T. KIMURA
S. KASAI
H. HOSOKAWA

Universal transitivity of simple and 2-simple prehomogeneous vector spaces

Annales de l'institut Fourier, tome 38, nº 2 (1988), p. 11-41
<http://www.numdam.org/item?id=AIF_1988__38_2_11_0>

Ann. Inst. Fourier, Grenoble
38, 2 (1988), 11-41.

UNIVERSAL TRANSITIVITY OF SIMPLE AND 2-SIMPLE PREHOMOGENEOUS VECTOR SPACES

by T. KIMURA, S. KASAI and H. HOSOKAWA

Introduction.

We denote by k a field of characteristic zero. Let $\tilde{G}$ be a connected k-split linear algebraic group acting on $X = \mathrm{Aff}^n$ rationally by ρ which is defined over k. If there exists a Zariski-dense $\tilde{G}$-orbit Y, we say that $(\tilde{G}, \rho, X)$ is a prehomogeneous vector space (abbrev. P.V.). When ρ is irreducible or $[\tilde{G}, \tilde{G}]$ is a simple algebraic group, or a product of two simple algebraic groups, they are completely classified over $\mathbf{C}$ (see [3] $\sim$ [6]). Put $G = \rho(\tilde{G})$. Let ℓ be the number of $G(k)$-orbits in $Y(k)$, i.e., $\ell = \ell_k(G,X) = |G(k)\backslash Y(k)|$. In this paper, we shall assume that there exists a nonsplit quaternion k-algebra. In other words, $H^1(k, \mathrm{Aut}(SL_2)) \neq 0$. This condition is satisfied by every local field k other than $\mathbf{C}$. We say that Y is a universally transitive open orbit if $\ell = \ell_k(G,X) = 1$ for all such fields k, i.e., $Y(k)$ is a $G(k)$-orbit. Note that $G(k) \neq \rho(\tilde{G}(k))$ in general. Professor J.-I. Igusa classified all irreducible regular P.V.'s with universally transitive open orbits ([1], [2]). He also proved in [2] that ℓ is invariant under castling transformations.

In this paper, we shall classify simple or 2-simple P.V.'s with universally transitive open orbits. We shall also prove that ℓ is invariant under some P.V.-equivalences such as (1) $(Sp_n \times G, \Lambda_1 \otimes \rho)$ $(\deg \rho \leqslant 2n) \leftrightarrow (G, \Lambda^2(\rho))$ (see Proposition 3.7) (2) $(G \times GL_n, \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*)$ $(n \geqslant \deg \rho_1 \geqslant \deg \rho_2) \leftrightarrow (G, \rho_1 \otimes \rho_2)$ (see Proposition 4.1), and

Key-words : Galois cohomology. — Universal transitivity prehomogeneous vector spaces.

others (cf. Lemma 4.3-Proposition 4.7). This paper consists of the following four sections :

1. Preliminaries.
2. Simple P.V.'s with Universally Transitive Open Orbits.
3. 2-Simple P.V.'s of Type I with Universally Transitive Open Orbits.
4. 2-Simple P.V.'s of Type II with Universally Transitive Open Orbits.

The results are given in Theorems 2.19 ; 3.20 ; 4.2 ; 4.18 ; 4.25 ; 4.26 ; and Corollaries 2.20 ; 3.21. Also we shall check universal transitivity for non-regular irreducible P.V.'s (see Corollary 3.22). The first author would like to express his hearty thanks to Professor J.-I. Igusa and other members at The Johns Hopkins University in U.S.A. for their mathematical stimulation and hospitality while he stayed there in 1986. The idea for this work was first obtained that time.

1. Preliminaries.

We shall use the same notations as in [2]. For $\xi \in Y(k)$, put $\tilde{G}_\xi = \{g \in \tilde{G}; \rho(g)\xi = \xi\}$ and $\tilde{G}_\xi = \rho(\tilde{G}_\xi)$. Let ℓ be a number of $G(k)$-orbits in $Y(k)$, i.e., $\ell = |G(k)\backslash Y(k)|$.

PROPOSITION 1.1. – *We have* $G(k)\backslash Y(k) = \alpha^{-1}(1)$, *where* $\alpha : H^1(k, G_\xi) \to H^1(k, G)$.

COROLLARY 1.2. – *Assume that* (1) $H^1(k, \tilde{G}) = \{1\}$, (2) $H^1(k, \tilde{G}_\xi) \to H^1(k, G_\xi)$ *is surjective. Then we have* $G(k)\backslash Y(k) = H^1(k, G_\xi)$.

Proof. – See [2].

COROLLARY 1.3. – *Assume that* (1) $H^1(k, \tilde{G}) = \{1\}$, (2) $\operatorname{Ker} \rho = \{1\}$. *Then we have* $G(k)\backslash Y(k) = H^1(k, \tilde{G}_\xi)$.

Proof. – If $\operatorname{Ker} \rho = \{1\}$, then we have $\tilde{G}_\xi \simeq G_\xi$ and hence $H^1(k, \tilde{G}_\xi) \to H^1(k, G_\xi)$ is bijective. Q.E.D.

COROLLARY 1.4. – *If* $\tilde{G}_\xi = \{1\}$, *then we have* $\ell = 1$, i.e., $Y(k)$ *is a* $G(k)$-*orbit.*

Proof. — We have $G_\xi = \rho(\tilde{G}_\xi) = \{1\}$ and hence $G(k)\backslash Y(k) = \alpha^{-1}(1) = \{1\}$ for $\alpha : H^1(k, G_\xi) = \{1\} \to H^1(k, G)$. Q.E.D.

PROPOSITION 1.5. — *We have $\ell = 1$ for $(\tilde{G}, \rho_1 \oplus \rho_2, X_1 \oplus X_2)$ if and only if (1) $\ell = 1$ for $(\tilde{G}, \rho_1, X_1)$ and (2) $\ell = 1$ for $(H, \rho_2|H, X_2)$ where H is a generic isotropy subgroup of $(\tilde{G}, \rho_1, X_1)$.*

Proof. — Let Y (resp. Y_1, Y_2') be the open orbit of $(\tilde{G}, \rho_1 \oplus \rho_2, X_1 \oplus X_2)$ (resp. $(\tilde{G}, \rho_1, X_1)$, $(H, \rho_2|H, X_2)$). ($\Rightarrow$) For any $\xi_1 \in Y_1(k)$ and $H = \tilde{G}_{\xi_1}$, take $\xi_2 \in Y_2'(k)$. Then we have $(\xi_1, \xi_2) \in Y(k)$ and hence the projection $Y(k) \to Y_1(k)$ is a $\tilde{G}$-equivariant surjective map. Since $Y(k)$ is a $G(k)$-orbit, $Y_1(k)$ must be a $G(k)$-orbit, i.e., $\ell = 1$ for $(\tilde{G}, \rho_1, X_1)$. Now take any two points $\xi_2, \xi_2' \in Y_2'(k)$ for $H = \tilde{G}_{\xi_1}$. Since (ξ_1, ξ_2) and (ξ_1, ξ_2') are elements of $Y(k)$, there exists $g \in G(k)$ satisfying $(g\xi_1, g\xi_2) = (\xi_1, \xi_2')$. This implies that $g \in G_{\xi_1}(k) = H(k)$ satisfying $g\xi_2 = \xi_2'$, i.e., $\ell = 1$ for $(H, \rho_2|H, X_2)$. ($\Leftarrow$) Take any two points (ξ_1, ξ_2) and (ξ_1', ξ_2') of $Y(k)$. Then there exists $g \in G(k)$ such that $g\xi_1' = \xi_1$. We have $g(\xi_1', \xi_2') = (\xi_1, g\xi_2')$, and two points ξ_2 and $g\xi_2'$ belong to $Y_2'(k)$ for $H = \tilde{G}_{\xi_1}$. Hence there exists $h \in H(k)$ satisfying $hg\xi_2' = \xi_2$, i.e., $hg(\xi_1', \xi_2') = (\xi_1, \xi_2)$, with $hg \in G(k)$ Q.E.D.

COROLLARY 1.6. — *Assume that $\ell = 1$ for $(\tilde{G}, \rho_1, X_1)$ and $(H^\circ, \rho_2|H^\circ, X_2)$ where H° is the connected component of a generic isotropy subgroup H of $(\tilde{G}, \rho_1, X_1)$. Then we have $\ell = 1$ for $(\tilde{G}, \rho_1 \oplus \rho_2, X_1 \oplus X_2)$.*

Remark 1.7. — Assume that $\ell = 1$ for (G, ρ, X). Then $\ell = 1$ for $(\tilde{G}, \tilde{\rho}, X)$ with $\tilde{\rho}(\tilde{G}) \supset \rho(G)$.

THEOREM 1.8 (J.-I. Igusa [1], [2]). — *A regular irreducible P.V. has a universally transitive open orbit* (i.e., $\ell = 1$) *if and only if it is castling-equivalent to one of the following P.V.'s :*

(1) *$(G \times GL_m, \rho \otimes \Lambda_1)$ where ρ is an m-dimensional irreducible representation of G.*

(2) (GL_{2m}, Λ_2).

(3) $(Sp_n \times GL_{2m}, \Lambda_1 \otimes \Lambda_1)$.

(4) *$(GL_1 \times SO_n, \Lambda_1 \otimes \Lambda_1)$ where n is even, and $n \geqslant 4$.*

(5) *$(GL_1 \times Spin_7, \Lambda_1 \otimes$ the spin rep.).*

(6) *$(GL_1 \times Spin_9, \Lambda_1 \otimes$ the spin rep.).*

(7) *$(Spin_{10} \times GL_2$, a half-spin rep.$\otimes \Lambda_1)$.*

(8) *$(GL_1 \times E_6, \Lambda_1 \otimes \Lambda_1)$ with* $\deg(\Lambda_1) = 27$ *for E_6.*

2. Simple P.V.'s with Universally Transitive Open Orbits.

THEOREM 2.1 ([4] with a correction [5]). – *All non-irreductible simple P.V.'s with scalar multiplications are given as follows :*

(1) $(GL_1^{k+1} \times SL_n, \overbrace{\Lambda_1 \oplus \cdots \oplus \Lambda_1}^{k} \oplus \Lambda_1^{(*)})$ $(1 \leqslant k \leqslant n, n \geqslant 2)$.

(2) $(GL_1^{k+1} \times SL_n, \Lambda_2 \oplus \overbrace{\Lambda_1^{(*)} \oplus \cdots \oplus \Lambda_1^{(*)}}^{k})$ $(1 \leqslant k \leqslant 3, n \geqslant 4)$ *except* $(GL_1^4 \times SL_n, \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1^*)$ *with* $n = odd$.

(3) $(GL_1^2 \times SL_{2m+1}, \Lambda_2 \oplus \Lambda_2)$ *for* $m \geqslant 2$.

(4) $(GL_1^2 \times SL_n, 2\Lambda_1 \oplus \Lambda_1^{(*)})$.

(5) $(GL_1^3 \times SL_5, \Lambda_2 \oplus \Lambda_2 \oplus \Lambda_1^*)$.

(6) $(GL_1^2 \times SL_n, \Lambda_3 \oplus \Lambda_1^{(*)})$. $(n = 6, 7)$

(7) $(GL_1^3 \times SL_6, \Lambda_3 \oplus \Lambda_1 \oplus \Lambda_1)$.

(8) $(GL_1^k \times Sp_n, \overbrace{\Lambda_1 \oplus \cdots \oplus \Lambda_1}^{k})$ $(k = 2, 3)$.

(9) $(GL_1^2 \times Sp_2, \Lambda_2 \oplus \Lambda_1)$.

(10) $(GL_1^2 \times Sp_3, \Lambda_3 \oplus \Lambda_1)$.

(11) $(GL_1^2 \times Spin_n$, *(half-)spin rep.*$\oplus$*the vector rep.), with* $n = 7$, 8, 10, 12.

(12) $(GL_1^2 \times Spin_{10}, \Lambda \oplus \Lambda)$ *where* $\Lambda =$ *the even half-spin representation.*

Here $\Lambda^{(*)}$ *stands for* Λ *or its dual* Λ^*. *Note that* $(G, \rho, X) \simeq (G, \rho^*, X^*)$ *as triplets if* G *is reductive.*

LEMMA 2.2. – *We have* $\ell = 1$ *for* $(GL_n, \overbrace{\Lambda_1 \oplus \cdots \oplus \Lambda_1}^{n}, M(n))$.

Proof. – Clearly the isotropy subgroup at $I_n \in M(n)$ is $\{I_n\}$, and hence $\ell = 1$ by Corollary 1.4. Q.E.D.

LEMMA 2.3. – *We have* $\ell = 1$ *for*

$$(GL_1^n \times GL_n, (\overbrace{\Lambda_1 \oplus \cdots \oplus \Lambda_1}^{n}) \oplus \Lambda_1^{(*)})$$

where GL_1^n *acts independently on each irreducible component of* $(\Lambda_1 \oplus \cdots \oplus \Lambda_1)$ *and it acts on* $\Lambda_1^{(*)}$ *trivially.*

Proof. — By Remark 1.7 and Lemma 2.2, we have $\ell = 1$ for

$$(GL_1^n \times GL_n, \overbrace{\Lambda_1 \oplus \cdots \oplus \Lambda_1}^{n}).$$

Its isotropy subgroup at I_n is

$$H = \{(\alpha_1, \cdots, \alpha_n, \begin{pmatrix} \alpha_1^{-1} & & \\ & \ddots & \\ & & \alpha_n^{-1} \end{pmatrix}); \alpha_1, \cdots, \alpha_n \in GL_1\}.$$

By Proposition 1.5 and Lemma 2.2 for $n = 1$, we have $\ell = 1$ for $(H, \Lambda_1^{(*)})$. Again by Proposition 1.5, we have $\ell = 1$ for our P.V.

Q.E.D.

PROPOSITION 2.4. — *We have* $\ell = 1$ *for*

$$(GL_1^{k+1} \times SL_n, \overbrace{\Lambda_1 \oplus \cdots \oplus \Lambda_1}^{k} \oplus \Lambda_1^{(*)}) \quad (1 \leqslant k \leqslant n,\ n \geqslant 2).$$

Proof. — By Proposition 1.5, Lemma 2.2 and Lemma 2.3, we have our result. Q.E.D.

PROPOSITION 2.5. — *We have* $\ell \geqslant 2$ *for following P.V.'s :*

(1) $(GL_1^2 \times SL_n, 2\Lambda_1 \oplus \Lambda_1^{(*)})$.

(2) $(GL_1^2 \times SL_n, \Lambda_3 \oplus \Lambda_1^{(*)})$. $(n = 6, 7)$.

(3) $(GL_1^3 \times SL_6, \Lambda_3 \oplus \Lambda_1 \oplus \Lambda_1)$.

(4) $(GL_1^2 \times Sp_2, \Lambda_2 \oplus \Lambda_1)$.

(5) $(GL_1^2 \times Sp_3, \Lambda_3 \oplus \Lambda_1)$.

(6) $(GL_1^2 \times Spin_n$, *(half-)spin rep.* $\oplus$ *the vector rep.), with* $n = 7$ *and* 12.

Proof. — By Theorem 1.8, we have $\ell \geqslant 2$ for $(GL_1 \times SL_2, 2\Lambda_1)$, (GL_n, Λ_3), $(n = 6,7)$ $(GL_1 \times Sp_2, \Lambda_2) \simeq (GL_1 \times SO_5, \Lambda_1)$, $(GL_1 \times Sp_3, \Lambda_3)$, $(GL_1 \times Spin_7$, the vector rep.$) \simeq (GL_1 \times SO_7, \Lambda_1)$, and $(GL_1 \times Spin_{12}$, a half-spin rep.). By proposition 1.5, we have our result. Q.E.D.

Remarks 2.6. — In [2], it is proved that, for (GL_7, Λ_3), $Y(k)$ is $G(k)$-transitive for any local field k other than $\mathbb{R}$. However, for $(GL_1^2 \times SL_7, \Lambda_3 \oplus \Lambda_1^{(*)})$, $Y(k)$ is not $G(k)$-transitive even when k is a p-adic field. Because its generic isotropy subgroup H is $(G_2) \times \{cI_7; c^3 = 1\}$ (see, p. 86 in [3]) and $(GL_1 \times (G_2), \Lambda_2) \subset (GL_1 \times SO_7, \Lambda_1)$, we have our result by Proposition 1.5 and [2].

LEMMA 2.7. — We have $\ell = 1$ for $(GL_1 \times Sp_n, 1 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_1)$.

Proof. — We have $X = M(2n,2)$ and $\rho(g)x = \tilde{A}x \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$ for $g = (\alpha, \tilde{A}) \in GL_1 \times Sp_n, x \in X$, $\rho = 1 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_1$. For

$$\tilde{A} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in GL_{2n}$$

with A, B, C $D \in M_n$, we have $\tilde{A} \in Sp_n$ if and only if (1) $A\,{}^tB$ and $C\,{}^tD$ are symmetric matrices, (2) $A\,{}^tD - B\,{}^tC = I_n$. We shall calculate the isotropy subgroup G_ξ at

$$\xi = \left[\begin{array}{c|c} 1 & \\ 0 & \\ \vdots & 0 \\ 0 & \\ \hline & 1 \\ & 0 \\ 0 & \vdots \\ & 0 \end{array}\right] \quad (= (e_1, e_{n+1})).$$

Put

$$A = \left[\begin{array}{c|c} a_1 & a_2 \\ \hline a_3 & A_4 \end{array}\right], \ldots, D = \left[\begin{array}{c|c} d_1 & d_2 \\ \hline d_3 & D_4 \end{array}\right].$$

Then

$$\tilde{A}\xi \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} = \left[\begin{array}{c|c} a_1 & b_1\alpha \\ a_3 & b_3\alpha \\ c_1 & d_1\alpha \\ c_3 & d_3\alpha \end{array}\right] = \xi$$

implies that $a_1 = 1$, $d_1 = \alpha^{-1}$, $b_1 = c_1 = 0$, and $a_3 = b_3 = c_3 = d_3 = 0$. By the condition $A \in Sp_n$, we get

$$(1) \begin{vmatrix} A_4 & B_4 \\ C_4 & D_4 \end{vmatrix} \in Sp_{n-1}$$

$$(2) \begin{vmatrix} A_4 & B_4 \\ C_4 & D_4 \end{vmatrix} \begin{vmatrix} {}^tb_2 \\ -\,{}^ta_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix},$$

$$(3) \begin{vmatrix} A_4 & B_4 \\ C_4 & D_4 \end{vmatrix} \begin{vmatrix} {}^td_2 \\ -\,{}^tc_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix},$$

(4) $\alpha^{-1} + a_2{}^td_2 - b_2{}^tc_2 = 1$.

Thus we have

$$\tilde{G}_\xi = \{(\alpha,\tilde{A}) \in GL_1 \times SP_n, \alpha = 1, \tilde{A} = \left[\begin{array}{cc|cc} 1 & & 0 & \\ & A_4 & & B_4 \\ \hline 0 & & 1 & \\ & C_4 & & D_4 \end{array}\right]\} \simeq Sp_{n-1}.$$

On the other hand, $\operatorname{Ker}\rho = \{1]$ and $H^1(k, GL_1 \times Sp_n) = \{1\}$, we have $G(k)\backslash Y(k) = H^1(k,\tilde{G}_\xi) = H^1(k, Sp_{n-1}) = \{1\}$ by Corollary 1.3. Q.E.D.

PROPOSITION 2.8. — *We have $\ell = 1$ for $(GL_1^2 \times Sp_n, \Lambda_1 \oplus \Lambda_1)$.*

Proof. — By Remark 1.7 and Lemma 2.7, we have our result. Q.E.D.

PROPOSITION 2.9. — *We have $\ell = 1$ for $(GL_1^3 \times Sp_n, \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$.*

Proof. — Similar calculation as Lemma 2.7 shows that a generic isotropy subgroup H of $(GL_1^2 \times Sp_n, \Lambda_1 \oplus \Lambda_1)$ is isomorphic to

$$\{(\alpha, \alpha^{-1}, \left[\begin{array}{cc|c} \alpha^{-1} & & \\ & \alpha & \\ \hline 0 & & A \end{array}\right]); A \in Sp_{n-1}, \alpha \in GL_1\}.$$

By Propositions 1.5 and 2.8, it is enough to show $\ell = 1$ for $(GL_1 \times H, \Lambda_1) \simeq (GL_1 \times GL_1 \times Sp_n, (\Lambda_1 \otimes \Lambda_1^* + \Lambda_1 \otimes \Lambda_1) \otimes 1 + \Lambda_1 \otimes 1 \otimes \Lambda_1)$. Since $\ell = 1$ for (Sp_n, Λ_1) by Lemma 2.7, it is enough to show $\ell = 1$ for $(GL_1 \times GL_1, \Lambda_1 \otimes \Lambda_1^* + \Lambda_1 \otimes \Lambda_1)$. Put

$$\tilde{G} = GL_1 \times GL_1, \rho = \Lambda_1 \otimes \Lambda_1^* + \Lambda_1 \otimes \Lambda_1,$$

i.e., $\rho(\alpha,\beta) = (\alpha\beta^{-1}, \alpha\beta)$ and $G = \rho(\tilde{G})$. Since $G = GL_1 \times GL_1$, we have $G(k) = GL_1(k) \times GL_1(k) (\supseteq \rho(\tilde{G}(k))$ and

$$Y(k) = \{(\alpha,\beta) \in k^2; \alpha\beta \neq 0\} = G(k).(1,1), \text{ i.e., } \ell = 1. \quad \text{Q.E.D.}$$

PROPOSITION 2.10. — *We have $\ell = 1$ for*

$$(GL_1^{k+1} \times SL_{2m}, \Lambda_2 \oplus \overbrace{\Lambda_1^{(*)} \oplus \cdots \oplus \Lambda_1^{(*)}}^{k}) \ (1 \leqslant k \leqslant 3, m \geqslant 2).$$

Proof. — By Proposition 1.5, it is enough to show $\ell = 1$ when $k = 3$, i.e., $(GL_1^3 \times GL_{2m}, \Lambda_2 \oplus \Lambda_1^{(*)} \oplus \Lambda_1^{(*)} \oplus \Lambda_1^{(*)})$ where GL_1^3 acts on $\Lambda_1^{(*)} \oplus \Lambda_1^{(*)} \oplus \Lambda_1^{(*)}$ as independent scalar multiplications. Since the isotropy subgroup of (GL_{2m}, Λ_2) is Sp_m, we have result by Proposition 2.9. Q.E.D.

LEMMA 2.11. – *We have* $\ell = 1$ *for* $(GL_{2m+1}, \Lambda_2 \oplus \Lambda_1)$.

Proof. – The isotropy subgroup

$$H = \{A \in GL_{2m+1}; (AJ''{}^tA, Ae_1) = (J', e_1)\}$$

at

$$\xi = \left(J' = \left|\begin{array}{c|c} 0 & 0 \\ \hline 0 & J \end{array}\right|, \; e_1 = \left|\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array}\right|\right) \text{ where } J = \left|\begin{array}{c|c} 0 & -I_m \\ \hline -I_m & 0 \end{array}\right|,$$

is given by

$$H = \left\{\left|\begin{array}{c|c} 1 & 0 \\ \hline 0 & A' \end{array}\right|; \; A' \in Sp_m\right\} \simeq Sp_m .$$

Since $\operatorname{Ker} \rho = \{1\}$ and $H^1(k, GL_{2m+1}) = \{1\}$, we have $G(k)\backslash Y(k) = H^1(k, Sp_m) = \{1\}$ by Corollary 1.3 Q.E.D.

PROPOSITION 2.12. – *We have* $\ell = 1$ *for* $(GL_1^4 \times SL_{2m+1}, \Lambda_2 \oplus \Lambda_1 \oplus (\Lambda_1 \oplus \Lambda_1)^{(*)})$.

Proof. – It is enough to show $\ell = 1$ when

$$\tilde{G} = GL_1 \times GL_1 \times GL_1 \times GL_{2m+1},$$

$$\rho = (1 \otimes 1 \otimes 1) \otimes \Lambda_2 + \Lambda_1 \otimes 1 \otimes 1 \otimes \Lambda_1 + (1 \otimes \Lambda_1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1) \otimes \Lambda_1^{(*)} .$$

A generic isotropy subgroup of $(1 \otimes 1 \otimes 1) \otimes \Lambda_2 + \Lambda_1 \otimes 1 \otimes 1 \otimes \Lambda_1$ is

$$\left\{\left(\alpha, \beta, \gamma; \left|\begin{array}{c|c} \alpha^{-1} & 0 \\ \hline 0 & A \end{array}\right|\right) \in \tilde{G}; \; A \in Sp_m\right\}$$

(cf. Lemma 2.11). Hence, by Proposition 1.5 and Lemma 2.11, it is enough to show $\ell = 1$ for $\tilde{G} = GL_1 \times GL_1 \times GL_1 \times Sp_m$, $\rho = \Lambda_1^{(*)} \otimes (\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1) \otimes 1 + 1 \otimes (\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1)$. One can prove that $\ell = 1$ for $(GL_1 \times Sp_m, \Lambda_1 \otimes (\Lambda_1 + \Lambda_1))$ similarly as Lemma 2.7. Note that $\tilde{G}_\xi \simeq Sp_{m-1} \times \operatorname{Ker} \rho$ in this case. Then our assertion is clear.

Q.E.D.

PROPOSITION 2.13. – *We have* $\ell = 1$ *for* $(GL_1^4 \times SL_{2m+1}, \Lambda_2 \oplus \Lambda_1^* \oplus \Lambda_1^* \oplus \Lambda_1^*)$.

Proof. – It is enough to show $\ell = 1$ when

$$\tilde{G} = GL_1 \times GL_1 \times GL_1 \times GL_{2m+1},$$

$\rho = (1\otimes1\otimes1) \otimes \Lambda_2 + (\Lambda_1\otimes1\otimes1+1\otimes\Lambda_1\otimes1+1\otimes1\otimes\Lambda_1) \otimes \Lambda_1^*$. The isotropy subgroup of $(1\otimes1\otimes1) \otimes \Lambda_2$ at

$$J' = \left|\begin{array}{c|c} 0 & 0 \\ \hline 0 & J \end{array}\right| \text{ is } \qquad H = \{\left|\begin{array}{c|c} \alpha & 0 \\ \hline A' & A \end{array}\right| \in GL_{2m+1}\ ;\ A \in Sp_m\}.$$

By Proposition 1.5 and Lemma 2.11, it is enough to show $\ell = 1$ for a P.V. given by

$$X \mapsto \left[\begin{array}{c|c} \alpha^{-1} & B \\ \hline 0 & A \end{array}\right] X \begin{bmatrix} \beta & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \delta \end{bmatrix} = \left|\begin{array}{c} (\alpha^{-1}x_1,\alpha^{-1}x_2,\alpha^{-1}x_3) + BZ \\ \hline AZ \begin{pmatrix} \beta & & \\ & \gamma & \\ & & \delta \end{pmatrix} \end{array}\right|$$

for $X = \left[\dfrac{x_1,x_2,x_3}{Z}\right] \in M(2m+1,3)$, $A \in Sp_m$. Now by Proposition 2.9, any point $X = \left[\dfrac{x_1,x_2,x_3}{Z}\right]$ of $Y(k)$ is $G(k)$-equivalent to

$$X_0 = \left[\frac{z_1,z_2,z_3}{Z_0}\right],$$

where

$$Z_0 = {}^t\left|\begin{array}{c|c} 1\,0\ldots0 & 0\ldots0 \\ \hline 0 & 1\,0\ldots0 \\ \hline 1\,1\,0\ldots0 & 1\,0\ldots0 \end{array}\right| (= (e_1, e_{m+1}, e_1+e_2+e_{m+1}),$$

cf. p. 81 in [4]) and $(z_1,z_2,z_3) \in k^3$. Put $B = (b_1,\ldots,b_{2m})$ with $b_1 = -z_1$, $b_2 = z_1 + z_2 - z_3$, $b_{m+1} = -z_2$, $b_j = 0$ for all $j \neq 1, 2, m+1$. Then we have

$$\left|\begin{array}{c|c} 1 & B \\ \hline 0 & I_{2m} \end{array}\right| X_0 I_3 = \left|\begin{array}{c} 0 \\ \hline Z_0 \end{array}\right|.$$

This implies that $G(k)$ acts on $Y(k)$ transitively. Q.E.D.

Proposition 2.14. – *We have* $\ell = 1$ *for* $(GL_1^3 \times SL_{2m+1}, \Lambda_2\oplus\Lambda_1^*\oplus\Lambda_1^*)$ *and* $(GL_1^2 \times SL_{2m+1}, \Lambda_2\oplus\Lambda_1^*)$.

Proof. – By Propositions 1.5 and 2.12, we have our result.

Q.E.D.

PROPOSITION 2.15. – *We have* $\ell = 1$ *for* $(GL_1^2 \times SL_{2m+1}, \Lambda_2 \oplus \Lambda_2)$.

Proof. – A direct calculation shows that the isotropy subgroup G_ξ of $(SL_{2m+1}, \Lambda_2 \oplus \Lambda_2)$ at

$$\xi = \left(\begin{array}{c|c|c} 0 & & I_m \\ \hline & & \\ \hline -I_m & & 0 \end{array} \quad \begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & -I_m \\ \hline 0 & I_m & 0 \end{array} \right)$$

is given by

$$G_\xi = \left\{ \begin{array}{c|c} I_{m+1} & 0 \\ \hline \begin{matrix} a_1 & a_2 & \cdots & a_{m+1} \\ a_2 & a_3 & \cdots & a_{m+2} \\ \vdots & & & \vdots \\ a_m & a_{m+1} & \cdots & a_{2m} \end{matrix} & I_m \end{array} \right\} \cong G_a^{2m}.$$

Since $H^1(k, SL_{2m+1}) = \{1\}$, $\operatorname{Ker} \rho = \{1\}$, and $H^1(k, G_a^{2m}) = \{1\}$, we have $\ell = 1$ for $(SL_{2m+1}, \Lambda_2 \oplus \Lambda_2)$ by Corollary 1.3. Hence we obtain our result. Q.E.D.

PROPOSITION 2.16. – *We have* $\ell = 1$ *for* $(GL_1^3 \times SL_5, \Lambda_2 \oplus \Lambda_2 \oplus \Lambda_1^*)$.

Proof. – Let H be the generic isotropy subgroup of $(GL_1^2 \times SL_5, \Lambda_2 \oplus \Lambda_2)$ at $\xi = (e_2 \wedge e_3 + e_1 \wedge e_4,\ e_1 \wedge e_3 + e_2 \wedge e_5)$. Clearly H contains $\{(\varepsilon_1, \varepsilon_2, \operatorname{diag}(\varepsilon_1^{-1}\varepsilon_2^{-2}, \varepsilon_1^{-2}\varepsilon_2^{-1}, \varepsilon_1\varepsilon_2, \varepsilon_2^2, \varepsilon_1^2)) \in GL_1^2 \times SL_5\}$ and

$$\{(1, 1, \left[\begin{array}{c|c} I_2 & A \\ \hline 0 & I_3 \end{array}\right]);\ A = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_1 & \gamma_4 \end{bmatrix}, (\gamma_1\ \ \gamma_2\ \ \gamma_3\ \ \gamma_4) \in G_a^4\}.$$

By Corollary 1.6 and Proposition 2.15, it is enough to show that $\ell = 1$ for $(GL_1 \times H, \Lambda_1 \otimes \Lambda_1^*)$. An element $x = {}^t(x_1, x_2, x_3, x_4, x_5) \in Aff^5$ is a generic point of $(GL_1 \times H, \Lambda_1 \otimes \Lambda_1^*)$ if and only if $x_1 x_2 \neq 0$ (cf. Proposition 1.1 in [5]). Assume that x is in $Y(k)$, then by the action of $g_1 = (\varepsilon, \operatorname{diag}(\varepsilon_1^{-1}\varepsilon_2^{-2}, \ldots, \varepsilon_1^2)) \in H(k)$ with $\varepsilon = x_1/x_2^2$, $\varepsilon_1 = 1$, $\varepsilon_2 = x_2/x_1$, we may assume that $x_1 = x_2 = 1$. Now it is transformed to $x_0 = {}^t(1,1,0,0,0)$ by the action of

$$g_2 = (1, \left[\begin{array}{c|c} I_3 & A \\ \hline 0 & I_2 \end{array}\right]) \in H(k)$$

with $A = \begin{bmatrix} x_3, & x_4 - x_3, & 0 \\ 0, & x_3, & x_5 \end{bmatrix}$. Thus $GL_1(k) \times H(k)$ acts on $Y(k)$ transitively. Q.E.D.

PROPOSITION 2.17. — *We have $\ell = 1$ for $(GL_1^2 \times Spin_n$, a half-spin rep. $\oplus$ the vector rep.) with $n = 8$ and 10.*

Proof. — Let n be 8 or 10. Then by Theorem 1.8, we have $l = 1$ for $(GL_1 \times Spin_n$, the vector rep.) and $(GL_1 \times Spin_{n-1}$, the spin rep.). Since the restriction of a half-spin representation of $Spin_n$ to a generic isotropy subgroup of $(GL_1 \times Spin_n$, the vector rep.) gives $(GL_1 \times Spin_{n-1}$, the spin rep.), we have our result by Corollary 1.6. Q.E.D.

PROPOSITION 2.18. — *We have $\ell = 1$ for $(GL_1^2 \times Spin_{10}, \Lambda \oplus \Lambda)$ where Λ = the even half-spin representation.*

Proof. — Prof. J.-I. Igusa proved that $\ell = 1$ for $(GL_1 \times Spin_{10}, \Lambda_1 \oplus (\Lambda \oplus \Lambda))$ (See p. 14 in [1]) and our assertion is obvious by Remark 1.7. Q.E.D.

THEOREM 2.19. — *All non-irreducible simple P.V.'s with universally transitive open orbits are given as follows :*

(1) $(GL_1^{k+1} \times SL_n, \Lambda_1 \oplus \overbrace{\cdots\cdots}^{k} \oplus \Lambda_1 \oplus \Lambda_1^{(*)})$ $(1 \leqslant k \leqslant n, n \geqslant 2)$,

(2) $(GL_1^{k+1} \times SL_n, \Lambda_2 \oplus \Lambda_1^{(*)} \oplus \overbrace{\cdots\cdots}^{k} \oplus \Lambda^{(*)1})$ $(1 \leqslant k \leqslant 3, n \geqslant 4)$, *except* $(GL_1^4 \times SL_n, \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1^*)$ *with n = odd.*

(3) $(GL_1^2 \times SL_{2m+1}, \Lambda_2 \oplus \Lambda_2)$ *for* $m \geqslant 2$.

(4) $(GL_1^3 \times SL_5, \Lambda_2 \oplus \Lambda_2 \oplus \Lambda_1^*)$.

(5) $(GL_1^k \times Sp_n, \Lambda_1 \oplus \overbrace{\cdots}^{k} \oplus \Lambda_1)$ $(k=2, 3)$.

(6) $(GL_1^2 \times Spin_n$, *a half-spin rep.* $\oplus$ *the vector rep.) with* $n = 8, 10$.

(7) $(GL_1^2 \times Spin_{10}, \Lambda \oplus \Lambda)$ *where* Λ = *the even half-spin representation.*

Proof. — By Proposition 2.4, 2.5, 2.8-2.10 ; 2.12-2.18, we have our result. Q.E.D.

COROLLARY 2.20. — *All non-irreducible regular simple P.V.'s with universally transitive open orbits are given as follows :*

(1) $(GL_1^2 \times SL_n, \Lambda_1 \oplus \Lambda_1^*)$.

(2) $(GL_1^n \times SL_n, \Lambda_1 \oplus \overset{n}{\widetilde{\cdots}} \oplus \Lambda_1)$.

(3) $(GL_1^{n+1} \times SL_n, \Lambda_1 \oplus \overset{n}{\widetilde{\cdots}} \oplus \Lambda_1 \oplus \Lambda_1^{(*)})$.

(4) $(GL_1^3 \times SL_{2m}, \Lambda_2 \oplus \Lambda_1^{(*)} \oplus \Lambda_1^{(*)})$.

(5) $(GL_1^2 \times SL_{2m+1}, \Lambda_2 \oplus \Lambda_1)$.

(6) $(GL_1^4 \times SL_{2m+1}, \Lambda_2 \oplus \Lambda_1 \oplus (\Lambda_1 \oplus \Lambda_1)^{(*)})$.

(7) $(GL_1^2 \times Sp_n, \Lambda_1 \oplus \Lambda_1)$.

(8) $(GL_1^2 \times Spin_n$, *a half-spin rep.* $\oplus$ *the vector rep.*) *with* $n = 8, 10$.

(9) $(GL_1^2 \times Spin_{10}, \Lambda \oplus \Lambda)$ *where* $\Lambda =$ *the even half-spin representation.*

3. 2-Simple P.V.'s of Type I with Universally Transitive Open Orbits.

THEOREM 3.1. ([5]). — *All non-irreducible 2-simple P.V.'s* $(GL_1^k \times G(=G_1 \times G_2), \rho(=\rho_1 \oplus \ldots \oplus \rho_k))$ *of Type I, which do not contain a regular irreducible P.V.'s with* $\ell \geqslant 2$, *are castling-equivalent to the following P.V.'s :*

(I) (1) $G = SL_{2m+1} \times SL_2$, $\rho = \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1 (+T)$ *with* $T = 1 \otimes \Lambda_1 (+1 \otimes \Lambda_1)$.

(2) $G = Spin_{10} \times SL_2$, $\rho =$ *a half-spin rep.* $\otimes \Lambda_1 + 1 \otimes \Lambda_1$ $(+T)$ *with* $T = 1 \otimes \Lambda_1 (+1 \otimes \Lambda_1)$.

(II) (3) $G = SO_n \times SL_{n-1}$, $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1^{(*)}$ $(n = even)$.

(4) $G = SL_4 \times SL_5$, $\rho = \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^*$.

(5) $G = Spin_7 \times SL_7$, $\rho =$ *the spin rep.* $\otimes \Lambda_1 + 1 \otimes \Lambda_1^*$.

(6) $G = Spin_8 \times SL_7$, $\rho =$ *the vector rep.* $\otimes \Lambda_1 +$ *a half spin rep.* $\otimes 1 + 1 \otimes \Lambda_1^*$.

(III) (7) $G = Sp_n \times SL_m$, $\rho = \Lambda_1 \otimes \Lambda_1 + T$, *with* $T = 1 \otimes (\Lambda_1^{(*)} + \overset{k}{\widetilde{\cdots}} + \Lambda_1^{(*)})$ $(1 \leqslant k \leqslant 3)$ *except* $1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1^*)$ *with* $m = odd$ $\Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1^{(*)} + \overset{k}{\widetilde{\cdots}} + \Lambda_1^{(*)})$ $(0 \leqslant k \leqslant 2)$ *except* $\Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1 + \Lambda_1^*)$ *with* $m = odd$, $1 \otimes \Lambda_2 (m = odd)$, $1 \otimes (\Lambda_2 + \Lambda_1^*) (m = 5)$.

(8) $G = Sp_n \times SL_{2m+1}$, $\rho = \Lambda_1 \otimes \Lambda_1 + (\Lambda_1 + \Lambda_1) \otimes 1$.

(9) $G = Sp_n \times SL_2$, $\rho = \Lambda_1 \otimes 2\Lambda_1 + 1 \otimes \Lambda_1$.

(IV) (10) $G=SL_5\times SL_2$, $\rho=\Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1(+1\otimes\Lambda_1;+\Lambda_1^*\otimes 1)$.

(11) $G=SL_5\times SL_2$, $\rho=\Lambda_2\otimes\Lambda_1+\Lambda_1^*\otimes 1(+1\otimes\Lambda_1(+1\otimes\Lambda_1))$.

(12) $G=SL_5\times SL_2$, $\rho=\Lambda_2\otimes\Lambda_1+(\Lambda_1^*+\Lambda_1^*)\otimes 1$.

(13) $G=SL_5\times SL_8$, $\rho=\Lambda_2\otimes\Lambda_1+1\otimes\Lambda_1^*$.

(14) $G=SL_5\times SL_9$, $\rho=\Lambda_2\otimes\Lambda_1+1\otimes\Lambda_1^*(+\Lambda_1^{(*)}\otimes 1)$.

(15) $G=SL_7\times SL_2$, $\rho=\Lambda_2\otimes\Lambda_1+\Lambda_1\otimes 1$, $\Lambda_2\otimes\Lambda_1+\Lambda_1^*\otimes 1(+1\otimes\Lambda_1)$.

(16) $G=SL_9\times SL_2$, $\rho=\Lambda_2\otimes\Lambda_1+\Lambda_1^*\otimes 1$.

(17) $G=Spin_{10}\times SL_n$, $(n=14, 15)$, $\rho=$ *a half-spin rep.* $\otimes\Lambda_1+1\otimes\Lambda_1^*$.

PROPOSITION 3.2. – *We have* $\ell=1$ *for P.V's in (I), i.e.,* (1) *and* (2) *in Theorem* 3.1.

Proof. – For (1), it is enough to show $\ell=1$ when $\rho=\Lambda_2\otimes\Lambda_1+1\otimes(\Lambda_1+\Lambda_1+\Lambda_1)$. Since we have $\ell=1$ for $(GL_1^3\times SL_2,\Lambda_1+\Lambda_1+\Lambda_1)$ and $(SL_{2m+1}\times\{I_2\},\ \Lambda_2\oplus\Lambda_1)=(SL_{2m+1},\Lambda_2\oplus\Lambda_2)$, we have our result by Corollary 1.6 and the proof of Proposition 2.15. For (2), one can prove similary as above by the proof of Proposition 2.18. Q.E.D.

PROPOSITION 3.3. – *We have* $\ell\geqslant 2$ *for P.V's in (II), i.e.,* (3)-(6) *in Theorem* 3.1.

Proof. – For (3), the GL_{n-1}-part of a generic isotropy subgroup H of $(SO_n\times GL_{n-1},\Lambda_1\otimes\Lambda_1)$ is $O_{(n-1)}$ (cf. p. 109 in [3]). Since $\ell\geqslant 2$ for $(GL_1\times O_{n-1},\Lambda_1\otimes\Lambda_1)$ $(n-1=\text{odd})$, we have our result by Proposition 1.5. For remaining P.V.'s, since $(Spin_7$, the spin rep.$)\subset(SO_8,\Lambda_1)\simeq(Spin_8$, the vector rep.$)$ and $(SL_4,\Lambda_2)\simeq(SO_6,\Lambda_1)$, we have our result.

Q.E.D.

SUBLEMMA 3.4. – *Let* $V=K^{2n}$ *with* $\langle v,v'\rangle={}^t vJv'$ *where* $J=\begin{bmatrix}0 & I_n\\ -I_n & 0\end{bmatrix}$. *Assume that* $\{v_1,\ldots,v_r\}$ *and* $\{u_1,\ldots,u_r\}$ *are linearly independent subsets of* V *satisfying* $\langle v_i,v_j\rangle=\langle u_i,u_j\rangle$ *for* $i,j=1,\ldots,r$ *with* $r<2n$. *Then there exist* v_{r+1} *and* u_{r+1} *such that* (1) $\{v_1,\ldots,v_{r+1}\}$ *and* $\{u_1,\ldots,u_{r+1}\}$ *are linearly independent,* (2) $\langle v_i,v_j\rangle=\langle u_i,u_j\rangle$ *for all* $i,j=1,\ldots,r+1$.

Proof. — (I) The case when $\langle v_1, \ldots, v_r\rangle^{\perp} \not\subset \langle v_1, \ldots, v_r\rangle$. Take u_{r-} such that $u_{r+1} \notin \langle u_1, \ldots, u_r\rangle$. Since $\{v_1, \ldots, v_r\}$ is linearly independen the linear equation ${}^t(v_1, \ldots, v_r)\ Jv = {}^t(u_1, \ldots, u_r)Ju_{r+1}$ (i.e $\langle v_i, v\rangle = \langle u_i, u_{r+1}\rangle$ *for* $i = 1, \ldots, r$) has a solution v_0, and the set c solution is given by $v_0 + \langle v_1, \ldots, v_r\rangle^{\perp} (\not\subset \langle v_1, \ldots, v_r\rangle)$. Hence ther exists $v_{r+1} \notin \langle v_1, \ldots, v_r\rangle$ such that $\langle v_i, v_{r+1}\rangle = \langle u_i, u_{r+1}\rangle$ fo $i = 1, \ldots, r$.

(II) The case when $\langle v_1, \ldots, v_r\rangle^{\perp} \subset \langle v_1, \ldots, v_r\rangle$. Tak $v_{r+1} \notin \langle v_1, \ldots, v_r\rangle$. Assume that any solution u o ${}^t(u_1, \ldots, u_r)Ju = {}^t(v_1, \ldots, v_r)Jv_{r+1}$ belongs to $\langle u_1, \ldots, u_r\rangle$. Le $u = a_1u_1 + \cdots + a_ru_r$ be a solution. Since ${}^tu_iJu_j = {}^tv_iJv_j$ fo $i, j = 1, \ldots, r$, we have ${}^t(v_1, \ldots, v_r)J(a_1v_1 + \cdots + a_rv_r) = {}^t(v_1, \ldots, v_r)Jv_{r+1}$, i.e., $v_{r+1} - a_1v_1 - \cdots - a_rv_r \in \langle v_1, \ldots, v_r\rangle^{\perp} \subset \langle v_1, \ldots, v_r\rangle$ and hence $v_{r+1} \in \langle v_1, \ldots, v_r\rangle$ a contradiction. Henc there exists $u_{r+1} \in V$ satisfying $u_{r+1} \notin \langle u_1, \ldots, u_r\rangle$ an ${}^t(u_1, \ldots, u_r)Ju_{r+1} = {}^t(v_1, \ldots, v_r)Jv_{r+1}$. Q.E.D

LEMMA 3.5. — *Let* $\{v_1, \ldots, v_r\}$ *and* $\{u_1, \ldots, u_r\}$ *are linearly independen subsets of* $V = K^{2n}$ *satisfying* $\langle v_i, v_j\rangle = \langle u_i, u_j\rangle$ *for* $i, j = 1, \ldots, r$. *The there exists an element* g *of the symplectic group* $Sp_n(K)$ *such tha* $gv_i = u_i$ *for* $i = 1, \ldots, r$.

Proof. — By Sublemma 3.4, there exist basis $\{v_1, \ldots, v_r, \ldots, v_{2n}$ and $\{u_1, \ldots, u_r, \ldots, u_{2n}\}$ satisfying $\langle v_i, v_j\rangle = \langle u_i, u_j\rangle$. Define an elemen g of GL_{2n} by $(v_1, \ldots, v_{2n})g = (u_1, \ldots, u_{2n})$. Then it is clear tha $g \in Sp_n(K)$. Q.E.D

LEMMA 3.6. — Let Ω be the universal domain and K a subfield For $2n \geqslant m$, put $W = \{v \in M_{2n,m}(\Omega)\ ;\ \text{rank}\, v = m\}$ an $W' = \{w \in Alt_m(\Omega)\ ;\ \text{rank}\ w \text{ is maximal}\}$. Define a map $\psi : W \to Alt_m(\Omega$ by $\psi(v) = (\langle v_i, v_j\rangle)$ for $v = (v_1, \ldots, v_{2m}) \in W$. Then $\psi(W) = W'$ and $\psi(W((K)) = W'(K)$.

Proof. — Note that W (resp. W') is the Zariski dense orbit o $(Sp_n \times GL_m, \Lambda_1 \otimes \Lambda_1,\ M_{2n,m}(\Omega))$ (resp. $(GL_m, \Lambda_2, Alt_m(\Omega)))$. Since $\psi(Av{}^tB) = B\psi(v){}^tB$ for any $(A,B) \in Sp_n \times GL_m$, $\psi(W)$ is an orbit of (GL_m, Λ_2). Let X_0 be the generic point of $(Sp_n \times GL_m,\ \Lambda_1 \otimes \Lambda_1)$ given in p. 101 in [3]. Then we have $\psi(X_0) = J(m = \text{even})$ o $\psi(X_0) = \begin{vmatrix} J & 0 \\ 0 & 0 \end{vmatrix}$ $(m = \text{odd})$, i.e., $\psi(X_0)$ is a generic point of (GL_m, Λ_2). Hence $\psi(W) = W'$. Since ψ is defined over the prime field, we have

$\psi(W(K)) \subset W'(K)$. Since $\ell = 1$ for (GL_m, Λ_2), $W'(K)$ is a single $\Lambda_2(GL_m)(K)$-orbit. Since $\psi(W(K))$ is $\Lambda_2(GL_m)(K)$-admissible, we have $\psi(W(K)) = W'(K)$. Q.E.D.

PROPOSITION 3.7. – *We have $\ell = 1$ for $(Sp_n \times G, \Lambda_1 \otimes \rho)$ $(m = \deg \rho \leqslant 2n)$ if and only if $\ell = 1$ for $(G, \Lambda^2(\rho))$.*

Proof. – Let $Y(\subset W \subset M_{2n,m}(\Omega))$ and $Y'(\subset W' \subset Alt_m(\Omega))$ be the Zariski-dense orbits of $(Sp_n \times G, \Lambda_1 \otimes \rho)$ and $(G, \Lambda^2(\rho))$ respectively. Then the map $\psi : W \to W'$ in Lemma 3.6 gives the surjective $Sp_n \times G$-equivariant map $\psi : Y \to Y'$. Clearly we have $\psi(Y(K)) \subset Y'(K)$. Take any element x of $Y'(K)$. Since $\psi(W(K)) = W'(K) \supset Y'(K)$, there exists $v = (v_1, \ldots, v_m) \in W(K)$ such that $\psi(v) = x$. On the other hand, we have $\psi(Y) = Y' \supset Y'(K)$ there exists $u = (u_1, \ldots, u_m) \in Y$ such that $\psi(u) = x$. By Lemma 3.5, there exists $g \in Sp_n$ satisfying $v = gu \in Y$, i.e., $v \in Y \cap W(K) = Y(K)$ Hence $\psi : Y(K) \to Y'(K)$ is surjective. By Lemma 3.5, each fibre is $Sp_n(K)$-homogeneous. Thus the orbits in $Y(K)$ and $Y'(K)$ correspond bijectively. Q.E.D.

COROLLARY 3.8. – (1) *We have $\ell = 1$ for $(Sp_n \times G, \Lambda_1 \otimes \rho + 1 \otimes \sigma)$ $(\deg \rho \leqslant 2n)$ if and only if $\ell = 1$ for $(G, \Lambda^2(\rho) + \sigma)$.*

(2) *We have $\ell = 1$ for $(Sp_n \times G \times GL_1, \Lambda_1 \otimes \rho \otimes 1 + \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \sigma \otimes 1)$ $(\deg \rho \leqslant 2n-1)$ if and only if $\ell = 1$ for $(G \times GL_1, \Lambda^2(\rho) \otimes 1 + \rho \otimes \Lambda_1 + \sigma \otimes 1)$.*

(3) *We have $\ell = 1$ for $(GL_1^2 \times Sp_n \times GL_{2m+1},\ 1 \otimes 1 \otimes \Lambda_1 \otimes \Lambda_1 + (\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1) \otimes \Lambda_1 \otimes 1)(2m + 3 \leqslant 2n)$ if and only if $\ell = 1$ for $(GL_1 \times GL_{2m+1},\ 1 \otimes \Lambda_2 + (\Lambda_1 + \Lambda_1^*) \otimes \Lambda_1)$.*

Proof. – (1) is obvious. Since $\Lambda^2(\rho \otimes 1 + 1 \otimes \Lambda_1) = \Lambda^2(\rho) \otimes 1 + \rho \otimes \Lambda_1$, we have (2). Since $\Lambda^2(1 \otimes 1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 \otimes 1 + 1 \otimes \Lambda_1 \otimes 1)$ for $GL_1^2 \times GL_{2m+1}$, is $GL_1^2 \times GL_{2m+1},\ 1 \otimes 1 \otimes \Lambda_2 + (\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1) \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_1 \otimes 1)$, we have (3) by Proposition 1.5. Q.E.D.

PROPOSITION 3.9. – *For P.V's in (III) in Theorem 3.1, we have $\ell = 1$ for (7), (8) and $\ell \geqslant 2$ for (9).*

Proof. – By Theorem 2.19 and Corollary 3.8, we have $\ell = 1$ for (7). By Lemma 2.7, the proof of Proposition 2.12, and (3) of Corollary 3.8, we have $\ell = 1$ for (8). Since $(SL_3, \Lambda^2(\Lambda_1)) = (SL_3, \Lambda_2) = (SL_3, \Lambda_1^*)$, we have $(SO_3, \Lambda^2(\Lambda_1)) = (SO_3, \Lambda_1)$. Hence we have $\ell \geqslant 2$ for $(Sp_n \times GL_2, \Lambda_1 \otimes 2\Lambda_1) = (Sp_n \times GO_3, \Lambda_1 \otimes \Lambda_1)$. Thus $\ell \geqslant 2$ for (9).

Q.E.D.

LEMMA 3.10. – *We have* $\ell \geqslant 2$ *for* $(GL_{2m+1} \times GL_2, \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes 1)(m=2,3)$.

Proof. – Assume that $\ell = 1$. Then, by Proposition 1.5, we have $\ell = 1$ for $(H \times GL_2, \Lambda_2 \otimes \Lambda_1)$ where

$$H = \left\{ \left| \begin{array}{c|c} 1 & A' \\ \hline 0 & A \end{array} \right| ; \ A \in GL_{2m} \right\}.$$

Since $\left| \begin{array}{c|c} 1 & A' \\ \hline 0 & A \end{array} \right| \left| \begin{array}{c|c} 0 & y \\ \hline -{}^ty & X \end{array} \right| \left| \begin{array}{c|c} 1 & 0 \\ \hline {}^tA' & {}^tA \end{array} \right| = \left| \begin{array}{c|c} 0 & * \\ \hline * & AX{}^tA \end{array} \right|$, this implies $\ell = 1$ for $(GL_{2m} \times GL_2, \Lambda_2 \otimes \Lambda_1)$, which is a contradiction by Theorem 1.8.

PROPOSITION 3.11. – *We have* $\ell \leqslant 2$ *for any P.V. in* (10) *in Theorem* 3.1.

Proof. – By Proposition 1.5 and Lemma 3.10, we have our result. Q.E.D.

PROPOSITION 3.12. – *We have* $\ell = 1$ *for any P.V. in* (11) *in Theorem* 3.1.

Proof. – It is enough to show $\ell = 1$ for $(GL_1^4 \times SL_5 \times SL_2, \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1 + 1 \otimes (\Lambda_1 + \Lambda_1))$. Since $\ell = 1$ for $(GL_1^2 \times SL_2, \Lambda_1 + \Lambda_1)$, it is enough to show $\ell = 1$ for

$$(GL_1^2 \times SL_5 \times \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}, \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1) \simeq (GL_1^3 \times SL_5, \Lambda_2 \otimes \Lambda_2 \otimes \Lambda_1^*).$$

Thus we have our result by Theorem 2.19. Q.E.D.

PROPOSITION 3.13. – *We have* $\ell = 1$ *for a P.V.* (12) *in Theorem* 3.1.

Proof. – We shall prove that a generic isotropy subgroup of $(GL_1 \times GL_5 \times GL_2, 1 \otimes \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^* \otimes 1 + \Lambda_1 \otimes \Lambda_1^* \otimes 1)$ is $\{1\}$. Then we have $\ell = 1$ by Corollary 1.4. The representation space V is given by $V = \{(X, Y), Z); X, Y \in M_5, {}^tX = -X, {}^tY = -Y, Z \in M_{5,2}\}$. Then the action is given by $\rho(g)x = \{(AX{}^tA, AY{}^tA){}^tB, {}^tA^{-1}Z({}^1\alpha)\}$ for $g = (\alpha, A, B) \in GL_1 \times GL_5 \times GL_2$ and $x = \{(X, Y), Z\} \in V$. Put $x_0 = \{(X_0, Y_0), Z_0\}$ with $X_0 = (-e_4, -e_5, 0, e_1 + e_5, e_2 - e_4)$, $Y_0 = (0, e_4, e_5, -e_2 + e_5, -e_3 - e_4)$, $Z_0 = (e_4, e_5)$ where $e_i = {}^t(0 \ldots \overset{\overset{i}{\smile}}{1} \ldots 0) \in \Omega^5$. We shall calculate the isotropy subgroup

$H = \{g \in GL_1 \times GL_5 \times GL_2; \rho(g)x_0 = x_0\}$. One can easily check that ${}^tA^{-1}Z_0({}^1\alpha) = Z_0$ if and only if A is of the form

$$\begin{array}{|c|c|} A_1 & A_2 \\ \hline 0 & ({}^t\alpha) \end{array} .$$

We shall determine (A, B) satisfying $(AX_0{}^tA, AY_0{}^tA){}^tB = (X_0, Y_0)$ where A is of the above form. By comparing the components of (1,4), (1,5), (2,4), (2,5), (3,4), (3,5), (4,5), we obtain $b_{12} = \alpha^{-1} - b_{11}$, $b_{21} = \alpha^{-1} - b_{22}$, $a_{12} = c - a_{11}$, $a_{13} = a_{11} - c$, $a_{14} = c - a_{11}$, $a_{15} = \alpha b_{22}c - a_{11}$, $a_{21} = c - a_{22}$, $a_{23} = c\alpha^{-1} - a_{22}$, $a_{24} = a_{22} - b_{22}c$, $a_{25} = a_{22} - \alpha b_{11}c$, $a_{31} = a_{33} - c\alpha^{-1}$, $a_{32} = c\alpha^{-1} - a_{33}$, $a_{34} = b_{11}c - a_{33}$, $a_{35} = c\alpha^{-1} - a_{33}$, where $c(b_{11} + b_{22} - \alpha^{-1}) = 1$, $A = (a_{ij})$ and $B = (b_{ij})$. Then, by comparing the (1,2), (1,3), (2,3) components, we obtain $a_{11} = a_{22} = a_{33} = b_{11} = b_{22} = c = \alpha = 1$ Thus we have $H = \{1\}$.

Q.E.D.

PROPOSITION 3.14. – *We have* $\ell \geqslant 2$ *for a P.V.* (13) *in Theorem* 3.1.

Proof. – Let $\mathfrak{H}$ be the $\mathfrak{sl}_8$-part of the generic isotropy subalgebra of $(GL_1 \times SL_5 \times SL_8, \Lambda_1 \otimes \Lambda_2 \otimes \Lambda_1)$ at $x_0 = (\omega_1, 2\omega_3, 2\omega_2, \omega_{10}, \omega_5 - \omega_8, \omega_4 - \omega_9, \omega_6, \omega_7)$ (see P.95 in [3]). Then its image by Λ_1^* is given by

$$\Lambda_1^*(\mathfrak{H} = \{\tilde{A} = \begin{pmatrix} A & B & 0 \\ 0 & A' & B' \\ 0 & 0 & a \end{pmatrix} \in M_8 \ ; B = \begin{pmatrix} -2d_3, & 4d_2, & -2d_1, & 0 \\ -d_4, & d_3, & d_2, & -d_1 \\ 0, & -d_4, & 2d_3, & -d_2 \end{pmatrix},$$

$B' = 2{}^t(d_1, d_2, d_3, d_4)$, $a = -25t$, $A = 15tI_3 + (2\Lambda_1)(C)$,

$$A' = -5tI_4 + (3\Lambda_1)(C) \quad \text{for} \quad C \in \mathfrak{sl}_2\}.$$

Let H be any algebraic subgroup of GL_8 with Lie (H) $= \Lambda_1^*(\mathfrak{H})$. It is enough to show $\ell \geqslant 2$ for $(GL_1 \times H, \Lambda_1 \otimes \Lambda_1, \Omega^8)$. Since $hAh^{-1} \in \Lambda_1^*(\mathfrak{H})$, for any $h \in H$ and $A \in \Lambda_1^*(\mathfrak{H})$, we have

$$H \subset \{\begin{pmatrix} h_1 & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}\}$$

by Schur's lemma. Since the normalizer of GO_3 is GO_3, we may assume that $h_1 \in GO_3$. Let $x = {}^t(x_1, \ldots, x_8)$ be a point of $Y(k)$ for

$(GL_1 \times H, \Lambda_1 \otimes \Lambda_1, \Omega^8)$. Clearly we may assume that $x_8 = 1$. By the action of one parameter subgroups obtained from B and B' in $\Lambda_1^*(\mathfrak{H})$, we may also assume that $x_4 = x_5 = x_6 = x_7 = 0$. Let H_1 be the subgroup of H fixing $x_4 = x_5 = x_6 = x_7 = 0$ and $x_8 = 1$. Then the corresponding Lie subalgebra $\mathfrak{H}_1$ of $\Lambda_1^*(\mathfrak{H})$ consists of A of $\Lambda_1^*(\mathfrak{H})$ satisfying $B = B' = 0$. Since H_1 normalizes $\mathfrak{H}_1$, we have

$$H_1 \subset \left\{ \left(\begin{array}{c|c|c} A & 0 & 0 \\ \hline 0 & * & 0 \\ \hline 0 & 0 & * \end{array} \right) ; \ A \in GO_3 \right\}$$

and hence the action on (x_1, x_2, x_3)-space is (GO_3, Λ_1) which $\ell \geqslant 2$ by Theorem 1.8. Q.E.D.

PROPOSITION 3.15. — *We have $\ell \geqslant 2$ for any P.V. in* (14) *in Theorem* 3.1.

Proof. — The generic isotropy subgroup of (GL_5, Λ_2) is connected (see P.76 in [3]). Hence the generic isotropy subgroup H of its castling transform $(SL_5 \times GL_9, \Lambda_2 \otimes \Lambda_1)$ is connected and it is contained in

$$\left\{ \left(\begin{array}{c|c} A & * \\ \hline 0 & * \end{array} \right) ; \ A \in GO_5 \right\}$$

(see the proof of Lemma 2.6 in [5]). Since $l \geqslant 2$ for (GO_5, Λ_1), we have $\ell \geqslant 2$ for $(GL_1 \times H, \Lambda_1 \otimes \Lambda_1^*, \Omega^9)$. This proves our assertion.

Q.E.D.

PROPOSITION 3.16. — *We have $\ell \geqslant 2$ for any P.V. in* (15) *in Theorem* 3.1.

Proof. — For the first P.V. in (15), we have $\ell \geqslant 2$ by Lemma 3.10. Now let H be the SL_7 – part of a generic isotropy subgroup of $(GL_1 \times SL_7 \times SL_2,\ \Lambda_2 \otimes \Lambda_1)$. Then we have

$$\mathrm{Lie}\,(H) = \left\{ \left(\begin{array}{c|c} A_1 & 0 \\ \hline * & A_2 \end{array} \right) ; \ A_1 = 3\Lambda_1^*(C) + 3tI_4, \right.$$
$$\left. A_2 = 2\Lambda_1(C) - 4tI_3 \ \text{for} \ C \in \mathfrak{sl}_2 \right\}$$

(see Lemma 1.4 in [5]). By the fact that the normalizer of GO_3 is GO_3 and by Schur's lemma, we have

$$H \subset \{\left(\begin{array}{c|c} * & 0 \\ \hline * & A \end{array}\right);\ A \in GO_3\}.$$

Since $\ell \geqslant 2$ for (GO_3, Λ_1), we have $\ell \geqslant 2$ for $(GL_1 \times H, \Lambda_1 \otimes \Lambda_1^*, \Omega^7)$ and hence $\ell \geqslant 2$ for the latter P.V.'s in (15). Q.E.D.

PROPOSITION 3.17. – *We have $\ell \geqslant 2$ for a P.V.* (16) *in Theorem* 3.1.

Proof. – Let H be the SL_9-part of a generic isotropy subgroup of $(GL_1 \times SL_9 \times SL_2, \Lambda_1 \otimes \Lambda_2 \otimes \Lambda_1)$. Then, similarly as the proof of Proposition 3.16, we have

$$H \subset \{\left|\begin{array}{c|c} * & 0 \\ \hline * & A \end{array}\right|;A \in 3\Lambda_1(GL_2)\}.$$

Since $\ell \geqslant 2$ for $(GL_2, 3\Lambda_1)$, we have $\ell \geqslant 2$ for $(GL_1 \times H, \Lambda_1 \otimes \Lambda_1^*, \Omega^7)$ and hence we obtain our result. Q.E.D.

PROPOSITION 3.18. – *We have $\ell \geqslant 2$ for $(GL_1^2 \times Spin_{10} \times SL_{15}$, a half-spin rep. $\otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$.*

Proof. – Let H be the SL_{15}-part of the generic isotropy subgroup of $(GL_1 \times Spin_{10} \times SL_{15}$, a half-spin rep. $\otimes \Lambda_1)$ at $X_0 = (e_1e_5, e_2e_5, e_3e_5, e_4e_5, e_2e_3e_4e_5, -e_1e_3e_4e_5, e_1e_2e_4e_5, -e_1e_2e_3e_5, -1+e_1e_2e_3e_4, e_1e_2, e_1e_3, e_1e_4, -e_3e_4, e_2e_4, -e_2e_3)$. Then we have

$$\mathrm{Lie}(H) = \{\left|\begin{array}{c|c} A_1 & 0 \\ \hline * & A_2 \end{array}\right| \in M_{15}, A_1 = \Lambda(B), A_2 = \Lambda'(B) \text{ for } B \in \mathfrak{o}_7\}$$

where Λ(resp. Λ') is the spin (resp. the vector) representation of $\mathfrak{o}_7$. By the fact that the normalizer of GO_7 is GO_7 and by Schur's lemma, we have $H \subset \{\begin{bmatrix} * & 0 \\ * & A \end{bmatrix}; A \in GO_7\}$. Since $\ell \geqslant 2$ for (GO_7, Λ_1), we have $\ell \geqslant 2$ for $(GL_1 \times H, \Lambda_1 \otimes \Lambda_1^*)$. This implies our assertion. Q.E.D.

PROPOSITION 3.19. – *We have $\ell \geqslant 2$ for $(GL_1^2 \times Spin_{10} \times SL_{14}$, a half-spin rep. $\otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$.*

Proof. – Let H be the SL_{14}-part of a generic isotropy subgroup of $(GL_1 \times Spin_{10} \times SL_{14}, \Lambda_1 \otimes$ a half-spin rep. $\otimes \Lambda_1)$. By checking the weights, one obtains $\mathrm{Lie}(H) = \mathrm{Lie}(G_2 \otimes SL_2)$. Let G be the image of $(GL_1 \times H, \Lambda_1 \otimes \Lambda_1^*)$. Then we have $1 \to GL_1 \to G \to \mathrm{Aut}(G_2 \otimes SL_2) \to 1$ (exact) and hence G is connected. Since $G \supset G_2 \otimes GL_2$ and $\dim G = \dim G_2 \otimes GL_2$, we have $(GL_1 \times H, \Lambda_1 \otimes \Lambda_1^*) \simeq (G_2 \times GL_2, \Lambda_2 \otimes \Lambda_1, \Omega^7 \otimes \Omega^2)$ which has $\ell \geqslant 2$ by Theorem 1.8. This completes the proof. Q.E.D.

THEOREM 3.20. — *All non-irreducible 2-simple P.V.'s* $(GL_1^k \times G, \rho\ (= \rho_1 \oplus \cdots \oplus \rho_k))$ *of type I with universally transitive open orbits are given as follows :*

(1) $G = SL_{2m+1} \times SL_2$, $\rho = \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1(+T)$ *with* $T = 1 \otimes \Lambda_1(+1 \otimes \Lambda_1)$.

(2) $G = SL_5 \times SL_2$, $\rho = \Lambda_2 \otimes \Lambda_1 + \Lambda_1^* \otimes 1(+1 \otimes \Lambda_1(+1 \otimes \Lambda_1))$.

(3) $G = SL_5 \times SL_2$, $\rho = \Lambda_2 \otimes \Lambda_1 + (\Lambda_1^* + \Lambda_1^*) \otimes 1$.

(4) $G = Sp_n \times SL_m$, $\rho = \Lambda_1 \otimes \Lambda_1 + T$, with $T = 1 \otimes (\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)})(1 \leqslant k \leqslant 3)$ *except* $1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1^*)$ *with* $m = odd$, $\Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1{}_1^{(*)} + \cdots + \Lambda_1{}_1^{(*)}))$ $(0 \leqslant k \leqslant 2)$ *except* $\Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1 + \Lambda_1^*)$ *with* $m =$ odd, $1 \otimes \Lambda_2 (m = \text{odd})$, $1 \otimes (\Lambda_2 + \Lambda_1^*)(m = 5)$.

(5) $G = Sp_n \times SL_{2m+1}$, $\rho = \Lambda_1 \otimes \Lambda_1 + (\Lambda_1 + \Lambda_1) \otimes 1$.

(6) $G = Spin_{10} \times SL_2$, $\rho = a$ *half-spin rep.* $\otimes \Lambda_1 + 1 \otimes \Lambda_1(+T)$ *with* $T = 1 \otimes \Lambda_1(+1 \otimes \Lambda_1)$.

COROLLARY 3.21. — *All non-irreducible regular 2-simple P.V.'s of type I with universally transitive orbits are given as follows :*

(1) $(GL_1^3 \times SL_5 \times SL_2, \Lambda_2 \otimes \Lambda_1 + (\Lambda_1^* + \Lambda_1^*) \otimes 1)$.

(2) $(GL_1^3 \times Sp_n \times SL_{2m}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^{(*)} + \Lambda_1^{(*)})$.

(3) $(GL_1^2 \times Sp_n \times SL_{2m+1}, \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1)$.

(4) $(GL_1^4 \times Sp_n \times SL_{2m+1}, \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1 + \Lambda_1)^{(*)})$.

(5) $(GL_1^3 \times Spin_{10} \times SL_2$, *a half-spin rep* $\otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1))$.

(6) $(GL_1^4 \times Spin_{10} \times SL_2$, *a half-spin rep* $\otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1))$.

COROLLARY 3.22. — *Any non-regular irreducible P.V., which is not castling-equivalent to* $(Sp_n \times GO_3, \Lambda_1 \otimes \Lambda_1)$, *has the universally transitive open orbit.*

Proof. — By Theorem 2.19 and the proof of Proposition 3.9, we have our result. Note that $\ell = 1$ for any trivial P.V.

$(G\times GL_n, \rho\otimes\Lambda_1, \Omega^m\otimes\Omega^n)$ $(\deg\rho=m\leqslant n)$ since we have $\ell=1$ for $(\{I_m\}\times GL_n, \rho\otimes\Lambda_1)\simeq(GL_n, \Lambda_1\oplus\cdots\oplus\Lambda_1)(m\leqslant n)$ by Proposition 1.5 and Lemma 2.2. Q.E.D.

4. 2-Simple P.V.'s of Type II with Universally Transitive Open orbits.

PROPOSITION 4.1. – *For $n\geqslant m_1\geqslant m_2$, we have $\ell=1$ for a P.V. $(G\times GL_n, \rho_1\otimes\Lambda_1+\rho_2\otimes\Lambda_1^*, M_{m_1,n}\oplus M_{m_2,n})$ if and only if $\ell=1$ for a P.V. $(G,\rho_1\otimes\rho_2, M_{m_1,m_2})$.*

Proof. – Define a map $\psi: M_{m_1,n}\oplus M_{m_2,n}\to M_{m_1,m_2}$ by $\psi(X,Y)=X{}^tY$ for $(X,Y)\in M_{m_1,n}\oplus M_{m_2,n}$. Since $\psi(\rho_1(A)X{}^tB, \rho_2(A)YB^{-1})=\rho_1(A)\psi(X,Y){}^t\rho_2(A)$ for $(A,B)\in G\times GL_n$, it is $G\times GL_n$-equivariant. Let W(resp. W') be the Zariski-dense orbit of the first P.V. (resp. the latter P.V.). By Theorems 1.4 and 1.6 in [6], we have $\psi(W)=W'$. It is enough to show that $\psi: W(k)\to W'(k)$ is surjective with $GL_n(k)$-homogeneous fibres. Clearly we have $W\subset U=\{(X,Y)\in M_{m_1,n}\oplus M_{m_2,n}\,;\ \text{rank } X=m_1,\ \text{rank } Y=\text{rank } X{}^tY=m_2\}$ and $W'\subset U'=\{Z\in M_{m_1,m_2}; \text{rank } Z=m_2\}$. Since $\psi((I_{m_1},0),({}^tZ,0))=Z$, the maps $\psi: U\to U'$ and $\psi: U(k)\to U'(k)$ are surjective. For any $(X,Y)\in\psi^{-1}(Z)\cap U$, there exists $B\in GL_n$ satisfying $X{}^tB=(I_{m_1},0)$ and $YB^{-1}=({}^tZ,Z')$. Since rank ${}^tZ=m_2$, we have ${}^tZC'=Z'$ for some $C'\in M_{m_1,n-m_1}$. Put $C=\left(\begin{array}{c|c} I & C' \\ \hline 0 & I\end{array}\right)\in GL_n$. Then we obtain $X{}^tB{}^tC=(I_{m_1},0)$ and $YB^{-1}C^{-1}=({}^tZ,-{}^tZC'+Z')=({}^tZ,0)$, i.e., $(X,Y)\sim((I_{m_1},0),({}^tZ,0))$. This implies that each fibre of $\psi: U\to U'$(resp. $\psi: U(k)\to U'(k)$) is GL_n(resp. $GL_n(k)$)-homogeneous. Hence $GL_n(k)$ acts on each fibre of $\psi: W(k)\to W'(k)$ transitively. For any $Z\in W'(k)=U'(k)\cap W'$, there exists (X,Y) in $U(k)$ satisfying $\psi(X,Y)=Z$. Since $\psi(W)=W'\ni Z$, there exists (X',Y') in W satisfying $\psi(X',Y')=Z$. Hence $(X,Y)=(X'{}^tB, Y'B^{-1})\in U(k)\cap W=W(k)$ for some $B\in GL_n$, i.e., $\psi(W(k))=W'(k)$. Q.E.D.

THEOREM 4.2. – *We have $\ell=1$ for the following 2-simple P.V.'s (4.a)-(4.c) of type II if and only if $\ell=1$ for a simple P.V. $(GL_1^r\times G, \rho_1\oplus\cdots\oplus\rho_r)$ ($\deg\rho_i\geqslant 2$ for $i=1,\cdots,r$) (see Theorem 2.19).*

(4.a) $(GL_1^{s+r}\times G\times SL_n, (\sigma_1+\cdots+\sigma_s)\otimes\Lambda_1+(\rho_1+\cdots+\rho_r)\otimes 1)$ *for any representation $\sigma_1+\cdots+\sigma_s$ of G and any natural number n satisfying $n\geqslant\deg\sigma_1+\cdots+\deg\sigma_s$.*

(4.b) $(GL_1^{t+r}\times G\times SL(\Sigma\deg\rho_i+r-1),\ (\rho_1+\cdots+\rho_k)\otimes\Lambda_1+(\rho_{k+1}^*+\cdots+\rho_r^*)\otimes 1+1\otimes(\Lambda_1+\cdots+\Lambda_1))(1\leqslant k\leqslant r)$ *for any* $t\geqslant 0$.

(4.c) $(GL_1^{t+r}\times G\times SL_n,\ (\rho_1+\cdots+\rho_k)\otimes\Lambda_1+(\rho_{k+1}+\cdots+\rho_r)\otimes 1+1\otimes(\Lambda_1+\overbrace{\cdots}^{t-1}+\Lambda_1+\Lambda_1^*)(1\leqslant k\leqslant r))$ *for any pair of natural number* (t,n) *satisfying* $t\geqslant 1$ *and* $n\geqslant t-1+\deg\rho_1+\cdots+\deg\rho_k$.

Proof. — For (4.a), we have our result by Proposition 1.5 and the remark in the proof of Corollary 3.22. A P.V. (4.b) is a castling transform of $(GL_1^{t+r}\times G,\rho_1^*+\cdots+\rho_r^*+1+\cdots+1)$. Clearly it has $\ell=1$ if and only if $\ell=1$ for $(GL_1^r\times G,\rho_1+\cdots+\rho_r)$ (see § 2 in [2]). By proposition 4.1, we have $\ell=1$ for (4.c) if and only if $\ell=1$ for $(GL_1^{t+r+1}\times G,\rho_1+\cdots+\rho_r+1+\overbrace{\cdots}^{t-1}+1)$, i.e., $\ell=1$ for $(GL_1^r\times G,\rho_1+\cdots+\rho_r)$. Q.E.D.

From now on, for simplicity, we shall write $(G,\rho)'$ instead of $(GL_1^k\times G,\rho(=\rho_1\oplus\cdots\oplus\rho_k))$ where GL_1^k acts on each irreducible component $\rho_i(1\leqslant i\leqslant k)$ independently.

LEMMA 4.3. — *We have* $\ell=1$ *for* $(GL_{2m+1}\times H,\Lambda_2\otimes 1+\rho\otimes\rho'(\text{resp. }\Lambda_2^*\otimes 1+\rho\otimes\rho'))$ *if and only if* $\ell=1$ *for* $(Sp_m\times GL_{2m+1}\times H,\Lambda_1\otimes\Lambda_1\otimes 1+1\otimes\rho\otimes\rho'(\text{resp. }\Lambda_1\otimes\Lambda_1^*\otimes 1+1\otimes\rho\otimes\rho'))$.

Proof. — Let H' be a generic isotropy subgroup of $(GL_{2m+1},\Lambda_2$ (resp. Λ_2^*)). Then the GL_{2m+1}-part of a generic isotropy subgroup of $(Sp_m\times GL_{2m+1},\Lambda_1\otimes\Lambda_1(\text{resp. }\Lambda_1\otimes\Lambda_1^*))$ is H'. Since $\ell=1$ for $(GL_{2m+1},\Lambda_2^{(*)})$ and $(Sp_m\times GL_{2m+1},\Lambda_1\otimes\Lambda_1^{(*)}))$, by Proposition 1.5, both of ℓ coincide with ℓ for $(H\times H',\rho\otimes\rho')$. Q.E.D.

PROPOSITION 4.4. — *We have* $\ell=1$ *for* $(G\times GL_{2m+1},\rho\otimes\Lambda_1+1\otimes\Lambda_2+\sigma\otimes 1)'$ *with* $\deg\rho\leqslant 2m+1$, *if and only if* $\ell=1$ *for* $(G\times GL(\deg\rho-1),\ \rho^*\otimes\Lambda_1+1\otimes\Lambda_2+\sigma\otimes 1)'$.

Proof. — By Lemma 4.3, ℓ for the first P.V. coincides with ℓ for $(G\times Sp_m\times GL_{2m+1},\ \sigma\otimes 1\otimes 1+(\rho\otimes 1+1\otimes\Lambda_1)\otimes\Lambda_1)$, which is castling-equivalent to $(G\times Sp_m\times GL(\deg\rho-1),\ \sigma\otimes 1\otimes 1+(\rho^*\otimes 1+1\otimes\Lambda_1)\otimes\Lambda_1)$. Then, by Proposition 3.7, we have our result. Q.E.D.

PROPOSITION 4.5. — *We have* $\ell=1$ *for* $(G\times GL_{2m+1},\rho\otimes\Lambda_1+1\otimes\Lambda_2^*+\sigma\otimes 1)'$ *with* $\deg\rho\leqslant 2m+1$, *if and only if* $\ell=1$ *for* $(G\times Sp_m,\rho\otimes\Lambda_1+\sigma\otimes 1)'$.

Proof. — By Lemma 4.3, the number ℓ for the first P.V. coincides with ℓ for $(G \times Sp_m \times GL_{2m+1},\ \sigma\otimes 1\otimes 1 + \rho\otimes 1\otimes \Lambda_1 + 1\otimes\Lambda_1\otimes\Lambda_1^*)'$. By Proposition 4.1, it has the same ℓ as $(G\times Sp_m, \sigma\otimes 1 + (\rho\otimes 1)\otimes(1\otimes\Lambda_1)(=\sigma\otimes 1+\rho\otimes\Lambda_1))'$. Q.E.D.

PROPOSITION 4.6. — *The following P.V.'s* (1), (2), (3) *have* $\ell = 1$ *if and only if* $\ell = 1$ *for* $(G, \Lambda^2(\rho)+\rho+\sigma)'$:

(1) $(G\times GL_{2n'+1}, \rho\otimes\Lambda_1 + 1\otimes(\Lambda_2+\Lambda_1)+\sigma\otimes 1)'(\deg\rho \leqslant 2n')$.

(2) $(G\times GL_{2n'+1}, \rho\otimes\Lambda_1 + 1\otimes(\Lambda_2^*+\Lambda_1)+\sigma\otimes 1)'(\deg\rho \leqslant 2n'-1)$.

(3) $(G\times GL_{2n'+1}, \rho\otimes\Lambda_1 + 1\otimes(\Lambda_2^*+\Lambda_1^*)+\sigma\otimes 1)'(\deg\rho \leqslant 2n')$.

Proof. — By Proposition 4.4, (1) is equivalent to $(G\times GL(\deg\rho), (\rho^*+1)\otimes\Lambda_1 + 1\otimes\Lambda_2+\sigma\otimes 1)'$. Since $\ell = 1$ for $(G\times GL(\deg\rho), \rho^*\otimes\Lambda_1)$ which has a generic isotropy subgroup $\{(g,\rho(g)); g\in G\}$, we have our result for (1). Since $\ell = 1$ for $(GL_{2n'+1}, (\Lambda_2+\Lambda_1)^{(*)})$ and their generic isotropy subgroups coincide, we have our result for (3). By Proposition 4.5, (2) is equivalent to $(G\times Sp_{n'}, (\rho+1)\otimes\Lambda_1+\sigma\otimes 1)'$, which is equivalent to $(G,\Lambda^2(\rho+1)+\sigma)' = (G,\Lambda^2(\rho)+\rho+\sigma)'$ by Proposition 3.7. Q.E.D.

PROPOSITION 4.7. — *Assume that* $\deg\rho = \text{odd} < 2n'+1$. *Then we have* $\ell = 1$ *for* $(G\times GL_{2n'+1}, \rho\otimes\Lambda_1 + 1\otimes(\Lambda_2+\Lambda_1^*)+\sigma\otimes 1)'$ *if and only if* $\ell = 1$ *for* $(G, \Lambda^2(\rho)^*+\rho+\sigma)'$.

Proof.. — Let $\left(W, \begin{array}{|c|c|} \hline X & Y \\ \hline -{}^tY & Z \\ \hline \end{array}\right)$ be a k-rational generic point of $(G\times GL_{2n'+1}, \rho\otimes\Lambda_1 + 1\otimes\Lambda_2, M_{2m'+1,2n'+1}\oplus \mathrm{Alt}_{2n'+1})'(\deg\rho = 2m'+1)$. Since $\ell = 1$ for a trivial P.V. $(G\times GL_{2n'+1}, \rho\otimes\Lambda_1)$, we may assume that $W = (I_{2m'+1}, 0)$. Then the fixer at W acts on Z-spaces as $(GL_{2(n'-m')}, \Lambda_2, \mathrm{Alt}_{2(n'-m')})$ which has $\ell = 1$. Hence we may take

$$Z = J = \begin{array}{|c|c|} \hline 0 & I_{n'-m'} \\ \hline -I_{n'-m'} & 0 \\ \hline \end{array}\,.$$

By the action of

$$\begin{array}{|c|c|} \hline I_{2m'+1} & YJ \\ \hline 0 & I_{2(n'-m')} \\ \hline \end{array} \ (\in GL_{2n'+1}),$$

we may assume that $Y = 0$. The generic isotropy subgroup of $(GL_1\times G\times$

$GL_{2n'+1}, 1\otimes\rho\otimes\Lambda_1+\Lambda_1\otimes 1\otimes\Lambda_2)$ at this point is given by

$$\left\{\left(\alpha, A, \begin{array}{|c|c|} {}^t\rho(A)^{-1} & 0 \\ \hline 0 & B \end{array}\right) \in GL_1\times G\times GL_{2n'+1}\,;\ \alpha{}^t\rho(A)^{-1}X \rho(A)^{-1}=X, \alpha BJ{}^tB=J\right\}.$$

Since

$$\Lambda_1^* \begin{array}{|c|c|} {}^t\rho(A)^{-1} & 0 \\ \hline 0 & B \end{array} = \begin{array}{|c|c|} \rho(A) & 0 \\ \hline 0 & {}^tB^{-1} \end{array}$$

and $\ell = 1$ for (Sp_{m-n}, Λ_1^*), our P.V. has $\ell = 1$ if and only if $(G, \Lambda^2(\rho)+\rho+\sigma)'$ has $\ell = 1$. Q.E.D.

THEOREM 4.8. – *We have* $\ell=1$ for $(GL_1^4\times SL_m\times SL_n, \rho)$ $(m<n=odd)$ *for the following* ρ*'s*:

(4.1) $\rho = \Lambda_1\otimes\Lambda_1 + 1\otimes(\Lambda_2^*+\Lambda_1+\Lambda_1)$ $(m=odd,$ *or* $m=2n', n=2n'+1)$.

(4.2) $\rho = \Lambda_1\otimes\Lambda_1 + 1\otimes(\Lambda_2+\Lambda_1+\Lambda_1)$ $(m=odd)$.

(4.3) $\rho = \Lambda_1\otimes\Lambda_1 + 1\otimes(\Lambda_2^*+\Lambda_1^*+\Lambda_1)$ $(m=odd)$.

(4.4) $\rho = \Lambda_1\otimes\Lambda_1 + 1\otimes(\Lambda_2+\Lambda_1^*+\Lambda_1^*)$ $(m=even)$.

(4.5) $\rho = \Lambda_1\otimes\Lambda_1 + 1\otimes(\Lambda_2^*+\Lambda_1^*+\Lambda_1^*)$ $(m=even)$.

(4.6) $\rho = \Lambda_1\otimes\Lambda_1 + 1\otimes(\Lambda_2+\Lambda_1+\Lambda_1^*)$ $(m=even)$.

Proof. – For (4.1) with $m = 2n'$, $n = 2n'+1$, it is castling-equivalent to $(GL_1^4\times SL_n, \Lambda_2^*+\Lambda_1+\Lambda_1+\Lambda_1^*)$, which has $\ell = 1$. When m = odd, by Lemma 4.3 and Proposition 4.1, it is equivalent to $(GL_1^3\times SL_m\times Sp_{n'}, \Lambda_1\otimes\Lambda_1+1\otimes(\Lambda_1+\Lambda_1))$, which has $\ell = 1$ by (5) in Theorem 3.20. By Proposition 4.4 (with $\rho=\Lambda_1+1+1$) and by a castling transformation, (4.2) is equivalent to $(GL_1^4\times SL_{m+1}, \Lambda_2+\Lambda_1+\Lambda_1+\Lambda_1^*)$, which has $\ell = 1$. Since the generic isotropy subgroups of $(GL_{2n'+1}, (\Lambda_2+\Lambda_1)^{(*)})$ coincide, we have (4.3) from (4.2). Now (4.4) (resp. (4.5), (4.6)) is a castling transform of (4.1) (resp. (4.2), 4.3)). Q.E.D.

LEMMA 4.9. – *We have* $\ell = 1$ *for* $(GL_1\times GL_{2m'}\times GL_{2n'+1}, 1\otimes(\Lambda_1^*\otimes 1+\Lambda_1\otimes\Lambda_1+1\otimes\Lambda_2)+\Lambda_1\otimes 1\otimes\Lambda_1^*)$ *with* $2m' < 2n'+1$.

Proof. – Since $\ell = 1$ for $(GL_{2m'}, \Lambda_1^*)$ and $(GL_{2n'+1}, \Lambda_2)$, we may assume that a k-rational generic point of

$$(GL_{2m'}\times GL_{2n'+1}, \Lambda_1^*\otimes 1+\Lambda_1\otimes\Lambda_1+1\otimes\Lambda_2, \Omega^{2m'}\oplus M_{2m',2n'+1}\oplus\Omega^{2n'+1})$$

is $\left({}^t(1,0,\ldots,0),\ \begin{array}{|c|c|}\hline x & Y \\ \hline Z & W \\ \hline\end{array},\ {}^t(1,0,\ldots,0)\right)$. By the action of

$$\left(\begin{array}{|c|c|}\hline x^{-1} & 0 \\ \hline x^{-1}Z & I \\ \hline\end{array},\ \begin{array}{|c|c|}\hline 1 & 0 \\ \hline x^{-1\,t}Y & I \\ \hline\end{array}\right)(\in (GL_{2m'} \times GL_{2n'+1})(k)),$$

we may assume that $x = 1$ and $Y = Z = 0$. The isotropy subgroup at this point is

$$\{(\begin{array}{|c|c|}\hline \alpha & 0 \\ \hline 0 & A \\ \hline\end{array},\ \begin{array}{|c|c|}\hline \alpha^{-1} & 0 \\ \hline 0 & B \\ \hline\end{array}) \in GL_{2m'} \times GL_{2n'+1};\alpha \in GL_1, A \in GL_{2m'-1},$$
$$B \in Sp_n, AW^tB = W\}.$$

Since

$$\Lambda_1^* \begin{array}{|c|c|}\hline \alpha^{-1} & 0 \\ \hline 0 & B \\ \hline\end{array} = \begin{array}{|c|c|}\hline \alpha & 0 \\ \hline 0 & B \\ \hline\end{array}$$

and $\ell = 1$ for $(GL_1^2 \times Sp_{n'} \times SL_{2m'-1}, \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1)$ by Theorem 3.20, we have our result. Q.E.D.

THEOREM 4.10. – *We have* $\ell = 1$ *for*

$$(GL_1^k \times SL_m \times SL_n, \rho(=\rho_1 \oplus \ldots \oplus \rho_k))(m<n=odd)$$

where ρ *is one of* (4.7) $\sim$ (4.13). *Here* T *stands for any one of* $\Lambda_2 \oplus \Lambda_1$, $\Lambda_2^* \oplus \Lambda_1^{(*)}$:

(4.7) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes T + (\Lambda_1 + \Lambda_1)^{(*)} \otimes 1$.

(4.8) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes T + (\Lambda_1 + \Lambda_1^*) \otimes 1$ $(m = even)$.

(4.9) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1^*) + (\Lambda_1 + \Lambda_1^*) \otimes 1$ $(m = odd)$.

(4.10) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1^*) + (\Lambda_1 + \Lambda_1) \otimes 1$.

(4.11) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2^{(*)} + \Lambda_1^{(*)})(+\Lambda_1^{(*)} \otimes 1)$.

(4.12) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1^*) + \Lambda_2^* \otimes 1$ $(m = 5)$.

(4.13) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_1^*) + \Lambda_2 \otimes 1$ $(m = 4)$.

Proof. – By Theorem 2.19 and Proposition 4.6, we have (4.7) and (4.8). By Proposition 4.7, we have (4.9) and (4.12). Now (4.10) is a castling transform of (one of) (4.7). From (4.7), (4.9), (4.10) and Lemma 4.8, we have (4.11). By (4.12) and Lemma 4.3, we have $\ell = 1$ for $(SL_4 \times SL_{2n'+1} \times SL_5,\ \Lambda_2 \otimes 1 \otimes 1 + 1 \otimes (\Lambda_2 + \Lambda_1^*) \otimes 1 + (1 \otimes \Lambda_1) \otimes \Lambda_1 + (\Lambda_1 \otimes 1) \otimes \Lambda_1^*)'$. Now the proof of Proposition 4.1 shows that if $\ell = 1$ for $(G \times GL_n, \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*)$ with $m_1 > n \geqslant m_2$, then we have $\ell = 1$ for $(G, \rho_1 \otimes \rho_2)$. In our case, we have $\ell = 1$ for (4.13). Q.E.D.

THEOREM 4.11. — *We have* $\ell = 1$ *for the following P.V.'s* $(GL_1^k \times SL_m \times SL_n, \rho(=\rho_1 \oplus \ldots \oplus \rho_k))$ *with* $m < n = odd$

(4.14) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + \sigma \otimes 1$ $(m = odd)$ *with* $\sigma = \Lambda_2^*$, $(\Lambda_1 + \Lambda_1 + \Lambda_1)^{(*)}$, $(\Lambda_1 + \Lambda_1 + \Lambda_1^*)$, $(\Lambda_2^* + \Lambda_1)$ $(m = 5)$.

(4.15) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + \sigma \otimes 1$ $(m = even)$ *with* $\sigma = \Lambda_2^{(*)}$, $(\Lambda_1 + \Lambda_1 + \Lambda_1)^{(*)}$, $(\Lambda_2 + \Lambda_1^{(*)})$ $(m = 4)$.

(4.16) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^* + \sigma \otimes 1$ *with* $\sigma = \Lambda_2$ $(m = odd)$, $\Lambda_1 + \Lambda_1 + \Lambda_1^*$ $(m = even)$, $\Lambda_1 + \Lambda_1^* + \Lambda_1^*$, $\Lambda_2 + \Lambda_1^*$ $(m = 5)$.

(4.17) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^{(*)}(+\Lambda_1^{(*)} \otimes 1(+\Lambda_1^{(*)} \otimes 1))'$.

Proof. — By Proposition 4.4, (4.14) is equivalent to $(SL^{2m'+1} \times Sp_{m'}, \Lambda_1^* \otimes \Lambda_1 + \sigma \otimes 1)'(m = 2m'+1)$, which is equivalent to $(SL_{2m'+1}\Lambda_2 + \sigma^*)'$ by Lemma 4.3. Hence, by Theorem 2.19, we have $l = 1$ for (4.14). For (4.15) with $\sigma = \Lambda_2^{(*)}$, it is equivalent to $Sp_{m'} \times SL_{2n'+1}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2)'$ since $\ell = 1$ for $(GL_m, \Lambda_2^{(*)})$ $(m = 2m')$. Then, by Propositions 4.4 and 3.7, it is equivalent to $(SL_{2m'-1}, \Lambda_2 \otimes \Lambda_2)'$ which has $\ell = 1$ by Theorem 2.19. For (4.15) with $\sigma = (\Lambda_1 + \Lambda_1 + \Lambda_1)^{(*)}$, by Proposition 4.4, it is equivalent to $(SL_{2m'} \times SL_{2m'-1}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + (\Lambda_1 + \Lambda_1 + \Lambda_1)^{(*)} \otimes 1)'$. When $\sigma = \Lambda_1 + \Lambda_1 + \Lambda_1$, it is castling-equivalent to $(SL_2 \times SL_{2m'-1}, (\Lambda_1 + \Lambda_1 + \Lambda_1) \otimes 1 + \Lambda_1 \otimes \Lambda_1^* + 1 \otimes \Lambda_2)$. Since $\ell = 1$ for $(GL_1^3 \times SL_2, \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$ with a generic isotropy subgroup $\{1\}$, it is equivalent to $(GL_1^3 \times SL_{2m'-1}, \Lambda_2 \oplus \Lambda_1^* \oplus \Lambda_1^*)$ which has $\ell = 1$. When $\sigma = \Lambda_1^* + \Lambda_1^* + \Lambda_1^*$, by Proposition 4.1, it is equivalent to $(GL_1^3 \times SL_{2m'-1}, \Lambda_2 \oplus \Lambda_1 \oplus \Lambda_1 \oplus \Lambda_1)$ which has $\ell = 1$. For (4.15) with $\sigma = \Lambda_2 + \Lambda_1^{(*)})$ $(m = 4)$, it is equivalent to $(SL_4 \times SL_3, \Lambda_1^* \otimes \Lambda_1 + 1 \otimes \Lambda_1^* + (\Lambda_2 + \Lambda_1^{(*)}) \otimes 1)'$ by Proposition 4.4. Clearly it is also equivalent to $(Sp_2 \times SL_3, \Lambda_1 \otimes \Lambda_1 + \Lambda_1^{(*)} \otimes 1 + 1 \otimes \Lambda_1^*)'$ which has $\ell = 1$ by Theorem 3.20. For (4.16), by Proposition 4.5, it is equivalent to $(SL_m \times SP_{n'}, \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1)'$, which is again equivalent to $(SL_m, \Lambda_2 + \sigma)'$ by Proposition 3.7. Hence we have our result by Theorem 2.19. By above results and Theorem 4.10, we have (4.17). Q.E.D.

THEOREM 4.12. — *We have* $l = 1$ *for the following P.V.'s :*

(4.18) $(GL_1^3 \times Sp_{m'} \times SL_{2n'+1}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2)$ *with* $2m' \leqslant 2n' + 1$.

(4.19) $(GL_1^3 \times Sp_2 \times SL_{2n'+1}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + \Lambda_1 \otimes 1)$.

(4.20) $(GL_1^3 \times Sp_2 \times SL_{2n'+1}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2 + 1 \otimes \Lambda_1^*)$.

Proof. — We have $l = 1$ for (4.18) (resp. (4.19), (4.20)) by (4.15) with $\sigma = \Lambda_2^{(*)}$ (resp. (4.15) with $\sigma = (\Lambda_2 + \Lambda_1^{(*)})$ $(m = 4)$, (4.13))) Q.E.D.

THEOREM 4.13. — *We have $\ell = 1$ for the following P.V.'s :*

(4.21) $(GL_1^3 \times SL_2 \times SL_5, \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2^* + \Lambda_2^*))$.

(4.22) $(GL_1^3 \times SL_3 \times SL_5, \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_2))$.

(4.23) $(GL_1^3 \times SL_4 \times SL_5, \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2 + \Lambda_2))$.

Proof. — Since (4.23) is castling-equivalent to $(GL_1^3 \times SL_5, \Lambda_2 \oplus \Lambda_2 \oplus \Lambda_1^*)$, we have $\ell = 1$ for (4.23). Since (4.21) is a castling transform of (4.22), it is enough to show $\ell = 1$ for (4.22), namely, for $(GL_1 \times GL_5 \times GL_3, (1 \otimes \Lambda_2 + \Lambda_1 \otimes \Lambda_2) \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1)$. The isotropy subalgebra $\mathfrak{H}$ of $(GL_1 \times GL_5, 1 \otimes A_2 + \Lambda_1 \otimes \Lambda_2)$ at ξ with $m = 2$ in the proof of Proposition 2.15 is given by

$$\{(\alpha, A) \in \mathfrak{gl}_1 \oplus \mathfrak{gl}_5 ; A = \left|\begin{array}{c|c} A_1 & 0 \\ \hline A_3 & A_2 \end{array}\right| ; A_1 = \mathrm{diag}\,(a, a-\alpha, a-2\alpha),$$

$$A_2 = \mathrm{diag}\,(-a, \alpha - a), A_3 = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{bmatrix}\}.$$

Therefore the GL_5-part H of the isotropy subgroup at ξ contains $\{\mathrm{diag}\,(\varepsilon, \varepsilon^{\eta-1}, \varepsilon^{\eta-2}, \varepsilon^{-1}, \varepsilon^{-1\eta}); \varepsilon, \eta \in GL_1\}$ and G_ξ with $m = 2$ in the proof of Proposition 2.15. We shall show $\ell = 1$ for $(H \times GL_3, \Lambda_1 \otimes \Lambda_1, M_{5,3})$. Let $X = \begin{bmatrix} Y \\ Z \end{bmatrix}$ be a k-rational generic point where $Y \in M_3(k)$ and $Z = \begin{pmatrix} u_1, & u_2, & u_3, \\ z_1, & z_2, & u_4 \end{pmatrix} \in M_{2,3}(k)$. *Since det* $Y \neq 0$, we may assume that $Y = I_3$ by the action of GL_3. Similarly we have $u_i = 0 (1 \leqslant i \leqslant 4)$ by the action of G_ξ. In this case, we have $z_1 z_2 \neq 0$ since otherwise it cannot be a generic point. For example, one can check this by calculation of the isotropy subalgebra. By the action of

$$g = \mathrm{diag}\,(\varepsilon, \varepsilon\eta^{-1}, \varepsilon\eta^{-2}, \varepsilon^{-1}, \varepsilon^{-1}\eta) \times \mathrm{diag}\,(\varepsilon^{-1}, \varepsilon^{-1}\eta, \varepsilon^{-1}\eta^2) \in H \times GL_3$$

with $\varepsilon^2 = z_1^2 z_2^{-1}$ and $\eta = z_1 z_2^{-1}$, we have $z_1 = z_2 = 1$, i.e., $\ell = 1$. Note that $\Lambda_1 \otimes \Lambda_1(g)$ is k-rational even if $g \notin (H \times GL_3)(k)$. Q.E.D.

THEOREM 4.14. — *We have $\ell = 1$ for the following P.V.'s $(GL_1^k \times SL_m \times SL_n, \rho (= \rho_1 \oplus \ldots \oplus \rho_k))$ where $n = 2n'$ $(=$even$)$:*

(4.24) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^{(*)} (+ \sigma \otimes 1)$ *with*
$\sigma = \Lambda_1^{(*)}, \Lambda_1^{(*)} + \Lambda_1^{(*)}, \Lambda_1 + \Lambda_1^* + \Lambda_1^*,$
$(\Lambda_1 + \Lambda_1 + \Lambda_1)^{(*)}, \Lambda_1 + \Lambda_1 + \Lambda_1^*$ *(m=even)*, Λ_2 *(m=odd)*.

(4.25) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2^{(*)} + \Lambda_1^{(*)})(+\sigma\otimes 1)$ *with* $\sigma = \Lambda_1^{(*)}$,

(4.26) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_2^{(*)} + \Lambda_1^{(*)} + \Lambda_1^{(*)})$ $(m = odd)$.

(4.27) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^{(*)} + (\Lambda_2 + \Lambda_1^*) \otimes 1$ $(m=5)$.

Proof. — Since $\ell = 1$ for $(GL_n, \Lambda_2^{(*)})$ with a generic isotropy subgroup $Sp_{n'}$, (4.24) $\sim$ (4.27) reduce to the case of type I, and we have our result by Theorem 3.20. Q.E.D.

PROPOSITION 4.15. — *We have* $\ell \geqslant 1$ *for* $(SL_2 \times SL_n, 2\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_2^{(*)})(+\Lambda_1 \otimes 1))'$.

Proof. — If $n = 2n'$, it is equivalent to $(SL_2 \times Sp_{n'}, 2\Lambda_1 \otimes \Lambda_1 (+\Lambda_1 \otimes 1))'$ which has $\ell \geqslant 2$ by Corollary 3.22. If n = odd, we have $\ell \geqslant 2$ by Propositions 4.4 and 4.5. Q.E.D.

THEOREM 4.16. — *We have* $\ell = 1$ *for the following P.V.'s :*

(4.28) $(GL_1^{s+t+1} \times SL_n \times SL_n, \Lambda_1 \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_t))$ *where* $(GL_1^{s+t} \times SL_n, \sigma_1^* + \cdots + \sigma_s^* + \tau_1 + \cdots + \tau_t)$ *is a simple P.V. with* $l = 1$ (*See Theorem* 2.19).

Proof. — It is obvious. Q.E.D.

THEOREM 4.17. — *A P.V. of the type*

$$(GL_1^{k+s+t} \times G \times SL_n, (\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\tau_1 + \cdots + \tau_t))$$

with $2 \leqslant \deg \rho_i \leqslant n$ $(i=1, \ldots, k)$ *and*

$$(\tau_1 + \cdots + \tau_t) \neq (\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)})$$

has $\ell = 1$ *if and only if it is one of* (4.1) $\sim$ (4.28).

Proof. — We can find the table of all P.V.'s of this type in §§ 5-2 in [6]. From Lemma 4.3 to Theorem 4.16, we have investigated the number ℓ for all P.V.'s in §§ 5-2 in [6] except P.V.'s which have an irreducible component with $\ell \geqslant 2$. Q.E.D.

PROPOSITION 4.18. — *We have* $\ell \geqslant 2$ *for* $(GL_1^2 \times SL_4 \times SL_8, (\Lambda_2 + \Lambda_1) \otimes \Lambda_1)$.

Proof. – It is castling-equivalent to $(GL_1^2 \times SL_4 \times SL_2, (\Lambda_2 + \Lambda_1) \otimes \Lambda_1)$ where $(SL_4 \times GL_2, \Lambda_2 \otimes \Lambda_1) = (SO_6 \times GL_2, \Lambda_1 \otimes \Lambda_1)$ has $\ell \geqslant 2$. Hence we have our result. Q.E.D.

LEMMA 4.19. – *We have* $\ell = 1$ *for the following P.V.'s* (1) $(GL_1 \times Sp_m, \Lambda_1 \otimes (\Lambda_1 + \Lambda_1))$, (2) $(GL_1 \times GL_{2m}, 1 \otimes \Lambda_2^{(*)} + \Lambda_1 \otimes (\Lambda_1 + \Lambda_1))$, (3) $(GL_1 \times GL_{2m+1}, 1 \otimes \Lambda_2^* + \Lambda_1 \otimes (\Lambda_1 + \Lambda_1))$.

Proof. – Applying Proposition 3.7 for $(G, \Lambda^2(\rho)) = (GL_1, \Lambda_1 \oplus \Lambda_1)$, we have $\ell = 1$ for $(G, \Lambda^2(\rho)) = (GL_1, 2\Lambda_1)$, i.e. (1). Hence we have (2). By Proposition 4.5, (3) is equivalent to (1). Note that $(GL_1 \times GL_{2m+1}, 1 \otimes \Lambda_2 + \Lambda_1 \otimes (\Lambda_1 + \Lambda_1))$ is a non P.V. since it has a non-constant absolute invariant. Q.E.D.

THEOREM 4.20. – *We have* $\ell = 1$ *for the following P.V.'s* $(GL_1^k \times SL_m \times SL_n, \rho (= \rho_1 \oplus \ldots \oplus \rho_k)) (n \geqslant m+1)$.

(4.29) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^* + \Lambda_1^*) + \sigma \otimes 1$ *with* $\sigma = \Lambda_2^{(*)}$, $\Lambda_2^{(*)} + \Lambda_1$, $\Lambda_2^* + \Lambda_1^*$, $\Lambda_2 + \Lambda_1^*$ $(m = even)$.

(4.30) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^* + \Lambda_1^* + \Lambda_1^*) + \Lambda_2^{(*)} \otimes 1$,

(4.31) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^* + \Lambda_1^* + \Lambda_1) + \sigma \otimes 1$ *with* $\sigma = \Lambda_2^*$, Λ_2 $(m = even)$.

Proof. – By Proposition 4.1, (4.29) (resp. (4.30)) is equivalent to $(SL_m, \Lambda_1 \oplus \Lambda_1 \oplus \sigma)'$ (resp. $(GL_1^4 \times SL_m, \Lambda_2^{(*)} + \Lambda_1 + \Lambda_1 + \Lambda_1))$ and hence, by Theorem 2.19, we have our result. For (4.31), it is equivalent to (2) or (3) in Lemma 4.20 by Proposition 4.1 and hence $\ell = 1$. Q.E.D.

THEOREM 4.21. – *We have* $\ell = 1$ *for the following P.V. :*

(4.32) $(GL_1^5 \times SL_m \times SL_{m+1}, \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1) + \sigma \otimes 1)$ *with* $\sigma = \Lambda_2$, Λ_2^* $(m = even)$.

Proof. – It is castling-equivalent to $(GL_1^5 \times SL_m \times SL_2, \Lambda_1^* \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1) + \sigma \otimes 1))$. Since $\ell = 1$ for $(GL_1^3 \times SL_2, \Lambda_1 + \Lambda_1 + \Lambda_1)$ with a generic isotropy subgroup $\{1\}$, it is equivalent to $(GL_1 \times GL_1 \times SL_m, \Lambda_1 \otimes 1 \otimes (\Lambda_1^* + \Lambda_1^*) + 1 \otimes \Lambda_1 \otimes \sigma)$. By Lemma 4.19, we have our result.

THEOREM 4.22. – *We have* $\ell = 1$ *for the following P.V.'s* $(GL_1^k \times SL_m \times SL_n, \rho (= \rho_1 \oplus \ldots \oplus \rho_k))$ $(m = odd)$:

(4.33) $\rho = \Lambda_2 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^* + \Lambda_1^*) (+ \Lambda_1^* \otimes 1$ *when* $m = 5)$ $(n \geqslant 1/2m(m-1))$.

(4.34) $\rho = \Lambda_2 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^* + \Lambda_1^* + \Lambda_1)$ $(n \geqslant 1/2m(m-1)+1)$.

(4.35) $\rho = \Lambda_2 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1)(+\Lambda_1 \otimes 1$ *when* $m=5)$ $(n = 1/2m(m-1))$.

(4.36) $\rho = \Lambda_2 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1)$ $(n = 1/2m(m-1)+1)$.

Proof. – Since $\ell = 1$ for $(SL_m, \Lambda_2 \oplus \Lambda_2)$ (see the proof of Proposition 2.15), we have $\ell = 1$ for (4.33) and (4.34) by Propositions 4.1 and 2.16. A castling transform of (4.35) and (4.36) has $\ell = 1$ by Theorem 3.20. Q.E.D.

THEOREM 4.23. – *We have* $\ell = 1$ *for the following P.V.'s* $(GL_1^k \times Sp_m \times SL_n, \rho(=\rho_1 \oplus \ldots \oplus \rho_k))$ $(n \geqslant 2m)$:

(4.37) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^* + \Lambda_1^*)(+T)$ with $T = \Lambda_1 \otimes 1$, $1 \otimes \Lambda_1^*$, $1 \otimes \Lambda_1$ $(n \geqslant 2m+1)$.

(4.38) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^{(*)} + \Lambda_1^{(*)})(+T)$ $(n = 2m)$ *with* $T = \Lambda_1 \otimes 1$, $1 \otimes \Lambda_1^{(*)}$,

(4.39) $\rho = \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1)$ $(n = 2m+1)$.

Proof. – By Propositions 4.1 ; 2.9 and Lemma 4.19, we have (4.37). Since $\Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1 + 1 \otimes \tau (n=m)$ is equivalent to $(Sp_m, \sigma + \tau)'$, we have (4.38). A castling transform of (4.39) has $\ell = 1$ by Theorem 3.20. Q.E.D.

THEOREM 4.24. – *We have* $\ell = 1$ *for the following P.V.'s* $(GL_1^k \times Spin_{10} \times SL_n$, *a half-spin rep.* $\otimes \Lambda_1 + \rho'(=\rho_2 \oplus \ldots \oplus \rho_k))$ $(n \geqslant 16)$.

(4.40) $\rho' = 1 \otimes (\Lambda_1^* + \Lambda_1^*)$, $1 \otimes (\Lambda_1^* + \Lambda_1^* + \Lambda_1)$ $(n \geqslant 17)$.

(4.41) $\rho' = 1 \otimes (\Lambda_1 + \Lambda_1)$ $(n=16)$, $1 \otimes (\Lambda_1 + \Lambda_1 + \Lambda_1)$ $(n=17)$.

Proof. – Since $\ell = 1$ for $(GL_1 \times Spin_{10}, \Lambda_1 \otimes (\Lambda + \Lambda))$ (see P.14 in [1]) where Λ is the even half-spin representation, we have (4.40) by Proposition 4.1. A castling transform of (4.41) has $\ell = 1$ by Theorem 3.20. Q.E.D.

THEOREM 4.25. – *A P.V. of the type* $(GL_1^{k+s+t} \times G \times SL_n$, $(\rho_1 + \cdots + \rho_k) \otimes \Lambda_1 + (\sigma_1 + \cdots + \sigma_s) \otimes 1 + 1 \otimes (\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{t}$ *with* $2 \leqslant \deg \rho_i \leqslant n (i = 1, \ldots, k)$ *and* $(G; \rho_1 + \cdots + \rho_k; \sigma_1 + \cdots + \sigma_s) \neq (SL_m, \Lambda_1 + \cdots + \Lambda_1; \Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)})$ has $\ell = 1$ *if and only if it is one of* (4.29)-(4.41).

Proof. — By §§ 5-3 in [6] and Proposition 4.18 $\sim$ Theorem 4.24, we have our result. Q.E.D.

THEOREM 4.26. — *A P.V. of the type* $(GL_1^{k+s+t} \times SL_m \times SL_n, \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{k}) \otimes \Lambda_1 + (\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s} \otimes 1 + 1 \otimes (\overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{t})$ *has always the universally transitive open orbit, i.e.,* $\ell = 1$.

Proof. — *P.V.'s* of such type are completely classified in § 4 in [6]. *P.V.*-equivalences used there keep ℓ invariant (cf. Proposition 4.1, etc.). They are essentially reduced to trivial *P.V.'s* or simple *P.V.'s* of type $(GL_1^s \times SL_n, \overbrace{\Lambda_1^{(*)} + \cdots + \Lambda_1^{(*)}}^{s}$ wich have $\ell = 1$, and hence we obtain our result. Q.E.D.

BIBLIOGRAPHY

[1] J. IGUSA, On functional equations of complex powers, Invent. Math., 85 (1986), 1-29.

[2] J. IGUSA, On a certain class of prehomogeneous vector spaces, to appear in Journal of Algebra.

[3] M. SATO and T. KIMURA, A classification of irreducible prehomogeneous vector spaces and their relative invariants, Nagoya Math. J., 65 (1977), 1-155.

[4] T. KIMURA, A classification of prehomogeneous vector spaces of simple algebraic groups with scalar multiplications, Journal of Algebra, Vol. 83, N° 1, July (1983), 72-100.

[5] T. KIMURA, S. KASAI, M. INUZUKA and O. YASUKURA, A classification of 2-simple prehomogeneous vector spaces of type I, to appear in Journal of Algebra.

[6] T. KIMURA, S. KASAI, M. TAGUCHI and M. INUZUKA, Some *P.V.*-equivalences and a classification of 2-simple prehomogeneous vector spaces of type II, to appear in Transaction of A.M.S.

[7] J. SERRE, Cohomologie Galoisienne, Springer Lecture Note, 5 (1965).

[8] H. RUBENTHELER, Formes réelles des espaces préhomogènes irréductibles de type parabolique, Annales de l'Institut Fourier, Grenoble, 36-1 (1986), 1-38.

Manuscrit reçu le 22 décembre 1986.

T. KIMURA, S. KASAI and H. HOSOKAWA,
University of Tsukuba,
Institute of Mathematics
Tsukuba-shi
Ibaraki 305 (Japan).