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GREEN FUNCTIONS AND SPECTRA ON FREE PRODUCTS OF CYCLIC GROUPS

by K. AOMOTO and Y. KATO

0.

Recently there have been considerable interests in spectral theory of free groups or finite free products of cyclic groups, in relation to the theory of C^*-algebras or irreducible unitary representations of them (see [F], [H2], [I1], [I2], [S1]). On the other hand, one of the authors has developed in his previous articles a spectral theory of free groups, as a natural extension of periodic Jacobi matrices for left-invariant convolution operators with respect to certain subgroups of finite index by proving that the Green functions are algebraic (see [A1], [A2]). In this article we want to develop a similar theory for left-invariant convolution operators which are all nearest neighbour cases in the l^2-space of a free product Γ of cyclic groups. But the problems are much more complicated and delicate.

We note that in spherical symmetric cases the operators considered here are different from the ones in [I1] where the Green functions have very simple forms. Even in these cases the Green functions constructed in our article do not seem elementary.

Algebraicity of the Green functions has been also proved in [S1]

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under slightly general conditions. Theorem 1 here has been proved in Theorem 1 in [C] (see also [P]).

The content of this article is as follows :

First we show that the Green functions $G(\gamma, \gamma' \mid z)$ in question, where $\gamma, \gamma' \in \Gamma$ and z is the spectral parameter, can be constructed as fixed points of certain mappings which are generally algebraic (see Theorem 1). Next we show that a detection of the spectra of the operator can be reduced to elementary Morse theoretic arguments for the mapping

$$\Phi : W = \frac{1}{G(e, e \mid z)} \longrightarrow z, \tag{0.1}$$

where e denotes the identity of Γ (see Theorem 2). By this method we shall compute the spectra of a few simple operators which seem interesting (see **4**). Some results here overlap with the ones in [S1].

Let Γ be a free product of a finite number of cyclic groups Γ_j of order n_j :

$$\Gamma = \mathbf{Z}_{n_1} * \cdots * \mathbf{Z}_{n_m} .$$

Γ is generated by each generator a_j of $\mathbf{Z}_{n_j}$ such that $a_j^{n_j} = e$. We consider the symmetric random walk defined by a self-adjoint operator A :

$$Au(\gamma) = \sum_{i=1}^{m} p_i(u(\gamma a_i) + u(\gamma a_i^{-1})),$$

in $\ell^2(\Gamma)$, where $p_i \in \mathbf{R}$ satisfy $2\sum_{i=1}^{m} p_i = 1,\ p_i > 0$.

Our main theorems which we want to prove can be stated as follows :

THEOREM 1. — *The Green functions $G(\gamma, \gamma' \mid z)$ for A are algebraic in z. The function $W(z) = 1/G(e, e|z)$ satisfies the equations*

$$\begin{aligned} z - W &= \psi^{(0,\dots,0)}(W) && \text{for Re } z > 1, \\ &= \psi^{(\overline{n}_1,\dots,\overline{n}_m)}(W) && \text{for Re } z < -1, \end{aligned}$$

where $\psi^{(\nu_1,\ldots,\nu_m)}(W)$, $0 \leq \nu_j \leq \overline{n}_j = \left[\frac{n_j}{2}\right]$, *are the algebraic functions in* W *defined in* **2**.

We shall prove this in **2**.

THEOREM 2. — (i) *If a)* $m \geq 3$, *or b)* $m = 2, n_1, n_2 \neq 3$ *then the spectra* $\sigma(A)$ *of* A *consists of a finite number of bands of continuous spectra included in* $[-1, 1]$. *The complement* $\sigma(A)^c \mathbf{R} - \sigma(A)$ *coincides with the set of unstable bands :*

$$\left\{z = W + \psi^{(\nu_1,\ldots,\nu_m)}(W) \mid W \in \mathbf{R},\ \frac{d\psi^{(\nu_1,\ldots,\nu_m)}}{dW}(W) > -1 \right.$$
$$\left. \text{for some sequence } (\nu_1, \ldots, \nu_m)\right\}.$$

The norm of A *is equal to* MAX $(\alpha, -\beta)$ *where* α *and* β *denote the unique values* $\psi^{(0,\ldots,0)}(W_c')$ *and* $\psi^{(\overline{n}_1,\ldots,\overline{n}_m)}(W_c'')$ *(see* **2***) such that*

$$\frac{d\psi^{(0,\ldots,0)}}{dW}(W_c') = -1, \tag{0.2}$$

and

$$\frac{d\psi^{(\overline{n}_1,\ldots,\overline{n}_m)}}{dW}(W_c'') = -1, \tag{0.3}$$

respectively.

(ii) *If* $m = 2$, n_1 *or* $n_2 = 3$, *then the discrete spectra consist of the set*

$$\left\{z = \psi^{(\nu_1,\nu_2)}(0) \middle| \frac{d\psi^{(\nu_1,\nu_2)}}{dW}(0) < -1 \right.$$
$$\left. \text{for some } (\nu_1, \nu_2),\ 0 \leq \nu_1 \leq \overline{n}_1,\ 0 \leq \nu_2 \leq \overline{n}_2\right\}.$$

Continuous spectra appear in the same manner as in (i).

Proof of Theorem 2. — (i)-a) follows from Proposition 3, (i)-b) and (ii) follow from the arguments in **3**.

1.

First we assume that Γ is a free product of finite groups Γ_j, $1 \le j \le m : \Gamma = \Gamma_1 * \cdots * \Gamma_m$. Let Γ_j have a system of generators $\{a_{j,1}, \ldots, a_{j,g_j}\}$ such that each $a_{j,\nu}$ is different from the identity of Γ_j. Then Γ has a system of generators $a_{j,\nu}$, $1 \le j \le m$, $1 \le \nu \le g_j$. Consider the operators A_j and A in $\ell^2(\Gamma_j)$ and $\ell^2(\Gamma)$, such that all $p_{j,\nu} > 0$ and $2 \sum_{j=1}^{m} \sum_{\nu=1}^{g_j} p_{j,\nu} = 1$:

$$A_j u_j(\gamma) = \sum_{\nu=1}^{g_j} p_{j,\nu}(u_j(\gamma a_{j,\nu}) + u_j(\gamma a_{j,\nu}^{-1})), \tag{1.1}$$

$$Au(\gamma) = \sum_{j=1}^{m} \sum_{\nu=1}^{g_j} p_{j,\nu}(u(\gamma a_{j,\nu}) + u(\gamma a_{j,\nu}^{-1})). \tag{1.2}$$

Since the operator A on $\ell^2(\Gamma)$ is self-adjoint, there exists the unique Green function $G(\gamma, \gamma' \mid z)$, for $\gamma, \gamma' \in \Gamma$ and $\mathrm{Im}\, z \neq 0$, which represents the (γ, γ')-component of the resolvent kernel $(z-A)^{-1}$. Since A is left-invariant, we have the invariance property of $G(\gamma, \gamma' \mid z)$:

$$G(\gamma, \gamma' \mid z) = G(\gamma'^{-1}\gamma, e \mid z). \tag{1.3}$$

So, in order to find the spectra of A, we have only to study the behaviour of the spectral function

$$\frac{-1}{2\pi i}(G(\gamma'^{-1}\gamma, e \mid \lambda + i0) - G(\gamma'^{-1}\gamma, e \mid \lambda - i0)), \tag{1.4}$$

for $\lambda \in \mathbf{R}$, which is our main subject in this note. We denote by $G_j(\gamma, \gamma' \mid z)$ the Green functions for A_j :

$$G_j(\gamma, \gamma' \mid z) = (z - A_j)^{-1}_{\gamma,\gamma'} \ , \ \gamma, \gamma' \in \Gamma_j. \tag{1.5}$$

Let γ be an element of Γ which has a minimal expression $\gamma = \gamma_{i_j} \cdots \gamma_{i_\ell}$, $\gamma_{i_\nu} \in \Gamma_{i_\nu}$. Then the following quotient does not depend on γ for $\gamma_j \in \Gamma_j$, $j \neq i_\ell$:

$$G(\gamma\gamma_j, e \mid z)/G(\gamma, e \mid z) = G(\gamma_j, e \mid z)/G(e, e \mid z). \tag{1.6}$$

This follows from a general property of Green functions on a free product of groups. In fact we restrict the operator A to the subspace $\ell^2(\Omega_j)$ of $\ell^2(\Gamma)$, where Ω_j consists of elements $\gamma' \in \Gamma$ having minimal expressions initiating from any $\gamma_j \in \Gamma_j - \{e\}$. Then (1.6) is equal to the quotient of the corresponding Green functions on Ω_j. The situation is completely similar to the case of free groups (see [A1]).

We put $F_{j,\nu}(z)$ and $f_{j,\nu}(z)$ to be the multipliers for A_j and A :

$$F_{j,\nu}(z) = \frac{G_j(a_{j,\nu}^{\pm 1}, e \mid z)}{G_j(e, e \mid z)}, \tag{1.7}$$

$$f_{j,\nu}(z) = \frac{G(a_{j,\nu}^{\pm 1}, e \mid z)}{G(e, e \mid z)}, \tag{1.8}$$

for $1 \leq j \leq m,\ 1 \leq \nu \leq g_j$. Remark that at $z = \infty$, we have

$$F_{j,\nu}(z) = \frac{p_{j,\nu}}{z} + O\Big(\frac{1}{z^2}\Big) \quad \text{or} \quad \frac{2p_{j,\nu}}{z} + O\Big(\frac{1}{z^2}\Big), \tag{1.9}$$

$$f_{j,\nu}(z) = \frac{p_{j,\nu}}{z} + O\Big(\frac{1}{z^2}\Big) \quad \text{or} \quad \frac{2p_{j,\nu}}{z} + O\Big(\frac{1}{z^2}\Big), \tag{1.10}$$

according as $a_{j,\nu}^2 \neq e$ or $= e$, because we have the expansions $(z - A)^{-1} = \sum_{k=0}^{\infty} A^k / z^{k+1}$ and $(z - A_j)^{-1} = \sum_{k=0}^{\infty} (A_j)^k / z^{k+1}$.

Then we have the following equations which are fundamental in the sequel :

LEMMA 1.1.

$$f_{j,\nu}(z) = F_{j,\nu}\Big(z - 2\sum_{k \neq j}\sum_{\mu=1}^{g_k} p_{k,\mu} f_{k,\mu}(z)\Big), \tag{1.11}$$

for $1 \leq j \leq m$ *and* $1 \leq \nu \leq g_j$.

Proof. — Owing to a property of the Green functions, the following equations hold :

$$(1.12)\qquad \begin{aligned} zG(\gamma, e \mid z) &= \sum_{k=1}^{m}\sum_{\nu=1}^{g_k} p_{k,\nu}\{G(\gamma a_{k,\nu}, e \mid z) + G(\gamma a_{k,\nu}^{-1}, e \mid z)\},\\ zG(e, e \mid z) &= \sum_{k=1}^{m}\sum_{\nu=1}^{g_k} p_{k,\nu}\{G(a_{k,\nu}, e \mid z) + G(a_{k,\nu}^{-1}, e \mid z)\} + 1, \end{aligned}$$

where $\gamma \in \Gamma - \{e\}$. Therefore if γ has a minimal expression $\gamma = \gamma' \cdot \gamma_j$ such that $\gamma_j \in \Gamma_j - \{e\}$ and $\gamma' = \gamma_{i_1} \cdots \gamma_{i_\ell}$ for $i_\ell \neq j$, then

$$(1.13)\qquad \begin{aligned} &(z - 2\sum_{s\neq j}\sum_{\mu=1}^{g_s} p_{s,\mu} f_{s,\mu}(z))G(\gamma, e \mid z)\\ &\qquad = \sum_{\nu=1}^{g_j} p_{j,\nu}\{G(\gamma a_{j,\nu}, e \mid z) + G(\gamma a_{j,\nu}^{-1}, e \mid z)\},\\ &\Big(z - 2\sum_{s=1}^{m}\sum_{\mu=1}^{g_s} p_{s,\mu} f_{s,\mu}(z)\Big)G(e, e \mid z) = 1. \end{aligned}$$

Hence $G(\gamma'\gamma_j, e \mid z)$ as a function of γ_j satisfies the equation of Green functions for A_j in Γ_j with a spectral parameter $z - 2\sum_{s\neq j}\sum_{\mu=1}^{g_s} p_{s,\mu} f_{s,\mu}(z)$:

$$(1.14)\qquad \Big(z - 2\sum_{s\neq j}\sum_{\mu=1}^{g_s} p_{s,\mu}(z)\Big)u - A_j u = 0\ ,$$

except where $\gamma_j = e$. Therefore by uniqueness of the ratios of the Green functions,

$$(1.15)\qquad f_{j,\nu}(z) = \frac{G(\gamma' a_{j,\nu}, e \mid z)}{G(\gamma', e \mid z)} = \frac{G(\gamma' a_{j,\nu}^{-1}, e \mid z)}{G(\gamma', e \mid z)},$$

must be equal to $F_{j,\nu}\Big(z - 2\sum_{s\neq j}\sum_{\mu=1}^{g_s} p_{s,\mu} f_{s,\mu}(z)\Big)$. This implies Lemma 1.1.

The argument in the proof of Lemma 1.1 also proves the following :

COROLLARY. — *For any* $\gamma_j \in \Gamma_j - \{e\}$,

$$\frac{G(\gamma_j, e \mid z)}{G(e, e \mid z)} = \frac{G_j(\gamma_j, e \mid z - 2\sum_{s\neq j}\sum_{\nu=1}^{g_s} p_{s,\nu} f_{s,\nu}(z))}{G_j(e, e \mid z - 2\sum_{s\neq j}\sum_{\nu=1}^{g_s} p_{s,\nu} f_{s,\nu}(z))}.$$

We denote by $X_j(z)$ and $X(z)$ the values $2\sum_{\mu=1}^{g_j} p_{j,\mu} f_{j,\mu}(z)$ and $\sum_{j=1}^{m} X_j(z)$ respectively. Then the equations (1.11) can be rewritten as follows :

$$f_{j,\nu}(z) = F_{j,\nu}(z - X(z) + X_j(z)). \tag{1.16}$$

Therefore we have

$$\begin{aligned} X_j(z) &= F_j(z - X(z) + X_j(z)) \\ &= F_j(W(z) + X_j(z)), \end{aligned} \tag{1.17}$$

where

$$W(z) = z - X(z) = \frac{1}{G(e, e)'},$$

$$F_j(z) = 2\sum_{\nu=1}^{g_j} p_{j,\nu} F_{j,\nu}(z) = z - \frac{1}{G_j(e, e \mid z)}\,. \tag{1.18}$$

$F_j(z)$ has an asymptotic expansion :

$$F_j(z) = \frac{2\sum_{\nu=1}^{g_j} p_{j,\nu}^2}{z} + O\Big(\frac{1}{z^2}\Big), \tag{1.19}$$

at $z = \infty$. Hence we have

LEMMA 1.2. — *The equations* (1.17) *with respect to* $X_j(z)$ *and* $X(z)$ *have the unique Laurent series solutions at* $z = \infty$ *such that*

$$X_j(z) = \frac{2\sum_{\nu=1}^{g_j} p_{j,\nu}^2}{z} + O\left(\frac{1}{z^2}\right), \tag{1.20}$$

$$X(z) = \frac{2\sum_{j=1}^{m}\sum_{\nu=1}^{g_j} p_{j,\nu}^2}{z} + O\left(\frac{1}{z^2}\right).$$

Proof. — $F_j(z)$ has a power series expansion in $1/z$ beginning from the term $1/z$. If we put $\omega = 1/W(z)$ then the equations (1.17) can be expressed in the following way :

$$X_j = F_j((1 + X_j\omega)/\omega). \tag{1.21}$$

The right hand side is holomorphic at the origin with respect to ω and X_j. The implicit function theorem implies Lemma 1.2.

2.

We now assume that for all j, g_j is equal to 1 and thus Γ_j is a cyclic group of order n_j. Then $F_j(z) = 2p_jF_{j,1}(z)$, where we put $p_{j,1} = p_j$, has the following expression :

$$F_j(z) = \frac{\frac{1}{2}\varphi_{n_j-2}\left(\frac{z}{2p_j}\right) + \left(\frac{1}{2}\right)^{n_j-1}}{\varphi_{n_j-1}\left(\frac{z}{2p_j}\right)} 2p_j. \tag{2.1}$$

$\varphi_n(z)$ denotes the determinant $|z - J_n|$ where J_n is a n-th order

Jacobi matrix as follows :

$$\begin{pmatrix} 0 & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & \frac{1}{2} \\ & & & \frac{1}{2} & 0 \end{pmatrix}.$$

$\varphi_n(z)$ is equal to the Tchebyscheff 2nd polynomial of degree n and is expressed as : $2^{-n}\dfrac{\sin(n+1)\theta}{\sin\theta}$ for $z = \cos\theta$ (see [S2]). Then $F_j(2p_j\cos\theta)$ is equal to $2p_j\cos\left(\frac{n_j}{2}-1\right)\theta/\cos\frac{n_j}{2}\theta$. So

LEMMA 2.1. — *$F_j(z)$ has a partial fraction as follows :*

$$F_j(z) = \sum_{k=1}^{\bar{n}_j} \frac{4p_j \sin^2 \dfrac{(2k-1)\pi}{n_j}}{n_j\left(\dfrac{z}{2p_j} - \cos\dfrac{(2k-1)\pi}{n_j}\right)}, \tag{2.2}$$

where $\overline{n}_j \left[\frac{n_j}{2}\right]$.

From now on we put $\beta_j^{(\nu)} = 2p_j\cos((2\nu-1)\pi/n_j), 1 \le \nu \le \overline{n}_j$, and $\beta_j^{(0)} = \beta_j^{(\overline{n}_j+1)} = 0$.

LEMMA 2.2. — *For any $W \in \mathbf{R}$, the equation (1.17) with respect to X_j has $\overline{n}_j + 1$ real solutions $\psi_j^{(0)}(W), \ldots, \psi_j^{(\overline{n}_j)}(W)$ which are decreasing in W and satisfy*

$$\begin{aligned} &\beta_j^{(1)} < \psi_j^{(0)}(W) + W, \\ &\beta_j^{(\nu+1)} < \psi_j^{(\nu)}(W) + W < \beta_j^{(\nu)}, 1 \le \nu \le \overline{n}_j - 1, \\ &\psi_j^{(\overline{n}_j)}(W) + W < \beta_j^{(\overline{n}_j)}, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\psi_j^{(0)}(W) \sim 0 \quad \text{for } W \to +\infty, \\ &\psi_j^{(0)}(W) \sim \beta_j^{(1)} - W \quad \text{for } W \to -\infty, \end{aligned}$$

$$
\begin{aligned}
(2.4)\qquad & \psi_j^{(\nu)}(W) \sim \beta_j^{(\nu)} - W \quad \text{for } W \to +\infty, \\
& \psi_j^{(\nu)}(W) \sim \beta_j^{(\nu+1)} - W \quad \text{for } W \to -\infty,\ 1 \le \nu \le \overline{n}_j - 1, \\
& \psi_j^{(\overline{n}_j)}(W) \sim \beta_j^{(\overline{n}_j)} - W \quad \text{for } W \to +\infty, \\
& \psi_j^{(\overline{n}_j)}(W) \sim 0 \quad \text{for } W \to -\infty.
\end{aligned}
$$

Proof. — $X_j = \psi_j^{(\nu)}(W)$ is obtained as the X-coordinate of the intersection of the graphs $Y = X$ and $Y = F_j(X + W)$. $F_j(X)$ is decreasing in each interval where $F_j(X)$ is defined. Moreover $F_j(+\infty) = F_j(-\infty) = 0$, and $F_j(\beta_j^{(k)} \pm 0) = \pm\infty$, $1 \le k \le \overline{n}_j$. Thus we have only to apply the middle value theorem to (1.17) for $X = X_j$ on each interval in $\mathbb{R} - \bigcup_{k=1}^{\overline{n}_j} \{\beta_j^{(k)}\}$. Since $F_j(X)$ is decreasing, all $\psi_j^{(\nu)}(W)$ are decreasing, too.

Definition. — For any sequence $(\nu_1, \ldots, \nu_m)$ such that $0 \le \nu_j \le \overline{n}_j$, we denote by $\psi^{(\nu_1,\ldots,\nu_m)}(W)$ the function $\sum_{j=1}^{m} \psi_j^{(\nu_j)}(W)$ which is a real algebraic function without singularities for $W \in \mathbb{R}$. They are all decreasing.

Remark that $\psi^{(\nu_1,\ldots,\nu_m)}(W)$ has the following asymptotic lines :

$$
\begin{aligned}
(2.5)\qquad \psi^{(\nu_1,\ldots,\nu_m)}(W) &\sim \sum_{j=1}^{m} \beta_j^{(\nu_j)} - m'W \qquad \text{for } W \to +\infty, \\
&\sim \sum_{j=1}^{m} \beta_j^{(\nu_j+1)} - m''W \quad \text{for } W \to -\infty,
\end{aligned}
$$

where m' (or m'') denotes the number of ν_j different from 0 (or $\overline{n}_j$). Hence $W(z) = 1/G(e, e \mid z)$ satisfies the algebraic equations :

$$
\begin{aligned}
(2.6)\qquad z - W &= \psi^{(0,\ldots,0)}(W) \qquad \text{for } z \to +\infty, \\
&= \psi^{(\overline{n}_1,\ldots,\overline{n}_m)}(W) \quad \text{for } z \to -\infty.
\end{aligned}
$$

These equations have the unique meromorphic solution $W = W(z)$ in a neighbourhood of $z = \infty$ such that

$$
(2.7)\qquad W(Z) = z + O(1).
$$

Proof of Theorem 1. — We have seen that for any $\gamma \in \Gamma$, $G(\gamma,\gamma \mid z) = G(e,e \mid z)$ is algebraic in z. Since $G(\gamma,\gamma' \mid z)$ is left-invariant, we have only to prove the algebraicity of $G(\gamma, e \mid z)$. We assume that γ has a minimal expression $\gamma_{i_1} \dots \gamma_{i_s}$ for $\gamma_{i_j} \in \Gamma_{i_j}$, then from Lemma 1.1 and its corollary

$$\frac{G(\gamma,e \mid z)}{G(e,e \mid z)} = \frac{G_{i_1}(\gamma_{i_1}, e \mid W + X_{i_1})}{G_{i_1}(e,e \mid W + X_{i_1})} \cdot \frac{G_{i_2}(\gamma_{i_2}, e \mid W + X_{i_2})}{G_{i_2}(e,e \mid W + X_{i_2})} \cdots \cdots \frac{G_{i_s}(\gamma_{i_s}, e \mid W + X_{i_s})}{G_{i_s}(e,e \mid W + X_{i_s})}. \tag{2.8}$$

Each factor in the right hand side is algebraic in z because $G_j(\gamma, e \mid z)$, W and X_j are algebraic in z. Theorem 1 has thus been proved.

Remark.— We have

$$\begin{aligned} \psi^{(0,\cdots,0)}(0) &= 1, \\ \psi^{(\overline{n}_1,\dots,\overline{n}_m)}(0) &\geq -1, \end{aligned} \tag{2.9}$$

because

$$\begin{aligned} \psi_j^{(0)}(0) &= 2p_j, \\ \psi_j^{(\overline{n}_j)}(0) &\geq -2p_j, \end{aligned} \tag{2.10}$$

hold for $1 \leq j \leq m$, in view of the relations

$$\begin{aligned} F_j(2p_j) &= 2p_j \quad \text{for all } n_j, \\ F_j(-2p_j) &= -2p_j \quad \text{if } n_j \text{ is even}, \\ &= -2p_j(2\overline{n}_j - 1)/(2\overline{n}_j + 1) \quad \text{if } n_j \text{ is odd}. \end{aligned} \tag{2.11}$$

LEMMA 2.3. — *$\psi^{(0,\dots,0)}(W)$ (or $\psi^{(\overline{n}_1,\dots,\overline{n}_m)}(W)$) is a convex (or concave) function of W, namely*

$$\begin{aligned} \frac{d^2\psi^{(0,\dots,0)}}{dW^2}(W) &> 0, \\ \frac{d^2\psi^{(\overline{n}_1,\dots,\overline{n}_m)}}{dW^2}(W) &< 0 \quad \text{for } W \in \mathbb{R}. \end{aligned} \tag{2.12}$$

Proof. — By two times of differentiation with respect to W, the equation (1.17) implies

$$X_j'' = F_j''(W+X_j)(1+X_j')^2 + F_j'(W+X)X_j'',$$

namely

$$X_j''(1-F_j'(W+X_j)) = F_j''(W+X_j)(1+X_j')^2.$$

$F_j'(W+X_j)$ is negative for $X_j = \psi_j^{(0)}(W)$ and $\psi_j^{(\overline{n}_j)}(W)$. $F_j''(W+X_j)$ is positive for $X_j = \psi_j^{(0)}(W)$ and negative for $X_j = \psi_j^{(\overline{n}_j)}(W)$. Hence Lemma 2.3 follows.

LEMMA 2.4. — *The following inequalities hold for* $m \geq 2$:

$$\frac{d\psi^{(0,\ldots,0)}}{dW}(0) < -1,$$

(2.13)

$$\frac{d\psi^{(\overline{n}_1,\ldots,\overline{n}_m)}}{dW}(0) < -1.$$

Proof. — In view of (2.10) and (2.11), we have

$$\frac{d\psi_j^{(0)}}{dW}(0) = \frac{F_j'(2p_j)}{1-F_j'(2p_j)} = \frac{1-n_j}{n_j} \qquad \text{for all } n_j, \tag{2.14}$$

$$\frac{d\psi_j^{(\overline{n}_j)}}{dW}(0) = \frac{F_j'(-2p_j)}{1-F_j'(-2p_j)} = \frac{1-n_j}{n_j} \quad \text{if } n_j \text{ is even,} \tag{2.15}$$

$$\leq \frac{F_j'(-2p_j)}{1-F_j'(-2p_j)} < -\frac{1}{2} \qquad \text{if } n_j \text{ is odd,}$$

by using the equalities :

$$F_j'(2p_j\cos\theta) = \frac{-1}{\sin\theta}\cdot\frac{d}{d\theta}\left(2p_j\cos\left(\frac{n_j}{2}-1\right)\theta/\cos\frac{n_j\theta}{2}\right)$$

(2.16)

$$= -\frac{(n_j-1)\sin\theta + \sin(n_j-1)\theta}{2\sin\theta\cdot\cos^2\frac{n_j\theta}{2}},$$

in particular

$$F_j'(2p_j) = 1 - n_j, \tag{2.17}$$

$$\begin{aligned} F_j'(-2p_j) &= 1 - n_j \quad \text{if } n_j \text{ is even,} \\ &= -\frac{2}{3}\overline{n}_j \frac{\overline{n}_j - \frac{1}{2}}{\overline{n}_j + \frac{1}{2}} \quad \text{if } n_j \text{ is odd.} \end{aligned} \tag{2.18}$$

Lemma 2.4 follows.

Moreover we have

$$\begin{aligned} &\psi^{(0,\dots,0)}(W) \sim 0, \\ &\frac{d\psi^{(0,\dots,0)}}{dW}(W) \sim 0 \quad \text{for } W \to +\infty, \\ &\psi^{(0,\dots,0)}(W) \sim \sum_{j=1}^{m} \beta_j^{(1)} - mW, \\ &\frac{d\psi^{(0,\dots,0)}}{dW}(W) \sim -m \quad \text{for } W \to -\infty, \\ &\psi^{(\overline{n}_1,\dots,\overline{n}_m)}(W) \sim \sum_{j=1}^{m} \beta_j^{(\overline{n}_j)} - mW, \\ &\frac{d\psi^{(\overline{n}_1,\dots,\overline{n}_m)}}{dW}(W) \sim -m \quad \text{for } W \to +\infty, \\ &\psi^{(\overline{n}_1,\dots,\overline{n}_m)}(W) \sim 0, \\ &\frac{d\psi^{(\overline{n}_1,\dots,\overline{n}_m)}}{dW}(W) \sim 0 \quad \text{for } W \to -\infty. \end{aligned} \tag{2.19}$$

Hence due to Lemma 2.3, there exists only one $W = W_c' > 0$ such that $d\psi^{(0,\dots,0)}(W)/dW = -1$. In the same way there exists only one $W = W_c'' < 0$ such that $d\psi^{(\overline{n}_1,\dots,\overline{n}_m)}(W)/dW = -1$. We denote the corresponding values of z by α and β respectively. Then $-1 < \beta < \alpha < 1$ from (2.10), (2.12) and (2.14). We have proved the following :

Proposition 1. — *$W(z) = 1/G(e,e \mid z)$ is an algebraic function which is holomorphic in $\mathbb{C} - [\beta,\alpha]$. α and β are real branch points of $W(z)$ of second order.*

3.

To investigate the spectra of the operator A, it is sufficient to study the structure of $\{G(\gamma,\gamma \mid z)\}_{\gamma\in\Gamma}$ and so $G(e,e \mid z) = 1/W(z)$. To see the behaviour of $W(z)$ in (β,α) in more detail, we consider the following equations $\varepsilon(\nu_1,\nu_2,\ldots,\nu_m)$ which are all possible analytic continuations of (2.6) :

$$z - W = \psi^{(\nu_1,\ldots,\nu_m)}(W), \quad 0 \le \nu_j \le \overline{n}_j. \tag{3.1}$$

We denote by ε the union $\cup_{\nu_1,\ldots,\nu_m}\varepsilon(\nu_1,\ldots,\nu_m)$. Since each $\psi_j^{(\nu)}(W)$ is an algebraic function of degree $\overline{n}_j + 1$ in W, the set $\mathcal{C} = \{(z,W) \in \mathbb{C}^2 \mid \Pi_{j=1}^m \Pi_{\nu_j=1}^{\overline{n}_j}(z - W - \psi^{(\nu_1,\ldots,\nu_m)}(W)) = 0\}$ defines a possibly reduced affine algebraic curve of degree less than or equal to $(\overline{n}_1+1)\ldots(\overline{n}_m+1)$. We denote by $\hat{\mathcal{C}}$ the compactification of $\mathcal{C}$ in $\mathbb{C}P^2$ which becomes a projective algebraic curve possibly with singularities. The intersection of $\hat{\mathcal{C}}$ and the line $\{W = 0\}$ in $\mathbb{C}P^2$ consists of $(\overline{n}_1+1)\ldots(\overline{n}_m+1)$ points. Therefore the degree of $\hat{\mathcal{C}}$ is exactly $(\overline{n}_1+1)\cdots(\overline{n}_m+1)$.

The crucial fact is the following :

Proposition 2. — *Assume $m \ge 3$. Then for $z \in (\beta,\alpha)$ the equations $\varepsilon(\nu_1,\ldots,\nu_m)$ with respect to W have at most two non-real solutions (denoted by $W_\pm(z)$) in the complex domain of W. These two coincide with analytic continuations in $\mathbb{C} - [\beta,\alpha]$ of the two real solutions of $\varepsilon(0,\ldots,0)$ (or $\varepsilon(\overline{n}_1,\ldots,\overline{n}_m)$) from $z > \alpha$ (or $z < \beta$).*

To prove Proposition 2, we need a few lemmas.

Lemma 3.1. — *When m' and m'' are greater than 1, then the equation $\varepsilon(\nu_1,\ldots,\nu_m)$ for a fixed real z has at least one real solution.*

Proof. — The function $\psi^{(\nu_1,\dots,\nu_m)}(W)$ has the asymptotic lines (2.5). Hence if $m' > 1$ and $m'' > 1$, the equation $\varepsilon(\nu_1,\dots,\nu_m)$ has always at least one real solution.

LEMMA 3.2. — *When $m' = 1$ or $m'' = 1$, among $\varepsilon(0,\dots,\underset{j-th}{0,\nu,0},\dots,0)$ and $\varepsilon(\overline{n}_1,\dots,\overline{n}_{j-1},\nu-1,\overline{n}_{j+1},\dots,\overline{n}_m)$, $1 \leq \nu \leq \overline{n}_j$, there exists at least one real solution for a fixed real z.*

Proof. — The functions $\psi^{(0,\dots,0,\nu,0,\dots,0)}(W)$ and $\psi^{(\overline{n}_1,\dots,\overline{n}_{j-1},\nu-1,\overline{n}_{j+1},\dots,\overline{n}_m)}(W)$ have the same asymptotic lines for $W \to +\infty$ and $W \to -\infty$ respectively :

$$\begin{aligned} \psi^{(0,\dots,0,\nu,0,\dots,0)}(W) &\sim \beta_j^{(\nu)} - W \text{ for } W \to +\infty, \\ &\sim \beta_j^{(\nu+1)} + (m-1)\beta_j^{(1)} - mW \text{ if } \nu \neq \overline{n}_j, \\ &\sim (m-1)\beta_j^{(1)} - (m-1)W \text{ if } \nu = \overline{n}_j \text{ for } W \to -\infty. \end{aligned} \tag{3.2}$$

$$\begin{aligned} \psi^{(\overline{n}_1,\dots,\overline{n}_{j-1},\nu-1,\overline{n}_{j+1}\dots,\overline{n}_m)}(W) & \\ &\sim \beta_j^{(\nu-1)} + \sum_{k\neq j} \beta_k^{(\overline{n}_k)} - mW \quad \text{for } W \to +\infty, \\ &\sim \beta_j^{(\nu)} - W \quad \text{for } W \to -\infty. \end{aligned}$$

Hence there exists at least one real solution.

Proof of Proposition 2. — From Lemma 3.1 and Lemma 3.2, we see that there exist at least

$$\begin{aligned} &(1+\overline{n}_1)\dots(1+\overline{n}_m) - \sum_{j=1}^{m} \overline{n}_j \quad \text{for } z > \alpha \text{ or } z < \beta, \\ &(1+\overline{n}_1)\dots(1+\overline{n}_m) - \sum_{j=1}^{m} \overline{n}_j - 2 \quad \text{for } z \in (\beta,\alpha), \end{aligned} \tag{3.3}$$

real solutions for ε. Moreover since the two functions $\psi^{(0,\dots,0,\nu,0,\dots,0)}(W)$ and $\psi^{(\overline{n}_1,\dots,\overline{n}_{j-1},\nu-1,\overline{n}_{j+1},\dots,\overline{n}_m)}(W)$ have the same asymptotic lines with tangent -1, the pair $\varepsilon(0,\dots,0,\nu,0,\dots,0)$, $\varepsilon(\overline{n}_1,\dots,\overline{n}_{j-1},\nu-1,\overline{n}_{j+1},\dots,\overline{n}_m)$ have one real solution $W = \infty$ such that $(z,W) \in \hat{\mathcal{C}} - \mathcal{C}$. Hence ε has other $\sum_{j=1}^{m} \overline{n}_j$ real different solutions. The total number of real solutions, multiplicities being counted, is at least equal to

$$(3.4)\qquad \begin{aligned} &(1+\overline{n}_1)\dots(1+\overline{n}_m) && \text{for } z>\alpha \text{ or } z<\beta,\\ &(1+\overline{n}_1)\dots(1+\overline{n}_m)-2 && \text{for } z\in(\beta,\alpha). \end{aligned}$$

We have seen that the degree of $\hat{C}$ is $(1+\overline{n}_1)\dots(1+\overline{n}_m)$. As a result, for $z>\alpha$ or $<\beta$ all solutions are real, while for $z\in(\beta,\alpha)$, besides the real solutions found above, we have only two remaining solutions $W_\pm(z)$ which may or may not be real. Proposition 2 has now been proved.

When $z\in\sigma(A)^c$, we have

$$(3.5)\qquad \begin{aligned} -\frac{dW(z)}{dz}W^{-2}(z) &= \frac{dG(e,e\mid z)}{dz}\\ &= -\sum_{\gamma\in\Gamma} G(e,\gamma\mid z)G(e,\gamma\mid z)<0, \end{aligned}$$

because $G(e,\gamma\mid z)=G(\gamma,e\mid z)\in\mathbf{R}$. In other words $W(z)$ is increasing in each component. The functions $\psi^{(\nu_1,\dots,\nu_m)}(W)+W$ have minimal (maximal) values at W corresponding to the right(left) end point z of the component. This fact shows that if there exist non-real solutions of ε for $\lambda\in\mathbf{R}$, then these must coincide with $W_\pm(\lambda)=W(\lambda\pm i0)$. Hence continuous spectra appear if and only if $W(\lambda+i0)\neq W(\lambda-i0)$ for $\lambda\in\mathbf{R}$.

The set

$$(3.6)\qquad \{\psi^{(\nu_1,\dots,\nu_m)}(W)+W \mid W\in\mathbf{R}, \frac{d\psi^{(\nu_1,\dots,\nu_m)}(W)}{dW}>-1 \text{ for some } (\nu_1,\dots,\nu_m)\},$$

consists of a finite number of open intervals :

$$(3.7)\qquad (-\infty,\lambda_1)\cup(\lambda_2,\lambda_3)\cup\dots\cup(\lambda_{2k-2},\lambda_{2k-1})\cup(\lambda_{2k},+\infty),$$

where $\lambda_1=\beta,\lambda_{2k}=\alpha$. If there appears a discrete spectrum then W vanishes. As a result, Proposition 2 implies the following :

PROPOSITION 3. — (i) *The continuous spectra $\sigma_c(A)$ of the operator A consist of the bands $[\lambda_1,\lambda_2]\cup[\lambda_3,\lambda_4]\cup\dots\cup[\lambda_{2k-1},\lambda_{2k}]$.*

(ii) *The discrete spectra are disjoint from any of the bands. There exist discrete spectra* $z = \psi^{(\nu_1,\dots,\nu_m)}(0)$ *if and only if*

$$\frac{d\psi^{(\nu_1,\dots,\nu_m)}(0)}{dW} < -1, \tag{3.8}$$

for some $(\nu_1,\dots,\nu_m)$.

Furthermore,

PROPOSITION 4. — *If* $m \geq 3$, *then there exists no discrete spectrum.*

Proof. — This follows from the following two lemmas.

LEMMA 3.3.

$$\frac{d\psi_j^{(\nu)}(0)}{dW} \leq -\frac{1}{2} \quad \text{for } 1 \leq \nu \leq \overline{n}_j - 1. \tag{3.9}$$

The strict inequality holds for $n_j > 4$.

Proof. — Since (1.17) implies

$$\frac{d\psi_j^{(\nu)}}{dW}(W) = \frac{F_j'(W + \psi_j^{(\nu)}(W))}{1 - F_j'(W + \psi_j^{(\nu)}(W))}, \tag{3.10}$$

this lemma is a consequence of the inequality

$$F_j'(W) \leq -1 \quad \text{for } \beta_j^{(\overline{n}_j)} < W < \beta_j^{(1)}, \tag{3.11}$$

which can be shown from (2.15) by the following computation. If $n_j > 4$, we can verify the inequality

$$\frac{1}{2}(n_j - 1)\sin\theta \geq |\sin(n_j - 1)\theta|, \tag{3.12}$$

for $\pi/n_j < \theta < \pi - \pi/n_j$, namely for $\beta_j^{(\overline{n}_j)} < W = 2p_j\cos\theta < \beta_j^{(1)}$. Hence

$$F_j'(W) \leq -\frac{(n_j - 1)\sin\theta}{4\sin\theta \cdot \cos^2\dfrac{n_j\theta}{2}} < -\frac{(n_j - 1)}{4} \leq -1\ . \tag{3.13}$$

When $n_j = 4$, then for $\beta_j^{(2)} = -\sqrt{2p_j} < W < \beta_j^{(0)} = \sqrt{2p_j}$,

$$(3.14)\quad F_j'(W) = -\frac{1}{4}\left\{\left(\frac{1}{\frac{W}{2p_j} - \frac{1}{\sqrt{2}}}\right)^2 + \left(\frac{1}{\frac{W}{2p_j} + \frac{1}{\sqrt{2}}}\right)^2\right\} \leq -1.$$

LEMMA 3.4. — *We have*

$$(3.15)\quad \frac{d\psi_j^{(0)}}{dW}(0) \leq -\frac{1}{2} \quad \text{for } n_j \geq 2,$$
$$\frac{d\psi_j^{(\overline{n}_j)}}{dW}(0) < -\frac{1}{2} \quad \text{for } n_j \geq 6,$$

while

$$(3.16)\quad \begin{aligned} \frac{d\psi_j^{(\overline{n}_j)}}{dW}(0) &= -\frac{3}{5} < -\frac{1}{2} \quad \text{for } n_j = 5,\\ &= -\frac{3}{4} < -\frac{1}{2} \quad \text{for } n_j = 4,\\ &= -\frac{1}{3} \quad \text{for } n_j = 3,\\ &= -\frac{1}{2} \quad \text{for } n_j = 2\ . \end{aligned}$$

Proof. — We first prove

$$(3.17)\quad F_j'(\psi_j^{(0)}(0)) = 1 - n_j \leq -1 \quad \text{for all } n_j,$$

and

$$(3.18)\quad F_j'(\psi_j^{(\overline{n}_j)}(0)) < -\frac{1}{2} \quad \text{for } n_j \geq 6\ .$$

(3.17) is derived from (2.15) in view of the equality $\psi_j^{(0)}(0) = 2p_j$. To prove (3.18) we remark that $F_j'(-2p_j) = 1 - n_j$ or $-\frac{2}{3}\overline{n}_j \dfrac{\overline{n}_j - \frac{1}{2}}{\overline{n}_j + \frac{1}{2}}$ according as n_j is even or odd and $\psi_j^{(\overline{n}_j)}(0)$ lies in $(-2p_j, \beta_j^{(\overline{n}_j)})$ from (2.10). Since $F_j'(W)$ is decreasing there, we have $-1 > F_j'(-2p_j) \geq$

$F_j'(\psi_j^{(\overline{n}_j)}(0))$ which implies (3.18). For $n_j = 5, 4, 3$ and 2 we can compute directly $\psi_j^{(\overline{n}_j)}(0)$ and $F_j'(\psi_j^{(\overline{n}_j)}(0))$, which turns out to be $2p_j \cos\frac{4}{5}\pi,\ -\frac{3}{2};\ -2p_j,\ -3;\ -p_j,\ -\frac{1}{2}$ and $-2p_j,\ -1$ respectively.

We consider now the case where $m = 2$. Owing to Lemmas 3.3 and 3.4 we have

$$\frac{d\psi^{(\nu_1,\nu_2)}}{dW}(0) < -1, \tag{3.19}$$

except for the following two cases :

i) n_1 or n_2 is equal to 3 ii) $n_1 = n_2 = 2$.

Hence there exists no discrete spectrum except for the above two cases. In case ii) there is no discrete spectrum. In case $n_1 = 3$ and $n_2 = 2,\ 3$ there always appears a discrete spectrum, since

$$-1 < \frac{d\psi^{(1,\overline{n}_2)}}{dW}(0) < 0.$$

A simple computation shows that there is no discrete spectrum when $n_2 = 4,\ 5$.

It is conjectured that the same holds for $n_2 \geq 6$.

4.

We give some examples (see also [S1]).

Example 1. — $\Gamma = \underbrace{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2}_{m \text{ products}},\ m \geq 3.$

Then the first part of the equation (2.6) becomes

$$2z + (m-2) = \sqrt{W^2 + 16p_1^2} + \cdots + \sqrt{W^2 + 16p_m^2} \text{ for } z \to +\infty,$$

$$= -\sqrt{W^2 + 16p_1^2} - \cdots - \sqrt{W^2 + 16p_m^2} \text{ for } z \to -\infty, \tag{4.1}$$

$$\psi_j^{(0)}(W) = \frac{-W + \sqrt{W^2 + 16p_j^2}}{2}, \tag{4.2}$$
$$\psi_j^{(1)}(W) = \frac{-W - \sqrt{W^2 + 16p_j^2}}{2},$$

where we put $p_j = p_{j,1}$ for simplicity. We assume that $p_1 > p_2 > \cdots > p_m > 0$.

i) Case where $p_1^2 > p_2^2 + \cdots + p_m^2$.

There exist the unique real values $W_{\pm}^{(1)}$ and $W_{\pm}^{(2)} (W_{+}^{(i)} = -W_{-}^{(i)})$ which are solutions of the equations :

$$\pm(m-2) = \frac{W}{\sqrt{W^2 + 16p_1^2}} + \cdots + \frac{W}{\sqrt{W^2 + 16p_m^2}}, \tag{4.3}$$

$$\pm(m-2) = \frac{-W}{\sqrt{W^2 + 16p_1^2}} + \cdots + \frac{W}{\sqrt{W^2 + 16p_m^2}}, \tag{4.4}$$

respectively. The existence of $W_{\pm}^{(1)}$ is easily seen. To prove the existence of $W_{\pm}^{(2)}$, we consider the behaviour of the right hand side of (4.4) as a function of real W. If $W \to \pm\infty$, then it has an asymptotic form :

$$\sim \pm\left\{\frac{8(p_1^2 - p_2^2 - \cdots - p_m^2)}{W^2} + m - 2\right\}. \tag{4.5}$$

Since $p_1^2 - p_2^2 - \cdots - p_m^2 > 0$, there exists the unique $W = W_{+}^{(2)} > 0$ such that the right hand side of (4.4) equals $m-2$ at $W_{+}^{(2)}$. It is greater than $m-2$ for $W > W_{+}^{(2)}$ and smaller than $m-2$ for $W < W_{+}^{(2)}$. In fact

$$\left\{\frac{-W}{\sqrt{W^2+16p_1^2}}+\cdots+\frac{W}{\sqrt{W^2+16p_m^2}}-(m-2)\right\}\cdot\frac{\sqrt{W^2+16p_1^2}}{\sqrt{W^2+16p_1^2}-W}$$

$$(4.6)\qquad =1-\sum_{j=2}^{m}\frac{\sqrt{W^2+16p_1^2}(W+\sqrt{W^2+16p_1^2})p_1^2}{\sqrt{W^2+16p_j^2}(W+\sqrt{W^2+16p_j^2})p_j^2}.$$

But $\dfrac{\sqrt{W^2+16p_1^2}}{\sqrt{W^2+16p_j^2}}$ and $\dfrac{W+\sqrt{W^2+16p_1^2}}{W+\sqrt{W^2+16p_j^2}}$ are strictly decreasing for $W \geq 0$ and (4.6) vanishes for $W = W_+^{(2)}$. Hence (4.6) is positive for $W > W_+^{(2)}$ and negative for $0 \leq W < W_+^{(2)}$. The same reasoning shows the unique existence of $W_-^{(2)}$. There is no critical value of W from other equations $\varepsilon(\nu_1, \ldots, \nu_m)$.

We denote by $\lambda_\pm^{(1)}$, $\lambda_\pm^{(2)}(\lambda_-^{(i)} = -\lambda_+^{(i)})$ the corresponding values of z. Then $\beta = \lambda_-^{(1)} < \lambda_-^{(2)} < 0 < \lambda_+^{(2)} < \lambda_+^{(1)} = \alpha$. There is no discrete spectrum. Thus $\sigma(A)$ coincides with $[\lambda_-^{(1)}, \lambda_-^{(2)}] \cup [\lambda_+^{(1)}, \lambda_+^{(2)}]$.

ii) Case where $p_1^2 < p_2^2 + \cdots + p_m^2$.

The equations (4.3) have the two real solutions $W_+^{(1)}$ and $W_-^{(1)}$.

The equations (4.4) have no real solutions. For suppose that W_c be a real solution. Then the same argument as above shows that the right hand side of (4.4) is greater than $m-2$ for $W > W_c$ and smaller than $m-2$ for $W < W_c$. But for $W \to +\infty$, (4.5) shows that it is smaller than $m-2$. This is a contradiction. Let $\lambda_+^{(1)}$ and $\lambda_-^{(1)}$ be the corresponding values of z. There is one band of continuous spectra $[\lambda_-^{(1)}, \lambda_+^{(1)}]$ and there is no discrete spectrum.

Example 2. — $\Gamma = \mathbb{Z}_3 * \mathbb{Z}_3$.

From (2.6) we have

$$(4.7)\quad z-W=\frac{1}{4}-W+\frac{1}{2}\sqrt{(W-p_1)^2+8p_1^2}+\frac{1}{2}\sqrt{(W-p_2)^2+8p_2^2}.$$

This equation can be explicitly solved with respect to W :

$$(4.8) \qquad W(z) = \frac{\left(2z - \frac{1}{2}\right)^2 + 9(p_2 - p_1)^2 + \sqrt{D(z)}}{\left(2z - \frac{1}{2}\right)^2 = (p_2 - p_1)^2},$$

where $D(z)$ denotes

$$(4.9) \qquad D(z) = \left(z - \frac{1}{4}\right)^2 (z - \lambda_1)(z - \lambda_2)(z - \lambda_3)(z - \lambda_4),$$

where $\lambda_1 = \frac{1}{4} - \frac{1}{4}\sqrt{(4p_1 - 1)^2 + 8}$, $\lambda_2 = \frac{1}{4} - \frac{3}{2}|p_1 - p_2|$, $\lambda_3 = \frac{1}{2} - \lambda_2$, and $\lambda_4 = \frac{1}{2} - \lambda_1$. There are two bands of continuous spectra $[\lambda_1, \lambda_2] \cup [\lambda_3, \lambda_4]$ and one discrete spectrum $z = -\frac{1}{2}$. (See Fig. 1.)

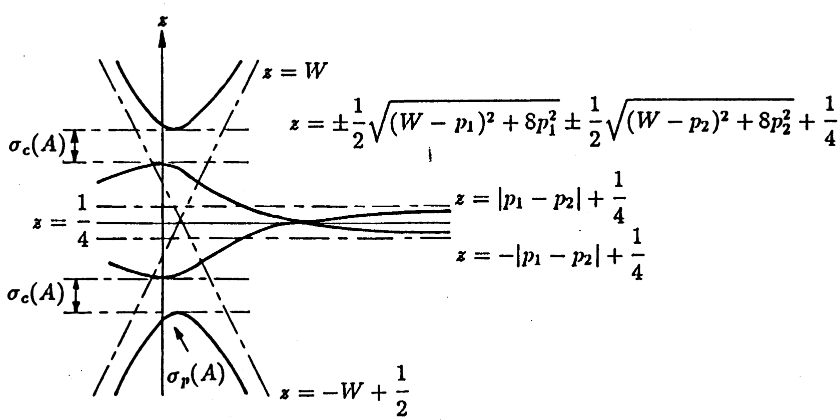


Fig. 1 (See ex. 2).

Example 3. — $\Gamma = \mathbf{Z}_3 * \cdots * \mathbf{Z}_3, \ m \geq 3$.

From (2.6) we have

$$(4.10) \qquad (m-2)W + 2z - \frac{1}{2} = \sum_{j=1}^{m} \pm\sqrt{(W - p_j)^2 + 8p_j^2}\ .$$

If $p_j^2 < \frac{2\sqrt{2}}{3} \sum_{k \neq j} p_k^2$ for any j, we have

$$\frac{d\psi^{(\nu_1,\ldots,\nu_m)}}{dW}(W) < -1 \quad \text{for} -\infty < W < +\infty, \tag{4.11}$$

except for $(\nu_1, \ldots, \nu_m) = (0, \ldots, 0)$ and $(1, \ldots, 1)$. In this case the spectra $\sigma(A)$ consists of one band of continuous spectra $[\beta, \alpha]$, where α and β denote the two solutions z of

$$\pm(m-2) = \sum_{j=1}^{m} \frac{W - p_j}{\sqrt{(W - p_j)^2 + 8p_j^2}}, \tag{4.12}$$

respectively.

We assume now $m = 3$. Then the functions $W + \psi^{(\nu_1,\nu_2,\nu_3)}(W)$ have asymptotic forms as follows :

(a) $W + \psi^{(0,0,0)}(W)$

$$\sim W + \frac{2(p_1^2 + p_2^2 + p_3^2)}{W} \quad \text{for } W \to +\infty,$$

$$\sim -2W + \frac{1}{2} - \frac{2(p_1^2 + p_2^2 + p_3^2)}{W} \quad \text{for } W \to -\infty,$$

(b) $W + \psi^{(0,0,1)}(W)$

$$\sim \frac{1}{2}(-p_1 - p_2 + p_3) + \frac{1}{4} + \frac{2(p_1^2 + p_2^2 - p_3^2)}{W} \quad \text{for } W \to +\infty,$$

$$\sim -W + \frac{1}{2}(p_1 + p_2 - p_3) + \frac{1}{4} - \frac{2(p_1^2 + p_2^2 - p_3^2)}{W} \quad \text{for } W \to -\infty,$$

(c) $W + \psi^{(0,1,0)}(W)$

$$\sim \frac{1}{2}(-p_1 + p_2 - p_3) + \frac{1}{4} + \frac{2(p_1^2 - p_2^2 + p_3^2)}{W} \quad \text{for } W \to +\infty,$$

$$\sim -W + \frac{1}{2}(p_1 - p_2 + p_3) + \frac{1}{4} - \frac{2(p_1^2 - p_2^2 + p_3^2)}{W} \quad \text{for } W \to -\infty,$$

(d) $W + \psi^{(0,1,1)}(W)$

$$\sim -W + \frac{1}{2}(-p_1 + p_2 + p_3) + \frac{1}{4} + \frac{2(p_1^2 - p_2^2 - p_3^2)}{W} \quad \text{for } W \to +\infty,$$

$$\sim \frac{1}{2}(p_1 - p_2 - p_3) + \frac{1}{4} - \frac{2(p_1^2 - p_2^2 - p_3^2)}{W} \quad \text{for } W \to -\infty,$$

(e) $W + \psi^{(1,0,0)}(W)$

$$\sim \frac{1}{2}(p_1 - p_2 - p_3) + \frac{1}{4} + \frac{2(-p_1^2 + p_2^2 + p_3^2)}{W} \quad \text{for } W \to +\infty,$$

$$\sim -W + \frac{1}{2}(-p_1 + p_2 + p_3) + \frac{1}{4} + \frac{2(+p_1^2 - p_2^2 - p_3^2)}{W} \quad \text{for } W \to -\infty,$$

(f) $W + \psi^{(1,0,1)}(W)$

$$\sim -W + \frac{1}{2}(p_1 - p_2 + p_3) + \frac{1}{4} + \frac{2(-p_1^2 + p_2^2 - p_3^2)}{W} \quad \text{for } W \to +\infty,$$

$$\sim \frac{1}{2}(-p_1 + p_2 - p_3) + \frac{1}{4} - \frac{2(-p_1^2 + p_2^2 - p_3^2)}{W} \quad \text{for } W \to -\infty,$$

(g) $W + \psi^{(1,1,0)}(W)$

$$\sim -W + \frac{1}{2}(p_1 + p_2 - p_3) + \frac{1}{4} + \frac{2(-p_1^2 - p_2^2 + p_3^2)}{W} \quad \text{for } W \to +\infty,$$

$$\sim \frac{1}{2}(-p_1 - p_2 + p_3) + \frac{1}{4} - \frac{2(-p_1^2 - p_2^2 + p_3^2)}{W} \quad \text{for } W \to -\infty,$$

(h) $W + \psi^{(1,1,1)}(W)$

$$\sim -2W + \frac{1}{2}(p_1 + p_2 + p_3) + \frac{1}{4} + \frac{2(-p_1^2 - p_2^2 - p_3^2)}{W} \quad \text{for } W \to +\infty,$$

$$\sim W - \frac{1}{2}(p_1 + p_2 + p_3) + \frac{1}{4} + \frac{2(p_1^2 + p_2^2 + p_3^2)}{W} \quad \text{for } W \to -\infty.$$

We further assume that the inequality holds : $p_1^2 + p_2^2 < p_3^2$. Then as is seen from the asymptotic forms $(a) \sim (h)$, the function $W + \psi^{(0,0,0)}(W)$ (or $W + \psi^{(0,0,1)}(W)$) has the unique minimal value $\lambda_{0,0,0}$ (or $\lambda_{0,0,1}$). For, if there exists another minimal $\lambda'_{0,0,1}$, then the equation $\varepsilon(0,0,1)$ would have more than 3 real solutions for z, smaller than $\lambda_{0,0,0}$ or $\lambda_{0,0,1}$. This contradicts that the number of real solutions of ε is at most equal to 8. In the same way the function $W + \psi^{(1,1,1)}(W)$ (or $W + \psi^{(1,1,0)}(W)$) has the unique maximal value $\lambda_{1,1,1}$(or $\lambda_{1,1,0}$). Similar arguments show that :

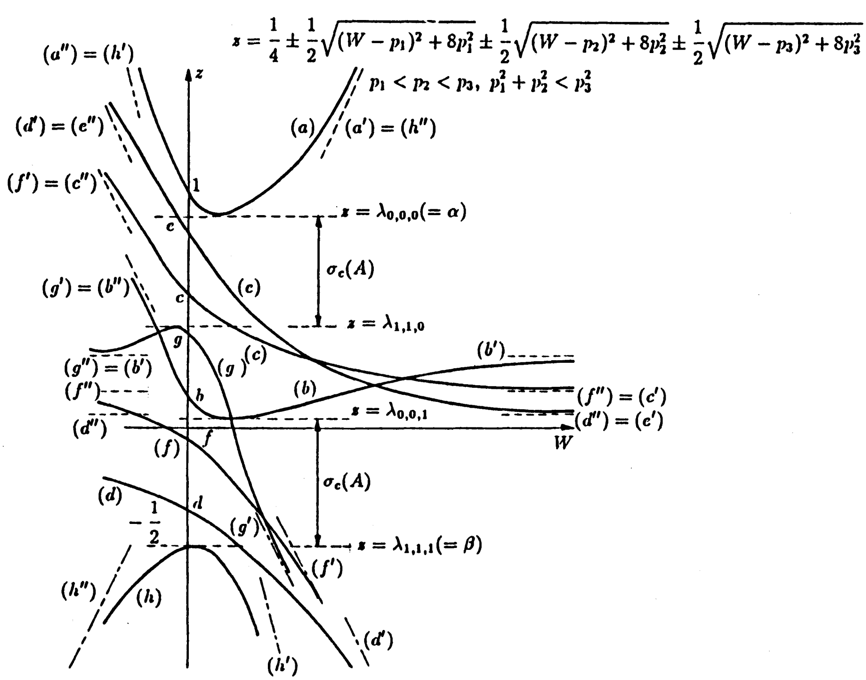
$$\lambda_{1,1,1} < \lambda_{0,0,1} < \lambda_{1,1,0} < \lambda_{0,0,0} \,. \tag{4.13}$$

For $z \in (\lambda_{0,0,1}, \lambda_{1,1,0})$, there exist just eight real solutions of ε, so

$$\frac{d\psi^{(\nu_1,\nu_2,\nu_3)}}{dW}(0) < -1, \tag{4.14}$$

except for $\dfrac{d\psi^{(1,1,1)}}{dW}(0)$, which is equal to -1.

Consequently the spectra $\sigma(A)$ consists of two bands of continuous spectra $[\lambda_{1,1,1}, \lambda_{0,0,1}] \cup [\lambda_{1,1,0}, \lambda_{0,0,0}]$. (See Fig. 2). If $p_1^2 + p_2^2 \geq p_3^2$, this fact does not hold for all (p_1, p_2, p_3). In fact if we take $p_1 = p_2 = p_3 = \frac{1}{6}$, then $\sigma(A)$ consists of one band $[\lambda_{1,1,1}, \lambda_{0,0,1}]$ because there exist neither maximal nor minimal values $\lambda_{1,1,0}, \lambda_{0,0,1}$.



(a'), (a'') denote the asymptotic lines for $W = \pm\infty$ of the curve (a) : $z = W + \psi^{(0,0,0)}(W)$ and so on.

Fig. 2 (See ex. 3).

Example 4. — $\Gamma = \mathbb{Z}_4 * \cdots * \mathbb{Z}_4,\ m \geq 2$.

$W_j = \psi_j^{(0)}(W)$ and the equation (2.6) can be explicitly written by using Cardano's formula for cubic equations :

$$W_j^3 + 2WW_j^2 + (W^2 - 4p_j^2)W_j - 2p_j^2 W = 0, \tag{4.15}$$

$$z = \left(1 - \frac{m}{3}\right)W + \frac{1}{6}\sum_{j=1}^{m}\left\{\sqrt[3]{\alpha_j(W) + \sqrt{\beta_j(W)}} + \sqrt[3]{\alpha_j(W) - \sqrt{\beta_j(W)}}\right\}, \tag{4.16}$$

where $\alpha_j(W)$ and $\beta_j(W)$ denote

$$\begin{aligned} \alpha_j(W), &= W^3 - 9p_j^2 W, \\ \beta_j(W) &= -6p_j^2 W^4 - 39p_j^4 W^2 - 192p_j^6. \end{aligned} \tag{4.17}$$

For $W = 0$, there correspond $z = \pm p_{j,1} \pm \cdots \pm p_{j,s}$. But none of them are eigenvalues for A in view of Corollary of Proposition 3.

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