z = (z_n) \in \Sigma.$$

It is known that Σ is a totally disconnected compact metric space. Let $G(x) = g_{A(x)}(x)$ for brevity. Note that the function $G: I \rightarrow I$ is not necessarily continuous. We then define the *itinerary* $v(x)$ of a point $x \in I$ by setting

$$v(x) = (A_0(x), A_1(x), \dots, A_n(x), \dots)$$

where $A_n(x) = A(G^n(x))$ for $n \geq 0$. Put $e_0 = \{0, 1\}$ and define the set e_{n+1} inductively by setting $e_{n+1} = \{0 < x < 1; G(x) \in e_n\}$ for $n \geq 0$. Obviously $\# e_n = m^{n-1}(m-1)$ for $n \geq 1$. Let $e = \bigcup_{n \geq 0} e_n$. Then it is easily verified that the set of discontinuity points of v is precisely equal to the set $e - e_0$.

Put $\Lambda_0 = \{v(x); x \in e_0\}$. For $N \geq 1$, let Λ_N be the set of words $w = (w_n) \in \Sigma$ such that either $w_n = 1$ for $n > N$, $w_N \neq 1$ or $w_n = m$ for $n > N$, $w_N \neq m$. Let $\Lambda = \bigcup_{n \geq 0} \Lambda_n$. Then it is easily seen that for $x \in e - e_0$ there exist the limits

$$\lim_{\varepsilon \rightarrow 0 \pm} v(x \pm \varepsilon) = (A_0(x \pm), A_1(x \pm), \dots)$$

in $\Lambda - \Lambda_0$ respectively. Note that $v(x)$ is equal to either $v(x+)$ or $v(x-)$. Thus the set Λ_n consists of the following $2m^{n-1}(m-1)$ distinct words:

$$\{v(x+); x \in e_n\} + \{v(x-); x \in e_n\}$$

for $n \geq 1$. Therefore we have $\Lambda = \Lambda_0 + \Sigma_+ + \Sigma_-$, where $\Sigma_+ = \{v(x+); x \in e - e_0\}$ and $\Sigma_- = \{v(x-); x \in e - e_0\}$.

We assume further that each function $h_j = g_j^{-1}: I \rightarrow I_j$ is a contraction; namely the Lipschitz constant

$$\text{Lip}(h_j) = \sup_{x \neq y \in I} \left| \frac{h_j(x) - h_j(y)}{x - y} \right|$$

satisfies $\text{Lip}(h_j) < 1$. Let $\gamma = \max_{1 \leq j \leq m} \text{Lip}(h_j) \in [1/m, 1)$. We then define the mapping $\mu: \Sigma \rightarrow I$ by setting

$$\mu(w) = \lim_{n \rightarrow \infty} h_{w_1} \circ h_{w_2} \circ \dots \circ h_{w_n}(I) \quad \text{for } w = (w_n) \in \Sigma.$$

Clearly μ is continuous. Then it follows that $X = \mu(\Sigma)$ is a compact subset of I and satisfies the following equality:

$$X = h_1(X) \cup h_2(X) \cup \dots \cup h_m(X).$$

It is known that the above equation possesses a unique non-empty compact solution [4, p. 384]; thus we have $\mu(\Sigma) = X = I$, since $h_j(I) = I_j$ for $1 \leq j \leq m$. It also follows that the set e is a dense subset of I ; therefore the mapping v is one to one.

Let $S_n = \bigcup_{0 \leq j \leq n} e_j$ for $n \geq 1$ and let

$$H_{n,x}(y) = h_{A_0(x)} \circ h_{A_1(x)} \circ \dots \circ h_{A_{n-1}(x)}(y)$$

for $n \geq 1$ and $x, y \in I$. Obviously $H_{n,x}$ is a contraction satisfying $\text{Lip}(H_{n,x}) \leq \gamma^n$. We first consider an arbitrary point $x \in I - e$. Put $K_{n,x} = H_{n,x}(\bar{I})$ for $n \geq 1$. Since $K_{n,x}$ is the connected component of $I - S_n$ containing x and $|K_{n,x}| \leq \gamma^n$, we have

$$\lim_{n \rightarrow \infty} \bar{K}_{n,x} = x;$$

that is, $\mu \circ v(x) = x$. Thus v maps $I - e$ homeomorphically onto

$v(I-e)$. We next consider an arbitrary point $x \in e_N$, $N \geq 1$. Put $K_{n,x}^\pm = H_{n,x \pm}(I)$ for $n \geq N$, respectively. Since $K_{n,x}^\pm$ are the two consecutive connected components of $I - S_n$ such that the left end point of $K_{n,x}^+$ is x and the right end point of $K_{n,x}^-$ is also x , we have

$$\lim_{n \rightarrow \infty} \bar{K}_{n,x}^+ = \lim_{n \rightarrow \infty} \bar{K}_{n,x}^- = x;$$

so $\mu \circ v(x) = \mu \circ v(x \pm) = x$. Similarly we can define $K_{n,0}^+$ and $K_{n,1}^-$ for $n \geq 1$; thus $\mu \circ v(0) = 0$ and $\mu \circ v(1) = 1$. Then we have

LEMMA 3.1. — $v(I-e) = \Sigma - \Lambda$; namely, $w = (w_n) \in v(I-e)$ if and only if

$$\# \{n \geq 1; w_n \neq 1\} = \infty = \# \{n \geq 1; w_n \neq m\}.$$

Proof. — Suppose that $w = v(x) \in \Lambda$ for some $x \in I - e$. Since v is one to one, we have $v(I-e) \cap v(e) = \emptyset$; thus $w \in \Sigma_+ + \Sigma_-$. Hence there exists $y \in e - e_0$ such that either $w = v(y+)$ or $w = v(y-)$. Therefore $x = \mu \circ v(x) = \mu(w) = \mu \circ v(y \pm) = y$. This contradiction implies that $\Lambda \cap v(I-e) = \emptyset$; that is, $v(I-e) \subset \Sigma - \Lambda$. Thus it suffices to show that $\Sigma - \Lambda \subset v(I-e)$.

Suppose now that there exists a word $w = (w_n) \in \Sigma - \Lambda$ such that $w \notin v(I-e)$. Put $z = (z_n) \equiv v \circ \mu(w)$. Then it follows that $w \neq z$. For otherwise, we have $\mu(w) \in e$; thus, $w \in v(e) \subset \Lambda$, contrary to $w \in \Sigma - \Lambda$. Let $N \geq 1$ be the smallest integer such that $w_N \neq z_N$. Since $\mu(w) = \mu \circ v \circ \mu(w) = \mu(z)$, it follows that

$$h_{w_N} \circ h_{w_{N+1}} \circ \cdots = h_{z_N} \circ h_{z_{N+1}} \circ \cdots, \text{ say } p.$$

Then we have $p \in e_1$ and $w, z \in \Lambda_N$, contrary to $w \in \Sigma - \Lambda$. This completes the proof. $\square$

4. Functional Equations.

Let $f_j: I \rightarrow I$ be a contraction for $1 \leq j \leq m$. We assume that $c_0 = 0$ and $c_m = 1$ are unique fixed points of $f_1(x)$ and $f_m(x)$ respectively. The following lemma is a special case of the general theorem obtained by the author [4, p. 397], but we include the proof for completeness.

LEMMA 4.1. — *The functional equations*

$$(4.1) \quad \psi(x) = f_j(\psi(g_j(x))) \quad \text{for} \quad x \in I_j, \quad 1 \leq j \leq m$$

possess a unique continuous solution $\psi(x)$ if and only if

$$(4.2) \quad f_j\left(\frac{1+\varepsilon_j}{2}\right) = f_{j+1}\left(\frac{1-\varepsilon_{j+1}}{2}\right) \quad \text{for} \quad 1 \leq j \leq m-1.$$

Remark. — This is a generalization of the theorem obtained by G. de Rham [10]; indeed he has shown that the equations

$$M\left(\frac{x}{2}\right) = F_0(M(x)), \quad M\left(\frac{1+x}{2}\right) = F_1(M(x)) \quad \text{for} \quad x \in I$$

possess a unique continuous solution $M(x)$ if and only if $F_1(p_0) = F_0(p_1)$ where p_0, p_1 are unique fixed points of the contractions F_0, F_1 respectively. Lebesgue's singular functions and Pólya's space-filling curves satisfy the above equations for certain affine contractions F_0 and F_1 .

Proof. — The conditions (4.2) are obviously necessary; thus it suffices to show the sufficiency. Let $\mathcal{F}$ be the set of continuous functions $u(x)$ defined on I satisfying $u(0) = 0$ and $u(1) = 1$; obviously $\mathcal{F}$ is a closed subset of the Banach space $C([0,1])$ with the usual uniform norm. We now consider the following operator:

$$Tu(x) = f_{A(x)}(u(G(x))).$$

Then it is easily seen that the conditions (4.2) imply that $T(\mathcal{F}) \subset \mathcal{F}$; moreover T is a contraction, since

$$\|Tu - Tv\| \leq \lambda \max_{x \in I} |u(G(x)) - v(G(x))| \leq \lambda \|u - v\|,$$

where $\lambda = \max_{1 \leq j \leq m} \text{Lip}(f_j) \in [1/m, 1)$, for any $u, v \in \mathcal{F}$. Hence T has a unique fixed point ψ in $\mathcal{F}$; namely

$$\psi(x) = f_j(\psi(g_j(x))) \quad \text{for} \quad c_{j-1} \leq x < c_j, \quad 1 \leq j \leq m.$$

Obviously this equality holds also true for $x = c_j$. This completes the proof. $\square$

For $n \geq 1$ and $x, y \in I$, we define

$$F_{n,x}(y) = f_{A_0(x)} \circ f_{A_1(x)} \circ \cdots \circ f_{A_{n-1}(x)}(y).$$

The function $F_{n,x}$ is a contraction satisfying $\text{Lip}(F_{n,x}) \leq \lambda^n$. Put $\beta = \max_{1 \leq j \leq m} \text{Lip}(g_j) \in [m, \infty]$. Then we have

LEMMA 4.2. — Suppose that $\{f_j\}$ satisfy the conditions (4.2). If $\beta < \infty$, then the continuous solution $\psi(x)$ satisfies Hölder's condition of order $\log(1/\lambda)/\log \beta$.

Proof. — Consider arbitrary two points $x < y$ in I . Let $N \geq 0$ be the smallest integer satisfying $\# \{S_{N+1} \cap (x, y)\} \geq 2$. We now distinguish two cases: (a) $S_N \cap (x, y) = \emptyset$; (b) $S_N \cap (x, y)$ consists of a single point, say p . In case (a), it follows that

$$\begin{aligned} |\psi(x) - \psi(y)| &= \lim_{\varepsilon \rightarrow 0+} |\psi(x+\varepsilon) - \psi(y-\varepsilon)| \\ &= \lim_{\varepsilon \rightarrow 0+} |F_{N,x+\varepsilon}(\psi(G^N(x+\varepsilon))) - F_{N,x+\varepsilon}(\psi(G^N(y-\varepsilon)))| \leq \lambda^N. \end{aligned}$$

Similarly we have $|\psi(x) - \psi(y)| \leq 2\lambda^N$ in case (b), since $(x, p) \cap S_N = (p, y) \cap S_N = \emptyset$. Now let $s < t$ be any two consecutive points of e_{N+1} contained in (x, y) . Then it follows that $|x - y| > |s - t| \geq \beta^{-N-1}$; thus

$$|\psi(x) - \psi(y)| \leq 2\lambda^N = \frac{2}{\lambda} \beta^{-\xi(N+1)} \leq \frac{2}{\lambda} |x - y|^\xi$$

where $\xi = \log(1/\lambda)/\log \beta$, which obviously completes the proof. $\square$

5. Some Properties.

The continuous solution $\psi(x)$ of the equations (4.1) is not necessarily singular in general; for example, if we take

$$g_j(x) = mx - j + 1 \quad \text{and} \quad f_j(x) = \frac{x}{m} + \frac{j-1}{m}$$

for $1 \leq j \leq m$, then obviously $\psi(x) \equiv x$ is a smooth solution of (4.1). In this paper, to discuss the singularities of $\psi(x)$, we shall restrict ourselves to the following case:

$$(5.1) \quad \varepsilon_j = 1 + 2 \left[\frac{j}{4} \right] - 2 \left[\frac{j+1}{4} \right]$$

and

$$f_j(x) = \frac{1}{2k} \left\{ (-1)^{[j/2]} x + \left[\frac{j}{2} \right] - \left[\frac{j}{4} \right] + \left[\frac{j-1}{4} \right] \right\}$$

for $1 \leq j \leq m = 4k$, where k is a positive integer; so $\lambda = 1/2k$. Then it is easily seen that the functions $\{f_j\}$ satisfy the conditions (4.2); therefore the equations (4.1) possess a unique continuous solution $\psi(x)$, which depends only on the functions $\{g_j\}$ satisfying the conditions (5.1). Let η_j be the sign of the function f_j ; namely $\eta_j = (-1)^{[j/2]}$, for $1 \leq j \leq 4k$. For brevity, put

$$\varepsilon_{n,x} = \prod_{j=0}^{n-1} \varepsilon_{A_j(x)} \quad \text{and} \quad \eta_{n,x} = \prod_{j=0}^{n-1} \eta_{A_j(x)}$$

for $n \geq 1$, $x \in I$.

Consider now an arbitrary point $x \in I - e$. We define

$$p_{j,n,x} = H_{n,x}(c_j) \quad \text{for} \quad n \geq 1, \quad 0 \leq j \leq 4k.$$

Obviously $p_{j,n,x} \neq x$. Since $p_{j,n,x} \in G^{-n}(c_j) \subset e_{n+1}$ for $1 \leq j \leq 4k - 1$, we have

$$G^n(p_{j,n,x}) = c_j \quad \text{for} \quad 1 \leq j \leq 4k - 1.$$

The points $p_{0,n,x}$ and $p_{4k,n,x}$ are two end points of $K_{n,x}$ and do not satisfy the above equality in general; however,

$$\lim_{\substack{y \rightarrow p_{j,n,x} \\ y \in K_{n,x}}} G^n(y) = c_j \quad \text{for} \quad j = 0, 4k.$$

Note that $0 < |x - p_{j,n,x}| < \gamma^n$ for any $n \geq 1$. Then we have

LEMMA 5.1. — Suppose that $x \in I - e$. Then the points $\{p_{j,n,x}\}$ satisfy the following properties :

- (1) $\text{sign}(x - p_{j,n,x}) = \varepsilon_{n,x} \text{sign} \left\{ A_n(x) - j - \frac{1}{2} \right\},$
- (2) $\psi(x) - \psi(p_{j,n,x}) = \frac{\eta_{n,x}}{(2k)^n} \left\{ \psi(G^n(x)) - \frac{1 - (-1)^j}{4k} - \frac{1}{k} \left[\frac{j}{4} \right] \right\}$

for $n \geq 1$ and $0 \leq j \leq 4k$.

Proof. — Since $p_{j,n,x} = H_{n,x}(c_j)$, we have

$$\text{sign}(x - p_{j,n,x}) = \text{sign} \{ H_{n,x}(G^n(x)) - H_{n,x}(c_j) \} = \varepsilon_{n,x} \text{sign} \{ G^n(x) - c_j \};$$

thus the property (1) follows immediately. Since $K_{n,x} \cap S_n = \phi$,

$$\psi(p_{j,n,x}) = \lim_{\substack{y \rightarrow p_{j,n,x} \\ y \in K_{n,x}}} \psi(y) = \lim_{\substack{y \rightarrow p_{j,n,x} \\ y \in K_{n,x}}} F_{n,x}(\psi(G^n(y))) = F_{n,x}(\psi(c_j))$$

for $0 \leq j \leq 4k$; hence

$$\psi(x) - \psi(p_{j,n,x}) = F_{n,x}(\psi(G^n(x))) - F_{n,x}(\psi(c_j)) \frac{\eta_{n,x}}{(2k)^n} \{\psi(G^n(x)) - \psi(c_j)\},$$

which obviously completes the proof. $\square$

We now consider an arbitrary point $x \in e_N$, $N \geq 1$. Then it is easily seen that, for $1 \leq j \leq 4k - 1$, each of the sets $K_{n,x}^\pm$ contains exactly one point of $G^{-n}(c_j) \subset e_{n+1}$, say $q_{j,n,x}^\pm$ respectively. Obviously $q_{j,n,x}^\pm \neq x$. Similarly we can define $\{q_{j,n,0}^+\}$ and $\{q_{j,n,1}^-\}$ for $n \geq 0$, $1 \leq j \leq 4k - 1$. Note that $0 < |x - q_{j,n,x}^\pm| < \gamma^n$ for any $n \geq N$. It also follows that

$$\lim_{\varepsilon \rightarrow 0^\pm} G^n(x + \varepsilon) = \frac{1}{2} (1 \mp \varepsilon_{N,x^\pm})$$

for every $n \geq N$, respectively. We, of course, adopt the rule: $\varepsilon_{0,0^+} = \varepsilon_{0,1^-} = \eta_{0,0^+} = \eta_{0,1^-} = 1$. Then we have

LEMMA 5.2. — Suppose that $x \in e_N$, $N \geq 0$. Then the points $\{q_{j,n,x}^\pm\}$ satisfy the following:

$$\psi(x) - \psi(q_{j,n,x}^\pm) = \frac{\eta_{N,x^\pm}}{(2k)^n} \left\{ \frac{1}{2} (1 \mp \varepsilon_{N,x^\pm}) - \frac{1 - (-1)^j}{4k} - \frac{1}{k} \left[\frac{j}{4} \right] \right\}$$

for $n \geq N$ and $1 \leq j \leq 4k - 1$, respectively.

Proof. — Since $K_{n,x}^\pm \cap S_n = \phi$, we have

$$\begin{aligned} \psi(x) - \psi(q_{j,n,x}^\pm) &= \lim_{\varepsilon \rightarrow 0^\pm} \{\psi(x + \varepsilon) - \psi(q_{j,n,x}^\pm)\} = \\ &= \lim_{\varepsilon \rightarrow 0^\pm} \{F_{n,x+\varepsilon}(\psi(G^n(x + \varepsilon))) - F_{n,x+\varepsilon}(\psi(c_j))\} = \frac{\eta_{N,x^\pm}}{(2k)^n} \left\{ \frac{1}{2} (1 \mp \varepsilon_{N,x^\pm}) - \psi(c_j) \right\} \end{aligned}$$

for every $n \geq N$, respectively. This completes the proof. $\square$

6. Singularities.

For any $x \neq y \in I$, we define $\Delta\psi(x, y) = (\psi(x) - \psi(y))/(x - y)$. Let W be the set of points $x \in I$ at which $A_n(x) \equiv 2$ or $3 \pmod{4}$ for infinitely many n 's. Obviously $W \subset I - e$. First of all, we have

THEOREM 6.1. — *Suppose that $\gamma \leq 1/2k$. Then we have*

$$D^{\pm}\psi(x) \geq 0 \geq D_{\pm}\psi(x) \quad \text{and} \quad D^{\pm}\psi(x) - D_{\pm}\psi(x) \geq 1/4k$$

respectively, for every $x \in W$.

Proof. — We distinguish two cases (not exclusive) as follows :

Case A. $A_n(x) \equiv 3 \pmod{4}$ for infinitely many n 's.

Let $0 < n_1 < n_2 < \dots$ be the subsequence of integers such that $A_{n_i}(x) = 4N_i + 3$, where $0 \leq N_i < k$. From the functional equations (4.1), we have

$$\frac{N_i}{k} \leq \psi(G^{n_i}(x)) \leq \frac{2N_i + 1}{k};$$

therefore $\{\psi(x) - \psi(P_{i,1})\}\{\psi(x) - \psi(P_{i,2})\} \leq 0$ by (2) of Lemma 5.1, where $p_{i,j} = p_{4N_i+j, n_i, x}$ for $0 \leq j \leq 4$. On the other hand, we have $\text{sign}(x - P_{i,1}) = \text{sign}(x - P_{i,2}) = \varepsilon_{n_i, x}$ by (1) of Lemma 5.1. Since $\varepsilon_{n_i, x}$ changes the sign infinitely many times as i increases, it follows that $D^{\pm}\psi(x) \geq 0 \geq D_{\pm}\psi(x)$. It also follows that

$$|\Delta\psi(x, P_{i,1})| + |\Delta\psi(x, P_{i,2})| \geq \frac{(2k)^{-n_i-1}}{|x - P_{i,1}|} > \frac{1}{2k} (2k\gamma)^{-n_i} \geq \frac{1}{2k};$$

therefore $D^{\pm}\psi(x) - D_{\pm}\psi(x) \geq 1/4k$ respectively, as required.

Case B. $A_n(x) \equiv 2 \pmod{4}$ for infinitely many n 's.

Let $0 < n_1 < n_2 < \dots$ be the subsequence of integers such that $A_{n_i}(x) = 4N_i + 2$, where $0 \leq N_i < k$. Since

$$\frac{N_i}{k} \leq \psi(G^{n_i}(x)) \leq \frac{2N_i + 1}{k},$$

it is easily seen that $\{\psi(x) - \psi(P_{i,0})\}\{\psi(x) - \psi(P_{i,1})\} \leq 0$ and $\{\psi(x) - \psi(P_{i,2})\}\{\psi(x) - \psi(P_{i,3})\} \leq 0$. On the other hand, we have

$\text{sign}(x - P_{i,0}) = \text{sign}(x - P_{i,1}) = \text{sign}(P_{i,2} - x) = \text{sign}(P_{i,3} - x)$; therefore $D^\pm \psi(x) \geq 0 \geq D_\pm \psi(x)$. Moreover,

$$|\Delta\psi(x, P_{i,0})| + |\Delta\psi(x, P_{i,1})| \geq \frac{(2k)^{-n_i-1}}{|x - P_{i,0}|} > \frac{1}{2k} (2k\gamma)^{-n_i} \geq \frac{1}{2k}.$$

The same estimate holds true if we replace $P_{i,0}, P_{i,1}$ by $P_{i,2}, P_{i,3}$, respectively; thus $D^\pm \psi(x) - D_\pm \psi(x) \geq 1/4k$ respectively. This completes the proof. $\square$

Let $W_0 \subset W$ be the set of points $x \in I$ at which $A_n(x) \equiv 2$ or $3 \pmod{4}$ and $A_{n+1}(x) \equiv 2$ or $3 \pmod{4}$ for infinitely many n 's. Then we have

THEOREM 6.2. — Suppose that $\gamma \leq 1/2k$. Then W_0 is contained in the set $\text{Knot}(\psi)$ except for a set of measure zero.

Proof. — We consider an arbitrary point x of W_0 . Let $0 \leq n_1 < n_2 < \dots$ be the subsequence of integers such that $A_{n_i}(x) = 4N_i + \delta_i$ and $A_{n_i+1}(x) = 4L_i + \omega_i$, where $0 \leq N_i, L_i < k$ and $2 \leq \delta_i, \omega_i \leq 3$. Then it is easily seen that

$$\frac{2N_i + 1}{2k} - \frac{2L_i + 1}{(2k)^2} \leq \psi(G^{n_i}(x)) \leq \frac{2N_i + 1}{k} - \frac{L_i}{2k^2};$$

therefore by (2) of Lemma 5.1,

$$\eta_{n_i, x}(2k)^{n_i} \{\psi(x) - \psi(P_{i,0})\} = \psi(G^{n_i}(x)) - \frac{N_i}{k} \geq \frac{1}{2k} - \frac{2L_i + 1}{(2k)^2} \geq (2k)^{-2}.$$

Similarly we have

$$\eta_{n_i, x}(2k)^{n_i} \{\psi(P_{i,4}) - \psi(x)\} = \frac{N_i + 1}{k} - \psi(G^{n_i}(x)) \geq \frac{1}{2k} + \frac{L_i}{2k^2} \geq \frac{1}{2k}.$$

Therefore, since $\text{sign}(x - P_{i,0}) = \text{sign}(P_{i,4} - x)$, it follows that

$$\text{sign}(\Delta\psi(x, P_{i,0})) = \text{sign}(\Delta\psi(x, P_{i,4}))$$

and

$$|\Delta\psi(x, P_{i,0})| > (2k)^{-2}, \quad |\Delta\psi(x, P_{i,4})| > \frac{1}{2k}.$$

Hence the set $[D_+ \psi(x), D^+ \psi(x)] \cap [D_- \psi(x), D^- \psi(x)]$ contains an interval of length $(2k)^{-2}$ by Theorem 6.1. Thus it follows from Denjoy's theorem

[2, p. 105] that except for a set of measure zero, every point of W_0 is a knot point of $\psi(x)$. This completes the proof. $\square$

For $N \geq 0$, let Y_N be the set of points $x \in I$ at which $A_n(x) \equiv 0$ or $1 \pmod{4}$ for all $n \geq N$ and $A_{N-1}(x) \equiv 2$ or $3 \pmod{4}$. Obviously $I - W = \bigcup_{n \geq 0} Y_n$. For brevity, put $Y_n^* = Y_n \cap (I - e)$ for $n \geq 0$. Then the unit interval I is decomposed as follows :

$$I = W + e + \bigcup_{n \geq 0} Y_n^*.$$

For $n \geq 1$, let Ξ_n be the set of finite words $(w_1, \dots, w_n)$ of length n such that $1 \leq w_j \leq 4k$ and $w_j \equiv 0$ or $1 \pmod{4}$ for $1 \leq j \leq n$. Then we have

THEOREM 6.3. — *Suppose that there exists a positive constant C_0 , independent of n , satisfying*

$$\min_{(w_1 \dots w_n) \in \Xi_n} |h_{w_1} \circ \dots \circ h_{w_n}(I)| \geq C_0(2k)^{-n}$$

for all $n \geq 1$. Suppose further that $\beta < \infty$. Then we have

$$D^\pm \psi(x) - D_\pm \psi(x) \geq \frac{1}{2k}$$

respectively, for every $x \in I - W$.

Proof. — We distinguish two cases as follows :

Case A. $x \in Y_N^*$ for some $N \geq 0$.

By Lemma 3.1, we have $A_n(x) \neq 1$ for infinitely many n 's. Let $N \leq n_1 < n_2 < \dots$ be the subsequence of integers such that $A_{n_i}(x) \geq 4$. Put $Q_{i,j} = p_{j,n_i,x}$ for $0 \leq j \leq 2$. Since

$$\psi(G^{n_i}(x)) \geq \frac{1}{2k}$$

and $\text{sign}(x - Q_{i,1}) = \text{sign}(x - Q_{i,2}) = \text{sign}(Q_{i,2} - Q_{i,1}) = \varepsilon_{N,x}$, we have

$$|\Delta\psi(x, Q_{i,1}) - \Delta\psi(x, Q_{i,2})| =$$

$$(2k)^{-n_i} \left| \psi(G^{n_i}(x)) \left\{ \frac{1}{x - Q_{i,2}} - \frac{1}{x - Q_{i,1}} \right\} + \frac{1}{2k(x - Q_{i,1})} \right| \geq \frac{(2k)^{-n_i-1}}{|x - Q_{i,1}|} > \frac{1}{2k}.$$

On the other hand, it follows that

$$|x - Q_{i,0}| > |Q_{i,1} - Q_{i,0}| \geq \beta^{-N} \left| h_{A_N(x)} \circ \dots \circ h_{A_{n_i-1}(x)} \circ h_1(I) \right| \geq C_0 \beta^{-N} (2k)^{-n_i+N-1};$$

therefore

$$|\Delta\psi(x, Q_{i,0})| = (2k)^{-n_i} \left| \frac{\psi(G^{n_i}(x))}{x - Q_{i,0}} \right| \leq \frac{2k}{C_0} \left(\frac{\beta}{2k} \right)^N.$$

Since $\text{sign}(x - Q_{i,0}) = \varepsilon_{N,x}$, we conclude that either $[D_+\psi(x), D^+\psi(x)]$ or $[D_-\psi(x), D^-\psi(x)]$ contains an interval of length $1/2k$ according as $\varepsilon_{N,x} = -1$ or $+1$.

It also follows from Lemma 3.1 that $A_n(x) \neq 4k$ for infinitely many n 's. Let $N \leq n_1 < n_2 < \dots$ be the subsequence of integers such that $A_{n_i}(x) \leq 4k - 3$. Put $R_{i,j} = p_{4k-j, n_i, x}$ for $0 \leq j \leq 3$. Since

$$\psi(G^{n_i}(x)) \leq \frac{2k-1}{2k}$$

and $\text{sign}(x - R_{i,2}) = \text{sign}(x - R_{i,3}) = \text{sign}(R_{i,3} - R_{i,2}) = -\varepsilon_{N,x}$, we have

$$\begin{aligned} |\Delta\psi(x, R_{i,2}) - \Delta\psi(x, R_{i,3})| &= \\ (2k)^{-n_i} \left| \left\{ \frac{2k-1}{2k} - \psi(G^{n_i}(x)) \right\} \left\{ \frac{1}{x - R_{i,3}} - \frac{1}{x - R_{i,2}} \right\} + \frac{1}{2k(x - R_{i,2})} \right| &\geq \\ \frac{(2k)^{-n_i-1}}{|x - R_{i,2}|} &> \frac{1}{2k}. \end{aligned}$$

On the other hand, $|x - R_{i,0}| > |R_{i,1} - R_{i,0}| \geq C_0 \beta^{-N} (2k)^{-n_i+N-1}$; thus

$$|\Delta\psi(x, R_{i,0})| = (2k)^{-n_i} \left| \frac{\psi(G^{n_i}(x)) - 1}{x - R_{i,0}} \right| \leq \frac{2k}{C_0} \left(\frac{\beta}{2k} \right)^N.$$

Since $\text{sign}(x - R_{i,0}) = -\varepsilon_{N,x}$, it follows that either $[D_+\psi(x), D^+\psi(x)]$ or $[D_-\psi(x), D^-\psi(x)]$ contains an interval of length $1/2k$ according as $\varepsilon_{N,x} = +1$ or -1 . Hence $D^\pm\psi(x) - D_\pm\psi(x) \geq 1/2k$ respectively.

Case B. $x \in e_N$ for some $N \geq 0$.

For $n \geq N$, let $Q_n^+ = \max\{q_{1,n,x}^+, q_{3,n,x}^+\}$, $Q_n^- = \min\{q_{1,n,x}^-, q_{3,n,x}^-\}$ and let $R_n^+ = q_{2,n,x}^+$ respectively. Then $Q_n^- < R_n^- < x < R_n^+ < Q_n^+$.

Since $\text{sign}(x - Q_n^\pm) = \text{sign}(Q_n^\pm - R_n^\pm) = \pm 1$ respectively, it follows from Lemma 5.2 that

$$|\Delta\psi(x, R_n^\pm) - \Delta\psi(x, Q_n^\pm)| = (2k)^{-n} \left| \frac{1}{2} (1 \mp \varepsilon_{N, x^\pm}) \left\{ \frac{1}{x - R_n^\pm} - \frac{1}{x - Q_n^\pm} \right\} + \frac{1}{2k(x - Q_n^\pm)} \right| \geq \frac{(2k)^{-n-1}}{|x - Q_n^\pm|} > \frac{1}{2k},$$

respectively. On the other hand, we have

$$|x - R_n^\pm| > |K_{n+1, x}^\pm| \geq \beta^{-N} |h_{A_N(x^\pm)} \circ \dots \circ h_{A_n(x^\pm)}(I)| \geq C_0 \beta^{-N} (2k)^{-n+N-1};$$

therefore

$$|\Delta\psi(x, R_n^\pm)| \leq \frac{(2k)^{-n}}{|x - R_n^\pm|} < \frac{2k}{C_0} \left(\frac{\beta}{2k} \right)^N.$$

Hence $D^+\psi(x) - D_+\psi(x) \geq 1/2k$ respectively. This completes the proof. $\square$

Let $Y^* = \bigcup_{n \geq 0} Y_n^*$ for brevity. Then we have

THEOREM 6.4. — Knot $(\psi) \cap Y^* = \emptyset$.

Proof. — We consider an arbitrary point x of Y_N^* for some $N \geq 0$. Let $s_n = p_{0, n, x}$ for $n \geq N$. Since $\text{sign}(x - s_n) = \varepsilon_{N, x}$ is independent of $n \geq N$, the sequence $\{s_n\}$ is monotone, either increasing or decreasing, and converges to x . Note that $s_n = s_{n+1}$ if and only if $A_n(x) = 1$. Put $J_n = [s_n, s_{n+1}] \subset \bar{K}_{n, x}$ for $n \geq N$. Then it is easily seen that

$$(x, s_N] = \bigcup_{n \geq N} J_n.$$

Since the function $G^n(x)$ maps $K_{n, x}$ homeomorphically onto $(0, 1)$, we have $A_n(x) > A_n(y)$ for all $y \in J_n$. Therefore

$$\psi(G^n(x)) \geq f_{A_n(x)}(0) \geq \max_{j < A_n(x)} \|f_j\| \geq \psi(G^n(y));$$

thus

$$\eta_{N, x} \text{sign} \{\psi(x) - \psi(y)\} = \eta_{N, x} \text{sign} \{F_{n, x}(\psi(G^n(x))) - F_{n, x}(\psi(G^n(y)))\} = \text{sign} \{\psi(G^n(x)) - \psi(G^n(y))\} \geq 0.$$

By the continuity of ψ , we conclude that

$$\eta_{N, x} \text{sign} \{\psi(x) - \psi(y)\} \geq 0 \quad \text{for every } y \in [x, s_N].$$

This means that x is not a knot point of $\psi(x)$. $\square$

7. Proof of Theorem 2.1.

First of all, for any integer $k \geq 1$ and positive numbers σ , τ , ρ satisfying

$$(7.1) \quad 2k(\sigma + \tau) < 1 \quad \text{and} \quad \sigma \geq \rho,$$

we shall construct two Cantor sets $E_0 \equiv E_0(k, \sigma, \tau)$ and $E_1 \equiv E_1(k, \sigma, \rho)$. The set $E_0(k, \sigma, \tau)$ is obtained from the unit interval I by a sequence of deletions of open intervals known as middle thirds, as follows: First divide I into k equal parts, say

$$I_{1,1} = \left[0, \frac{1}{k}\right], \quad I_{1,2} = \left[\frac{1}{k}, \frac{2}{k}\right], \quad \dots, \quad I_{1,k} = \left[\frac{k-1}{k}, 1\right],$$

and remove from each closed interval $I_{1,j}$ the open interval $U_{1,j}$ centered at $(2j-1)/2k$ and of length 2σ . We subdivide each of the $2k$ remaining closed intervals into k equal parts, say $I_{2,j}$, $1 \leq j \leq 2k^2$, ordered from left to right, each of length $(1-2k\sigma)/(2k^2)$. Then remove from each closed interval $I_{2,j}$ the middle open interval $U_{2,j}$ of length $2\sigma\tau$, leaving the $4k^2$ closed intervals, each of length $(1-2k\sigma-4k^2\sigma\tau)/(4k^2)$. This process is permitted to continue indefinitely. At the n th stage of deletion, each length of the $2^{n-1}k^n$ open intervals $U_{n,j}$ is $2\sigma\tau^{n-1}$, and therefore the measure of the union of the open intervals removed in the entire sequence of removal operations is $2k\sigma/(1-2k\tau)$. The set E_0 is defined to be the closed set remaining; thus

$$|E_0| = \frac{1 - 2k(\sigma + \tau)}{1 - 2k\tau}.$$

We next define the set $E_1(k, \sigma, \rho)$, which is slightly different from E_0 defined above, as follows: First divide the unit interval I into k equal parts, say

$$J_{1,1} = \left[0, \frac{1}{k}\right], \quad J_{1,2} = \left[\frac{1}{k}, \frac{2}{k}\right], \quad \dots, \quad J_{1,k} = \left[\frac{k-1}{k}, 1\right].$$

Then remove from each closed interval $J_{1,j}$ the two intervals

$$V_{1,j}^- = \left[\frac{j-1}{k}, \frac{2j-1-2k\sigma}{2k}\right), \quad V_{1,j}^+ = \left(\frac{2j-1+2k\sigma}{2k}, \frac{j}{k}\right],$$

each of length $(1-2k\sigma)/2k$. We subdivide each of the k remaining closed intervals into $2k$ equal parts, say $J_{2,j}$, $1 \leq j \leq 2k^2$, ordered

from left to right, each of length σ/k . Then delete from each closed interval $J_{2,j}$ the two intervals $V_{2,j}^\pm$ of length $\rho(1-2k\sigma)/2k$, leaving the $2k^2$ middle closed intervals, each of length $(\sigma-\rho+2k\sigma\rho)/k$. At the n th stage of deletion, we have $|V_{n,j}^\pm| = \rho^{n-1}(1-2k\sigma)/2k$; therefore the measure of the union of the removed intervals in the entire sequence of removal operations is $(1-2k\sigma)/(1-2k\rho)$. The set E_1 is defined to be the closed set remaining; thus

$$|E_1| = \frac{2k(\sigma-\rho)}{1-2k\rho}.$$

Note that the set E_1 is contained in $\left[\frac{1-2k\sigma}{2k(1-\rho)}, 1 - \frac{1-2k\sigma}{2k(1-\rho)}\right]$.

We now define the continuous function $\zeta_0(x) \equiv \zeta_0(k, \sigma, \tau; x)$ by setting

$$\zeta_0(x) = \int_0^x d_0(s) ds \quad \text{for} \quad 0 \leq x \leq 1,$$

where $d_0(s) = 1/2k$ if $s \in E_0(k, \sigma, \tau)$ and $d_0(s) = \tau$ otherwise. We also define the continuous function $\zeta_1(x) \equiv \zeta_1(k, \sigma, \rho; x)$ by setting

$$\zeta_1(x) = \frac{1}{2k} - \sigma + \int_0^x d_1(s) ds \quad \text{for} \quad 0 \leq x \leq 1,$$

where $d_1(s) = 1/2k$ if $s \in E_1(k, \sigma, \rho)$ and $d_1(s) = \rho$ otherwise. Then it is easily seen that $\zeta_0(I) = [0, (1-2k\sigma)/2k]$, $\zeta_1(I) = [(1-2k\sigma)/2k, 1/2k]$ and $\zeta_i(E_i) = E_i \cap \zeta_i(I)$ for $i = 0, 1$.

We next define, for $0 \leq i < k$,

$$\begin{aligned} g_{4i+1}(x) &= \zeta_0^{-1}\left(x - \frac{i}{k}\right) & \text{for } x \in I_{4i+1} &= \left[\frac{i}{k}, \frac{2i+1}{2k} - \sigma\right], \\ g_{4i+2}(x) &= \zeta_1^{-1}\left(x - \frac{i}{k}\right) & \text{for } x \in I_{4i+2} &= \left[\frac{2i+1}{2k} - \sigma, \frac{2i+1}{2k}\right], \\ g_{4i+3}(x) &= \zeta_1^{-1}\left(\frac{i+1}{k} - x\right) & \text{for } x \in I_{4i+3} &= \left[\frac{2i+1}{2k}, \frac{2i+1}{2k} + \sigma\right], \\ g_{4i+4}(x) &= \zeta_0^{-1}\left(x - \frac{2i+1}{2k} - \sigma\right) & \text{for } x \in I_{4i+4} &= \left[\frac{2i+1}{2k} + \sigma, \frac{i+1}{k}\right]; \end{aligned}$$

thus the unit interval I is divided into $m = 4k$ subintervals $I_j = [c_{j-1}, c_j]$. We have $|I_{4i+1}| = |I_{4i+4}| = (1-2k\sigma)/2k$ and $|I_{4i+2}| = |I_{4i+3}| = \sigma$. Obviously the functions $g_j(x)$ satisfy the conditions (5.1) and we denote

by $\psi(k, \sigma, \tau, \rho; x)$ the corresponding continuous solution of the equations (4.1).

It follows from Theorems 6.1 and 6.3 that $\psi(k, \sigma, \tau, \rho; x)$ has nowhere a unilateral derivative finite or infinite for any integer k and positive numbers σ, τ, ρ satisfying (7.1), since we have

$$\gamma = \frac{1}{2k}, \quad \beta = \max \left\{ \frac{1}{\rho}, \frac{1}{\tau} \right\}$$

and

$$|h_{w_1} \circ \cdots \circ h_{w_n}(I)| = \frac{1}{(2k)^n} - \frac{\sigma}{(2k)^{n-1}} - \frac{\sigma\tau}{(2k)^{n-2}} - \cdots - \sigma\tau^{n-1} > \frac{|E_0|}{(2k)^n},$$

for every finite word $(w_1 \dots w_n) \in \Xi_n$.

Since the Cantor set E_0 is a unique compact subset of I satisfying

$$E_0 = h_1(E_0) \cup h_4(E_0) \cup h_5(E_0) \cup \cdots \cup h_{4k}(E_0)$$

and since the mapping v maps Y_0^* homeomorphically onto $v(Y_0^*)$, it follows that $\bar{Y}_0^* = E_0$. On the other hand, for every $x \in W + \bigcup_{n \geq 1} Y_n^*$, there exist $n = n(x)$ and $j = j(x)$ such that $x \in U_{n,j}$; thus $E_0 \subset Y_0^* + e$. Therefore $|Y_0^*| = |E_0|$, since e is countable. Let Ω_n be the set of finite words $(w_1 \dots w_n)$ of length n such that $1 \leq w_j \leq 4k$ for $1 \leq j \leq n$. Then for any $n \geq 0$, the set Y_{n+1}^* is decomposed as follows:

$$Y_{n+1}^* = \bigcup_{\substack{(w_1 \dots w_n) \in \Omega_n \\ j \in \Omega_1 - \Xi_1}} h_{w_1} \circ \cdots \circ h_{w_n} \circ h_j(Y_0^*).$$

On each interval $V_{1,j}^\pm$, for any $(w_1 \dots w_n) \in \Omega_n$ and $j \in \Omega_1 - \Xi_1$, the function $h_{w_1} \circ \cdots \circ h_{w_n} \circ h_j(x)$ is a linear contraction; more precisely we have

$$\left| \frac{d}{dx} (h_{w_1} \circ \cdots \circ h_{w_n} \circ h_j)(x) \right| = \rho^{n+1-r(w)} \tau^{r(w)} \quad \text{for } x \in V_{1,j}^\pm$$

where $r(w) \equiv r(w_1, \dots, w_n) = \frac{1}{2} \sum_{j=1}^n (1 + \eta_{w_j})$. Since $Y_0^* \cap U_{1,j} = \emptyset$ for all j , we have

$$|Y_{n+1}^*| = 2k |Y_0^*| \sum_{(w_1 \dots w_n) \in \Omega_n} \rho^{n+1-r(w)} \tau^{r(w)} = 2k\rho |E_0| (2k(\rho + \tau))^n.$$

Therefore it follows that

$$|Y^*| = \sum_{n=0}^{\infty} |Y_n^*| = |E_0| + 2k\rho|E_0| \sum_{n=0}^{\infty} (2k(\rho+\tau))^n = \frac{1 - 2k(\sigma+\tau)}{1 - 2k(\rho+\tau)}.$$

For $N \geq 0$, let Z_N be the set of points $x \in I$ at which $A_n(x) \equiv 2$ or $3 \pmod{4}$ for all $n \geq N$ and $A_{N-1}(x) \equiv 0$ or $1 \pmod{4}$. Put $Z = \bigcup_{n \geq 0} Z_n$. Obviously $Z \subset W_0 \subset I - e$. Then it is easily seen that the set Z_0 is a compact subset of I satisfying

$$Z_0 = h_2(Z_0) \cup h_3(Z_0) \cup h_6(Z_0) \cup \dots \cup h_{4k-1}(Z_0);$$

therefore $Z_0 = E_1$. For any $n \geq 0$, the set Z_{n+1} is decomposed as follows:

$$Z_{n+1} = \bigcup_{\substack{(w_1 \dots w_n) \in \Omega_n \\ j \in \Xi_1}} h_{w_1} \circ \dots \circ h_{w_n} \circ h_j(Z_0).$$

On each open interval $U_{1,j}$, for any $(w_1 \dots w_n) \in \Omega_n$ and $j \in \Xi_1$, the function $h_{w_1} \circ \dots \circ h_{w_n} \circ h_j(x)$ is a linear contraction such that

$$\left| \frac{d}{dx} (h_{w_1} \circ \dots \circ h_{w_n} \circ h_j) \right| = \rho^{n-r(w)} \tau^{1+r(w)} \quad \text{for } x \in U_{1,j}.$$

Since $Z_0 \cap V_{1,j}^\pm = \emptyset$ for all j , we have

$$|Z_{n+1}| = 2k|Z_0| \sum_{(w_1 \dots w_n) \in \Omega_n} \rho^{n-r(w)} \tau^{1+r(w)} = 2k\tau|E_1|(2k(\rho+\tau))^n;$$

therefore

$$|Z| = \sum_{n=0}^{\infty} |Z_n| = |E_1| + 2k\tau|E_1| \sum_{n=0}^{\infty} (2k(\rho+\tau))^n = \frac{2k(\sigma-\rho)}{1 - 2k(\rho+\tau)} \leftarrow 1 - |Y^*|.$$

Then it follows from Theorems 6.2 and 6.4 that

$$|Z| \leq |W_0| \leq |\text{Knot}(\psi)| \leq 1 - |Y^*| = |Z|;$$

hence we obtain

$$|\text{Knot}(\psi)| = \frac{2k(\sigma-\rho)}{1 - 2k(\rho+\tau)}.$$

Thus if we take, for a fixed number $\alpha \in [0, 1)$,

$$\sigma_0 = \frac{1 + \alpha}{8k}, \quad \tau_0 = \frac{1}{4k} \quad \text{and} \quad \rho_0 = \frac{1}{8k},$$

then the function $\psi_0(x) \equiv \psi(k, \sigma_0, \tau_0, \rho_0; x)$ satisfies $|\text{Knot}(\psi_0)| = \alpha$ and Hölder's condition of order $\log(2k)/\log(8k)$ by Lemma 4.2, which obviously converges to 1 as k tends to infinity. This completes the proof of Theorem 2.1. $\square$

Remark. — Besicovitch's function $B(x)$ illustrated in Figure 1 is precisely equal to the function $\psi(1, 1/8, 1/4, 1/8; x)$; thus $B(x)$ satisfies Hölder's condition of order $1/3$.

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