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DIFFERENTIATION IN LACUNARY DIRECTIONS AND AN EXTENSION OF THE MARCINKIEWICZ MULTIPLIER THEOREM

par Anthony CARBERY*

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1. On the Marcinkiewicz multiplier theorem.

Let ϕ be a nonnegative C^∞ bump function on $\mathbf{R}^n$, identically one on $\{1 \leq |\xi_i| \leq 2\}$ and vanishing off $\{\frac{1}{2} \leq |\xi_i| \leq 4\}$. For $\alpha > 0$, let L^2_α denote the n-parameter Sobolev space of functions g for which

$$\|g\|^2_{L^2_\alpha} = \int |\hat{g}(\xi)|^2 \prod_{i=1}^{n} (1+\xi_i^2)^\alpha d\xi$$

is finite. One formulation of the multiplier theorem of Marcinkiewicz is as follows :

THEOREM A. — *Suppose m is a function which satisfies*

$$\sup_{k_1,\ldots,k_n \in \mathbf{Z}} \|m(2^{k_1}\xi_1, \ldots, 2^{k_n}\xi_n)\phi(\xi)\|_{L^2_\alpha} < \infty$$

for some $\alpha > \frac{1}{2}$. Then m is a Fourier multiplier of $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ and moreover the operator T associated to m satisfies a weighted L^2 inequality

$$\int |Tf|^2 w \leq C_r \int |f|^2 (Mw^r)^{\frac{1}{r}}$$

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for each $r > 1$, M denoting an appropriate iterate of the strong maximal function.

While the hypothesis of the Hörmander multiplier theorem (see for example Stein, [6]) is rotationally invariant, clearly that of the Marcinkiewicz theorem is not, since the definition of L^2_α gives the directions $\{e_j\}$ parallel to the co-ordinate axes a special rôle. In this paper we explore the situation when this special rôle is weakened by the introduction of an arbitrary linear change of variables. Thus for $A \in GL(n, \mathbf{R})$ we let

$$L^{2,A}_\alpha = \{g : \|g\|_{L^{2,A}_\alpha} = \|g(A^{-1}\cdot)\|_{L^2_\alpha} < \infty\},$$

and we ask whether the condition

$$\sup_{k_1,\dots,k_n \in \mathbf{Z}} \|m(2^{k_1}\xi_1, \dots, 2^{k_n}\xi_n)\phi(\xi)\|_{L^{2,A}_\alpha} = Q(m) < \infty,$$

for some $\alpha > \frac{1}{2}$, for some $A \in GL(n, \mathbf{R})$ is still sufficient to imply that m is a multiplier of L^p, $1 < p < \infty$.

To answer this question, we attempt to adapt the proof of the Marcinkiewicz theorem to our setting, and so we introduce g- and g^*_λ- type functions. Thus if ϕ is any function of the type described above, we let

$$\hat{\beta}_{k_1,\dots,k_n}(\xi) = \phi(2^{k_1}\xi_1, \dots, 2^{k_n}\xi_n)$$

and

$$g_\phi(f)^2(x) = \sum_{k_1,\dots,k_n \in \mathbf{Z}} |\beta_{k_1,\dots,k_n} * f(x)|^2.$$

Let

$$w(x) = \prod_{i=1}^n (1 + x_i^2)^{-\alpha},$$

and let

$$Aw(x) = (\det A)^{-1} w(A^{-1}x).$$

LEMMA.. — *If $v \geq 0$ is a test function, and K is a convolution kernel, then*

$$\int g_{\phi^2}(K * f)^2(x)v(x)dx \leq Q(\widehat{K}) \int g_\phi(f)^2(x) \sup_{k_1,\dots,k_n \in \mathbf{Z}} (Aw)_{k_1,\dots,k_n} * v(x)dx.$$

The proof of this lemma is well-known, but we include it for completeness.

Proof.

$$\int g_{\phi^2}(K * f)^2(x)v(x)dx$$

$$= \sum_{k_1,\dots,k_n} \int |(\beta * \beta)_{k_1,\dots,k_n} \star K * f(x)|^2 v(x)dx$$

$$= \sum_{k_1,\dots,k_n} \int \left| \int \beta_{k_1,\dots,k_n} * K(y)\beta_{k_1,\dots,k_n} * f(x-y)dy \right|^2 v(x)dx$$

$$\leq \sum_{k_1,\dots,k_n} \iint |\beta_{k_1,\dots,k_n} * K(y)|^2 \frac{dy}{(Aw)_{k_1,\dots,k_n}(y)}$$

$$\int \beta_{k_1,\dots,k_n} * f(x-z)|^2 (Aw)_{k_1,\dots,k_n}(z)dzv(x)dx$$

$$\leq \left(\sup_{k_1,\dots,k_n} \int |\beta * K_{-k_1,\dots,-k_n}(y)|^2 \frac{dy}{(Aw)(y)} \right)$$

$$\sum_{k_1,\dots,k_n} \int |\beta_{k_1,\dots,k_n} * f(z)|^2 \ (Aw)_{k_1,\dots,k_n} * v(z)dz$$

$$\leq Q(\widehat{K}) \int g_\phi(f)^2(x) \sup_{k_1,\dots,k_n} (Aw)_{k_1,\dots,k_n} * v(x)dx.$$

□

Invoking the Littlewood-Paley theory, ($\|g_\phi(f)\|_p \approx \|f\|_p$, $1 < p < \infty$,) we see that $\|K*f\|_p \leq C\|g_{\phi^2}(K*f)\|_p \leq C'Q(\widehat{K})\|f\|_p$, $2 \leq p < \infty$, provided that $v \to \sup_{k_1,\dots,k_n} |(Aw)_{k_1,\dots,k_n} * v|$ is bounded on L^q for $1 < q \leq \infty$.

Thus the answer to our question is affirmative provided we can control a certain maximal function. This maximal function, the so-called "differentiation in lacunary directions" operator has been studied in the case $n = 2$ and shown to be bounded on L^q, $1 < q \leq \infty$, by Nagel, Stein and Wainger [5]. Here we take up the case of higher dimensions, and work, for ease of exposition, with the case $n = 3$, although the method readily extends to all dimensions.

Another reason for being interested in this maximal operator for $n \geq 3$ is that it is the lacunary analogue of the "equally-spaced" Kakeya maximal operator which "controls" the Bochner-Riesz multipliers $(1-|\xi|^2)_+^\lambda$ for λ small and positive. Optimal results for the Kakeya maximal operator are known only when $n = 2$, with partial results when $n \geq 3$ in [4]. As is usual in Fourier Analysis, the lacunary operator is easier to handle, and, in this context, the moral of our theorem below is that we have not uncovered any new obstacles to boundedness of the Kakeya maximal operator in the optimal range for $n \geq 3$.

We give a formal statement and proof of a maximal theorem in §3 below; in §2 we give a general principle for maximal functions which is useful in §3.

2. An almost-orthogonality principle.

The original argument of Nagel, Stein and Wainger [5] used to prove the 2-dimensional maximal theorem contained a bootstrapping argument which required some geometrical considerations at each stage. M. Christ has observed that it is possible to separate the geometry from the analysis, and once the geometry is removed we are in the following situation. We have a doubly indexed family of subadditive operators $\{T_{j\nu}\}$, $j \in \mathbf{Z}$, $\nu \in S$ (with S any set). We shall assume

$$(1) \qquad \sup_j \| \sup_\nu |T_{j\nu} f| \|_p \leq A \|f\|_p$$

and, to be able to assert that the $T_{j\nu}$ for different j act independently, that there is some sequence of operators $\{R_j\}$ satisfying

$$(2) \qquad \| \left(\sum_j |R_j f|^2 \right)^{\frac{1}{2}} \|_p \leq A \|f\|_p .$$

We shall say that $\{T_{j\nu}\}$ is *essentially positive* if $T_{j\nu} = P_{j\nu} - Q_{j\nu}$ with $|f| \leq g \Rightarrow |P_{j\nu} f| \leq P_{j\nu} g$ and $|Q_{j\nu} g| \leq S_{j\nu}|g|$ for some $\{S_{j\nu}\}$ satisfying $0 \leq f \leq g \Rightarrow 0 \leq S_{j\nu} f \leq S_{j\nu} g$ and $\| \sup_{j,\nu} S_{j\nu} |f| \|_p \leq A \|f\|_p$.

THEOREM B (M. Christ, [3], unpublished). — *Let $1<p<\infty$; suppose $\{R_j\}$ is a sequence of operators for which (2) holds and that $\{T_{j\nu}\}$ is a family of subadditive operators for which (1) holds. If $p<2$ we also assume that $\{T_{j\nu}\}$ is essentially positive. Suppose furthermore that*

$$\| \sup_{j,\nu} |T_{j\nu}(I-R_j)f| \,\|_p \le A\|f\|_p. \tag{3}$$

Then there exists a constant C depending only on A and p such that

$$\| \sup_{j,\nu} |T_{j\nu}f| \|_p \le C\|f\|_p.$$

Proof. — i) $\underline{p \ge 2}$. By the subadditivity of the $T_{j\nu}$,

$$\sup_{j,\nu} |T_{j\nu}f| \le \sup_{j,\nu} |T_{j\nu}(I-R_j)f| + \left(\sum_j \sup_\nu |T_{j\nu}R_jf|^p \right)^{\frac{1}{p}}.$$

Therefore,

$$\begin{aligned}
\| \sup_{j,\nu} |T_{j\nu}f| \,\|_p &\le A\|f\|_p + \left(\sum_j \| \sup_\nu |T_{j\nu}R_jf| \,\|_p^p \right)^{\frac{1}{p}} \\
&\le A\|f\|_p + A\left(\sum_j \|R_jf\|_p^p \right)^{\frac{1}{p}} \\
&\le A\|f\|_p + A\|(\sum |R_jf|^2)^{\frac{1}{2}}\|_p \\
&\le A(A+1)\|f\|_p\,.
\end{aligned}$$

ii) $\underline{p<2}$. We first suppose that $T_{j\nu}\equiv 0$ for all but N j's; by (1) there exists a least constant $C(N)$ such that

$$\| \sup_{j,\nu} |T_{j\nu}f| \,\|_p \le C(N)\|f\|_p.$$

Hence

$$\begin{aligned}
\| \sup_{j,\nu} |T_{j\nu}g_j| \,\|_p &\le \| \sup_{j,\nu} |P_{j\nu}g_j| \,\|_p + \| \sup_{j,\nu} |Q_{j\nu}g_j| \,\|_p \\
&\le \| \sup_{j,\nu} P_{j\nu}(\sup_k |g_k|)\|_p + \| \sup_{j,\nu} S_{j\nu}(\sup_k |g_k|)\|_p \\
&\le \| \sup_{j,\nu} |T_{j,\nu}(\sup_k |g_k|)\|_p + 2\| \sup_{j,\nu} S_{j\nu}(\sup_k |g_k|)\|_p \\
&\le (C(N)+2A)\| \sup_j |g_j| \,\|_p.
\end{aligned}$$

But (1) implies that

$$\| \| \sup_{\nu} |T_{j\nu} g_j| \|_{\ell^p} \|_p \leq A \| \|g_j\|_{\ell^p} \|_p.$$

Thus by interpolation,

$$\| \left(\sum_j \sup_{\nu} |T_{j\nu} g_j|^2 \right)^{\frac{1}{2}} \|_p \leq (2A + C(N)^{\theta} A^{1-\theta}) \| (\sum |g_j|^2)^{\frac{1}{2}} \|_p$$

where $0 < \theta = \theta(p) < 1$. Proceeding now as in case (i),

$$\begin{aligned} \| \sup_{j,\nu} |T_{j\nu} f| \|_p &\leq \| \sup_{j,\nu} |T_{j\nu} (I - R_j) f| \|_p \\ &\quad + \left\| \left(\sum_j \sup_{\nu} |T_{j\nu} R_j f|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\leq A \|f\|_p + (2A + C(N)^{\theta} A^{1-\theta}) \| (\sum |R_j f|^2)^{\frac{1}{2}} \|_p \\ &\leq (A + 2A^3 + C(N)^{\theta} A^{2-\theta} \|f\|_p. \end{aligned}$$

Now by the definition of $C(N), C(N) \leq (A + 2A^3 + C(N)^{\theta} A^{2-\theta})$. Consequently $C(N) \leq C = C(p, A)$, independently of N. Removing the restriction that $T_{j\nu} \equiv 0$ for all but N j's yields the desired result. □

For a variant of this theorem where condition (3) is replaced by an assumption that the maximal operator $\sup_{j,\nu} |T_{j\nu} f|$ is "strongly bounded" on L^2, see [1]. At least when the $T_{j\nu}$ are essentially positive, this strong boundedness condition is *a posteriori* stronger than (3) but is sometimes easier to verify. An interesting question is to what extent some positivity hypothesis on $\{T_{j\nu}\}$ is necessary when $p < 2$. In certain special cases it may be dispensed with provided that there is some control over "where $T_{j\nu}$ lives", or some mild smoothness hypothesis is satisfied. See [2].

3. A maximal theorem.

We fix an even function $\psi : \mathbf{R} \to \mathbf{R}$ which satisfies $\psi(0) > 0$, $\psi \geq 0$, $\int \psi = 1$, ψ decreasing polynomially at ∞, $\widehat{\psi}$ compactly supported in a small neighbourhood of 0 - for example a many-fold convolution of the Fejér kernel with itself.

THEOREM. — *Let ψ be as above, and let u be any nonzero vector in $\mathbf{R}^3$. Let*

$$(T_{k_1,k_2,k_3,h}f)\widehat{\ }(\xi) = \widehat{\psi}((2^{k_1}\xi_1, 2^{k_2}\xi_2, 2^{k_3}\xi_3)\cdot hu)\hat{f}(\xi).$$

Then for each $p > 1$ there exists a constant C_p such that

$$(4) \qquad \| \sup_{k_1,k_2,k_3\in\mathbf{Z}\, h>0} |T_{k_1,k_2,k_3,h}f| \|_p \leq C_p\|f\|_p.$$

COROLLARY. — *Let $w(x) = \prod_{i=1}^{3}(1+x_i^2)^{-\alpha}$, $A \in GL(3,\mathbf{R})$ and $Aw(x) = (\det A)^{-1}w(A^{-1}x)$. Let*

$$(Aw)\widehat{\ }_{k_1,k_2,k_3}(\xi) = (Aw)\widehat{\ }(2^{k_1}\xi_1, 2^{k_2}\xi_2, 2^{k_3}\xi_3).$$

Then if $\alpha > \frac{1}{2}$,

$$\| \sup_{k_1,k_2,k_3} |(Aw)_{k_1,k_2,k_3} * f| \|_p \leq C_p\|f\|_p \quad \text{for} \quad 1 < p \leq \infty.$$

Proof of Corollary. — When $\alpha > \frac{1}{2}$, w is dominated by

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{\chi_{\pi[-t_i,t_i]}(x)}{2^3t_1t_2t_3}d\mu(t_1,t_2,t_3),$$

with $\int d\mu \leq C$, which is itself dominated by

$$\int_0^\infty\int_0^\infty\int_0^\infty \psi_{t_1}\otimes\psi_{t_2}\otimes\psi_{t_3}d\mu(t_1,t_2,t_3)$$

with ψ as above. Thus $(Aw)_{k_1,k_2,k_3}$ is dominated by

$$\int_0^\infty\int_0^\infty\int_0^\infty \{A([\psi\otimes\psi\otimes\psi]_{t_1,t_2,t_3})\}_{k_1,k_2,k_3}d\mu(t_1,t_2,t_3),$$

where $\{A([\psi\otimes\psi\otimes\psi]_{t_1,t_2,t_3}\}_{k_1,k_2,k_3}$ has Fourier multiplier

$$\prod_{i=1}^{3}\widehat{\psi}((2^{k_1}\xi_1, 2^{k_2}\xi_2, 2^{k_3}\xi_3)\cdot t_iu_i),$$

u_i being the i'th row of the matrix A. Hence convolution with $(Aw)_{k_1,k_2,k_3}$ is dominated by 3 applications of a maximal operator of the type described in the theorem. So once the theorem is proved, the corollary follows. $\square$

Proof of Theorem. — First of all, we may asume that no component of u is zero, for if one is zero, then we are taking maximal averages over lines in a 1-parameter family of lacunary directions, and so the result follows from the two-dimensional theorem of Nagel, Stein and Wainger. So, without loss of generality, $u = 1 = (1,1,1)$, and $m(\xi) = \widehat{\psi}(\xi \cdot 1) = \widehat{\psi}(\xi_1+\xi_2+\xi_3)$ is supported in $\{|\xi_1+\xi_2+\xi_3| \leq \alpha\}$ with α as small as we please.

Let now τ be a C^∞ function of 2 variables, of compact support, with $\tau \equiv 1$ on a 4α-neighbourhood of 0 in $\mathbf{R}^2$. Let $n_3(\xi) = m(\xi)\tau(\xi_1+\xi_2,\xi_3)$. Then if

$$(V^{(3)}_{k_1,k_2,k_3,h}g)\widehat{\ }(\xi) = n_3(h2^{k_1}\xi_1, h2^{k_2}\xi_2, h2^{k_3}\xi_3)\hat{g}(\xi),$$

$\sup\limits_{k_3,h}|V^{(3)}_{k_1,k_2,k_3,nh}g|$ is dominated by the strong maximal function of g defined with respect to axes pointing in directions $\{e_3,(1,-2^{k_2-k_1},0),(2^{k_2-k_1},1,0)\}$. Thus $\sup\limits_{k_1,k_2,k_3,h}|V^{(3)}_{k_1,k_2,k_3,h}g|$ is dominated by a maximal function associated to a 1-parameter lacunary family of directions of the type already controlled by Nagel, Stein and Wainger. The same goes when we permute the coordinate variables in the definition of n_3 to obtain $n_1(\xi) = m(\xi)\tau(\xi_2+\xi_3,\xi_1)$ and $n_2(\xi) = m(\xi)\tau(\xi_1+\xi_3,\xi_2)$. If $\sigma : \mathbf{R}^3 \to \mathbf{R}$ is a C^∞ function of compact support containing 0, then $U_{k_1,k_2,k_3,h}$ defined by $(U_{k_1,k_2,k_3,h}g)\widehat{\ }(\xi) = (\sigma m)(h2^{k_1}\xi_1, h2^{k_2}\xi_2, h2^{k_3}\xi_3)\hat{g}(\xi)$ is dominated by the strong maximal function.

Hence, the family of operators whose multipliers are the dilates of $q = m - n_1 - n_2 - n_3 - \sigma m$ is essentially positive in the sense of §2, and, for an appropriate choice of σ, q is supported in $\{|\xi_1+\xi_2+\xi_3| \leq \alpha\} \bigcap \{|\xi_1+\xi_2| \geq 2\alpha \text{ or } |\xi_3| \geq 2\alpha\} \bigcap \{|\xi_1+\xi_3| \geq 2\alpha \text{ or } |\xi_2| \geq 2\alpha\} \bigcap \{|\xi_2+\xi_3| \geq 2\alpha \text{ or } |\xi_1| \geq 2\alpha\}$; this set is contained in the cone

$$\Gamma = \left\{\frac{1}{\beta} \le \frac{\xi_1+\xi_2}{-\xi_3} \le \beta\right\} \bigcap \left\{\frac{1}{\beta} \le \frac{\xi_1+\xi_3}{-\xi_2} \le \beta\right\} \bigcap \left\{\frac{1}{\beta} \le \frac{\xi_2+\xi_3}{-\xi_1} \le \beta\right\}$$

for an appropriate choice of $\beta > 1$.

Thus to prove (4), the assertion of the theorem, it suffices to prove that

$$\left\| \sup_{k_1,k_2,h} |Q_{k_1,k_2,h} f| \right\|_p \le C_p \|f\|_p \tag{5}$$

for $1 < p \le \infty$, where

$$(Q_{k_1,k_2,h} f)\widehat{\ }(\xi) = q(h2^{k_1}\xi_1, h2^{k_2}\xi_2, h\xi_3)\hat{f}(\xi).$$

Now the 1-dimensional Hardy-Littlewood maximal theorem ensures that $\sup_{k_1,k_2} \| \sup_h |T_{k_1,k_2,0,h} f| \|_p \le C_p \|f\|_p$, $1 < p \le \infty$, and $T_{k_1,k_2,0,h} - Q_{k_1,k_2,h}$ is controlled by a maximal function already known to be bounded on all L^p; so

$$\sup_{k_1,k_2} \| \sup_h |Q_{k_1,k_2,h} f| \|_p \le C_p \|f\|_p, \ 1 < p \le \infty.$$

Thus by theorem B, since the operators $Q_{k_1,k_2,h}$ are essentially positive, (5) holds if we can construct operators $R_{k_1 k_2}$ such that $Q_{k_1 k_2 h} = Q_{k_1 k_2 h} R_{k_1 k_2}$ for all k_1, k_2 and h, and

$$\left\| \left(\sum_{k_1 k_2} |R_{k_1 k_2} f|^2 \right)^{\frac{1}{2}} \right\|_p \le C_p \|f\|_p \tag{6}$$

for $1 < p < \infty$.

Let $\theta : \mathbf{R} \to \mathbf{R}$ be a nonnegative C^∞ function, identically equal to one on $[\frac{1}{A}, A]$ and vanishing outside $[\frac{1}{2A}, 2A]$. Let $\gamma_1(\xi) = \theta(^{(\xi_2+\xi_3)}/ - \xi_1)$ if $\xi_1 \ne 0$ and $\gamma_1(0,\xi_2,\xi_3) = 0$. Define γ_2 and γ_3 similarly by permuting the coordinate variables. Let $\gamma(\xi) = \gamma_1(\xi)\gamma_2(\xi)\gamma_3(\xi)$. If $A > 1$ is chosen appropriately, γ is identically 1 on Γ. Let $(R_{jk} g)\widehat{\ }(\xi) = \gamma(2^j\xi_1, 2^k\xi_2, \xi_3)\hat{g}(\xi)$. Thus $Q_{jkh} = Q_{jkh} R_{jk}$ for all j, k and h. If we can show that the function $\sum_{j,k} \pm\gamma(2^j\xi_1, 2^k\xi_2, \xi_3)$

is a Fourier multiplier of $L^p(\mathbf{R}^3)$, $1 < p < \infty$, uniformly in the random choice of $\pm$, the usual argument with Rademacher functions (see for example Stein, [6]) yields (6).

We first of all decompose γ into four pieces, a typical one of which is $\tilde{\gamma}(\xi) = \gamma(\xi)\chi_{\xi_1,\xi_2>0,\xi_3<0}$, and then smoothly decompose $\tilde{\gamma}$ into three pieces,

$$\begin{aligned}\tilde{\gamma}(\xi) &= \tilde{\gamma}(\xi)\theta(\frac{\xi_1}{\xi_2})\theta(\frac{\xi_1}{\xi_3})\theta(\frac{\xi_2}{-\xi_3})+ \\ &\quad \tilde{\gamma}(\xi)\left[1-\theta(\frac{\xi_1}{\xi_2})\theta(\frac{\xi_1}{-\xi_3})\theta(\frac{\xi_2}{-\xi_3}\right]\delta(\frac{\xi_1}{\xi_2}) \\ &\quad +\tilde{\gamma}(\xi)\left[1-\theta(\frac{\xi_1}{\xi_2})\theta(\frac{\xi_1}{-\xi_3})\theta(\frac{\xi_2}{-\xi_3})\right]\left[1-\delta(\frac{\xi_1}{\xi_2})\right] \\ &= \rho_1(\xi)+\rho_2(\xi)+\rho_3(\xi),\end{aligned}$$

where A in the definition of θ is chosen to be large and $\delta : (0,\infty) \to \mathbf{R}$ is a nonnegative C^∞ function, identically zero on $(0, \frac{1}{2}]$ and one on $[2,\infty)$. Now ρ_1 is a C^∞ function which is homogeneous of degree zero and is supported in a cone contained in $\{\xi_1,\ \xi_2 > 0\ ,\ \xi_3 < 0\}$ which stays away from the coordinate hyperplanes, and so $\sum_{j,k} \pm\rho_1(2^j\xi_1, 2^k\xi_2, \xi_3)$ satisfies the hypotheses of the Marcinkiewicz theorem, theorem A, for all $\alpha > 0$. By symmetry in ξ_1 and ξ_2, it suffices to deal with either ρ_2 on ρ_3; we choose ρ_2.

Now $\rho_2(\xi) = \theta((\xi_1+\xi_3)/-\xi_2)\lambda(\xi)$ where λ is a C^∞ function which is homogeneous of degree zero and is supported in a small conical neighbourhood of the line $\xi_1+\xi_3 = 0$, $\xi_2 = 0$. We fix $j = 0$ and examine $m_0(\xi) = \sum_k \pm\rho_2(\xi_1, 2^k\xi_2, \xi_3)$. After performing the linear change of variables $\xi_1 \mapsto \frac{\xi_1+\xi_3}{2}$, $\xi_2 \mapsto \xi_2$, $\xi_3 \mapsto \frac{\xi_1-\xi_3}{2}$, this multiplier becomes one which satisfies the hypothesis of theorem A, for all $\alpha > 0$. Consequently, if $\widehat{S_0 g}(\xi) = m_0(\xi)\hat{g}(\xi)$, we have

$$\int |S_0 g|^2 w \le C_r \int |g|^2 (M_0 w^r)^{\frac{1}{r}}$$

for each $r > 1$, M_0 denoting an appropriate iterate of the strong maximal function defined with respect to the directions

$$\{(0,1,0),(1,0,1),(1,0,-1)\}.$$

Similarly, if

$$m_j(\xi) = \pm \sum_k \pm\rho_2(2^j\xi_1, 2^k\xi_2, \xi_3),$$

and S_j is the corresponding operator, then

$$\int |S_j g|^2 w \le C_r \int |g|^2 (M_j w^r)^{\frac{1}{r}},$$

where M_j is the iterate of the strong maximal function defined with respect to the directions $\{(0,1,0),(2^j,0,1),(1,0,-2^j),\}$. Hence,

$$\int |S_j g|^2 w \le C_r \int |f|^2 (\mathcal{M} w^r)^{\frac{1}{r}}$$

for all j, where $\mathcal{M}$ is an appropriate power of a 1-parameter Nagel, Stein and Wainger type lacunary maximal operator.

Observe that $m_j(\xi)$ is supported in $\{\xi : B^{-1} \le 2^j\xi_1/-\xi_3 \le B\}$, and if $p_j(\xi)$ is a smooth function which is homogeneous of degree zero, identically one on the support of m_j but vanishing on $\bigcup_{j\neq k}$ supp m_j, and if we define $(P_j g)\widehat{\ }(\xi) = p_j(\xi)\hat{g}(\xi)$, the Marcinkiewicz multiplier theorem and the usual argument with Rademacher functions give

$$\|(\sum |P_j f|^2)^{\frac{1}{2}}\| \approx \|f\|_p, \quad 1 < p < \infty.$$

Thus, to prove that $\sum_{j,k} \pm\rho_2(2^j\xi_1, 2^k\xi_2, \xi_3)$ is a Fourier multiplier of $L^p(\mathbf{R}^3)$, $1 < p < \infty$, it suffices to show that

$$\left\| \left(\sum_j |P_j X f|^2 \right)^{\frac{1}{2}} \right\|_p \le C_p \|(\sum |P_j f|^2)^{\frac{1}{2}}\|_p$$

for $2 \le p < \infty$, where $\widehat{X}g(\xi) = \sum_{j,k} \pm\rho_2(2^j\xi_1, 2^k\xi_2, \xi_3)\hat{g}(\xi)$. To see this, we take $w \in L^{(\frac{p}{2})'}$ and examine

$$\begin{aligned}\int \sum_j |P_j X f|^2 w &= \sum_j \int |S_j P_j f|^2 w \\ &\le C_r \sum_j \int |P_j f|^2 (\mathcal{M} w^r)^{\frac{1}{r}} \\ &= C_r \int \sum_j |P_j f|^2 (\mathcal{M} w^r)^{\frac{1}{r}}.\end{aligned}$$

The use of the Nagel Stein and Wainger theorem concerning $\mathcal{M}$ then establishes the required result. $\square$

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