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## GRADED MORPHISMS OF G-MODULES

by H. KRAFT and C. PROCESI

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### 1. Introduction.

During the 1987 meeting in honor of J. K. Koszul, Steve Halperin explained to us the following conjecture (motivated by the study of the spectral sequence associated to a homogeneous space).

**1.1. CONJECTURE.** — *If  $f_1, f_2, \dots, f_n$  is a regular sequence in the polynomial ring  $C[x_1, x_2, \dots, x_n]$ , the connected component of the automorphism group of the (finite dimensional) algebra  $C[x_1, \dots, x_n]/(f_1, \dots, f_n)$  is solvable.*

In this paper we prove a weak form of this (Corollary 4.3) which implies the conjecture at least when the  $f_i$ 's are homogeneous (Remark 4.4).

### 2. Preliminaries.

Our base field is  $C$ , the field of complex numbers, or any other algebraically closed field of characteristic zero.

**2.1. DEFINITION.** — *A morphism  $\varphi : V \rightarrow W$  between finite dimensional vector spaces  $V$  and  $W$  is called graded if there is a basis of  $W$  such that the components of  $\varphi$  are all homogeneous polynomials.*

Let us denote by  $\mathcal{O}(V)$ ,  $\mathcal{O}(W)$  the ring of regular functions on  $V$  and  $W$ . These  $C$ -algebras are naturally graded by degree:  $\mathcal{O}(V) = \bigoplus_i \mathcal{O}(V)_i$ . A subspace  $S \subset \mathcal{O}(V)$  is called *graded* if  $S = \bigoplus_i S \cap \mathcal{O}(V)_i$ .

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If  $\varphi: V \rightarrow W$  is a morphism and  $\varphi^*: \mathcal{O}(W) \rightarrow \mathcal{O}(V)$  the corresponding comorphism we have the following equivalence:

$\varphi$  is graded  $\Leftrightarrow \varphi^*(W^*)$  is a graded subspace of  $\mathcal{O}(V)$ .

**2.2. LEMMA.** — *For any graded morphism  $\varphi: V \rightarrow W$  there is a unique decomposition  $W = \bigoplus_{v \geq 0} W_v$  and homogeneous morphisms  $\varphi_v: V \rightarrow W_v$  of degree  $v$  such that*

$$\varphi = (\varphi_0, \varphi_1, \varphi_2, \dots): V \rightarrow W_0 \oplus W_1 \oplus W_2 \oplus \dots$$

(This is clear from the definitions.)

**2.3. Remark.** — Let  $G$  be an algebraic group. Assume that  $V$  and  $W$  are  $G$ -modules and that  $\varphi: V \rightarrow W$  is graded and  $G$ -equivariant. Then in the notations of lemma 2.2 all  $W_v$  are submodules and all components  $\varphi_v$  are  $G$ -equivariant.

**2.4. Remark.** — If  $\varphi: V \rightarrow W$  is graded and dominant with  $\varphi^{-1}(0) = \{0\}$ , then  $\varphi$  is a finite surjective morphism. In fact given a finitely generated graded algebra  $A = \bigoplus_{i \geq 0} A_i$  with  $A_0 = \mathbb{C}$  and a graded subspace  $S \subset A$  such that the radical  $\text{rad}(S)$  of the ideal generated by  $S$  is the homogeneous maximal ideal  $\bigoplus_{i > 0} A_i$  of  $A$ , then  $A$  is a finitely generated module over the subalgebra  $\mathbb{C}[S]$  generated by  $S$  (see [1, II.4.3 Satz 8]).

### 3. The Main Theorem.

**3.1. THEOREM.** — *Let  $G$  be a connected reductive algebraic group and let  $V, W$  be two  $G$ -modules. Assume that  $V$  and  $W$  do not contain 1-dimensional submodules. Then any graded  $G$ -equivariant dominant morphism with finite fibres is a linear isomorphism.*

We first prove this for  $G = \text{SL}_2$  and then reduce to this situation.

For any  $\mathbb{C}^*$ -module  $V$  we have the weight decomposition

$$V = \bigoplus_j V_j, \quad V_j := \{v \in V \mid t(v) = t^j \cdot v\}.$$

We say that  $V$  has *only positive weights* if  $V = \bigoplus_{j > 0} V_j$ .

**3.2. LEMMA.** — *Let  $V, W$  be two  $C^*$ -modules with only positive weights, and let  $\varphi: V \rightarrow W$  be a  $C^*$ -equivariant graded morphism with finite fibres. For all  $k \geq 0$  we have*

$$\varphi^{-1} \left( \bigoplus_{j \leq k} W_j \right) \subseteq \bigoplus_{j \leq k} V_j,$$

and the inclusion is strict for at least one  $k$  in case  $\varphi$  is not linear.

*Proof.* — By lemma 2.2 and remark 2.3 we have  $\varphi = \sum_{v \geq 1} \varphi_v$  where  $\varphi_v: V \rightarrow W_v$  is homogeneous of degree  $v$  and  $C^*$ -equivariant. Let  $v = \sum_{j=1}^k v_j \in \bigoplus_{j > 0} V_j = V$  with  $v_k \neq 0$ . Then

$$\lim_{\lambda \rightarrow 0} \lambda^k \cdot t_\lambda^{-1}(v) = v_k.$$

(Here  $t_\lambda$  denotes the action of  $C^*$ .) Since  $\varphi_v$  is homogeneous of degree  $v$  and  $C^*$ -equivariant we obtain

$$(1) \quad \lim_{\lambda \rightarrow 0} \lambda^{vk} \cdot t_\lambda^{-1}(\varphi_v(v)) = \varphi_v(v_k).$$

This implies that  $\varphi_v(v) \in \bigoplus_{j \leq vk} W_j$  for all  $v$ , proving the first claim.

If  $\varphi$  is not linear, i.e.  $\varphi \neq \varphi_1$ , then there is a  $v > 1$ , an index  $k$  and an element  $v \in V_k$  such that  $\varphi_v(v) \neq 0$ . But  $\varphi_v(v) \in W_{vk}$  by (1) and so  $v \notin \varphi^{-1} \left( \sum_{j \leq k} W_j \right)$ .  $\square$

**3.3. COROLLARY.** — *Under the assumptions of lemma 3.2 suppose that  $\varphi$  is surjective. Put  $\lambda_j := \dim V_j$  and  $\mu_j := \dim W_j$ . Then for all  $k \geq 1$  we have*

$$(2) \quad \lambda_1 + \lambda_2 + \dots + \lambda_k \geq \mu_1 + \mu_2 + \dots + \mu_k.$$

If  $\varphi$  is not linear the inequality is strict for at least one  $k$ .

(This is clear.)

**3.4. PROPOSITION.** — *Let  $V, W$  be two  $SL_2$ -modules containing no fixed lines. Let  $\varphi: V \rightarrow W$  be a graded  $SL_2$ -equivariant morphism, which is dominant and has finite fibres. Then  $\varphi$  is a linear isomorphism.*

*Proof.* — Consider the maximal unipotent subgroup

$$U := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subset \mathrm{SL}_2$$

and the maximal torus

$$T := \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{C}^* \right\} \simeq \mathbb{C}^*.$$

By assumption  $\varphi$  is finite and surjective (Remark 2.4), and  $\varphi^{-1}(W^U) = V^U$ . Hence the induced morphism

$$\varphi|_{V^U} : V^U \rightarrow W^U$$

is graded,  $T$ -equivariant, finite and surjective too. Furthermore all weights  $\lambda_j$  of  $V^U$  and  $\mu_j$  of  $W^U$  are positive. It follows from (2) that

$$\lambda_k + \lambda_{k+1} + \dots \leq \mu_k + \mu_{k+1} + \dots$$

for all  $k$ , because  $\sum_j \lambda_j = \dim V^U = \dim W^U = \sum_j \mu_j$ . From this we get

$$\begin{aligned} \dim V &= 2\lambda_1 + 3\lambda_2 + \dots + (n+1)\lambda_n \\ &\leq 2\mu_1 + 3\mu_2 + \dots + (n+1)\mu_n = \dim W \end{aligned}$$

for all  $n$  which are big enough. (Remember that an irreducible  $\mathrm{SL}_2$ -module of highest weight  $j$  is of dimension  $j+1$ ). If  $\varphi$  is not linear this inequality is strict (Corollary 3.3), contradicting the fact that  $\varphi$  is finite and surjective.  $\square$

**3.5. Proof of the Theorem.** — Assume that  $\varphi : V \rightarrow W$  is not linear, i.e. there is a  $v_0 > 1$  such that the component  $\varphi_{v_0} : V \rightarrow W_{v_0}$  is non-zero. Then there is a homomorphism  $\mathrm{SL}_2 \rightarrow G$  and a non-trivial irreducible  $\mathrm{SL}_2$ -submodule  $M \subseteq V$  such that  $\varphi_j|_M \neq 0$ . (In fact the intersection of the fixed point sets  $V^{(\mathrm{SL}_2)}$  for all homomorphisms  $\iota : \mathrm{SL}_2 \rightarrow G$  is zero.) Now consider the  $G$ -stable decompositions  $V = V^{\mathrm{SL}_2} \oplus V'$  and  $W = W^{\mathrm{SL}_2} \oplus W'$  and the following morphism :

$$\varphi' : V' \hookrightarrow V \xrightarrow{\varphi} W \xrightarrow{\mathrm{pr}} W'.$$

Since  $V'$  and  $W'$  are sums of isotypic components the morphism  $\varphi'$  is again graded. Furthermore  $\varphi^{-1}(W^{\mathrm{SL}_2}) = V^{\mathrm{SL}_2}$ , hence  $\varphi^{-1}(0) = V^{\mathrm{SL}_2} \cap V' = \{0\}$ . This implies that  $\varphi' : V' \rightarrow W'$  is dominant

with finite fibres and satisfies therefore the assumptions of proposition 3.4. As a consequence  $\varphi'$  is linear. Since  $\varphi|_{V'}: V' \rightarrow W$  is graded too we have  $\varphi_v|_{V'} = 0$  for all  $v > 1$ . This contradicts the facts that  $M \subseteq V'$  and  $\varphi_{v_0}|_M \neq 0$  (see the construction above).

#### 4. Some Consequences.

We add some corollaries of the theorem. Let  $G$  be a connected reductive group. For every  $G$ -module  $V$  we have the canonical  $G$ -stable decomposition  $V = V^\circ \oplus V'$  where  $V^\circ$  is the sum of all 1-dimensional representations (i.e.  $V^\circ = V^{(G,G)}$ ) and  $V'$  the sum of all others. The proof of the theorem above easily generalizes to obtain the following result :

**4.1. THEOREM.** — *Let  $\varphi: V \rightarrow W$  be a graded  $G$ -equivariant dominant morphism with finite fibres. Then  $\varphi$  induces a linear isomorphism*

$$\varphi|_{V'}: V' \xrightarrow{\sim} W'.$$

**4.2. COROLLARY.** — *Let  $\mathcal{O}(V)$  be the ring of regular functions on a  $G$ -module  $V$ , and let  $f_1, \dots, f_n$  be a regular sequence of homogenous elements of  $\mathcal{O}(V)$  such that the linear span  $\langle f_1, \dots, f_n \rangle$  is  $G$ -stable. Then  $\langle f_1, \dots, f_n \rangle$  contains all non-trivial representations of  $(G, G)$  in  $\mathcal{O}(V)_1$ , the linear part of  $\mathcal{O}(V)$ .*

*Proof.* — The regular sequence  $f_1, \dots, f_n$  defines a  $G$ -equivariant finite morphism  $\varphi: V \rightarrow W$ ,  $W := \langle f_1, \dots, f_n \rangle^*$ . By the theorem above the restriction  $\varphi'|_{V'}: V' \rightarrow W'$  is a linear isomorphism which means that every non-trivial  $(G, G)$ -submodule of  $\langle f_1, \dots, f_n \rangle$  is contained in the linear part  $\mathcal{O}(V_1)$  of  $\mathcal{O}(V)$ .  $\square$

**4.3.** Recall that a finite dimensional  $\mathbb{C}$ -algebra is called a *complete intersection* if it is of the form  $\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_n)$  with a regular sequence  $f_1, \dots, f_n$ .

**COROLLARY.** — *Let  $A$  be a finite dimensional local  $\mathbb{C}$ -algebra with maximal ideal  $\mathfrak{m}$  and let  $\text{gr}_{\mathfrak{m}}A$  be the associated graded algebra (with respect to the  $\mathfrak{m}$ -adic filtration). If  $\text{gr}_{\mathfrak{m}}A$  is a complete intersection then the connected component of the automorphism group of  $A$  is solvable.*

*Proof.* — Let  $G$  and  $\bar{G}$  be the connected components of the automorphism groups of  $A$  and of  $\text{gr}_m A$  respectively. Since the  $m$ -adic filtration of  $A$  is  $G$ -stable we have a canonical homomorphism  $\rho: G \rightarrow \bar{G}$ . It is easy to see that  $\ker \rho$  is unipotent, so it remains to show that  $\bar{G}$  is solvable.

Assume that  $\bar{G}$  is not solvable. Then  $\bar{G}$  contains a (non-trivial) semisimple subgroup  $H$ . By assumption we have an isomorphism

$$\text{gr}_m A \simeq \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_n)$$

with a regular sequence  $f_1, \dots, f_n$  where all  $f_i$  are homogeneous of degree  $\geq 2$ . Clearly the action of  $\bar{G}$  on  $\text{gr}_m A$  is induced from a (faithful) linear representation on  $\mathbb{C}[x_1, \dots, x_n]_1 \subset \mathbb{C}[x_1, \dots, x_n]$ . Hence it follows from corollary 4.2 that  $\langle f_1, \dots, f_n \rangle$  contains all non-trivial  $H$ -submodules of  $\mathbb{C}[x_1, \dots, x_n]_1$ , contradicting the fact that all  $f_i$  have degree  $\geq 2$ .  $\square$

**4.4. Remark.** — The corollary above implies that conjecture 1.1 is true in case all  $f_i$  are homogeneous, i.e. if the algebra

$$A = \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_n)$$

is finite dimensional and graded with all  $x_i$  of degree 1.

**4.5. Remark.** — Another formulation of our result is the following: Let  $V$  be a representation of a connected algebraic group  $G$  and  $Z \subset V$  a  $G$ -stable graded subscheme, which is a complete intersection supported in  $\{0\}$ . Then  $(G, G)$  acts trivially on  $Z$ .

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