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ON CONTINUOUS FUNCTIONS WITH NO UNILATERAL DERIVATIVES

by Masayoshi HATA

1. Introduction.

It is known that A. S. Besicovitch in 1925 gave the first example of a continuous function $B(x)$ which has nowhere a unilateral derivative *finite or infinite* by geometrical process. E. D. Pepper [9] has examined this same function $B(x)$, giving a different exposition. The graph of his function is illustrated in Figure 1. Later, A. N. Singh [12, 13] gave the arithmetical definition of $B(x)$ and constructed an infinite class of such non-differentiable functions. On the other hand, A. P. Morse [8] gave an example of a continuous function $f(x)$ satisfying

$$\liminf_{s\to x\pm}\left|\frac{f(s)-f(x)}{s-x}\right| < \limsup_{s\to x\pm}\left|\frac{f(s)-f(x)}{s-x}\right| = \infty$$

respectively, for every $x\in(0,1)$, by arithmetical process.

It seems, however, that their methods are somewhat complicated and inappropriate to the study concerning further properties of such functions. In the present paper we shall develop a simple but powerful method to construct and analyze such singular functions by using certain one-dimensional dynamical systems.

The difficulties of finding such functions may be explained by the fact that the set of functions which have nowhere a unilateral derivative finite or infinite is of only the first category in the space of continuous functions (S. Saks [11]), while the set of functions which have nowhere a finite unilateral derivative is of the second category (S. Banach [1], S. Mazurkiewicz [7] and V. Jarnik [5]).

Key-words : Non-differentiable functions - Knot points - Functional equations.

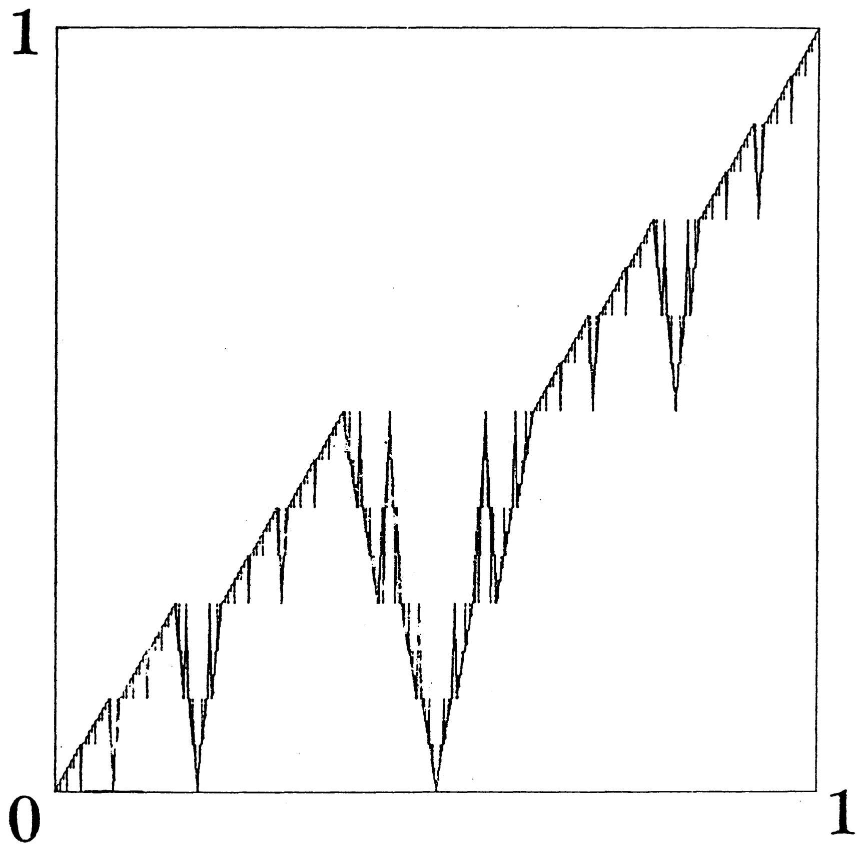

Fig. 1.

2. Main Result.

To state our main theorem, we need some definitions and notations. We denote, as usual, the upper and lower derivatives at x of a real-valued function $f(x)$ on the right by $D^+f(x)$, $D_+f(x)$ respectively. Similarly the upper and lower derivatives, on the left, are denoted by $D^-f(x)$, $D_-f(x)$ respectively. A point x is said to be a knot point of $f(x)$ provided that

$$D^+f(x) = D^-f(x) = \infty \qquad \text{and} \qquad D_+f(x) = D_-f(x) = -\infty .$$

The set of knot points of $f(x)$ is denoted by Knot (f). For a measurable set E, we denote by $|E|$ the Lebesgue measure of E. Our theorem can now be stated as follows :

THEOREM 2.1. — *For any* $\alpha \in [0,1)$ *and* $\varepsilon \in (0,1)$, *there exists a continuous function* $\psi_{\alpha,\varepsilon}(x)$ *defined on the unit interval* I *sayisfying the following properties :*

(1) $\psi_{\alpha,\varepsilon}(x)$ *has nowhere a unilateral derivative finite or infinite ;*

(2) $|\mathrm{Knot}\,(\psi_{\alpha,\varepsilon})| = \alpha$;

(3) $\psi_{\alpha,\varepsilon}(x)$ *satisfies Hölder's condition of order* $1 - \varepsilon$.

Remark. — K. M. Garg [3] has shown that the set of knot points of Besicovitch's function is of measure zero. He also showed that, for every continuous function defined on I which has nowhere a unilateral derivative finite or infinite, the set of points at which the upper derivative on one side is $+\infty$, the lower derivative on the other side is $-\infty$, and the other two derivatives are finite and equal has a positive measure in every subinterval of I; therefore the constant α in our theorem can not be taken to be 1. Note that the set $\mathrm{Knot}\,(f)$ is of the second category if $f(x)$ is a continuous function which has nowhere a finite or infinite derivative (W. H. Young [14]).

As a corollary, we have immediately

COROLLARY 2.2. — *For any* $\alpha \in [0,2\pi)$ *and* $\varepsilon \in (0,1)$, *there exists an absolutely convergent cosine Fourier series*

$$\Psi_{\alpha,\varepsilon}(x) = \sum_{n=0}^{\infty} a_{\alpha,\varepsilon,n} \cos\, nx$$

satisfying the following properties :

(1) $\Psi_{\alpha,\varepsilon}(x)$ *has nowhere a unilateral derivative finite or infinite;*

(2) $|\mathrm{Knot}\,(\Psi_{\alpha,\varepsilon}|_{[0,2\pi]})| = \alpha$;

(3) $\sum_{n=1}^{\infty} |a_{\alpha,\varepsilon,n}|^2 n^{2-\varepsilon} < \infty$.

For the proof of Theorem 2.1, we shall introduce a symbol space in section 3 and certain functional equations in section 4. The fundamental properties of the solution are investigated in sections 5 and 6. We then prove Theorem 2.1 in section 7 using Cantor sets of positive measure.

3. Preliminaries.

We first divide the unit interval I into m subintervals

$$I_1 = [c_0, c_1],\ I_2 = [c_1, c_2],\ \ldots,\ I_m = [c_{m-1}, c_m]$$

where $0 = c_0 < c_1 < c_2 < \cdots < c_m = 1$, $m \geqslant 2$ and define the *address* $A(x)$ of a point $x \in I$ by setting $A(x) = j$ for $c_{j-1} \leqslant x < c_j$, $1 \leqslant j \leqslant m$ and $A(c_m) = m$. Let $g_j(x)$ be a strictly monotone, either increasing or decreasing, continuous function defined on the subinterval I_j such that $g_j(I_j) = I$ for $1 \leqslant j \leqslant m$. Define the *sign* ε_j to be either $+1$ or -1 according as g_j is monotone increasing or monotone decreasing on I_j. We assume, in addition, that $g_1(x)$ and $g_m(x)$ are monotone increasing; so $\varepsilon_1 = \varepsilon_m = +1$.

Let $\Sigma = \{1,2,\ldots,m\}^{\mathbf{N}}$ be the one-sided symbol space endowed with the metric

$$d(w,z) = \sum_{n=1}^{\infty} 2^{-n}|w_n - z_n| \qquad \text{for} \qquad w = (w_n),\quad z = (z_n) \in \Sigma.$$

It is known that Σ is a totally disconnected compact metric space. Let $G(x) = g_{A(x)}(x)$ for brevity. Note that the function $G : I \to I$ is not necessarily continuous. We then define the *itinerary* $\nu(x)$ of a point $x \in I$ by setting

$$\nu(x) = (A_0(x), A_1(x), \ldots, A_n(x), \ldots)$$

where $A_n(x) = A(G^n(x))$ for $n \geqslant 0$. Put $e_0 = \{0,1\}$ and define the set e_{n+1} inductively by setting $e_{n+1} = \{0<x<1; G(x) \in e_n\}$ for $n \geqslant 0$. Obviously $\# e_n = m^{n-1}(m-1)$ for $n \geqslant 1$. Let $e = \bigcup_{n \geqslant 0} e_n$. Then it is easily verified that the set of discontinuity points of ν is precisely equal to the set $e - e_0$.

Put $\Lambda_0 = \{\nu(x); x \in e_0\}$. For $N \geqslant 1$, let Λ_N be the set of words $w = (w_n) \in \Sigma$ such that either $w_n = 1$ for $n > N$, $w_N \neq 1$ or $w_n = m$ for $n > N$, $w_N \neq m$. Let $\Lambda = \bigcup_{n \geqslant 0} \Lambda_n$. Then it is easily seen that for $x \in e - e_0$ there exist the limits

$$\lim_{\varepsilon \to 0\pm} \nu(x+\varepsilon) = (A_0(x\pm), A_1(x\pm), \ldots)$$

in $\Lambda - \Lambda_0$ respectively. Note that $\nu(x)$ is equal to either $\nu(x+)$ or $\nu(x-)$. Thus the set Λ_n consists of the following $2m^{n-1}(m-1)$ distinct words:

$$\{\nu(x+); x \in e_n\} + \{\nu(x-); x \in e_n\}$$

for $n \geqslant 1$. Therefore we have $\Lambda = \Lambda_0 + \Sigma_+ + \Sigma_-$, where $\Sigma_+ = \{\nu(x+); x \in e - e_0\}$ and $\Sigma_- = \{\nu(x-); x \in e - e_0\}$.

We assume further that each function $h_j = g_j^{-1} : I \to I_j$ is a contraction; namely the Lipschitz constant

$$\mathrm{Lip}\,(h_j) = \sup_{x \neq y \in I} \left| \frac{h_j(x) - h_j(y)}{x - y} \right|$$

satisfies $\mathrm{Lip}\,(h_j) < 1$. Let $\gamma = \max_{1 \leqslant j \leqslant m} \mathrm{Lip}\,(h_j) \in [1/m, 1)$. We then define the mapping $\mu : \Sigma \to I$ by setting

$$\mu(w) = \lim_{n \to \infty} h_{w_1} \circ h_{w_2} \circ \cdots \circ h_{w_n}(I) \qquad \text{for} \qquad w = (w_n) \in \Sigma.$$

Clearly μ is continuous. Then it follows that $X = \mu(\Sigma)$ is a compact subset of I and satisfies the following equality:

$$X = h_1(X) \cup h_2(X) \cup \cdots \cup h_m(X).$$

It is known that the above equation possesses a unique non-empty compact solution [4, p. 384]; thus we have $\mu(\Sigma) = X = I$, since $h_j(I) = I_j$ for $1 \leqslant j \leqslant m$. It also follows that the set e is a dense subset of I; therefore the mapping ν is one to one.

Let $S_n = \bigcup_{0 \leqslant j \leqslant n} e_j$ for $n \geqslant 1$ and let

$$H_{n,x}(y) = h_{A_0(x)} \circ h_{A_1(x)} \circ \cdots \circ h_{A_{n-1}(x)}(y)$$

for $n \geqslant 1$ and $x,\ y \in I$. Obviously $H_{n,x}$ is a contraction satisfying $\mathrm{Lip}\,(H_{n,x}) \leqslant \gamma^n$. We first consider an arbitrary point $x \in I - e$. Put $K_{n,x} = H_{n,x}(\mathring{I})$ for $n \geqslant 1$. Since $K_{n,x}$ is the connected component of $I - S_n$ containing x and $|K_{n,x}| \leqslant \gamma^n$, we have

$$\lim_{n \to \infty} \bar{K}_{n,x} = x\,;$$

that is, $\mu \circ \nu(x) = x$. Thus ν maps $I - e$ homeomorphically onto

$\nu(I-e)$. We next consider an arbitrary point $x \in e_N$, $N \geqslant 1$. Put $K_{n,x}^{\pm} = H_{n,x\pm}(\mathring{I})$ for $n \geqslant N$, respectively. Since $K_{n,x}^{\pm}$ are the two consecutive connected components of $I - S_n$ such that the left end point of $K_{n,x}^{+}$ is x and the right end point of $K_{n,x}^{-}$ is also x, we have

$$\lim_{n\to\infty} \bar{K}_{n,x}^{+} = \lim_{n\to\infty} \bar{K}_{n,x}^{-} = x ;$$

so $\mu \circ \nu(x) = \mu \circ \nu(x\pm) = x$. Similarly we can define $K_{n,0}^{+}$ and $K_{n,1}^{-}$ for $n \geqslant 1$; thus $\mu \circ \nu(0) = 0$ and $\mu \circ \nu(1) = 1$. Then we have

LEMMA 3.1. – $\nu(I-e) = \Sigma - \Lambda$; *namely,* $w = (w_n) \in \nu(I-e)$ *if and only if*

$$\#\{n\geqslant 1; w_n \neq 1\} = \infty = \#\{n\geqslant 1; w_n \neq m\}.$$

Proof. – Suppose that $w = \nu(x) \in \Lambda$ for some $x \in I - e$. Since ν is one to one, we have $\nu(I-e) \cap \nu(e) = \phi$; thus $w \in \Sigma_+ + \Sigma_-$. Hence there exists $y \in e - e_0$ such that either $w = \nu(y+)$ or $w = \nu(y-)$. Therefore $x = \mu \circ \nu(x) = \mu(w) = \mu \circ \nu(y\pm) = y$. This contradiction implies that $\Lambda \cap \nu(I-e) = \phi$; that is, $\nu(I-e) \subset \Sigma - \Lambda$. Thus it suffices to show that $\Sigma - \Lambda \subset \nu(I-e)$.

Suppose now that there exists a word $w = (w_n) \in \Sigma - \Lambda$ such that $w \notin \nu(I-e)$. Put $z = (z_n) \equiv \nu \circ \mu(w)$. Then it follows that $w \neq z$. For otherwise, we have $\mu(w) \in e$; thus, $w \in \nu(e) \subset \Lambda$, contrary to $w \in \Sigma - \Lambda$. Let $N \geqslant 1$ be the smallest integer such that $w_N \neq z_N$. Since $\mu(w) = \mu \circ \nu \circ \mu(w) = \mu(z)$, it follows that

$$h_{w_N} \circ h_{w_{N+1}} \circ \cdots = h_{z_N} \circ h_{z_{N+1}} \circ \cdots, \text{ say } p.$$

Then we have $p \in e_1$ and w, $z \in \Lambda_N$, contrary to $w \in \Sigma - \Lambda$. This completes the proof. □

4. Functional Equations.

Let $f_j \colon I \to I$ be a contraction for $1 \leqslant j \leqslant m$. We assume that $c_0 = 0$ and $c_m = 1$ are unique fixed points of $f_1(x)$ and $f_m(x)$ respectively. The following lemma is a special case of the general theorem obtained by the author [4, p. 397], but we include the proof for completeness.

LEMMA 4.1. — *The functional equations*

$$(4.1) \quad \psi(x) = f_j(\psi(g_j(x))) \quad \text{for} \quad x \in I_j, \ 1 \leqslant j \leqslant m$$

possess a unique continuous solution $\psi(x)$ *if and only if*

$$(4.2) \quad f_j\left(\frac{1+\varepsilon_j}{2}\right) = f_{j+1}\left(\frac{1-\varepsilon_{j+1}}{2}\right) \quad \text{for} \quad 1 \leqslant j \leqslant m-1.$$

Remark. — This is a generalization of the theorem obtained by G. de Rham [10]; indeed he has shown that the equations

$$M\left(\frac{x}{2}\right) = F_0(M(x)), \ M\left(\frac{1+x}{2}\right) = F_1(M(x)) \quad \text{for} \quad x \in I$$

possess a unique continuous solution $M(x)$ if and only if $F_1(p_0) = F_0(p_1)$ where p_0, p_1 are unique fixed points of the contractions F_0, F_1 respectively. Lebesgue's singular functions and Pólya's space-filling curves satisfy the above equations for certain affine contractions F_0 and F_1.

Proof. — The conditions (4.2) are obviously necessary; thus it suffices to show the sufficiency. Let $\mathscr{F}$ be the set of continuous functions $u(x)$ defined on I satisfying $u(0) = 0$ and $u(1) = 1$; obviously $\mathscr{F}$ is a closed subset of the Banach space $C([0,1])$ with the usual uniform norm. We now consider the following operator:

$$Tu(x) = f_{A(x)}(u(G(x))).$$

Then it is easily seen that the conditions (4.2) imply that $T(\mathscr{F}) \subset \mathscr{F}$; moreover T is a contraction, since

$$\|Tu - Tv\| \leqslant \lambda \max_{x \in I} |u(G(x)) - v(G(x))| \leqslant \lambda \|u - v\|,$$

where $\lambda = \max_{1 \leqslant j \leqslant m} \operatorname{Lip}(f_j) \in [1/m, 1)$, for any u, $v \in \mathscr{F}$. Hence T has a unique fixed point ψ in $\mathscr{F}$; namely

$$\psi(x) = f_j(\psi(g_j(x))) \quad \text{for} \quad c_{j-1} \leqslant x < c_j, \ 1 \leqslant j \leqslant m.$$

Obviously this equality holds also true for $x = c_j$. This completes the proof. □

For $n \geqslant 1$ and x, $y \in I$, we define

$$F_{n,x}(y) = f_{A_0(x)} \circ f_{A_1(x)} \circ \cdots \circ f_{A_{n-1}(x)}(y).$$

The function $F_{n,x}$ is a contraction satisfying $\mathrm{Lip}(F_{n,x}) \leqslant \lambda^n$. Put $\beta = \max_{1 \leqslant j \leqslant m} \mathrm{Lip}(g_j) \in [m, \infty]$. Then we have

LEMMA 4.2. – *Suppose that* $\{f_j\}$ *satisfy the conditions* (4.2). *If* $\beta < \infty$, *then the continuous solution* $\psi(x)$ *satisfies Hölder's condition of order* $\log(1/\lambda)/\log\beta$.

Proof. – Consider arbitrary two points $x < y$ in I. Let $N \geqslant 0$ be the smallest integer satisfying $\#\{S_{N+1} \cap (x,y)\} \geqslant 2$. We now distinguish two cases: (*a*) $S_N \cap (x,y) = \phi$; (*b*) $S_N \cap (x,y)$ consists of a single point, say p. In case (*a*), it follows that

$$\begin{aligned}|\psi(x)-\psi(y)| &= \lim_{\varepsilon \to 0+} |\psi(x+\varepsilon)-\psi(y-\varepsilon)| \\ &= \lim_{\varepsilon\to 0+} |F_{N,x+\varepsilon}(\psi(G^N(x+\varepsilon)))-F_{N,x+\varepsilon}(\psi(G^N(y-\varepsilon)))| \leqslant \lambda^N.\end{aligned}$$

Similarly we have $|\psi(x)-\psi(y)| \leqslant 2\lambda^N$ in case (*b*), since $(x,p) \cap S_N = (p,y) \cap S_N = \phi$. Now let $s < t$ be any two consecutive points of e_{N+1} contained in (x,y). Then it follows that $|x-y| > |s-t| \geqslant \beta^{-N-1}$; thus

$$|\psi(x)-\psi(y)| \leqslant 2\lambda^N = \frac{2}{\lambda}\beta^{-\xi(N+1)} \leqslant \frac{2}{\lambda}|x-y|^{\xi}$$

where $\xi = \log(1/\lambda)/\log\beta$, which obviously completes the proof. □

5. Some Properties.

The continuous solution $\psi(x)$ of the equations (4.1) is not necessarily singular in general; for example, if we take

$$g_j(x) = mx - j + 1 \qquad \text{and} \qquad f_j(x) = \frac{x}{m} + \frac{j-1}{m}$$

for $1 \leqslant j \leqslant m$, then obviously $\psi(x) \equiv x$ is a smooth solution of (4.1). In this paper, to discuss the singularities of $\psi(x)$, we shall restrict ourselves to the following case:

$$\varepsilon_j = 1 + 2\left[\frac{j}{4}\right] - 2\left[\frac{j+1}{4}\right] \tag{5.1}$$

and

$$f_j(x) = \frac{1}{2k}\left\{(-1)^{[j/2]}x + \left[\frac{j}{2}\right] - \left[\frac{j}{4}\right] + \left[\frac{j-1}{4}\right]\right\}$$

for $1 \leqslant j \leqslant m = 4k$, where k is a positive integer; so $\lambda = 1/2k$. Then it is easily seen that the functions $\{f_j\}$ satisfy the conditions (4.2); therefore the equations (4.1) possess a unique continuous solution $\psi(x)$, which depends only on the functions $\{g_j\}$ satisfying the conditions (5.1). Let η_j be the sign of the function f_j; namely $\eta_j = (-1)^{[j/2]}$, for $1 \leqslant j \leqslant 4k$. For brevity, put

$$\varepsilon_{n,x} = \prod_{j=0}^{n-1} \varepsilon_{A_j(x)} \qquad \text{and} \qquad \eta_{n,x} = \prod_{j=0}^{n-1} \eta_{A_j(x)}$$

for $n \geqslant 1$, $x \in I$.

Consider now an arbitrary point $x \in I - e$. We define

$$p_{j,n,x} = H_{n,x}(c_j) \qquad \text{for} \qquad n \geqslant 1, \quad 0 \leqslant j \leqslant 4k.$$

Obviously $p_{j,n,x} \neq x$. Since $p_{j,n,x} \in G^{-n}(c_j) \subset e_{n+1}$ for $1 \leqslant j \leqslant 4k - 1$, we have

$$G^n(p_{j,n,x}) = c_j \qquad \text{for} \qquad 1 \leqslant j \leqslant 4k - 1.$$

The points $p_{0,n,x}$ and $p_{4k,n,x}$ are two end points of $K_{n,x}$ and do not satisfy the above equality in general; however,

$$\lim_{\substack{y \to p_{j,n,x} \\ y \in K_{n,x}}} G^n(y) = c_j \qquad \text{for} \qquad j = 0,\ 4k.$$

Note that $0 < |x - p_{j,n,x}| < \gamma^n$ for any $n \geqslant 1$. Then we have

LEMMA 5.1. — *Suppose that $x \in I - e$. Then the points $\{p_{j,n,x}\}$ satisfy the following properties :*

(1) $\operatorname{sign}\ (x - p_{j,n,x}) = \varepsilon_{n,x} \operatorname{sign} \left\{ A_n(x) - j - \frac{1}{2} \right\},$

(2) $\psi(x) - \psi(p_{j,n,x}) = \dfrac{\eta_{n,x}}{(2k)^n} \left\{ \psi(G^n(x)) - \dfrac{1-(-1)^j}{4k} - \dfrac{1}{k}\left[\dfrac{j}{4}\right] \right\}$

for $n \geqslant 1$ and $0 \leqslant j \leqslant 4k$.

Proof. — Since $p_{j,n,x} = H_{n,x}(c_j)$, we have

$$\operatorname{sign}\ (x - p_{j,n,x}) = \operatorname{sign}\ \{H_{n,x}(G^n(x)) - H_{n,x}(c_j)\} = \varepsilon_{n,x} \operatorname{sign}\ \{G^n(x) - c_j\};$$

thus the property (1) follows immediately. Since $K_{n,x} \cap S_n = \phi$,

$$\psi(p_{j,n,x}) = \lim_{\substack{y \to p_{j,n,x} \\ y \in K_{n,x}}} \psi(y) = \lim_{\substack{y \to p_{j,n,x} \\ y \in K_{n,x}}} F_{n,x}(\psi(G^n(y))) = F_{n,x}(\psi(c_j))$$

for $0 \leqslant j \leqslant 4k$; hence

$$\psi(x) - \psi(p_{j,n,x}) = \mathrm{F}_{n,x}(\psi(\mathrm{G}^n(x))) - \mathrm{F}_{n,x}(\psi(c_j)) \frac{\eta_{n,x}}{(2k)^n}\{\psi(G^n(x)) - \psi(c_j)\},$$

which obviously completes the proof. □

We now consider an arbitrary point $x \in e_N$, $N \geqslant 1$. Then it is easily seen that, for $1 \leqslant j \leqslant 4k-1$, each of the sets $K^{\pm}_{n,x}$ contains exactly one point of $G^{-n}(c_j) \subset e_{n+1}$, say $q^{\pm}_{j,n,x}$ respectively. Obviously $q^{\pm}_{j,n,x} \neq x$. Similarly we can define $\{q^{+}_{j,n,0}\}$ and $\{q^{-}_{j,n,1}\}$ for $n \geqslant 0$, $1 \leqslant j \leqslant 4k-1$. Note that $0 < |x - q^{\pm}_{j,n,x}| < \gamma^n$ for any $n \geqslant N$. It also follows that

$$\lim_{\varepsilon \to 0\pm} G^n(x+\varepsilon) = \frac{1}{2}(1 \mp \varepsilon_{N,x\pm})$$

for every $n \geqslant N$, respectively. We, of course, adopt the rule: $\varepsilon_{0,0+} = \varepsilon_{0,1-} = \eta_{0,0+} = \eta_{0,1-} = 1$. Then we have

LEMMA 5.2. — *Suppose that $x \in e_N$, $N \geqslant 0$. Then the points $\{q^{\pm}_{j,n,x}\}$ satisfy the following :*

$$\psi(x) - \psi(q^{\pm}_{j,n,x}) = \frac{\eta_{N,x\pm}}{(2k)^n}\left\{\frac{1}{2}(1 \mp \varepsilon_{N,x\pm}) - \frac{1-(-1)^j}{4k} - \frac{1}{k}\left[\frac{j}{4}\right]\right\}$$

for $n \geqslant N$ and $1 \leqslant j \leqslant 4k-1$, respectively.

Proof. — Since $K^{\pm}_{n,x} \cap S_n = \phi$, we have

$$\psi(x) - \psi(q^{\pm}_{j,n,x}) = \lim_{\varepsilon \to 0\pm} \{\psi(x+\varepsilon) - \psi(q^{\pm}_{j,n,x})\} =$$

$$\lim_{\varepsilon \to 0\pm} \{F_{n,x+\varepsilon}(\psi(G^n(x+\varepsilon))) - F_{n,x+\varepsilon}(\psi(c_j))\} = \frac{\eta_{N,x\pm}}{(2k)^n}\left\{\frac{1}{2}(1 \mp \varepsilon_{N,x\pm}) - \psi(c_j)\right\}$$

for every $n \geqslant N$, respectively. This completes the proof. □

6. Singularities.

For any $x \neq y \in I$, we define $\Delta\psi(x,y) = (\psi(x) - \psi(y))/(x-y)$. Let W be the set of points $x \in I$ at which $A_n(x) \equiv 2$ or $3 \pmod 4$ for infinitely many n's. Obviously $W \subset I - e$. First of all, we have

THEOREM 6.1. – *Suppose that* $\gamma \leqslant 1/2k$. *Then we have*

$$D^{\pm}\psi(x) \geqslant 0 \geqslant D_{\pm}\psi(x) \qquad \text{and} \qquad D^{\pm}\psi(x) - D_{\pm}\psi(x) \geqslant 1/4k$$

respectively, for every $x \in W$.

Proof. – We distinguish two cases (not exclusive) as follows :

Case A. $A_n(x) \equiv 3 \pmod 4$ *for infinitely many* n's.

Let $0 < n_1 < n_2 < \cdots$ be the subsequence of integers such that $A_{n_i}(x) = 4N_i + 3$, where $0 \leqslant N_i < k$. From the functional equations (4.1), we have

$$\frac{N_i}{k} \leqslant \psi(G^{n_i}(x)) \leqslant \frac{2N_i + 1}{k};$$

therefore $\{\psi(x)-\psi(P_{i,1})\}\{\psi(x)-\psi(P_{i,2})\} \leqslant 0$ by (2) of Lemma 5.1, where $p_{i,j} = p_{4N_i+j,n_i,x}$ for $0 \leqslant j \leqslant 4$. On the order hand, we have $\operatorname{sign}(x-P_{i,1}) = \operatorname{sign}(x-P_{i,2}) = \varepsilon_{n_i,x}$ by (1) of Lemma 5.1. Since $\varepsilon_{n_i,x}$ changes the sign infinitely many times as i increases, it follows that $D^{\pm}\psi(x) \geqslant 0 \geqslant D_{\pm}\psi(x)$. It also follows that

$$|\Delta\psi(x,P_{i,1})| + |\Delta\psi(x,P_{i,2})| \geqslant \frac{(2k)^{-n_i-1}}{|x-P_{i,1}|} > \frac{1}{2k}(2k\gamma)^{-n_i} \geqslant \frac{1}{2k};$$

therefore $D^{\pm}\psi(x) - D_{\pm}\psi(x) \geqslant 1/4k$ respectively, as required.

Case B. $A_n(x) \equiv 2 \pmod 4$ *for infinitely many* n's.

Let $0 < n_1 < n_2 < \cdots$ be the subsequence of integers such that $A_{n_i}(x) = 4N_i + 2$, where $0 \leqslant N_i < k$. Since

$$\frac{N_i}{k} \leqslant \psi(G^{n_i}(x)) \leqslant \frac{2N_i + 1}{k},$$

it is easily seen that $\{\psi(x) - \psi(P_{i,0})\}\{\psi(x) - \psi(P_{i,1})\} \leqslant 0$ and $\{\psi(x) - \psi(P_{i,2})\}\{\psi(x) - \psi(P_{i,3})\} \leqslant 0$. On the other hand, we have

$\text{sign}\,(x-P_{i,0}) = \text{sign}\,(x-P_{i,1}) = \text{sign}\,(P_{i,2}-x) = \text{sign}\,(P_{i,3}-x)$; therefore $D^{\pm}\psi(x) \geqslant 0 \geqslant D_{\pm}\psi(x)$. Moreover,

$$|\Delta\psi(x,P_{i,0})| + |\Delta\psi(x,P_{i,1})| \geqslant \frac{(2k)^{-n_i-1}}{|x-P_{i,0}|} > \frac{1}{2k}(2k\gamma)^{-n_i} \geqslant \frac{1}{2k}.$$

The same estimate holds true if we replace $P_{i,0}$, $P_{i,1}$ by $P_{i,2}$, $P_{i,3}$, respectively ; thus $D^{\pm}\psi(x) - D_{\pm}\psi(x) \geqslant 1/4k$ respectively. This completes the proof. □

Let $W_0 \subset W$ be the set of points $x \in I$ at which $A_n(x) \equiv 2$ or 3 (mod 4) and $A_{n+1}(x) \equiv 2$ or 3 (mod 4) for infinitely many n's. Then we have

THEOREM 6.2. — *Suppose that $\gamma \leqslant 1/2k$. Then W_0 is contained in the set* Knot(ψ) *except for a set of measure zero.*

Proof. — We consider an arbitrary point x of W_0. Let $0 \leqslant n_1 < n_2 < \cdots$ be the subsequence of integers such that $A_{n_i}(x) = 4N_i + \delta_i$ and $A_{n_i+1}(x) = 4L_i + \omega_i$, where $0 \leqslant N_i$, $L_i < k$ and $2 \leqslant \delta_i$, $\omega_i \leqslant 3$. Then it is easily seen that

$$\frac{2N_i+1}{2k} - \frac{2L_i+1}{(2k)^2} \leqslant \psi(G^{n_i}(x)) \leqslant \frac{2N_i+1}{k} - \frac{L_i}{2k^2};$$

therefore by (2) of Lemma 5.1,

$$\eta_{n_i,x}(2k)^{n_i}\{\psi(x)-\psi(P_{i,0})\} = \psi(G^{n_i}(x)) - \frac{N_i}{k} \geqslant \frac{1}{2k} - \frac{2L_i+1}{(2k)^2} \geqslant (2k)^{-2}.$$

Similarly we have

$$\eta_{n_i,x}(2k)^{n_i}\{\psi(P_{i,4})-\psi(x)\} = \frac{N_i+1}{k} - \psi(G^{n_i}(x)) \geqslant \frac{1}{2k} + \frac{L_i}{2k^2} \geqslant \frac{1}{2k}.$$

Therefore, since $\text{sign}\,(x-P_{i,0}) = \text{sign}\,(P_{i,4}-x)$, it follows that

$$\text{sign}\,(\Delta\psi(x,P_{i,0})) = \text{sign}\,(\Delta\psi(x,P_{i,4}))$$

and

$$|\Delta\psi(x,P_{i,0})| > (2k)^{-2}, \qquad |\Delta\psi(x,P_{i,4})| > \frac{1}{2k}.$$

Hence the set $[D_+\psi(x), D^+\psi(x)] \cap [D_-\psi(x), D^-\psi(x)]$ contains an interval of length $(2k)^{-2}$ by Theorem 6.1. Thus it follows from Denjoy's theorem

[2, p. 105] that except for a set of measure zero, every point of W_0 is a knot point of $\psi(x)$. This completes the proof. □

For $N \geqslant 0$, let Y_N be the set of points $x \in I$ at which $A_n(x) \equiv 0$ or 1 (mod 4) for all $n \geqslant N$ and $A_{N-1}(x) \equiv 2$ or 3 (mod 4). Obviously $I - W = \bigcup_{n \geqslant 0} Y_n$. For brevity, put $Y_n^* = Y_n \cap (I - e)$ for $n \geqslant 0$. Then the unit interval I is decomposed as follows :

$$I = W + e + \bigcup_{n \geqslant 0} Y_n^* .$$

For $n \geqslant 1$, let Ξ_n be the set of finite words $(w_1, \ldots w_n)$ of length n such that $1 \leqslant w_j \leqslant 4k$ and $w_j \equiv 0$ or 1 (mod 4) for $1 \leqslant j \leqslant n$. Then we have

THEOREM 6.3. — *Suppose that there exists a positive constant C_0, independent of n, satisfying*

$$\min_{(w_1 \ldots w_n) \in \Xi_n} |h_{w_1} \circ \ldots \circ h_{w_n}(I)| \geqslant C_0 (2k)^{-n}$$

for all $n \geqslant 1$. Suppose further that $\beta < \infty$. Then we have

$$D^{\pm}\psi(x) - D_{\pm}\psi(x) \geqslant \frac{1}{2k}$$

respectively, for every $x \in I - W$.

Proof. — We distinguish two cases as follows :

Case A. $x \in Y_N^*$ for some $N \geqslant 0$.

By Lemma 3.1, we have $A_n(x) \neq 1$ for infinitely many n's. Let $N \leqslant n_1 < n_2 < \cdots$ be the subsequence of integers such that $A_{n_i}(x) \geqslant 4$. Put $Q_{i,j} = p_{j,n_i,x}$ for $0 \leqslant j \leqslant 2$. Since

$$\psi(G^{n_i}(x)) \geqslant \frac{1}{2k}$$

and $\operatorname{sign}(x - Q_{i,1}) = \operatorname{sign}(x - Q_{i,2}) = \operatorname{sign}(Q_{i,2} - Q_{i,1}) = \varepsilon_{N,x}$, we have

$|\Delta\psi(x, Q_{i,1}) - \Delta\psi(x, Q_{i,2})| =$

$$(2k)^{-n_i} \left| \psi(G^{n_i}(x)) \left\{ \frac{1}{x - Q_{i,2}} - \frac{1}{x - Q_{i,1}} \right\} + \frac{1}{2k(x - Q_{i,1})} \right| \geqslant \frac{(2k)^{-n_i - 1}}{|x - Q_{i,1}|} > \frac{1}{2k} .$$

On the other hand, it follows that

$$|x-Q_{i,0}| > |Q_{i,1}-Q_{i,0}| \geqslant \beta^{-N}\left|h_{A_N(x)}\circ \dots \circ h_{A_{n_i-1}(x)}\circ h_1(I)\right| \geqslant C_0\beta^{-N}(2k)^{-n_i+N-1};$$

therefore

$$|\Delta\psi(x, Q_{i,0})| = (2k)^{-n_i}\left|\frac{\psi(G^{n_i}(x))}{x-Q_{i,0}}\right| \leqslant \frac{2k}{C_0}\left(\frac{\beta}{2k}\right)^N.$$

Since sign $(x-Q_{i,0}) = \varepsilon_{N,x}$, we conclude that either $[D_+\psi(x), D^+\psi(x)]$ or $[D_-\psi(x), D^-\psi(x)]$ contains an interval of length $1/2k$ according as $\varepsilon_{N,x} = -1$ or $+1$.

It also follows from Lemma 3.1 that $A_n(x) \neq 4k$ for infinitely many n's. Let $N \leqslant n_1 < n_2 < \dots$ be the subsequence of integers such that $A_{n_i}(x) \leqslant 4k-3$. Put $R_{i,j} = p_{4k-j,n_i,x}$ for $0 \leqslant j \leqslant 3$. Since

$$\psi(G^{n_i}(x)) \leqslant \frac{2k-1}{2k}$$

and sign $(x-R_{i,2})$ = sign $(x-R_{i,3})$ = sign $(R_{i,3}-R_{i,2}) = -\varepsilon_{N,x}$, we have

$$|\Delta\psi(x, R_{i,2}) - \Delta\psi(x, R_{i,3})| = (2k)^{-n_i}\left|\left\{\frac{2k-1}{2k}-\psi(G^{n_i}(x))\right\}\left\{\frac{1}{x-R_{i,3}}-\frac{1}{x-R_{i,2}}\right\}+\frac{1}{2k(x-R_{i,2})}\right| \geqslant \frac{(2k)^{-n_i-1}}{|x-R_{i,2}|} > \frac{1}{2k}.$$

On the other hand, $|x-R_{i,0}| > |R_{i,1}-R_{i,0}| \geqslant C_0\beta^{-N}(2k)^{-n_i+N-1}$; thus

$$|\Delta\psi(x,R_{i,0})| = (2k)^{-n_i}\left|\frac{\psi(G^{n_i}(x))-1}{x-R_{i,0}}\right| \leqslant \frac{2k}{C_0}\left(\frac{\beta}{2k}\right)^N.$$

Since sign $(x-R_{i,0}) = -\varepsilon_{N,x}$, it follows that either $[D_+\psi(x), D^+\psi(x)]$ or $[D_-\psi(x), D^-\psi(x)]$ contains an interval of length $1/2k$ according as $\varepsilon_{N,x} = +1$ or -1. Hence $D^\pm\psi(x) - D_\pm\psi(x) \geqslant 1/2k$ respectively.

Case B. $x \in e_N$ *for some* $N \geqslant 0$.

For $n \geqslant N$, let $Q_n^+ = \max\{q_{1,n,x}^+, q_{3,n,x}^+\}$, $Q_n^- = \min\{q_{1,n,x}^-, q_{3,n,x}^-\}$ and let $R_n^\pm = q_{2,n,x}^\pm$ respectively. Then $Q_n^- < R_n^- < x < R_n^+ < Q_n^+$.

Since $\operatorname{sign}(x-Q_n^{\pm}) = \operatorname{sign}(Q_n^{\pm}-R_n^{\pm}) = \pm 1$ respectively, it follows from Lemma 5.2 that

$$|\Delta\psi(x,R_n^{\pm})-\Delta\psi(x,Q_n^{\pm})| =$$
$$(2k)^{-n}\left|\frac{1}{2}(1\mp\varepsilon_{N,x\pm})\left\{\frac{1}{x-R_n^{\pm}}-\frac{1}{x-Q_n^{\pm}}\right\}+\frac{1}{2k(x-Q_n^{\pm})}\right| \geqslant \frac{(2k)^{-n-1}}{|x-Q_n^{\pm}|} > \frac{1}{2k},$$

respectively. On the other hand, we have

$$|x-R_n^{\pm}|>|K_{n+1,x}^{\pm}|\geqslant\beta^{-N}|h_{A_N(x\pm)}\circ\cdots\circ h_{A_n(x\pm)}(I)|\geqslant C_0\beta^{-N}(2k)^{-n+N-1};$$

therefore

$$|\Delta\psi(x,R_n^{\pm})| \leqslant \frac{(2k)^{-n}}{|x-R_n^{\pm}|} < \frac{2k}{C_0}\left(\frac{\beta}{2k}\right)^N.$$

Hence $D^{\pm}\psi(x) - D_{\pm}\psi(x) \geqslant 1/2k$ respectively. This completes the proof. □

Let $Y^* = \bigcup_{n\geqslant 0} Y_n^*$ for brevity. Then we have

THEOREM 6.4. – Knot $(\psi) \cap Y^* = \phi$.

Proof. – We consider an arbitrary point x of Y_N^* for some $N \geqslant 0$. Let $s_n = p_{0,n,x}$ for $n \geqslant N$. Since $\operatorname{sign}(x-s_n) = \varepsilon_{N,x}$ is independent of $n \geqslant N$, the sequence $\{s_n\}$ is monotone, either increasing or decreasing, and converges to x. Note that $s_n = s_{n+1}$ if and only if $A_n(x) = 1$. Put $J_n = [s_n, s_{n+1}] \subset \bar{K}_{n,x}$ for $n \geqslant N$. Then it is easily seen that

$$(x,s_N] = \bigcup_{n\geqslant N} J_n.$$

Since the function $G^n(x)$ maps $K_{n,x}$ homeomorphically onto $(0,1)$, we have $A_n(x) > A_n(y)$ for all $y \in \mathring{J}_n$. Therefore

$$\psi(G^n(x)) \geqslant f_{A_n(x)}(0) \geqslant \max_{j<A_n(x)} \|f_j\| \geqslant \psi(G^n(y));$$

thus

$$\eta_{N,x}\operatorname{sign}\{\psi(x)-\psi(y)\} = \eta_{N,x}\operatorname{sign}\{F_{n,x}(\psi(G^n(x)))-F_{n,x}(\psi(G^n(y)))\} =$$
$$\operatorname{sign}\{\psi(G^n(x))-\psi(G^n(y))\} \geqslant 0.$$

By the continuity of ψ, we conclude that

$$\eta_{N,x}\operatorname{sign}\{\psi(x)-\psi(y)\} \geqslant 0 \qquad \text{for every} \qquad y\in[x,s_N].$$

This means that x is not a knot point of $\psi(x)$. □

7. Proof of Theorem 2.1.

First of all, for any integer $k \geqslant 1$ and positive numbers σ, τ, ρ satisfying

$$2k(\sigma+\tau) < 1 \quad \text{and} \quad \sigma \geqslant \rho, \tag{7.1}$$

we shall construct two Cantor sets $E_0 \equiv E_0(k,\sigma,\tau)$ and $E_1 \equiv E_1(k,\sigma,\rho)$. The set $E_0(k,\sigma,\tau)$ is obtained from the unit interval I by a sequence of deletions of open intervals known as middle thirds, as follows : First divide I into k equal parts, say

$$I_{1,1} = \left[0,\frac{1}{k}\right], \quad I_{1,2} = \left[\frac{1}{k},\frac{2}{k}\right], \quad \ldots, \quad I_{1,k} = \left[\frac{k-1}{k},1\right],$$

and remove from each closed interval $I_{1,j}$ the open interval $U_{1,j}$ centered at $(2j-1)/2k$ and of length 2σ. We subdivide each of the $2k$ remaining closed intervals into k equal parts, say $I_{2,j}$, $1 \leqslant j \leqslant 2k^2$, ordered from left to right, each of length $(1-2k\sigma)/(2k^2)$. Then remove from each closed interval $I_{2,j}$ the middle open interval $U_{2,j}$ of length $2\sigma\tau$, leaving the $4k^2$ closed intervals, each of length $(1-2k\sigma-4k^2\sigma\tau)/(4k^2)$. This process is permitted to continue indefinitely. At the nth stage of deletion, each length of the $2^{n-1}k^n$ open intervals $U_{n,j}$ is $2\sigma\tau^{n-1}$, and therefore the measure of the union of the open intervals removed in the entire sequence of removal operations is $2k\sigma/(1-2k\tau)$. The set E_0 is defined to be the closed set remaining; thus

$$|E_0| = \frac{1 - 2k(\sigma+\tau)}{1 - 2k\tau}.$$

We next define the set $E_1(k,\sigma,\rho)$, which is slightly different from E_0 defined above, as follows : First divide the unit interval I into k equal parts, say

$$J_{1,1} = \left[0,\frac{1}{k}\right], \quad J_{1,2} = \left[\frac{1}{k},\frac{2}{k}\right], \quad \ldots, \quad J_{1,k} = \left[\frac{k-1}{k},1\right].$$

Then remove from each closed interval $J_{1,j}$ the two intervals

$$V^-_{1,j} = \left[\frac{j-1}{k},\frac{2j-1-2k\sigma}{2k}\right), \quad V^+_{1,j} = \left(\frac{2j-1+2k\sigma}{2k},\frac{j}{k}\right],$$

each of length $(1-2k\sigma)/2k$. We subdivide each of the k remaining closed intervals into $2k$ equal parts, say $J_{2,j}$, $1 \leqslant j \leqslant 2k^2$, ordered from left to right, each of length σ/k. Then delete from each closed interval $J_{2,j}$ the two intervals $V^{\pm}_{2,j}$ of length $\rho(1-2k\sigma)/2k$, leaving the $2k^2$ middle closed intervals, each of length $(\sigma-\rho+2k\sigma\rho)/k$. At the nth stage of deletion, we have $|V^{\pm}_{n,j}| = \rho^{n-1}(1-2k\sigma)/2k$; therefore the measure of the union of the removed intervals in the entire sequence of removal operations is $(1-2k\sigma)/(1-2k\rho)$. The set E_1 is defined to be the closed set remaining; thus

$$|E_1| = \frac{2k(\sigma-\rho)}{1-2k\rho}.$$

Note that the set E_1 is contained in $\left[\frac{1-2k\sigma}{2k(1-\rho)}, 1-\frac{1-2k\sigma}{2k(1-\rho)}\right]$.

We now define the continuous function $\zeta_0(x) \equiv \zeta_0(k,\sigma,\tau;x)$ by setting

$$\zeta_0(x) = \int_0^x d_0(s)\,ds \qquad \text{for} \qquad 0 \leqslant x \leqslant 1,$$

where $d_0(s) = 1/2k$ if $s \in E_0(k,\sigma,\tau)$ and $d_0(s) = \tau$ otherwise. We also define the continuous function $\zeta_1(x) \equiv \zeta_1(k,\sigma,\rho;x)$ by setting

$$\zeta_1(x) = \frac{1}{2k} - \sigma + \int_0^x d_1(s)\,ds \qquad \text{for} \qquad 0 \leqslant x \leqslant 1,$$

where $d_1(s) = 1/2k$ if $s \in E_1(k,\sigma,\rho)$ and $d_1(s) = \rho$ otherwise. Then it is easily seen that $\zeta_0(I) = [0,(1-2k\sigma)/2k], \zeta_1(I) = [(1-2k\sigma)/2k, 1/2k]$ and $\zeta_i(E_i) = E_i \cap \zeta_i(I)$ for $i = 0, 1$.

We next define, for $0 \leqslant i < k$,

$$\begin{aligned}
g_{4i+1}(x) &= \zeta_0^{-1}\left(x-\frac{i}{k}\right) && \text{for} \quad x \in I_{4i+1} = \left[\frac{i}{k}, \frac{2i+1}{2k}-\sigma\right],\\
g_{4i+2}(x) &= \zeta_1^{-1}\left(x-\frac{i}{k}\right) && \text{for} \quad x \in I_{4i+2} = \left[\frac{2i+1}{2k}-\sigma, \frac{2i+1}{2k}\right],\\
g_{4i+3}(x) &= \zeta_1^{-1}\left(\frac{i+1}{k}-x\right) && \text{for} \quad x \in I_{4i+3} = \left[\frac{2i+1}{2k}, \frac{2i+1}{2k}+\sigma\right],\\
g_{4i+4}(x) &= \zeta_0^{-1}\left(x-\frac{2i+1}{2k}-\sigma\right) && \text{for} \quad x \in I_{4i+4} = \left[\frac{2i+1}{2k}+\sigma, \frac{i+1}{k}\right];
\end{aligned}$$

thus the unit interval I is divided into $m = 4k$ subintervals $I_j = [c_{j-1}, c_j]$. We have $|I_{4i+1}| = |I_{4i+4}| = (1-2k\sigma)/2k$ and $|I_{4i+2}| = |I_{4i+3}| = \sigma$. Obviously the functions $g_j(x)$ satisfy the conditions (5.1) and we denote by $\psi(k,\sigma,\tau,\rho;x)$ the corresponding continuous solution of the equations (4.1).

It follows from Theorems 6.1 and 6.3 that $\psi(k,\sigma,\tau,\rho;x)$ has nowhere a unilateral derivative finite or infinite for any integer k and positive numbers σ, τ, ρ satisfying (7.1), since we have

$$\gamma = \frac{1}{2k}, \qquad \beta = \max\left\{\frac{1}{\rho}, \frac{1}{\tau}\right\}$$

and

$$|h_{w_1} \circ \cdots \circ h_{w_n}(I)| = \frac{1}{(2k)^n} - \frac{\sigma}{(2k)^{n-1}} - \frac{\sigma\tau}{(2k)^{n-2}} - \cdots - \sigma\tau^{n-1} > \frac{|E_0|}{(2k)^n},$$

for every finite word $(w_1 \ldots w_n) \in \Xi_n$.

Since the Cantor set E_0 is a unique compact subset of I satisfying

$$E_0 = h_1(E_0) \cup h_4(E_0) \cup h_5(E_0) \cup \cdots \cup h_{4k}(E_0)$$

and since the mapping ν maps Y_0^* homeomorphically onto $\nu(Y_0^*)$, it follows that $\bar{Y}_0^* = E_0$. On the other hand, for every $x \in W + \bigcup_{n\geqslant 1} Y_n^*$, there exist $n = n(x)$ and $j = j(x)$ such that $x \in U_{n,j}$; thus $E_0 \subset Y_0^* + e$. Therefore $|Y_0^*| = |E_0|$, since e is countable. Let Ω_n be the set of finite words $(w_1 \ldots w_n)$ of length n such that $1 \leqslant w_j \leqslant 4k$ for $1 \leqslant j \leqslant n$. Then for any $n \geqslant 0$, the set Y_{n+1}^* is decomposed as follows:

$$Y_{n+1}^* = \bigcup_{\substack{(w_1 \ldots w_n) \in \Omega_n \\ j \in \Omega_1 - \Xi_1}} h_{w_1} \circ \cdots \circ h_{w_n} \circ h_j(Y_0^*).$$

On each interval $V_{1,j}^{\pm}$, for any $(w_1 \ldots w_n) \in \Omega_n$ and $j \in \Omega_1 - \Xi_1$, the function $h_{w_1} \circ \cdots \circ h_{w_n} \circ h_j(x)$ is a linear contraction; more precisely we have

$$\left|\frac{d}{dx}(h_{w_1} \circ \cdots \circ h_{w_n} \circ h_j)(x)\right| = \rho^{n+1-r(w)}\tau^{r(w)} \qquad \text{for} \qquad x \in \mathring{V}_{1,j}^{\pm}$$

where $r(w) \equiv r(w_1, \ldots, w_n) = \frac{1}{2}\sum_{j=1}^{n} (1 + \eta_{w_j})$. Since $Y_0^* \cap U_{1,j} = \phi$ for all j, we have

$$|Y_{n+1}^*| = 2k|Y_0^*| \sum_{(w_1 \ldots w_n) \in \Omega_n} \rho^{n+1-r(w)}\tau^{r(w)} = 2k\rho|E_0|(2k(\rho+\tau))^n.$$

Therefore it follows that

$$|Y^*| = \sum_{n=0}^{\infty} |Y_n^*| = |E_0| + 2k\rho|E_0| \sum_{n=0}^{\infty} (2k(\rho+\tau))^n = \frac{1 - 2k(\sigma+\tau)}{1-2k(\rho+\tau)}.$$

For $N \geqslant 0$, let Z_N be the set of points $x \in I$ at which $A_n(x) \equiv 2$ or 3 (mod 4) for all $n \geqslant N$ and $A_{N-1}(x) \equiv 0$ or 1 (mod 4). Put $Z = \bigcup_{n \geqslant 0} Z_n$. Obviously $Z \subset W_0 \subset I - e$. Then it is easily seen that the set Z_0 is a compact subset of I satisfying

$$Z_0 = h_2(Z_0) \cup h_3(Z_0) \cup h_6(Z_0) \cup \cdots \cup h_{4k-1}(Z_0);$$

therefore $Z_0 = E_1$. For any $n \geqslant 0$, the set Z_{n+1} is decomposed as follows :

$$Z_{n+1} = \bigcup_{\substack{(w_1 \ldots w_n) \in \Omega_n \\ j \in \Xi_1}} h_{w_1} \circ \cdots \circ h_{w_n} \circ h_j(Z_0).$$

On each open interval $U_{1,j}$, for any $(w_1 \ldots w_n) \in \Omega_n$ and $j \in \Xi_1$, the function $h_{w_1} \circ \cdots \circ h_{w_n} \circ h_j(x)$ is a linear contraction such that

$$\left|\frac{d}{dx}(h_{w_1} \circ \cdots \circ h_{w_n} \circ h_j)(x)\right| = \rho^{n-r(w)}\tau^{1+r(w)} \qquad \text{for} \qquad x \in U_{1,j}.$$

Since $Z_0 \cap V_{1,j}^{\pm} = \phi$ for all j, we have

$$|Z_{n+1}| = 2k|Z_0| \sum_{(w_1 \ldots w_n) \in \Omega_n} \rho^{n-r(w)}\tau^{1+r(w)} = 2k\tau|E_1|(2k(\rho+\tau))^n;$$

therefore

$$|Z| = \sum_{n=0}^{\infty} |Z_n| = |E_1| + 2k\tau|E_1| \sum_{n=0}^{\infty} (2k(\rho+\tau))^n = \frac{2k(\sigma-\rho)}{1 - 2k(\rho+\tau)} = 1 - |Y^*|.$$

Then it follows from Theorems 6.2 and 6.4 that

$$|Z| \leqslant |W_0| \leqslant |\mathrm{Knot}(\psi)| \leqslant 1 - |Y^*| = |Z|;$$

hence we obtain

$$|\mathrm{Knot}(\psi)| = \frac{2k(\sigma-\rho)}{1 - 2k(\rho+\tau)}.$$

Thus if we take, for a fixed number $\alpha \in [0,1)$,

$$\sigma_0 = \frac{1 + \alpha}{8k}, \qquad \tau_0 = \frac{1}{4k} \qquad \text{and} \qquad \rho_0 = \frac{1}{8k},$$

then the function $\psi_0(x) \equiv \psi(k, \sigma_0, \tau_0, \rho_0; x)$ satisfies $|\mathrm{Knot}(\psi_0)| = \alpha$ and Hölder's condition of order $\log(2k)/\log(8k)$ by Lemma 4.2, which obviously converges to 1 as k tends to infinity. This completes the proof of Theorem 2.1. $\square$

Remark. — Besicovitch's function $B(x)$ illustrated in Figure 1 is precisely equal to the function $\psi(1, 1/8, 1/4, 1/8; x)$; thus $B(x)$ satisfies Hölder's condition of order $1/3$.

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