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FINAL FORMS FOR A THREE-DIMENSIONAL VECTOR FIELD UNDER BLOWING-UP

par Felipe CANO (*)

0. Introduction.

A plane vector field $D = a\partial/\partial x + b\partial/\partial y$, with g.c.d $(a,b) = 1$, defines a unidimensional saturated foliation $\mathcal{F}$ having singularities at the zeroes of D. It is well known that after a finite number of quadratic blowing-ups of the ambient space, we can obtain a foliation $\tilde{\mathcal{F}}$ which is given locally at each singular point by a vector field $\tilde{D}$ having a linear part with eigenvalues 1 and λ, where $\lambda \notin \mathbf{Q}_+$ (= strictly positive rational numbers). (See [2] and [12]). The above singularities may be thought of as final forms in the sense that they are preserved under new quadratic blowing-ups.

This paper is mainly devoted to identifying final forms in the above sense for a three-dimensional vector field.

This « stable » situation is described in paragraph 3. There it is proved that the situation is preserved under permissible blowing-ups. The main result is stated in paragraph 4 : there is a global sequence of permissible blowing-ups such that each sequence of infinitely near singular points stabilizes in final forms (if one begins with order zero).

In paragraph 5 the two-dimensional situation is revisited in order to study, in paragraph 6, the restriction of the vector field to the exceptional divisor. There it is shown that we can obtain the additional

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property that the restriction to each « non-dicritical » component of the exceptional divisor has non-nilpotent linear part. It is also shown that we cannot expect to obtain eigenvalues 1 and λ with $\lambda \notin \mathbf{Q}_+$.

Paragraph 7 is devoted to the study of the structure of the integral branches, which is closely related to the singular points contained in only one component of the exceptional divisor once they are in a final form.

Previously ([6],[7]) a procedure for reducing the order of a three-dimensional vector field was given. Here we shall restrict ourselves to the case of order zero (see paragraph 1 for more details). We shall end with an Appendix in which a more general case is studied.

1. Preliminaries.

Most of the concepts and results in this paragraph may be found in ([6],[7]) and we shall omit the proofs.

1.1. The ambient space will be a regular variety X, i.e. a regular integral separated scheme of finite type over an algebraically closed field k, and we shall assume that k is a field of characteristic zero (e.g. $k=\mathbf{C}$). The tangent sheaf Ξ_X is a locally free $\mathcal{O}_X$-module of rank $n = \dim X$.

Any invertible $\mathcal{O}_X$-submodule $\mathcal{D}$ of Ξ_X will be called a « unidimensional distribution over X ». Locally at each closed point P, $\mathcal{D}$ is generated by a germ of a vector field D. The unidimensional foliation $\mathcal{F}_{\mathcal{D}}$ given locally by D does not depend on the generator D of $\mathcal{D}_P$. In this way we obtain a global foliation $\mathcal{F}_{\mathcal{D}}$. Let us denote by $\alpha(\mathcal{D})$ the double orthogonal of $\mathcal{D}$ with respect to the natural pairing between Ξ_X and its dual sheaf (which may be identified with the cotangent sheaf Ω_X). $\alpha(\mathcal{D})$ is also a unidimensional distribution over X. If sat $(\mathcal{F}_{\mathcal{D}})$ is the saturated foliation of $\mathcal{F}_{\mathcal{D}}$ (see [10]), we have $\mathcal{F}_{\alpha(\mathcal{D})} = \text{sat}\ (\mathcal{F}_{\mathcal{D}})$. Let $P \in X$ be a closed point and let $P = (x_1, \ldots, x_n)$ be a regular system of parameters of the local ring $\mathcal{O}_{X,P}$. If $\mathcal{D}_P$ is generated by $D = \Sigma a_i \partial/\partial x_i$, then $\alpha(\mathcal{D})_P$ is generated by D/b, where $b = \text{g.c.d.}\,(a_i)$. We shall say that $\mathcal{D}$ is multiplicatively irreducible iff $\mathcal{D} = \alpha(\mathcal{D})$.

1.2. A closed subscheme $E \subset X$ is a « normal crossings divisor of X » iff for each closed point P of X, there is a regular system of parameters $p = (x_1, \ldots, x_n)$ of $\mathcal{O}_{X,P}$ and a subset $A \subset \{1, \ldots, n\}$ such that the ideal $I(E)_P$ of E is generated by $\Pi_{i \in A} x_i$. Such a regular system of parameters is called « suited for E at P». We shall denote $e(E,P) = \# A$.

Let us denote by $\Xi_X[E]$ the sheaf of the « germs of vector fields tangent to E », which is given by

$$\Xi_{X,P}[E] = \{D \in \Xi_{X,P};\ D(I(E)_P) \subset I(E)\} \tag{1.2.1}$$

for each closed point P. (The sheaf $\Xi_X[E]$ may be identified in a natural way with the dual sheaf of the sheaf Ω_X [log E] of the forms with poles logarithmic along E. The sheaf $\Xi_X[E]$ is a locally free $\mathcal{O}_X$-submodule of Ξ_X of rank n and a base of $\Xi_{X,P}[E]$ is given by

$$\{x_i^{\tau(i)} \partial/\partial x_i\}\ i = 1, \ldots, n \tag{1.2.2}$$

where $\tau(i) = 1$ if $i \in A$ and $\tau(i) = 0$ if $i \notin A$, for a suited regular system of parameters $p = (x_1, \ldots, x_n)$ for E at P.

A unidimensional distribution $\mathcal{D}$ over X will be called « adapted to E » iff $\mathcal{D} \subset \Xi_X[E]$. The « adaptation » $(\mathcal{D},E) = \mathcal{D} \cap \Xi_X[E]$ of a unidimensional distribution is also a unidimensional distribution. Let us assume that $\mathcal{D}$ is adapted to E and let us denote by $\alpha_E(\mathcal{D})$ the double orthogonal of $\mathcal{D}$ with respect to the natural pairing between $\Xi_X[E]$ and its dual sheaf. Let us remark that if $E = \varnothing$, then $\Xi_X[\varnothing] = \Xi_X$ and $\alpha = \alpha_\varnothing$. If E_1 is another normal crossings divisor with $E_1 \supset E$, then

$$\alpha_{E_1}(\mathcal{D},E_1) = (\alpha_E(\mathcal{D},E),E_1) \tag{1.2.3}$$

for each unidimensional distribution $\mathcal{D}$. Finally, $\mathcal{D}$ is said to be « multiplicatively irreducible and adapted to E » iff $\mathcal{D} = \alpha_E(\mathcal{D},E)$.

Assume that $\mathcal{D}$ is adapted to E, then $\mathcal{D} = \alpha_E(\mathcal{D})$ iff for each closed point P, $\mathcal{D}_P$ is generated by

$$D = \sum_{i \in A} a_i x_i/\partial x_i + \sum_{i \notin A} a_i \partial/\partial x_i \tag{1.2.4}$$

(for a suited regular system of parameters) in such a way that g.c.d. $(a_i) = 1$.

If $E \subset E_1$, we have $((\mathcal{D},E_1),E) = (\mathcal{D},E_1)$, but it is not true in general that $\alpha_E(\mathcal{D},E_1) = \alpha_{E_1}(\mathcal{D},E_1)$. For instance, take

$n = 2$, $D = x\partial/\partial x + x\partial/\partial y$, $E_1 = \hat{(x=0)}$ and $E = \varnothing$, then $\alpha_{E_1}(\mathscr{D}) = \mathscr{D} \neq \alpha_E(\mathscr{D}) = \mathscr{D}/x$.

1.3. Let us fix a normal crossings divisor E of X. A closed subscheme Y of X is said to have « normal crossings with E » iff for each closed point P of Y there is a regular system of parameters $p = (x_1, \ldots, x_n)$ suited for E at P and sets $B_j \subset \{1,\ldots,n\}$, $j = 1, \ldots, l$ (j runs over the irreducible components of Y) such that

$$\text{(1.3.1)} \qquad I(Y)_P = \bigcap_{j=1,\ldots,l} \left(\sum_{i \in B_j} x_i \mathcal{O}_{X,P} \right).$$

Such a regular system of parameters will be called « suited for the pair (E,Y) at P ». Note that Y must have a reduced structure.

Let Y be a regular subscheme of X having normal crossings with E and let $\pi : X' \to X$ be the blowing-up of X with center Y. Then $E' = \pi^{-1}(E \cup Y)$ is also a normal crossings divisor. Let $\mathscr{D}$ be a unidimensional distribution, multiplicatively irreducible and adapted to E over X. Then there is a unique unidimensional distribution $\mathscr{D}'$ multiplicatively irreducible and adapted to E' over X' such that $\mathscr{D}'|_{\pi^{-1}(X-Y)} = \mathscr{D}|_{X-Y}$ via the induced isomorphism between the tangent sheaves of $\pi^{-1}(X-Y)$ and $X - Y$. We shall say that $(X',\mathscr{D}',E')$ is the strict transform of $(X,\mathscr{D},E)$ by π.

There is a slight difference between the above procedure and the usual blowing-up of a foliation (for instance, if π is quadratic, i.e. centered at a closed point). Assume that $E = \varnothing$, then the strict transform of the foliation $\mathscr{F}_{\mathscr{D}}$ is not in general the foliation $\mathscr{F}_{\mathscr{D}'}$, but it is the foliation $\mathscr{F}_{\alpha(\mathscr{D}')}$. Moreover, if $\{(X(i),\mathscr{D}(i),E(i))\}_{i=0,\ldots,N}$ is a sequence of blowing-ups, (i.e. $(X(i),\mathscr{D}(i),E(i))$ is the strict transform of $(X(i-1),\mathscr{D}(i-1),E(i-1))$ by the blowing-up $\pi(i) : X(i) \to X(i-1)$ centered at $Y(i-1)$, $i = 1,\ldots,N$) we deduce easily that if $(X(N),\mathscr{D}'(N),E'(N))$ is the strict transform of $(X(N-1), \alpha(\mathscr{D}(N-1)),\varnothing)$ by $\pi(N)$, then $\alpha(\mathscr{D}(N)) = \alpha(\mathscr{D}'(N))$. Hence we have to consider only the last $\mathscr{D}(N)$ in order to compute the strict transform of $\mathscr{F}_{\mathscr{D}(0)}$ under $\pi(1) \circ \ldots \circ \pi(N)$.

Let $P \in Y$ be a closed point, let $p = (x_1, \ldots, x_n)$ be a regular system of parameters suited for (E,Y) at P and let us assume that $I(E)_P = \left(\prod_{i \in A} x_1\right) \cdot \mathcal{O}_{X,P}$ and that $I(Y)_P = \sum_{i \in B} x_i \mathcal{O}_{X,P}$. Let P' be a closed point of X′ such that $\pi(P') = P$. The morphism π induces an inclusion $\mathcal{O}_{X,P} \subset \mathcal{O}_{X',P'}$. Under this inclusion, we can find a regular system of parameters $p' = (x'_1, \ldots, x'_n)$ of $\mathcal{O}_{X',P'}$ such that there is an index $i_0 \in B$ and scalars $\zeta_i \in k$, $i \in B - \{i_0\}$ such that

$$(1.3.2)\qquad \begin{aligned} x_i &= x'_i, \qquad i \notin B - \{i_0\} \\ x_i &= (x'_i + \zeta_i)x'_{i_0}, \qquad i \in B - \{i_0\}. \end{aligned}$$

The exceptional divisor $\pi^{-1}(Y)$ is given locally by $x'_{i_0} = 0$ and E′ is given by $\prod_{i \in A'} x'_i = 0$, where $A' = (A - B) \cup \{i_0\} \cup \{i \in A \cap B; \zeta_i = 0\}$. If $\mathcal{D}_P$ is generated by $D = \sum_{i \in A} a_i x_i\, \partial/\partial x_i + \sum_{i \notin A} a_i \partial/\partial x_i$, then $\mathcal{D}'_{P'}$ is generated by

$$(1.3.3.)\qquad D' = (1/x'_{i_0})^{\mu}\left(\sum_{i \in A'} a'_i x'_i\, \partial/\partial x'_i + \sum_{i \notin A'} a'_i\, \partial/\partial x'_i\right)$$

for some $\mu \in \mathbf{Z}$, where

$$(1.3.4)\qquad \begin{aligned} a'_{i_0} &= a_{i_0} \quad \text{if} \quad i_0 \in A; \; a'_{i_0} = a_{i_0}/x'_{i_0} \quad \text{if} \quad i_0 \notin A \\ a'_i &= (a_i - a'_{i_0}) \quad \text{if} \quad i \in A' \cap B - \{i_0\} \\ a'_i &= (a_i - a'_{i_0})(x'_i + \zeta_i) \quad \text{if} \quad i \in (A - A') \cap B - \{i_0\} \\ a'_i &= a_i/x'_{i_0} - (x'_i + \zeta_i)a'_{i_0} \quad \text{if} \quad i \in B - A - \{i_0\} \\ a'_i &= a_i \quad \text{if } i \notin B. \end{aligned}$$

The integer μ may be negative and it is the maximum power of x'_{i_0} which divides $a'_i, i = 1, \ldots n$.

1.4. Let $\mathcal{D}$ be a unidimensional distribution adapted to a normal crossings divisor E. For each closed point P of X, the adapted order $\nu(\mathcal{D},E,P)$ is the maximum integer m such that $\mathcal{D}_P \subset \eta^m \Xi_{X,P}[E]$ where η is the maximal ideal of $\mathcal{O}_{X,P}$. The adapted order is the minimum of the orders of the coefficients of a generator of $\mathcal{D}_P$.

If $E_1 \subset E$, we have

$$(1.4.1)\qquad \nu(\mathcal{D},E,P) \leqslant \nu(\mathcal{D},E_1,P) \leqslant \nu(\mathcal{D},E,P) + 1.$$

If $(X',E',\mathscr{D}')$ is the strict transform of $(X,E,\mathscr{D})$, then for each closed point P' of X'

$$\nu(\mathscr{D}',E',P') \leqslant \nu(\mathscr{D},E,\pi(P')). \tag{1.4.2}$$

This result motivates considering the adapted situation in order to study the desingularization of a unidimensional distribution rather than the non-adapted case (see [5], [6], [7]).

1.5. [6] is devoted to the proof that if $n = 3$ we can obtain, at least punctually, that $\nu(\mathscr{D},E,P) \leqslant 1$ for all P, after a finite sequence of blowing-ups. [9] partially globalizes this result. The transition $\nu = 1$ to $\nu = 0$ has a special behaviour (see [5] for the case $n = 2$) and we shall treat it in the Appendix.

From now on, we shall assume that $n = 3$, that E is a normal crossings divisor of X and that $\mathscr{D}$ is a unidimensional distribution, multiplicatively irreducible and adapted to E, such that for each closed point P of X we have

$$\nu(\mathscr{D},E,P) = 0. \tag{1.5.1}$$

We can think of $(X,E,\mathscr{D})$ as a final situation after a procedure of reduction of the order by means of blowing-ups. In this way, E may be looked at as the exceptional divisor of the composition of all these blowing-ups.

2. Normal crossings for Sing $(\mathscr{D},\varnothing)$.

2.1. Let $(X,E,\mathscr{D})$ satisfy the assumptions of (1.5). Then $\mathrm{Sing}(\mathscr{D},\varnothing) = \mathrm{Sing}^1(\mathscr{D},\varnothing)$ is a Zariski closed subset of X. Let us assume that

$$\mathrm{Sing}(\mathscr{D},\varnothing) = S_2(\mathscr{D}) \cup S_1(\mathscr{D}) \cup S_0(\mathscr{D}) \tag{2.1.1.}$$

where $S_i(\mathscr{D})$ is the union of the irreducible components of $\mathrm{Sing}(\mathscr{D},\varnothing)$ of dimension i, $i = 0, 1, 2$. Note that we can have $S_2(\mathscr{D}) \neq \varnothing$ since $\mathscr{D}$ is not necessarily multiplicatively irreducible with respect to $\varnothing$. On the other hand, we always have

$$\mathrm{Sing}(\mathscr{D},\varnothing) \subset E \tag{2.1.2}$$

and hence the irreducible components of $S_2(\mathscr{D})$ are also irreducible components of E. Let us call them « dicritical components of $(X,E,\mathscr{D})$ ».

In this paragraph we shall prove that after a finite sequence of quadratic blowing-ups we can assume that Sing $(\mathcal{D},\emptyset)$ and E have normal crossings at the points P with $e(\mathrm{E},\mathrm{P}) \leqslant 2$ and this property is stable under « permissible » blowing-ups at the points P with $e(\mathrm{E},\mathrm{P}) = 1$.

2.2. LEMMA. — *Let Z, Y be closed subschemes of* X *of pure dimension* 1 *and let* $\mathrm{P} \in \mathrm{Z} \cap \mathrm{Y}$ *be a closed point such that both* Z *and* Y *have normal crossings with* E *at* P *and assume that there exists a normal crossings divisor* $\mathrm{E}' \subset \mathrm{E}$ *such that no component of* Z *passing through* P *is contained in* E′ *and each component of* Y *passing through* P *is contained in* E′. *Then* $\mathrm{Z} \cup \mathrm{Y}$ *has normal crossings with* E *at* P.

Proof. — A fixed suited regular system of parameters for the pair (E,Z) at P may be easily modified in order to obtain a regular system of parameters suited for (E,Z) and (E,Y) at P, hence suited for $(\mathrm{E},\mathrm{Z}\cup\mathrm{Y})$ at P.

2.3. PROPOSITION. — *Let* P *be a closed point of* X *and let* $(\mathrm{X}',\mathrm{E}',\mathcal{D}')$ *be the strict transform of* $(\mathrm{X},\mathrm{E},\mathcal{D})$ *by the quadratic blowing-up* $\pi : \mathrm{X}' \to \mathrm{X}$ *centered at* P. *Let us denote by* $\mathrm{S}_1(\mathcal{D})'$ *the strict transform of* $\mathrm{S}_1(\mathcal{D})$ *by* π. *Then we have*

$$\mathrm{S}_1(\mathcal{D}') = \mathrm{S}_1(\mathcal{D})' \cup \mathrm{Y} \tag{2.3.1}$$

where $\mathrm{Y} \subset \pi^{-1}(\mathrm{P})$ *is empty or there are some irreducible components* E_i, $i \in \mathrm{I}$, *of* E *at* P *and a projective line* $\bar{\mathrm{L}}$ in $\pi^{-1}(\mathrm{P})$ *such that*

$$\mathrm{Y} = \bar{\mathrm{L}} \cup \bigcup_{i \in \mathrm{I}} \mathrm{Proj}\,(\mathrm{T_P E}_i) \tag{2.3.2}$$

(hence Y *has normal crossings with* E′ *at each closed point* $\mathrm{P}' \in \pi^{-1}(\mathrm{P})$ *such that* $e(\mathrm{E}',\mathrm{P}') \leqslant 2$).

Remark. — L may coincide with one of the projective lines $\mathrm{Proj}\,(\mathrm{T_P}(\mathrm{E}_i))$.

Proof. — Necessarily $\mathrm{Y} = \cup$ {1-dimensional irreducible components of $\mathrm{S}_1(\mathcal{D}') \cap \pi^{-1}(\mathrm{P})$}. Thus, let us look at Sing $(\mathcal{D}',\emptyset) \cap \pi^{-1}(\mathrm{P})$.

If $e(\mathrm{E},\mathrm{P}) = 0$, then we can take a regular system of parameters (x,y,z) of $\hat{\mathcal{O}}_{\mathrm{X},\mathrm{P}}$ (formal completion of $\mathcal{O}_{\mathrm{X},\mathrm{P}}$) such that $\mathcal{D}_\mathrm{P}$ is generated by $\partial/\partial x$. Now, by (1.3.4), Sing $(\mathcal{D}',\emptyset) \cap \pi^{-1}(\mathrm{P})$ consists in exactly one closed point, hence $\mathrm{Y} = \emptyset$. Assume now that $e(\mathrm{E},\mathrm{P}) = 1$ and let (x,y,z)

be a regular system of parameters of $\mathcal{O}_{X,P}$ suited for (E,P), such that $I(E)_P = x.\mathcal{O}_{X,P}$. If $\nu(\mathcal{D},\varnothing,P) = 0$ then $(x,y,z) \subset \hat{\mathcal{O}_{X,P}}$ may be chosen, in addition to the above properties, to be such that $\mathcal{D}_P$ is generated by $\partial/\partial y$. Now, we can reason as above and hence $Y = \varnothing$. Assume now that $\nu(\mathcal{D},\varnothing,P) = 1$ and that $\mathcal{D}_P$ is generated by

$$D = x\partial/\partial x + D(y)\partial/\partial y + D(z)\partial/\partial z \tag{2.3.3}$$

with

$$\begin{array}{ll} cl^1(D(y)) = \alpha \underline{x} + \beta \underline{y} + \gamma \underline{z}, & \alpha, \beta, \gamma, \delta, \varepsilon, \psi \in k \\ cl^1(D(z)) = \delta \underline{x} + \varepsilon \underline{y} + \psi \underline{z} & \end{array} \tag{2.3.4}$$

where cl^i denotes the image in m^i/m^{i+1}, m = maximal ideal of $\mathcal{O}_{X,P}$, and $\underline{x}$, $\underline{y}$, $\underline{z}$ are the initial forms of x, y, z, respectively. Let us identify $\pi^{-1}(P) = \mathrm{Proj}(T_PX)$ and let $P' = [(1,\zeta,\xi)] \in \pi^{-1}(P)$. Now, a regular system of parameters (x',y',z') of $\mathcal{O}_{X',P'}$ is given by the equations

$$T(1,\zeta,\xi):\ x = x',\quad y = (y'+\zeta)x',\quad z = (z'+\xi)x'. \tag{2.3.5}$$

By (1.3.4), $\nu(D',\varnothing,P') = 1$ iff

$$\alpha + (\beta-1).\zeta + \gamma.\xi = \delta + \varepsilon.\zeta + (\psi-1)\xi = 0 \tag{2.3.6}$$

(otherwise $\nu(\mathcal{D}',\varnothing,P') = 0$). If (2.3.6) does not define a 1-dimensional linear variety L, then we are done, since then Y must be contained in $\mathrm{Proj}(T_PE)$. Otherwise, let $\bar{L}$ be the projective line given by L. We deduce easily that

$$\bar{L} \subset Y \subset \bar{L} \cup \mathrm{Proj}(T_PE). \tag{2.3.7}$$

If $e(E,P) = 2$ or 3, we can reason as above by replacing in (2.3.7) $\mathrm{Proj}(T_PE)$ by $\mathrm{Proj}(T_PE_i)$, where E_i are the components of E through P.

2.4. Corollary. — *There is a finite sequence of quadratic blowing-ups*

$$X = X(0) \xleftarrow{\pi(1)} X(1) \leftarrow \dots \xleftarrow{\pi(N)} X(N) \tag{2.4.1}$$

such that if $(X(i),E(i),\mathcal{D}(i))$, $i = 1,\dots,N$ *is the strict transform of* $(X(i-1),E(i-1)),\mathcal{D}(i-1))$ *and* $(X(0),E(0),\mathcal{D}(0)) = (X,E,\mathcal{D})$, *then :*

1° $\pi(i)$ *is centered at a closed point* $P(i-1)$ *of* $X(i-1)$ *such that* $e(E(i-1),P(i-1)) \leqslant 2$ *and* $S_1(\mathcal{D}(i-1))$ *does not have normal crossings with* $E(i-1)$ *at* $P(i-1)$.

2° $S_1(\mathscr{D}(N))$ *has normal crossings with* $E(N)$ *at each closed point* Q *with* $e(E(N), Q) \leqslant 2$.

Proof. – Let $Z(0) = S_1(\mathscr{D})$, $Y(0) = \varnothing$ and for each $i > 1$ let $Z(i)$ = strict transform of $Z(i-1)$ by $\pi(i)$ and let $Y(i)$ = strict transform of $Y(i-1)$ union the corresponding Y of (2.3.1). One sees inductively that $Z(i)$ does not have normal crossings at $P(i)$, $i < N$. Now, by standard results about desingularization of varieties (see e.g. [11]) in a finite number of steps $Z(N)$ has normal crossings with $E(N)$. Now one can apply Lemma (2.2) to $Z(N)$, $Y(N)$, $E(N)$.

2.5. Since Sing $(\mathscr{D},\varnothing)$ has normal crossings with E iff $S_1(\mathscr{D})$ has normal crossings with E (see (2.1.2)), then, after a finite number of quadratic blowing-ups we can assume that Sing $(\mathscr{D},\varnothing)$ has normal crossings with E at each point with $e(E,P) \leqslant 2$. We shall prove in Paragraph 4 that we can obtain that Sing $(\mathscr{D},\varnothing)$ and E have normal crossings after a finite number of quadratic blowing-ups, even if $e(E,P) = 3$.

2.6. Definition. – *A closed subscheme* Y *of* X *is said to be permissible for* $(X,E,\mathscr{D})$ *iff* Y *is a closed point of* Sing $(\mathscr{D},\varnothing)$ *or* Y *is an irreducible component of* $S_1(\mathscr{D})$ *having normal crossings with* E. (*Note that this definition is more restrictive than the one in* [6] I. (3.4.4)).

2.7. Let P be a closed point of Sing $(\mathscr{D},\varnothing)$ and let Y be $\{P\}$ or an irreducible component of $S_1(\mathscr{D})$ such that $P \in Y$ and such that Y has normal crossings with E at P. (This implies that Y is nonsingular at P). Let us fix a generator D of $\mathscr{D}_P$. Since $P \in \text{Sing}(\mathscr{D},\varnothing)$, for each $h \in I(Y)_P$, $cl^1(D(h))$ depends only of $cl^1(h)$ and thus one can write $cl^1(D(h)) = cl^1(D(cl^1(h)))$. Let us denote by $\text{In}(I(Y)_P)_1$ the k-vector space of the homogeneous elements of degree one of the initial ideal $\text{In}(I(Y)_P) \subset \text{Gr}(\mathscr{O}_{X,P})$ of $I(Y)_P$. One has a well defined k-linear map

$$(2.7.1) \qquad L(D;Y;P) : \text{In}(I(Y)_P)_1 \to \text{In}(I(Y)_P)_1, \quad \Phi \mapsto cl^1(D(\Phi)).$$

If D_1 is another generator of $\mathscr{D}_P$, one has that $D_1 = u.D$, where $u(P) \neq 0$, and $L(D_1;Y;P) = u(P).L(D;Y;P)$.

Note that $L(D;Y;P)$ can also be defined as above if one has the following condition (which corresponds to the fact that Y is an integral variety of $\mathscr{D}$):

$$(2.7.2.) \qquad \nu_Y(D(I(Y)_P)) \geqslant 1.$$

Some properties of the eigenvalues of L(D;Y;P) will serve us to define the final situations in the next paragraph. But first, let us study the effect of a permissible blowing-up over the normal crossings property of Sing $(\mathscr{D},\varnothing)$ with E.

2.8. Proposition. – *Let* P *be a closed point of* Sing $(\mathscr{D},\varnothing)$ *and assume that* Sing $(\mathscr{D},\varnothing)$ *has normal crossings with* E *at* P. *Let* $Y \ni P$ *be a permissible center for* $(X,E,\mathscr{D})$. *Let* $\pi : X' \to X$ *be the blowing-up centered at* Y *and let* $(X',E',\mathscr{D}')$ *be the strict transform of* $(X,E,\mathscr{D})$. *Then we have :*

a) $\nu(\mathscr{D}',E',Q') = 0$ *for each closed point* $Q' \in X'$.

b) Sing $(\mathscr{D}',\varnothing)$ *has normal crossings with* E' *at each closed point* $P' \in X'$ *such that* $\pi(P') = P$ *and* $e(E',P') = 1$.

c) If the eigenvalues of L(D;Y;P) *are distinct, then* Sing $(\mathscr{D}',\varnothing)$ *has normal crossings with* E' *at each closed point* $P' \in X'$ *such that* $\pi(P') = P$.

Proof. – If π is quadratic, then *a)* follows from (1.4.2) and *b)* follows from 2.2 and 2.3. If $e(E',P') \leqslant 2$, then *c)* also follows from 2.3. Now, assume that $e(E',P') = 3$. Taking the notations of the proof of 2.3, we have to prove that $P' \notin \bar{L}$, but if $P' \in \bar{L}$, then $\psi - 1 = 0$, hence $\psi = 1$ and we contradict the assumption of *c*).

Let us assume that π is monoidal. In order to prove *a*), let $Q = \pi(Q')$ and assume first that $e(E,Q) = 1$. Let us fix a regular system of parameters (x,y,z) suited for (E,Y) at Q such that $I(E)_Q = (x)$ and $I(Y)_Q = (x,z)$ (recall that $Y \subset E$). Let us use the notation of (2.3.3) and (2.3.4) for a generator D of $\mathscr{D}_Q$. Since $Y \subset S_1(\mathscr{D})$ we have $\beta = \varepsilon = 0$. Moreover, the eigenvalues of L(D;Y;Q) are 1 and ψ. Assume first that $Q' = [(1,\zeta)] \in \mathrm{Proj}\,(T_QX/T_QY) = \pi^{-1}(Q)$. Then a regular system of parameters (x',y',z') of $\mathcal{O}_{X',Q'}$ is given by the coordinate transformation (2.8.1) which we shall denote by $\mathrm{TM}((1,\zeta);(x,z))$.

$$(2.8.1) \quad \mathrm{TM}((1,\zeta);(x,z)) : x = x',\ y = y',\ z = (z'+\zeta)x'.$$

By (1.3.4), $\mathscr{D}'_{Q'}$ is generated by

$$(2.8.2) \qquad D' = x'\partial/\partial x' + D'(y')\partial/\partial y' + D'(z')\partial/\partial z'$$

where

(2.8.3) x' divides $D'(y')$

$$D'(z') = \delta + (\psi-1)(z'+\zeta) + y'^r\Phi(y',z') + x'\Psi(x',y',z')$$

where $r \geqslant 1$ and either $\Phi = 0$ or $\Phi(0,0) \neq 0$ or $cl^1(\Phi) = \lambda \underline{y}' + \mu \underline{z}'$ with $\mu \neq 0$. Now, *a*) follows immediately from (2.8.2).

Assume now that $Q' = [(0,1)] \in \pi^{-1}(Q)$, then we have the equations

(2.8.4) $\quad TM((0,1);(x,z)) : x = x'z',\ y = y',\ z = z'$

and $\mathscr{D}'_{Q'}$ is generated by

(2.8.5) $\quad D' = (D'(x')/x')x'\partial/\partial x' + D'(y')\partial/\partial y' + (D'(z')/z')z'\partial/\partial z'$

(here $I(E')_{Q'} = (x'z')$), where

$$\text{(2.8.6.)} \quad \begin{aligned} & D'(x')/x' = (1-\psi) - \delta x' + y'^r\Phi(x',y') + z'\Psi(x',y',z') \\ & z' \text{ divides } D'(y') \\ & D'(z')/z' = \psi + \text{terms of order} \geqslant 1 \end{aligned}$$

where $r \geqslant 1$ and either $\Phi = 0$ or $\Phi(0,0) \neq 0$ or $cl^1(\Phi) = \lambda x' + \mu y'$ with $\lambda \neq 0$. Since ψ and $1 - \psi$ are not both zero, *a*) follows from (2.8.6). (Note that in this case $\nu(\mathscr{D}',\varnothing,Q') = 1$).

Similar computations show *a*) in the cases $e(E,Q) = 2.3$.

Let us prove *b*) in the case $e(E,P) = 1$ (we can proceed in a similar way for the cases $e(E,P) = 2,3$). In view of 2.2 and since

$$\text{(2.8.7)} \quad \text{Sing}(\mathscr{D}',\varnothing) = \{\text{strict transform of Sing}(\mathscr{D},\varnothing) - Y\} \cup [\text{Sing}(\mathscr{D}',\varnothing) \cap \pi^{-1}(Y)],$$

it is enough to show that $\text{Sing}(\mathscr{D}',\varnothing) \cap \pi^{-1}(Y)$ has normal crossings with E' at P'. Since $e(E',P') = 1$, by (2.8.3) one has that $P' \in \text{Sing}(\mathscr{D}',\varnothing)$ in the following cases:

1° $\psi - 1 \neq 0$, $\zeta = -\delta/(\psi-1)$. Then $\text{Sing}(\mathscr{D}',\varnothing) \cap \pi^{-1}(Y)$ is given locally at P' by $x' = (\psi-1)z' + y'^r\Phi(y',z') = 0$, which defines a regular curve having normal crossings with E' at P'.

2° $\psi - 1 = \delta = 0$ (no restriction on ζ). Then $\text{Sing}(\mathscr{D}',\varnothing) \cap \pi^{-1}(Y)$ is given locally at P' by $x' = y'^r\Phi(y',z,pr) = 0$. If $\Phi = 0$, then $\pi^{-1}(Y) \subset \text{Sing}(\mathscr{D}',\varnothing)$ locally at P' (hence globally). If $\Phi(0,0) \neq 0$, then one has the « vertical » curve $x' = y' = 0$. If $\Phi(0,0) = 0$, one has two 1-dimensional components $x' = y' = 0$ and $x' = \Phi(y',z') = 0$ each one is regular and $\text{Sing}(\mathscr{D}',\varnothing) \cap \pi^{-1}(Y)$ has normal crossings with E' since $cl^1(\Phi) = \lambda y' + \mu z'$, $\mu \neq 0$. In this last case, one can say something more: there is at most one value of ζ for which this case can occur, since $\Phi = \mu z' + y'(\ldots)$.

Now, let us prove $c)$ in the case $e(\mathrm{E,P}) = 1$ (one can proceed in a similar way for the cases $e(\mathrm{E,P}) = 2,3$). In view of $b)$, it is enough to look at P′ with $e(\mathrm{E',P'}) = 2$. If $\psi - 1 \neq 0$, by (2.8.6) one has that $\mathrm{Sing}(\mathscr{D}',\emptyset) \cap \pi^{-1}(\mathrm{Y})$ is given locally at P′ by $z' = x' = 0$ and hence has normal crossings with E′ at P′.

2.9. *Remark.* – In the case $\psi = 1$, we can show that all the components of $\mathrm{Sing}(\mathscr{D}',\emptyset) \cap \pi^{-1}(\mathrm{Y})$ are nonsingular, but they may not have normal crossings with E′. For instance, if $\mathscr{D}_{\mathrm{P}}$ is generated by

$$\mathrm{D} = x\partial/\partial x + (z-x+y^2z)\,\partial/\partial z \tag{2.9.1}$$

one has that for $\mathrm{P'} = [(0,1)]$, $\mathscr{D}'_{\mathrm{P'}}$ is generated by

$$\mathrm{D'} = (x-y^2)x\partial/\partial x + (1-x+y^2)z\,\partial/\partial z \tag{2.9.2}$$

and $\mathrm{Sing}(\mathscr{D}',\emptyset) \cap \pi^{-1}(\mathrm{Y}) = (x=z=0) \cup (x-y^2=z=0)$ which are two nonsingular components, tangent one to another.

3. Final forms.

3.1. Let $\mathrm{P} \in \mathrm{X}$ be a closed point such that $e(\mathrm{E,P}) = 1$ and $\nu(\mathscr{D},\emptyset,\mathrm{P}) = 1$ (recall the assumptions of 1.5). Let us consider the k-linear map

$$\mathrm{L(D;\{P\};P)} : \mathrm{In}\,(\mathrm{I(P)_P})_1 \to \mathrm{In}\,(\mathrm{I(P)_P})_1 \tag{3.1.1}$$

given in 2.7. Since P is a closed point, $\mathrm{In}\,(\mathrm{I(P)_P})_1$ is exactly the k-vector space of the linear homogeneous forms in $\mathrm{Gr}\,(\mathcal{O}_{\mathrm{X,P}})$. Since $e(\mathrm{E,P}) = 1$, E is regular at P and we have $\mathrm{In}\,(\mathrm{I(E)_P})_1 \subset \mathrm{In}\,(\mathrm{I(P)_P})_1$. Moreover, since $\mathscr{D} \subset \Xi_{\mathrm{X}}[\mathrm{E}]$, $\mathrm{In}\,(\mathrm{I(E)_P})_1$ is invariant under L(D;{P};P) and the corresponding eigenvalue $\lambda_1(\mathrm{D})$ is nonzero. Indeed, if $\lambda_1(\mathrm{D}) = 0$ then $\nu(\mathscr{D},\emptyset,\mathrm{P}) = 0$ (recall that $\nu(\mathscr{D},\mathrm{E,P}) = 0$). Let $\lambda_2(\mathrm{D})$, $\lambda_3(\mathrm{D})$ be the remaining eigenvalues of L(D;{P};P). The set

$$\Lambda(\mathscr{D};\mathrm{E;P}) = \{\lambda_2(\mathrm{D})/\lambda_1(\mathrm{D}), \lambda_3(\mathrm{D})/\lambda_1(\mathrm{D})\} \tag{3.1.2}$$

is well-defined and does not depend on the generator D of $\mathscr{D}$.

3.2. Definition. – *Let* $\mathrm{P} \in \mathrm{X}$ *be a closed point such that* $e(\mathrm{E,P}) = 1$ *and* $\nu(\mathscr{D},\emptyset,\mathrm{P}) = 1$ (*we always assume* $\lambda(\mathscr{D},\mathrm{E,P}) = 0$). *The point* P *is* *said to be « simple » for* $(X,E,\mathscr{D})$ *iff*

$$\Lambda(\mathscr{D};E;P) \cap \mathbf{Q}_+ = \varnothing \tag{3.2.1}$$

where $\mathbf{Q}_+ \subset k$ *denotes the strictly positive rational numbers.*

3.3. Definition. — *Let* $P \in X$ *be a closed point such that* $e(E,P) \geqslant 2$ *and* $\nu(\mathscr{D},\varnothing,P) = 1$. *The point* P *is said to be a « simple corner » for* $(X,E,\mathscr{D})$ *iff there are two components* E_1 *and* E_2 *of* E *passing through* P *such that if* $Y = E_1 \cap E_2$, *then the k-linear map* $L(D;Y;P)$ (*see* (2.7.1) *and* (2.7.2)) *has two distinct eigenvalues* λ_1 *and* λ_2 ; *with* $\lambda_1 \neq 0$ *and* $\lambda_2/\lambda_1 \notin \mathbf{Q}_+$. (*Note that this property does not depend on the choice of the generator* D of $\mathscr{D}$).

3.4. Definition. — $(X,E,\mathscr{D})$ *is said to be in the « first final situation » at a closed point* $P \in X$ *iff* $\operatorname{Sing}(\mathscr{D},\varnothing)$ *has normal crossings with* E *at* P *in the case* $e(E,P) = 1$ *and one of the following properties holds :*

a) $\nu(\mathscr{D},\varnothing,P) = 0$, *i.e.* P *is a regular point of* $\mathscr{D}$.

b) P *is a simple point for* $(X,E,\mathscr{D})$.

c) P *is a simple corner for* $(X,E,\mathscr{D})$.

3.5. *Remark.* — 1. (*Dicritical situation*). If E_1 is a component of E such that $E_1 \subset \operatorname{Sing}(\mathscr{D},\varnothing)$, then each closed point of E_1 is a simple point or a simple corner (in both cases there is only one eigenvalue different from zero).

2. If $e(E,P) = 0$ (hence $\nu(\mathscr{D},\varnothing,P) = \nu(\mathscr{D},E,P) = 0$) and $\pi : X' \to X$ is the quadratic blowing-up centered at P, then exactly one point of $\pi^{-1}(P)$ is a simple point of $(X',E',\mathscr{D}')$, the other ones being regular points of $\mathscr{D}'$.

3. We shall see later (paragraph 5) that the above « first final situation » generalizes the known final situation for $\dim X = 2$ (see [12], [10]).

4. In paragraph 4 we shall see that the above first final situation may be reached punctually by blowing-ups, but it is not possible to reach it for all the points $P \in X$ at the same time.

3.6. Theorem. — *Let* Y *be a permissible center for* $(X,E,\mathscr{D})$ *and let* $\pi : X' \to X$ *be the blowing-up centered at* Y. *Then :*

a) If $P \in Y$ *is a simple point, then there is exactly one point* $P' \in \pi^{-1}(P)$ *with* $e(E',P') = 1$ *and* $\nu(\mathcal{D},\varnothing,P') = 1$. *Moreover,* P' *is also a simple point.*

b) If $P \in Y$ *is a simple point,* $P' \in \pi^{-1}(P)$, $e(E',P') \geqslant 2$ *and* $\nu(\mathcal{D}',\varnothing,P') = 1$, *then* P' *is a simple corner for* $(X',E',\mathcal{D}')$.

c) If $P \in Y$ *is a simple corner and* $P' \in \pi^{-1}(P)$, *then* $\nu(\mathcal{D}',\varnothing,P') = 0$ *or* P' *is a simple corner for* $(X',E',\mathcal{D}')$.

Proof. — Let us assume first that $Y = \{P\}$, hence π is a quadratic blowing-up.

1. *a)* Since P is a simple point, one can take a regular system of parameters (x,y,z) of $\mathcal{O}_{X,P}$ suited for (E,P), such that $I(E)_P = (x)$ and such that there is a generator D of $\mathcal{D}_P$ having one of the following properties :

(3.6.1) A) $D(x)/x = 1$, $cl^1(D(y)) = \beta.\underline{y}$, $cl^1(D(z)) = \psi.\underline{z}$, where $\beta, \psi \notin \mathbf{Q}_+$.

(3.6.2) B) $D(x)/x = 1$, $cl^1(D(y)) = \beta.\underline{y}$, $cl^1(D(z)) = \underline{y} + \beta\underline{z}$, where $\beta \notin \mathbf{Q}_+$

(see [4] and [9]). Let $P' \in \pi^{-1}(P)$ be such that $e(E',P') = 1$. Then $P' = [(1,\zeta,\xi)]$ and we have the equations $T(1,\zeta,\xi)$. (See (2.3.5)). A generator D′ of $\mathcal{D}'_{P'}$ satisfies

$$(3.6.3) \qquad \begin{aligned} D'(x')/x' &= 1 \\ D'(y') &= (\beta-1)(y'+\zeta) + x'(\ldots) \\ D'(z') &= (\psi-1)(z'+\xi) + x'(\ldots) \end{aligned}$$

in the case A, and

$$(3.6.4) \qquad \begin{aligned} D'(x')/x' &= 1 \\ D'(y') &= (\beta-1)(y'+\zeta) + x'(\ldots) \\ D'(z') &= (y'+\zeta) + (\beta-1)(z'+\xi) + x'(\ldots) \end{aligned}$$

in the case B. Now, if $(\zeta,\xi) \neq (0,0)$, then $\nu(\mathcal{D}',\varnothing,P') = 0$. If $(\zeta,\xi) = (0,0)$, then P′ is a simple point of the same type (A or B) as P. Note that in general one has to make a coordinate change $y_1 = y' + \lambda x'$, $z_1 = z' + \mu x'$ in order to obtain property A or B.

1. *b*) With the notation of 1. *a*), let us assume now that $e(\mathrm{E}',\mathrm{P}') \geqslant 2$ (hence $e(\mathrm{E}',\mathrm{P}') = 2$). Then $\mathrm{P}' = [(0,1,\xi)]$ or $\mathrm{P}' = [(0,0,1)]$. In the first case we have the equations

$$(3.6.5)\quad \mathrm{T}(0,1,\xi): \quad x = x'y', \qquad y = y', \qquad z = (z'+\xi)y'.$$

A generator D′ of $\mathscr{D}'_{\mathrm{P}'}$ satisfies

$$(3.6.6)\qquad \begin{aligned} \mathrm{D}'(x')/x' &= 1 - \beta + y'(\ldots) \\ \mathrm{D}'(y')/y' &= \beta + y'(\ldots) \\ \mathrm{D}'(z') &= (\psi-\beta)(z'+\xi) + y'(\ldots) \end{aligned}$$

in case A, and

$$(3.6.7)\qquad \begin{aligned} \mathrm{D}'(x')/x' &= 1 - \beta + y'(\ldots) \\ \mathrm{D}'(y')/y' &= \beta + y'(\ldots) \\ \mathrm{D}'(z') &= 1 + y'(\ldots) \end{aligned}$$

in case B. Assume that $\nu(\mathscr{D}',\varnothing,\mathrm{P}') = 1$ (this occurs iff $(\psi-\beta)\xi = 0$ in case A and never in case B), then the eigenvalues of $\mathrm{L}(\mathrm{D}';\mathrm{Z}';\mathrm{P}')$, where $\mathrm{I}(\mathrm{Z}')_{\mathrm{P}'} = (x',y')$, are $1 - \beta$ and β. Since $\beta/(1-\beta) \notin \mathbf{Q}_+$, we have a simple corner.

Assume now that $\mathrm{P}' = [(0,0,1)]$. Then we have the equations:

$$(3.6.8)\quad \mathrm{T}(0,0,1): \quad x = x'z', \qquad y = y'z', \qquad z = z'.$$

A generator D′ of $\mathscr{D}'_{\mathrm{P}'}$ satisfies

$$(3.6.9)\qquad \begin{aligned} \mathrm{D}'(x')/x' &= 1 - \psi + z'(\ldots) \\ \mathrm{D}'(y') &= (\beta-\psi).y' + z'(\ldots) \\ \mathrm{D}'(z')/z' &= \psi + z'(\ldots) \end{aligned}$$

in case A, and

$$(3.6.10)\qquad \begin{aligned} \mathrm{D}'(x')/x' &= 1 - \beta - y' + z'(\ldots) \\ \mathrm{D}'(y') &= -\, y'^2 + z'(\ldots) \\ \mathrm{D}'(z')/z' &= \beta + y' + z'(\ldots) \end{aligned}$$

in case B. We always have $\psi(\mathscr{D}',\varnothing,\mathrm{P}') = 1$. The eigenvalues of $\mathrm{L}(\mathrm{D}';(x'=z'=0);\mathrm{P}')$ are $1 - \psi$ and ψ in case A (resp. $1 - \beta, \beta$ in case B) and we can reason as above.

1. c) P is a simple corner iff there is a regular system of parameters (x,y,z) of $\mathcal{O}_{X,P}$ suited for (E,P) and a generator D of $\mathcal{D}_P$ having the following properties:

$$\text{(3.6.11)}\quad \begin{array}{l} 1^\circ\ (xy) \supset I(E)_P \\ 2^\circ\ D(x)/x = 1 \\ 3^\circ\ D(y)/y = \beta + \text{terms of order} \geqslant 1, \text{ where } \beta \notin \mathbf{Q}_+ . \end{array}$$

Now, let $P' = [(1,\zeta,\xi)] \in \pi^{-1}(P)$, in the coordinates obtained from (x,y,z). Applying $T(1,\zeta,\xi)$, a generator D′ of $\mathcal{D}'_{P'}$ satisfies

$$\text{(3.6.12)}\quad \begin{array}{l} D'(x')/x' = 1 \\ D'(y') = (\beta - 1 + x'(\ldots))(y'+\zeta) . \end{array}$$

If $\nu(\mathcal{D}',\varnothing,P') = 1$, then $\zeta = 0$ and D′ satisfies (3.6.11), hence P′ is a simple corner. Let $P' = [(0,1,\zeta)] \in \pi^{-1}(P)$. Applying $T(0,1,\zeta)$, a generator D′ of $\mathcal{D}'_{P'}$ satisfies

$$\text{(3.6.13)}\quad \begin{array}{l} D'(x')/x' = 1 - \beta + y'(\ldots) \\ D'(y')/y' = \beta + y'(\ldots) . \end{array}$$

Assume that $\nu(\mathcal{D}',\varnothing,P') = 1$. Since $\beta/(1-\beta) \notin \mathbf{Q}_+$, dividing by $D'(x')/x'$ we obtain (3.6.11), hence P′ is a simple corner. Let $P' = [(0,0,1)] \in \pi^{-1}(P)$. Applying $T(0,0,1)$, a generator D′ of $\mathcal{D}'_{P'}$ satisfies

$$\text{(3.6.14)}\quad \begin{array}{l} D'(x')/x' = 1 - \psi + z'(\ldots) \\ D'(y')/y' = \beta - \psi + z'(\ldots) \\ D'(z')/z' = \psi + z'(\ldots) \end{array}$$

for some $\psi \in k$. If $\psi = 1$, dividing D′ by $D'(z')/z'$ and interchanging x' and z', we obtain (3.6.11). If $\psi \neq 1$, then either $(\beta-\psi)/(1-\psi)$ or $\psi/(1-\psi)$ is not in $\mathbf{Q}_+$. Now, dividing D′ by $D'(x')/x'$ and interchanging z' and y' if $\psi/(1-\psi) \notin \mathbf{Q}_+$, we obtain (3.6.11) and hence P′ is a simple corner.

Let us assume now that Y is a permissible curve passing through P.

2. a), b). Let us assume first that P is a simple point. Since $Y \subset E$, we can take a regular system of parameters (x,y,z) suited for (E,Y) at P such that $I(E)_P = (x)$, $I(Y)_P = (x,y)$ and a generator D of $\mathcal{D}_P$ satisfying

$$\text{(3.6.15)}\quad c)\ \begin{array}{l} D(x')/x = 1 \\ cl^1(D(y)) = \beta \underline{y},\ \beta \notin \mathbf{Q}_+ \\ cl^1(D(z)) = \varepsilon \underline{y} \end{array}$$

(note that $Y \subset S_1(D)$ and that, possibly, $\beta = 0$, $\varepsilon = 1$). Let $P' = [(1,\zeta)] \in \pi^{-1}(P) = \mathrm{Proj}(T_PX/T_PY)$. Then we have the following equations for π at P':

$$\text{(3.6.16)} \quad TM((1,\zeta);(x,y)) : x = x', \ y = (y'+\zeta)x', \ z = z'.$$

A generator D' of $\mathscr{D}'_{P'}$ satisfies

$$\text{(3.6.17)} \quad \begin{aligned} D'(x')/x' &= 1 \\ D'(y') &= (\beta-1)(y'+\zeta) + z'.\Phi(y'+\zeta,z') \\ &\qquad + x'\Psi(x',y'+\zeta,z') \\ D'(z') &= x'\psi(x',y'+\zeta,z') \end{aligned}$$

where $\psi(0,0,0) = 0$. If $\zeta \neq 0$, then $\nu(\mathscr{D}',\varnothing,P') = 0$. If $\zeta = 0$, making the coordinate change

$$y'_1 = y' + (\Phi(0,0,0)/(\beta-1))z' + (\Psi(0,0,0)/(\beta-2))x'$$

(note that $\beta - 1 \neq 0$ and $\beta - 2 \neq 0$ since $\beta \notin \mathbf{Q}_+$) we see that P' is a simple point with $\Lambda(\mathscr{D}';E';P') = \{\beta-1,0\}$. Now, let $P' = [(0,1)] \in \pi^{-1}(P)$. We have the equations

$$\text{(3.6.18)} \quad TM((0,1);(x,y)) : x = x'y', \ y = y', \ z = z'.$$

A generator D' of $\mathscr{D}'_{P'}$ satisfies

$$\text{(3.6.19)} \quad \begin{aligned} D'(x')/x' &= 1 - \beta - z'\Phi(x',z') - y'\Psi(x',y'z') \\ D'(y')/y' &= \beta + z'\Phi(x',z') + y'\Psi(x',y',z') \\ D'(z') &= y'(\ldots) \end{aligned}$$

and we have a simple corner since $\beta/(1-\beta) \notin \mathbf{Q}_+$.

2. *c*) Let us assume that P is a simple corner. We can take a regular system of parameters (x,y,z) suited for (E,Y) at P, satisfying (3.6.11) and such that $I(Y)_P = (x,y)$, (x,z) or (y,z). If $I(Y)_P = (x,y)$, then we can apply the above computations (of 2 *a*), b)) and we obtain simple corners or regular points for $(X',E',\mathscr{D}')$. Let $I(Y)_P = (x,z)$ (this implies $\beta = 0$). After $TM((1,\zeta);(x,z))$ (see (2.8.1)), we have a generator D' of $\mathscr{D}'_{P'}$ satisfying

$$\text{(3.6.20)} \quad \begin{aligned} D'(x')/x' &= 1 \\ D'(y')/y' &= x'(\ldots) \end{aligned}$$

and if $\nu(\mathscr{D}',\emptyset,P') = 1$, then P′ is a simple corner. After TM((0,1); (x,z)) we have a generator D′ of $\mathscr{D}'_{P'}$ satisfying

$$\begin{aligned} D'(x')/x' &= 1 - \psi + \text{terms of order } \geq 1 \\ D'(y')/y' &= z'(\ldots) \qquad\qquad (3.6.21) \\ D'(z')/z' &= \psi + \text{terms of order } \geq 1 \end{aligned}$$

for some $\psi \in k$. If $\psi \neq 1$, dividing by $D'(x')/x'$ we obtain the conditions of (3.6.11), hence P′ is a simple corner. If $\psi = 1$, dividing by $D'(z')/z'$ and interchanging x' and z', we obtain (3.6.11). Finally, $I(Y)_P = (y,z)$ is not possible since $D(x)/x = 1$. This ends the proof of the theorem.

3.7. Corollary. — *If* $(X,E,\mathscr{D})$ *is in the first final situation at* $P \in X$ *and* $\pi : X' \to X$ *is a blowing-up with a permissible center, then the strict transform* $(X',E',\mathscr{D}')$ *is also in the first final situation for each* $P' \in X'$ *such that* $\pi(P') = P$.

4. Reduction to the final situation.

4.1. In this paragraph we shall construct a local invariant Inv $(\mathscr{D},E,P)$ which decreases strictly each time a permissible center containing P is blown-up. It also decreases strictly after finitely many steps of a global process. The property Inv $(\mathscr{D},E,P) = 0$ means « first final situation » and it will be reached « punctually » after a finite number of steps.

4.2. Let $(X,E,\mathscr{D})$ be as in 1.5 and let P be a closed point such that $P \in \mathrm{Sing}(\mathscr{D},\emptyset)$. Let us fix a component E_1 of E, $P \in E_1$, such that the order $\psi(\mathscr{D}(I(E_1)_P))$ is 1. Then, there is a generator D of $\mathscr{D}_P$ such that if $I(E_1)_P = f.\mathcal{O}_{X,P'}$, then

$$cl^0(D(f)/f) = 1 \in k \qquad (4.2.1)$$

(this property does not depend of the choice of f). Let $\Phi(T) \in k[T]$ be the characteristic polynomial of the k-linear map $L(D;\{P\};P)$ of (2.7.1). Let $\{\alpha_i\}_{i=1,2,3}$ be the roots of $\Phi(T) = 0$, counted with multiplicities, and let us assume that $\alpha_1 = 1$ (this is always possible in view of (4.2.1)). Let $J = \{i;\alpha_i \in \mathbf{Q}_+\}$ and let us define the invariant $c(\mathscr{D},E,E_1;P)$ by

$$c(\mathscr{D},E,E_1;P) = \# J - 1 \qquad (4.2.2)$$

where $\#$ = number of elements. The above invariant does not depend on the choice of D with property (4.2.1). Note that $c(\mathscr{D},E,E_1;P) \geq 0$

and P is a simple point iff $e(\mathrm{E,P}) = 1$ and $c(\mathscr{D},\mathrm{E,E;P}) = 0$. Moreover, if $e(\mathrm{E,P}) \geqslant 2$ and $c(\mathscr{D},\mathrm{E,E_1;P}) = 0$ for some $\mathrm{E_1}$ (having the property $\nu(\mathrm{D(I(E_1)_P)}) = 1$, then P is a simple corner. Let us define the invariant $c(\mathscr{D},\mathrm{E,P})$ by

(4.2.3) $c(\mathscr{D},\mathrm{E,P}) = \min\{c(\mathscr{D},\mathrm{E,E_1;P}); \mathrm{E_1}$ is an irreducible component of $\mathrm{E, P} \in \mathrm{E_1}$ and $\nu(\mathscr{D}(\mathrm{I(E_1)_P})) = 1\}$.

4.3. Assume now that $e(\mathrm{E,P}) = 1$. Let us define

(4.3.1) $$h = \min\{p \in \mathbf{Z}_+;\ p\alpha_i \in \mathbf{Z}_+ \text{ for all } i \in \mathrm{J})$$

with the above notation). In this case, the invariant $n(\mathscr{D},\mathrm{E,P})$ is defined to be

(4.3.2) $$n(\mathscr{D},\mathrm{E,P}) = h \sum_{i \in \mathrm{J}} \alpha_i .$$

Assume that $e(\mathrm{E,P}) \geqslant 2$ and $\mathrm{Y} = \cap \mathrm{E}_i$, where E_i runs over the irreducible components of E passing through P. Y is a curve or $\mathrm{Y} = \{\mathrm{P}\}$. Let $\mathrm{E_1}$ be a component of E passing through P such that $\nu(\mathrm{D(I(E_1)_P)}) = 1$ and let us fix a generator D of $\mathscr{D}_\mathrm{P}$ satisfying (4.2.1). Let $\{\beta_i\}_{i=1,\ldots,e(\mathrm{E,P})}$ be the roots of the characteristic polynomial of $\mathrm{L(D;Y;P)}$ counted with multiplicities. Assume $\beta_1 = 1$. Let $\mathrm{J} = \mathrm{J}(\mathscr{D},\mathrm{E,E_1,Y;P}) = \{i; \beta_i \in \mathbf{Q}_+\}$. As in (4.2.2), let us define

(4.3.3) $$c(\mathscr{D},\mathrm{E,E_1,Y;P}) = \#\,\mathrm{J} - 1$$

and

(4.3.4) $c(\mathscr{D},\mathrm{E,Y;P}) = \min\{c(\mathscr{D},\mathrm{E,E_1,Y;P}); \mathrm{E_1}$ is an irreducible component of E at P with $\nu(\mathscr{D}(\mathrm{I(E_1)_P})) = 1\}$.

We always have $c(\mathscr{D},\mathrm{E,Y;P}) \leqslant e(\mathrm{E,P}) - 1$ and the point P is a simple corner iff $c(\mathscr{D},\mathrm{E,Y;P}) < e(\mathrm{E,P}) - 1$. (Note also that if $e(\mathrm{E,P}) = 3 = \dim \mathrm{X}$, then $c(\mathscr{D},\mathrm{E,Y;P}) = c(\mathscr{D},\mathrm{E,P})$). As in (4.3.1), let us define

(4.3.5) $$h = h(\mathscr{D},\mathrm{E,E_1,Y;P}) = \min\{p \in \mathbf{Z}_+;\ p\beta_i \in \mathbf{Z}_+ \text{ for all } i \in \mathrm{J}\},$$

and

(4.3.6) $$n(\mathscr{D},\mathrm{E,E_1,Y;P}) = h \sum_{i \in \mathrm{J}} \beta_i .$$

If $e(\mathrm{E,P}) \geqslant 2$ and P is not a simple corner, then for any two components E_1 and E_2 of E such that $\nu(\mathscr{D}(\mathrm{I}(\mathrm{E}_1)_\mathrm{P})) = (\mathscr{D}(\mathrm{I}(\mathrm{E}_2)_\mathrm{P})) = 1$, we have

$$\text{(4.3.7)} \quad \begin{aligned} &c(\mathscr{D},\mathrm{E},\mathrm{E}_1,\mathrm{Y};\mathrm{P}) = c(\mathscr{D},\mathrm{E},\mathrm{E}_2,\mathrm{Y};\mathrm{P}) = e(\mathrm{E,P}) - 1 \\ &n(\mathscr{D},\mathrm{E},\mathrm{E}_1,,\mathrm{Y};\mathrm{P}) = n(\mathscr{D},\mathrm{E},\mathrm{E}_2,\mathrm{Y};\mathrm{P}) \end{aligned}$$

and in this case we shall define

$$\text{(4.3.8)} \qquad n(\mathscr{D},\mathrm{E,P}) = n(\mathscr{D},\mathrm{E},\mathrm{E}_1,\mathrm{Y};\mathrm{P}).$$

4.4. Definition. – *Let* P *be a closed point of* Sing $(\mathscr{D},\varnothing)$. *The invariant* Inv $(\mathscr{D},\mathrm{E,P})$ *is defined to be*

$$\text{(4.4.1)} \quad \mathrm{Inv}(\mathscr{D},\mathrm{E,P}) = (c(\mathscr{D},\mathrm{E,P}),\ 3 - e(\mathrm{E,P}),\ n(\mathscr{D},\mathrm{E,P})) \in \mathbf{N}^3,$$

if P *is neither a simple point nor a simple corner. Otherwise*

$$\text{(4.4.2)} \qquad \mathrm{Inv}(\mathscr{D},\mathrm{E,P}) = (0,0,0) \in \mathbf{N}^3.$$

Two such an invariants will always be compared in the lexicographic order of $\mathbf{N}^3$.

4.5. Theorem. – *Let* Y *be a permissible center for* $(\mathrm{X,E},\mathscr{D})$, *let* $\pi : \mathrm{X}' \to \mathrm{X}$ *be the blowing-up centered at* Y *and let* $\mathrm{P} \in \mathrm{Y}$ *be a closed point of* Sing $(\mathscr{D},\varnothing)$ *which is neither a simple point nor a simple corner. Then for each closed point* $\mathrm{P}' \in \mathrm{Sing}(\mathscr{D}',\varnothing) \cap \pi^{-1}(\mathrm{P})$

$$\text{(4.5.1)} \qquad \mathrm{Inv}(\mathscr{D}',\mathrm{E}',\mathrm{P}') < \mathrm{Inv}(\mathscr{D},\mathrm{E,P}) \quad (\textit{strictly}).$$

Proof. – *a*) Case $e(\mathrm{E,P}) = 1$. Then there is a regular system of parameters (x,y,z) of $\theta_{\mathrm{X,P}}$ suited for $(\mathrm{E,Y})$ and a generator D of $\mathscr{D}_\mathrm{P}$ satisfying

$$\text{(4.5.2)} \qquad \mathrm{I(E)_P} = (x);\ \mathrm{I(Y)_P} = (x,y) \text{ or } (x,y,z)$$

$$\text{(4.5.3)} \quad \begin{aligned} &\mathrm{D}(x)/x = 1 \\ &\mathrm{D}(y) = \alpha x + \beta y + \text{terms of order} \geqslant 2 \\ &\mathrm{D}(z) = \delta x + \varepsilon y + \psi z + \text{terms of order} \geqslant 2. \end{aligned}$$

With the notation of 4.2 and 4.3, we have $(\alpha_1,\alpha_2,\alpha_3) = (1,\beta,\psi)$ and either $c = c(\mathscr{D},\mathrm{E,P}) = 1$ (iff $\beta \in \mathbf{Q}_+ \ni \psi$ or $\beta \notin \mathbf{Q}_+ \ni \psi$), or $c(\mathrm{D,E,P}) = 2$

(iff $\beta \in \mathbf{Q}_+ \ni \psi$). Moreover, if $c = 1$ and $\beta = p/q \in \mathbf{Q}_+$ (resp. $\psi = r/s$) with g.c.d. $(p,q) = 1$ (resp. g.c.d. $(r,s) = 1$), then $n = n(\mathscr{D},\mathrm{E},\mathrm{P}) = p + q$ (resp. $r+s$). If $c = 2$ and $\beta = p/q$, $\psi = r/s$, where g.c.d. $(p,q) =$ g.c.d. $(r,s) = 1$, let $m =$ g.c.d. (q,s), $q = mq_1$, $s = ms_1$. Then

$$(4.5.4) \qquad n = m.q_1.s_1 + qs_1 + rq_1 .$$

Let us assume first that π is quadratic. Let $\mathrm{P}' = [(1,\zeta,\xi)] \in \pi^{-1}(\mathrm{P})$. Applying $\mathrm{T}(1,\zeta,\xi)$, a generator D′ of $\mathrm{D}'_{\mathrm{P}'}$ satisfies

$$(4.5.5) \qquad \begin{aligned} \mathrm{D}'(x')/x' &= 1 \\ \mathrm{D}'(y') &= \alpha + \alpha' x' + (\beta-1)(y'+\zeta) + \text{terms of order } \geqslant 2 \\ \mathrm{D}'(z') &= \delta + \delta' x' \\ &\quad + \varepsilon(y'+\zeta) + (\psi-1)(z'+\xi) + \text{terms of order } \geqslant 2. \end{aligned}$$

Since $\mathrm{P}' \in \operatorname{Sing}(\mathrm{D}',\varnothing)$, we have $\alpha + (\beta-1)\zeta = \delta + \varepsilon\zeta + (\psi-1)\xi = 0$. The new roots of the characteristic polynomial are $(1,\beta-1,\psi-1)$. We have $c' = c(\mathscr{D}',\mathrm{E}',\mathrm{P}') \leqslant c$ and if $c' = c$, then

$$(4.5.6) \qquad \begin{aligned} n' = n(\mathscr{D}',\mathrm{E}',\mathrm{P}') &\leqslant p + (p-q) < n && \text{if } c = 1 \\ n' &\leqslant mq_1s_1 + ps_1 + rq_1 - 2mq_1s_1 < n && \text{if } c = 2 \end{aligned}$$

and (4.5.1) is verified. Let $\mathrm{P}' = [(0,1,\xi)] \in \pi^{-1}(\mathrm{P})$. Applying $\mathrm{T}(0,1,\xi)$, a generator D′ of $\mathscr{D}'_{\mathrm{P}'}$ satisfies

$$(4.5.7) \qquad \begin{aligned} \mathrm{D}'(x')/x' &= 1 - \beta + \text{terms of order } \geqslant 1 \\ \mathrm{D}'(y')/y' &= \beta + \text{terms of order } \geqslant 1 \\ \mathrm{D}'(z') &= \varepsilon + (\psi-\beta)\xi + (\delta-\alpha\xi)x' + \varepsilon' y' + (\psi-\beta)z' \\ &\quad + \text{terms of order } \geqslant 2 \end{aligned}$$

(we have $\varepsilon + (\psi-\beta)\xi = 0$). If $\beta = 0$, clearly $c' = c$. If $\beta \neq 0$, we have that $((1-\beta)/\beta,(\psi-\beta)/\beta) \in \mathbf{Q}_+^2$ implies that $(\beta,\psi) \in \mathbf{Q}_+^2$, hence $c' \leqslant c$. Now, (4.5.1) follows from the fact that $e(\mathrm{E}',\mathrm{P}') > e(\mathrm{E},\mathrm{P})$. Let $\mathrm{P}' = [(0,0,1)] \in \pi^{-1}(\mathrm{P})$. Applying $\mathrm{T}(0,0,1)$, a generator D′ of $\mathscr{D}'_{\mathrm{P}'}$ satisfies

$$(4.5.8) \qquad \begin{aligned} \mathrm{D}'(x')/x' &= 1 - \psi + \text{terms of order } \geqslant 1 \\ \mathrm{D}'(y') &= \alpha x' + (\beta-\psi)y' + \gamma z' + \text{terms of order } \geqslant 2 \\ \mathrm{D}'(z')/z' &= \psi + \text{terms of order } > 1. \end{aligned}$$

The roots of the characteristic polynomial are $(1-\psi, \beta-\psi, \psi)$ and we can reason as above.

Let us assume now that π is monoidal. Note that $\psi = 0$, since $\gamma \subset S_1(\mathscr{D})$. Let $P' = [(1,\zeta)] \in \pi^{-1}(P)$. Applying $TM((1,\zeta);(x,y))$, a generator D' of $\mathscr{D}'_{P'}$ satisfies

$$(4.5.9)\quad \begin{aligned} &D'(x')/x' = 1 \\ &D'(y') = \alpha + \alpha' x' + (\beta - 1)(y' + \zeta) + \gamma' z' + \text{terms of order} \geqslant 2 \\ &D'(z')/x'(\ldots) \end{aligned}$$

$(\alpha + (\beta - 1)\zeta = 0)$. Clearly $c' \leqslant c = 1$. If $c' = 1$, $\beta = p/q$, g.c.d $(p.q) = 1$, then $n' \leqslant p < p + q = n$ and (4.5.1) follows immediately. Let $P' = [(0,1)] \in \pi^{-1}(P)$. Applying $TM((0,1);(x,y))$, a generator D' of $\mathscr{D}'_{P'}$ satisfies

$$(4.5.10)\quad \begin{aligned} D'(x')/x' &= 1 - \beta + \text{terms of order} \geqslant 1 \\ D'(y')/y' &= \beta + \text{terms of order} \geqslant 1 \\ D'(z') &= y'(\ldots). \end{aligned}$$

One has $c' \leqslant c = 1$ and $e(E',P') = 2 > e(E,P) = 1$, hence (4.5.1).

b) Case $e(E,P) = 2$. There is a regular system of parameters (x,y,z) of $\mathcal{O}_{X,P}$ suited for (E,Y) and a generator D of $\mathscr{D}_P$ satisfying

$$(4.5.11)\quad I(E)_P = (xy)\,;\; I(Y)_P = (x,y),\ (x,z) \text{ or } (x,y,z),$$

$$(4.5.12)\quad \begin{aligned} D(x)/x &= 1 \\ D(y)/y &= \beta + \text{terms of order} \geqslant 1 \\ D(z) &= \delta x + \varepsilon y + \psi z + \text{terms of order} \geqslant 2. \end{aligned}$$

Since P is not a simple corner, we have $\beta \in \mathbf{Q}_+$. If $\beta = p/q$, g.c.d $(p,q) = 1$, then $n = p + q$.

Let us assume first that π is quadratic. Let $P' = [(1,\zeta,\xi)] \in \pi^{-1}(P)$. Applying $T(1,\zeta,\xi)$, a generator D' of $\mathscr{D}'_{P'}$ satisfies

$$(4.5.13)\quad \begin{aligned} D'(x')/x' &= 1 \\ D'(y') &= (\beta - 1 + x'(\ldots))(y' + \zeta) \\ D'(z') &= \delta + \delta' x' + \varepsilon(y' + \zeta) + (\psi - 1)(z' + \xi) \\ &\qquad + \text{terms of order} \geqslant 2 \end{aligned}$$

$((\beta - 1)\zeta = \delta + \varepsilon\zeta + (\psi - 1)\xi = 0)$. The roots of the characteristic polynomial of $L(D';\{P'\};P')$ are $(1, \beta - 1, \psi - 1)$, hence $c' \leqslant c$. If $c' = c$, then $\beta - 1 \neq 0$, hence $\zeta = 0$ and $e(E',P') = e(E,P)$. In this case

$n' \leqslant p < p + q = n$. Now (4.5.1) is straightforward. Let $P' = [(0,1,\xi)] \in \pi^{-1}(P)$. Applying $T(0,1,\zeta)$, a generator D′ of $\mathscr{D}'_{P'}$ satisfies

$$
\begin{aligned}
& D'(x')/x' = 1 - \beta + \text{terms of order} \geqslant 1 \\
(4.5.14) \qquad & D'(y')/y' = \beta + \text{terms of order} \geqslant 1 \\
& D'(z') = \delta x' + \varepsilon + \varepsilon' y' + (\psi - \beta)(z' + \xi) \\
& \qquad\qquad + \text{terms or order} \geqslant 2
\end{aligned}
$$

$(\varepsilon + (\psi - \beta)\xi = 0)$. The roots of the characteristic polynomial of $L(D';\{P'\};P')$ are $(1-\beta, \beta, \psi - \beta)$, hence $c' \leqslant c$. If $c = c'$, we have as above, that $n' \leqslant p < p + q = n$ and (4.5.1) follows immediately. Let $P' = [(0,0,1)] \in \pi^{-1}(P)$. Applying $T(0,0,1)$, a generator D′ of $\mathscr{D}'_{P'}$ satisfies

$$
\begin{aligned}
& D'(x')/x' = 1 - \psi + \text{terms of order} \geqslant 1 \\
(4.5.15) \qquad & D'(y')/y' = \beta - \psi + \text{terms of order} \geqslant 1 \\
& D'(z')/z' = \psi + \text{terms of order} \geqslant 1 .
\end{aligned}
$$

One has $c' \leqslant c$ and since $e(E',P') = 3 > e(E,P)$, (4.5.1) is satisfied.

Let us assume now that π is monoidal centered at (x,y), hence $\psi = 0$. Let $P' = [(1,\zeta)] \in \pi^{-1}(P)$. Applying $TM((1,\zeta);(x,y))$, a generator D' of $\mathscr{D}'_{P'}$ satisfies

$$
\begin{aligned}
& D'(x')/x' = 1 \\
(4.5.16) \quad & D'(y') = (\beta - 1 + \alpha' x' + Y' z' + \text{terms of order} \geqslant 2)(y' + \zeta) \\
& D'(z') = x'(\ldots)
\end{aligned}
$$

$((\beta - 1)\zeta = 0)$. Clearly $c' \leqslant c$. If $c' = c = 1$, then $\zeta = 0$ (since if $\beta = 1$ then $c' = 0$), and $e(E,P) = e(E',P') = 2$ and $n' \leqslant p < p + q = n$, hence (4.5.1) is verified. Let $P' = [(0,1)] \in \pi^{-1}(P)$. Applying $TM((0,1);(x,y))$, a generator D′ of $\mathscr{D}'_{P'}$ satisfies (4.5.10). Hence $c' \leqslant c$, $e(E,P) = e(E',P')$ and if $c' = c = 1$, then $n' \leqslant p < p + q = n$.

If π is monoidal centered at (x,z), then $\varepsilon = \beta = 0$ and hence P is a simple corner.

c) Case $e(E,P) = 3$. Then there is a regular system of parameters (x,y,z) of $\mathcal{O}_{X,P}$, suited for (E,P) and a generator D of $\mathscr{D}_P$ satisfying

$$
\begin{aligned}
& D(x)/x = 1 \\
(4.5.17) \qquad & D(y)/y = \beta + \text{terms of order} \geqslant 1 \\
& D(z)/z = \psi + \text{terms of order} \geqslant 1 .
\end{aligned}
$$

Since P is not a simple corner, we have β, $\psi \in \mathbf{Q}_+$. In particular, $\beta \neq 0 \neq \psi$, and hence π is never monoidal. Writing $\beta = p/q$, $\psi = r/s$, $q = mq_1$, $s = ms_1$, then $n = mq_1s_1 + ps_1 + rq_1$ as in (4.5.4). If $P' = [(1,\zeta,\xi)] \in \pi^{-1}(P)$ or $P' = [(0,1,\xi)]$ with $(\zeta,\xi) \neq (0,0)$, resp. $\xi \neq 0$, then $c' \leqslant c$ by arguments as in (4.5.13). If $(\zeta,\xi) = (0,0)$, resp. $\xi = 0$ then $c' \leqslant c$, $e(E',P') = e(E,P) = 3$ and if $c' = c$, then $n' < n$ as in (4.5.6). Note that since β, $\psi \in \mathbf{Q}_+$, we have (4.3.7). This ends the proof of the theorem.

4.6. Now, we shall construct an invariant, called « date of birth », which will allow to run on the trees of infinitely near points « horizontally » rather than « vertically ». This kind of invariant has already been used in [7] and [9].

Let E be a normal crossings divisor of X. Put $E = E(0)$, $X = X(0)$. Let us fix an infinite sequence of blowing-ups

$$\mathscr{S} = \{\pi(i) : X(i) \to X(i-1)\}_{i \geqslant 1}, \tag{4.6.1}$$

such that for each $i \geqslant 1$, $\pi(i)$ has a non-singular center $Y(i-1)$, having normal crossings with $E(i-1)$, where $E(i) = \pi(i)^{-1}(E(i-1) \cup Y(i-1))$ with reduced structure (hence $E(i)$ is a normal crossings divisor of $X(i)$). We assume also that $\dim Y(i) \leqslant \dim X(i) - 2 = \dim X - 2$.

Assume that $E = E(0)$ has $n+1$ irreducible components and let

$$E(0) = E_{-n}(0) \cup \ldots \cup E_{-1}(0) \cup E_0(0) \tag{4.6.2}$$

be the decomposition of $E(0)$ into irreducible components. Now, let us write inductively

$$E(i) = E_{-n}(i) \cup \ldots \cup E_0(i) \cup E_1(i) \cup \ldots \cup E_{i-1}(i) \cup E_i(i) \tag{4.6.3}$$

where for $j \leqslant i-1$, $E_j(i)$ is the strict transform of $E_j(i-1)$ by $\pi(i)$ and $E_i(i)$ is the exceptional divisor $\pi(i)^{-1}(Y(i-1))$ of $\pi(i)$.

4.7. Définition. — *Let us fix an index $i \geqslant 0$ and let $Y \subset E(i)$ be an irreducible closed subscheme of $X(i)$ having normal crossings with $E(i)$. Then the « date of birth » $\mathrm{dat}(Y;E(i))$ is defined to be*

$$dat\,(Y;E(i)) = \max\{j; E_j(i) \supset Y\} \tag{4.7.1}$$

once we have fixed the decomposition (4.6.3).

4.8. DÉFINITION. – *Let* $(X,E,\mathscr{D})$ *be as in* 1.5. *Let us denote* $(X(0),E(0),\mathscr{D}(0)) = (X,E,\mathscr{D})$. *Let us fix a sequence* $\mathscr{S}$ *of blowing-ups as in* (4.6.1). *The sequence* $\mathscr{S}$ *is said to « respect the procedure of reduction » iff the following properties hold for each* $i \geqslant 1$:

a) The center $Y(i-1)$ *is a permissible center for* $(X(i-1),E(i-1),\mathscr{D}(i-1))$. *We shall denote by* $(X(i),E(i),\mathscr{D}(i))$ *the strict transform of* $(X(i-1),E(i-1),\mathscr{D}(i-1))$ *by* $\pi(i)$.

b) If $S_1(\mathscr{D}(i-1))$ *does not have normal crossings with* $E(i-1)$, *then the center* $Y(i-1)$ *of* $\pi(i)$ *is a closed point of* $X(i-1)$ *such that* $S_1(\mathscr{D}(i-1))$ *and* $E(i-1)$ *do not have normal crossings at* $Y(i-1)$. *Moreover, we assume that* $\operatorname{dat}(Y(i-1);E(i-1))$ *is minimal.*

c) Assume that $S_1(\mathscr{D}(i-1))$ *and* $E(i-1)$ *have normal crossings. If there is an irreducible component of* $S_0(\mathscr{D}(i-1)) \cup S_1(\mathscr{D}(i-1))$ *containing at least one point which is neither a simple point nor a simple corner, then the center* $Y(i-1)$ *of* $\pi(i)$ *is such an irreducible component having minimal* $\operatorname{dat}(Y(i-1);E(i-1))$.

d) Otherwise, the center $Y(i-1)$ *is an arbitrary closed point of* $X(i-1)$.

Remark. — A sequence which respects the procedure of reduction always exists. We have put *d*) above in order to assure the infiniteness of the sequence in the case that the « first final situation » is globally reached.

4.9. DEFINITION. – *Let us take the notation of* 4.8 *and let us fix a sequence* $\mathscr{S}$ *which respects the procedure of reduction. A « sequence of infinitely near singular points for* $(X,E,\mathscr{D})$ *in* S » *is a sequence*

$$\Sigma = \{P(i)\}_{i \geqslant 0} \tag{4.9.1}$$

satisfying :

a) $P(i) \in X(i)$, *for all* $i \geqslant 0$.
b) $P(i) \in \operatorname{Sing}(\mathscr{D}(i),\varnothing)$, *for all* $i \geqslant 0$.
c) $\pi(i)(P(i)) = P(i-1)$, *for all* $i \geqslant 1$.

We shall say that the sequence Σ *is « non-degenerate » iff for each index* i *such that* $P(i)$ *is neither a simple point nor a simple corner, or* $\operatorname{Sing}(\mathscr{D}(i),\varnothing)$ *does not have normal crossings with* $E(i)$ *at* $P(i)$, *then there is an index* $j \geqslant i$ *such that* $P(j) \in Y(j)$.

4.10. THEOREM (*reduction theorem*). — *Let us fix a sequence* $\mathscr{S}$ *which respects the procedure of reduction and let us fix a sequence of infinitely near singular points* Σ *for* (X,E,D) *in* $\mathscr{S}$. *Then :*

a) Σ *is non degenerate.*

b) There is an index N *such that for each* $i \geqslant N$, $(X(i), E(i), \mathscr{D}(i))$ *is in the first final situation at* P(i).

Proof. — First, let us prove that *a*) implies *b*). Assume that i is an index such that P(i) is neither a simple point nor a simple corner. Let $j \geqslant i$ be the first index such that $P(j) \in Y(j)$. By theorem 4.5 we have

$$(4.10.1) \quad \mathrm{Inv}\,(\mathscr{D}(j+1), E(j+1), P(j+1)) < \mathrm{Inv}\,(\mathscr{D}(i), E(i), P(i)) \quad \text{(strictly)}.$$

Hence there is an index M such that for each $i \geqslant M$, P(i) is a simple point or a simple corner (theorem 3.6 assures that this property is stable under permissible blowing-ups). Let us consider now P(M), which is a simple point or a simple corner. There are two possibilities : 1) $S_1(\mathscr{D}(M))$ has normal crossings with E(M) at P(M). 2) The above is not satisfied. In the first case P(M) is in the first final situation and by 3.7 this property is stable under permissible blowing-ups. Hence we can take $N = M$. Assume we are in the second case. Let $j \geqslant M$ be the first index such that $P(j) \in Y(j)$. We can assume without loss of generality that $j = M$. Now, by the priorities *b*), *c*), *d*) of 4.8 we have $Y(M) = \{P(M)\}$ and hence $\pi(M+1)$ is quadratic. By 2.3 we have

$$(4.10.2) \qquad S_1(\mathscr{D}(M+1)) = S_1(\mathscr{D}(M))' \cup Y$$

where $Y \subset \pi(M+1)^{-1}(P(M))$ and has normal crossings with $E(M+1)$ at $P(M+1)$ if $e(E(M+1), P(M+1)) \leqslant 2$. If $e(E(M+1), P(M+1)) = 3$, we are in the first final situation and this property is stable under permissible blowing-ups. Otherwise, we repeat the above arguments and by 2.4 in a finite number of steps we are in the first final situation.

Now, let us prove that Σ is non-degenerate. It is enough to prove that there is an index $j \geqslant 0$ such that $P(j) \in Y(j)$. We shall reason by contradiction, assuming that for each $j \geqslant 0$, $P(j) \notin Y(j)$. This is equivalent to

$$(4.10.3) \quad \{P(j)\} = (\pi(j) \circ \pi(j-1) \circ \ldots \circ \pi(1))^{-1}(P(0))$$

for each $j \geqslant 1$. Let us assume without loss of generality that $E_{-m}(0), \ldots, E_{-1}(0), E_0(0)$ are exactly the irreducible components of $E(0)$ containing $P(0)$ ($m=0$, 1 or 2). Then $E_{-m}(i), \ldots, E_{-1}(i), E_0(i)$ are exactly the irreducible components of $E(i)$ containing $P(i)$ for each $i \geqslant 0$.

Assume first that $S_1(\mathscr{D}(0))$ and $E(0)$ do not have normal crossings at $P(0)$. Hence, by the priorities of 4.8, each $\pi(i+1)$ is a quadratic blowing-up centered at a closed point $Y(i) \neq P(i)$ such that $S_1(\mathscr{D}(i))$ and $E(i)$ do not have normal crossings at $Y(i)$, for each $i \geqslant 0$. Let

$$(4.10.4) \qquad \{Q_s(0)\}_{s=0,\ldots,h}$$

be the closed points of $X(0)$ such that $S_1(\mathscr{D}(0))$ and $E(0)$ do not have normal crossings at $Q_s(0)$ and $\operatorname{dat}(Q_s(0)); E(0)) \leqslant 0 = \operatorname{dat}(P(0), E(0))$. If $h = 0$, we are done, since by (4.8) we must have $Y(0) = Q_0(0) = P(0)$. Assume that $h \geqslant 1$. Without loss of generality, we can assume that $Q_0(0) = P(0)$ and that $Y(0) = Q_h(0)$. Since each point over $Y(0)$ under $\pi(1)$ has date of birth $\geqslant 1$ (in fact $= 1$), we have that

$$(4.10.5) \qquad Q_s(1) = \pi(1)^{-1}(Q_s(0)), \qquad s = 0, \ldots, h-1$$

are exactly the points of $X(1)$ such that $S_1(\mathscr{D}(1))$ does not have normal crossings with $E(1)$ at $Q_s(1)$ and $\operatorname{dat}(Q_s(1); E(1)) \leqslant 0 = \operatorname{dat}(P(1); E(1))$. Hence in a finite number of steps we have $h = 0$, and we obtain the desired contradiction.

Let us assume now that $S_1(\mathscr{D}(0))$ and $E(0)$ have normal crossings at $P(0)$. Let

$$(4.10.6) \qquad \{R_s(0)\}_{s \in I(0)}$$

be the irreducible components of $S_0(\mathscr{D}(0)) \cup S_1(\mathscr{D}(0))$ containing at least one point which is neither a simple point nor a simple corner. Hence $I(0)$ is a finite set. Note that the $R_s(0)$, $s \in I(0)$, are exactly the irreducible components of $S_0(\mathscr{D}(0)) \cup S_1(\mathscr{D}(0))$ containing at least one point which is neither a simple point nor a simple corner and having the following property:

$$(4.10.7) \qquad \text{If } E_p(0) \supset R_s(0), \text{ then } p \leqslant 0$$

(this is trivial). Now, let us define by induction a family

$$(4.10.8) \qquad \{R_s(i)\}_{s \in I(i)}$$

putting

$$(4.10.9)\quad J(i) = \{s \in I(i-1) ; R_s(i-1) \neq Y(i-1)\} \subset I(i-1)$$

and if $s \in J(i)$, then $R_s(i)$ is the strict transform of $R_s(i-1)$ by $\pi(i)$. Now, define

(4.10.10) $I(i) = \{s \in J(i)$; $R_s(i)$ contains at least one point which is neither a simple point nor a simple corner$\} \subset J(i) \subset I(i-1)$.

Now it is easy to verify that the $R_s(i)$, $s \in I(i)$, are exactly the irreducible components of $S_0(\mathscr{D}(i)) \cup S_1(\mathscr{D}(i))$ containing at least one point which is neither a simple point nor a simple corner and satisfying the following property:

$$(4.10.11)\qquad \text{If } E_p(i) \supset R_s(i), \text{ then } p \leqslant 0.$$

Now, we shall proceed by induction on $\# I(0)$. In order to do this, let us assume that the following statement is true:

(4.10.12) « Let $\mathscr{S}$ be a sequence which respects the procedure of reduction. Then there is a step N such that $E(N)$ and $S_1(\mathscr{D}(N))$ have normal crossings ».

(By (4.10.12), given M, there is $N \geqslant M$ such that $E(N)$ and $S_1(\mathscr{D}(N))$ have normal crossings). Assume first that $\# I(0) = 1$. Then $\# I(N) = 1$ and hence we have only one $R(N)$ in (4.10.8). By assumption we have $P(N) \in R(N)$. Moreover, let Z be an irreducible component of $S_0(\mathscr{D}(N)) \cup S_1(\mathscr{D}(N))$ containing at least one point which is neither a simple point nor a simple corner. Then by (4.10.11) we have

$$(4.10.13)\qquad \operatorname{dat}(R(N); E(N)) < \operatorname{dat}(Z; E(N))$$

(strictly). Hence the priorities of 4.8 *c*) imply that $R(N) = Y(N)$ and we obtain the desired contradiction. If $\# I(0) \geqslant 2$ and $\# I(0) = \# I(N)$ we obtain as above that $Y(N) = R_s(N)$ for some $s \in I(N)$ and hence $\# I(N+1) < \# I(N)$.

Now, let us prove (4.10.12). We shall reason by contradiction, assuming that (4.10.12) is not true. Then by 4.8 *b*), each $\pi(i)$ is centered at a closed point $Y(i-1)$ such that $S_1(\mathscr{D}(i-1))$ does not have normal crossings with $E(i-1)$ at $Y(i-1)$. Since the set of points Q such that

$S_1(\mathscr{D}(i))$ and $E(i)$ do not have normal crossings at Q is a finite set, then we can easily find a sequence of infinitely near singular points

$$(4.10.14) \qquad \Sigma' = \{P'(i)\}_{i \geqslant 0}$$

which is non-degenerate and such that $S_1(\mathscr{D}(i))$ and $E(i)$ do not have normal crossings at $P'(i)$. By (4.10.1) we can assume without loss of generality that $P'(0)$ is a simple point or a simple corner. Moreover, we can assume that $P'(0) = Y(0)$. By 2.3 we have (as in (4.10.2))

$$(4.10.15) \qquad S_1(\mathscr{D}(1)) = S_1(\mathscr{D}(0))' \cup Y$$

where $Y \subset \pi(1)^{-1}(Y(0))$ and has normal crossings with $E(1)$ at $P'(1)$ if $e(E(1), P'(1)) \leqslant 2$. Let us prove that this assertion is also true in the case $e(E(1), P'(1)) = 3$. Then reasoning as in 2.4, after a finite number of steps we have normal crossings at $P'(i)$ and we obtain the desired contradiction. Let us take the notation of 2.3 and assume first that $e(E(0), P'(0)) = 2$. Hence we have $\alpha = \gamma = 0$ and $\beta \notin \mathbf{Q}_+$. Thus (2.3.6) is

$$(4.10.16) \qquad (\beta - 1)\zeta = \delta + \varepsilon.\zeta + (\psi - 1)\xi = 0$$

hence if it defines a 1-dimensional linear variety L, then $L = (\zeta = 0)$ and $\bar{L}$ is $\operatorname{Proj}(T_{P'(0)}E_p(0))$ for some $E_p(0)$. Same argument for the case $e(E(0), P'(0)) = 3$. This ends the proof of the theorem.

4.11. *Remark.* — We cannot expect that for some step N all the points of $X(N)$ are regular points, or simple points, or simple corners. Take, for instance, the following example: $X = \mathbf{A}^3(\mathbf{C})$, $\mathscr{D}$ is globally generated by

$$(4.11.1) \qquad D = x\partial/\partial x + zy\,\partial/\partial y$$

where $E = (x=0)$. We have infinitely many singular points $(0,0,p/q)$ which are not simple points. Except for a finite number of these points, we have (if we want to desingularize) to blow-up the line $x = y = 0$. Hence the desingularization is essentially the same as $x\partial/\partial x + (p/q)y\partial/\partial y$ in dimension two. But the number of blowing-ups needed for this depends on Euclides' algorithm for dividing p/q, which is not uniformly bounded.

5. Dimension two revisited.

5.1. In this paragraph we shall briefly recall the analogue of 4.10 for the case $\dim X = 2$ (without any restrictions on the initial situation $(X,E,\mathscr{D})$).

5.2. THEOREM ([5], § 3, cor. 2 and § 4, th. (4.2)). — *Let us assume that* $\dim X = 2$, E *is a normal crossings divisor of* X *and* $\mathscr{D}$ *is a unidimensional distribution over* X, *multiplicatively irreducible and adapted to* E. *Then there is a finite sequence of quadratic blowing-ups*

$$\mathscr{S} = \{\pi(i) : X(i) \to X(i-1)\}_{i=1,\dots,N} \tag{5.2.1}$$

with $X(0) = X$ *such that if* $(X(N), E(N), \mathscr{D}(N))$ *is the strict transform of* $(X,E,\mathscr{D})$ *by the successive* $\pi(i)$, *each closed point* P *of* X(N) *satisfies one of the following conditions :*

(5.2.2) *a)* $\nu(\mathscr{D}(N), E(N), P) = 0$

b) $\nu(\mathscr{D}(N), E(N), P) = 1$ *and there is a regular system of parameters* (x,y) *of* $\mathscr{O}_{X,P}$ *and a generator* D *of* $\mathscr{D}(N)_P$ *such that* $I(E(N))_P = (x)$, $\nu(D(x)) \geqslant 2$ *and* $cl^1(D(y)) = \underline{y}$ *(i.e. one has one of the final forms of* [12], *vgr.).*

5.3. Assume that $(X,E,\mathscr{D})$ satisfies the conditions *a)* or *b)* of (5.2.2) for each closed point $P \in X$. Let us define the invariant Inv (D,E,P). Put $\mathrm{Inv}(\mathscr{D},E,P) = 0$ if $\nu(\mathscr{D},\varnothing,P) = 0$. Let $\Phi(T) \in k[T]$ be the characteristic polynomial of the linear map $L(D;\{P\};P)$ defined as in (2.7.1) for a generator D of $\mathscr{D}_P$. Let (λ_1, λ_2) be the roots of $\Phi(T)$, with $\lambda_1 \neq 0$. The invariant $\mathrm{Inv}(\mathscr{D},E,P)$ is defined to be 0 if $\lambda_2/\lambda_1 \notin \mathbf{Q}_+$ and $p+q$ if $\lambda_2/\lambda_1 = p/q \in \mathbf{Q}_+$, $\mathrm{g.c.d}(p,q) = 1$.

Note that if $\nu(\mathscr{D},E,P) = 1$, then $\mathrm{Inv}(\mathscr{D},E,P) = 0$ since $\lambda_2/\lambda_1 = 0$ in view of *b)* of (5.2.2). The number of closed points $P \in X$ such that $\mathrm{Inv}(\mathscr{D},E,P) > 0$ is finite. (Note that if E_1 is a component of E with $E_1 \subset \mathrm{Sing}(\mathscr{D},\varnothing)$, then $\mathrm{Inv}(\mathscr{D},E,P) = 0$ for each $P \in E_1$). Let $a(\mathscr{D},E)$ be the maximum of the numbers $\mathrm{Inv}(\mathscr{D},E,P)$ and let $b(\mathscr{D},E)$ be the number of closed points P such that $\mathrm{Inv}(\mathscr{D},E,P) = a(\mathscr{D},E)$. The invariant

$$\mathrm{Inv}(\mathscr{D},E) = (a(\mathscr{D},E), b(\mathscr{D},E)) \tag{5.3.1}$$

decreases strictly each time one makes a quadratic blowing-up centered at a closed point P with $\mathrm{Inv}(\mathscr{D},E,P) = a(\mathscr{D},E) \neq 0$.

Thus, after a finite number of steps we can obtain the following « final situation » for $(X,E,\mathscr{D})$:

(5.3.2) 1° $(X,E,\mathscr{D})$ satisfies *a*) or *b*) of (5.2.2) for each closed point $P \in X$.

(5.3.3) 2° $a(\mathscr{D},E) = 0$.

The above situation agrees with the classical one (see [12]) except for two slight differences: first of all, the classical situation takes $\mathscr{D}$ multiplicatively irreducible and adapted to $\varnothing = E$, thus in order to obtain the classical situation one has to divide D by the equations of the components E_1 of E such that $E_1 \subset \mathrm{Sing}(\mathscr{D},\varnothing)$. Secondly, in the classical situation one does not distinguish whether an integral branch (i.e. a branch Γ which is tangent to $\mathscr{D}$; algebraically this is equivalent to saying that $\mathscr{D}_P(I(\Gamma)_P) \subset I(\Gamma)_P)$ of the final situation comes from an integral curve of the initial situation or has been created in the process. In our situation, the initial integral branches correspond exactly to the points of $\mathrm{Sing}(\mathscr{D},\varnothing)$ which are not « corners ». See below and paragraph 7 for more details.

5.4. Let us assume that $(X,E,\mathscr{D})$ is in the « final situation » defined in (5.3.2) and (5.3.3). A closed point $P \in X$ is a « corner point » iff $e(E,P) = 2$. Hence $\nu(\mathscr{D},E,P) = 0$, $\nu(\mathscr{D},\varnothing,P) = 1$ and $\mathscr{D}_P$ is generated by D, satisfying

$$(5.4.1) \quad \begin{aligned} D(x)/x &= 1 \\ D(y)/y &= \lambda + \text{terms of order} \geqslant 1,\ \lambda \notin \mathbf{Q}_+ \end{aligned}$$

for a certain regular system of parameters (x,y) of $\mathcal{O}_{X,P}$ suited for (E,P). Easy computations from (5.4.1) show that P is transformed only into corner points or into regular points under a quadratic blowing-up. The « simple corners » of 3.3 have the same property (see 3.6 *c*)). In particular, there is no integral branch passing through these corners, different from the components of the divisor. In fact, the desingularization of this branch would create a singular point of $\mathscr{D}$ which is not a corner. See also paragraph 7.

5.5. In the situations *a*) and *b*) of (5.2.2) the linear map $L(D;\{P\};P)$ is not nilpotent (i.e. there is at least one nonzero eigenvalue). We shall say that P is a « pre-simple » point iff $L(D;\{P\};P)$ is not nilpotent. Being pre-simple or regular is stable under blowing-up. Moreover, if $P \in E$ (i.e. $e(E,P) \geqslant 1$), then pre-simple is equivalent to *a*) or *b*) of (5.2.2). Hence, after one blowing-up of a pre-simple point, we always have *a*) or *b*) of (5.2.2) at the singular points.

6. Second final situation.

6.1. Thoughout this paragraph we shall assume that $\dim X = 3$ and that $(X,E,\mathscr{D})$ satisfies the assumptions of 1.5.

Let us fix an irreducible component F of E and let us denote by E/F the normal crossings divisor of F given by $(\overline{E-F}) \cap F$. Then we can define an $\mathcal{O}_F$-submodule of $\Xi_F[E/F]$, denoted by $\mathscr{D}/F$, which is given locally at each closed point $P \in F$ in the following way. Let (x,y,z) be a regular system of parameters of $\mathcal{O}_{X,P}$ suited for (E,P) such that $I(F)_P = (x)$. Let us denote also by (y,z) the induced regular system of parameters in $\mathcal{O}_{F,P}$ (which is also suited for (E/F,P)). Let D be a generator of $\mathscr{D}_P$. Then $(\mathscr{D}/F)_P$ is generated by an element $D/F \in \Xi_F[E/F]$ satisfying

$$(6.1.1) \qquad \begin{aligned} (D/F)(y) &= D(y)(\mathrm{mod}\, x) \\ (D/F)(z) &= D(z)(\mathrm{mod}\, x). \end{aligned}$$

Then $\mathscr{D}/F$ does not depend on (x,y,z) nor D. In fact, $\mathscr{D}/F$ is the inverse image of $\mathscr{D}$ via the inclusion $i: F \to X$, (see [6], I.(2.1)). Then we deduce easily that $\mathscr{D}/F = 0$ iff $F \subset \mathrm{Sing}(\mathscr{D},\varnothing)$ and if $\mathscr{D}/F \neq 0$ then $\mathscr{D}/F$ is a unidimensional distribution over F adapted to E/F, which may not be irreducible.

Assume that $\mathscr{D}/F \neq 0$. Then there is a locally principal ideal sheaf $\mathscr{H}_F(\mathscr{D},E)$ of $\mathcal{O}_F$ satisfying

$$(6.1.2) \qquad \mathscr{H}_F(\mathscr{D},E).\alpha_{E/F}(\mathscr{D}/F) = \mathscr{D}/F$$

(see 1.2 for the definition of $\alpha_{E/F}$).

Let us denote by $H_F(\mathscr{D},E)$ the closed subset of F given by $\mathscr{H}_F(\mathscr{D},E)$. (In general $\mathscr{H}_F(\mathscr{D},E)$ does not give a reduced structure on $H_F(\mathscr{D},E)$). If $\nu(\mathscr{D},\varnothing,P) = 0$, then $P \notin H_F(\mathscr{D},E)$, hence

$$(6.1.3) \qquad H_F(\mathscr{D},E) \subset S_1(\mathscr{D}) \cup S_0(\mathscr{D}).$$

6.2. DEFINITION. — $(X,E,\mathscr{D})$ *is in the « second final situation » at a closed point* $P \in X$ *iff the following conditions hold :*

a) $(X,E,\mathscr{D})$ *is in the first final situation at* P.

b) $\mathrm{Sing}(\mathscr{D},\emptyset)$ *has normal crossings with* E *at* P.

c) For each irreducible component F *of* E *at* P, *such that* $F \not\subset \mathrm{Sing}(\mathscr{D},\emptyset)$, *then* P *is a pre-simple point for* $(F,E/F,\alpha_{E/F}(\mathscr{D}/F))$. (*Note that b) implies that* $H_F(\mathscr{D},E)$ *has normal crossings with* E *at* P, *in view of* (6.1.3)).

6.3. DEFINITION. — *Let us consider a sequence*

$$\mathscr{S} = \{\pi(i) : X(i) \to X(i-1)\}_{i \geq 1} \tag{6.3.1}$$

as in (4.6.1). *Put* $(X(0),E(0),\mathscr{D}(0)) = (X,E,\mathscr{D})$. *The sequence* $\mathscr{S}$ *is said to « respect the second procedure of reduction » iff the following properties hold for each* $i \geq 1$:

a) Same as in (4.8).

b) Same as in (4.8).

c) Assume that $S_1(\mathscr{D}(i-1))$ *and* $E(i-1)$ *have normal crossings. Assume that there is an irreducible component* F *of* $E(i-1)$ *such that* $F \not\subset \mathrm{Sing}(\mathscr{D}(i-1),\emptyset)$ *and a closed point* P *of* F *such that* P *is not a pre-simple point for* $(F,E(i-1)/F,\alpha_{E(i-1)/F}(\mathscr{D}(i-1)/F))$. *Then the center* $Y(i-1)$ *of* $\pi(i)$ *is such a point* P *having minimal* $\mathrm{dat}(P;E(i-1))$.

d) Assume that $S_1(\mathscr{D}(i-1))$ *and* $E(i-1)$ *have normal crossings and there is no* F, P *as in c). If there is an irreducible component of* $S_0(\mathscr{D}(i-1)) \cup S_1(\mathscr{D}(i-1))$ *containing at least one point which is neither a simple point nor a simple corner, then the center* $Y(i-1)$ *of* $\pi(i)$ *is such an irreducible component having minimal* $\mathrm{dat}(Y(i-1);E(i-1))$.

e) Otherwise, the center $Y(i-1)$ *is an arbitrary closed point of* $X(i-1)$.

As in 4.9, we shall say that a sequence Σ of infinitely near singular points for $(X,E,\mathscr{D})$ in $\mathscr{S}$ is « non-degenerate » iff for each index i such that $(X(i),E(i),\mathscr{D}(i))$ is not in the second final situation at $P(i)$, there is $j \geq i$ such that $P(j) \in Y(j)$.

6.4. THEOREM (*Second theorem of reduction*). — *Let us fix a sequence* $\mathcal{S}$ *which respects the second procedure of reduction and a sequence* Σ *of infinitely near singular points for* $(X,E,\mathcal{D})$ *in* $\mathcal{S}$. *Then*

a) Σ *is non-degenerate.*

b) For each index N *there is an index* $M \geqslant N$ *such that* $(X(M), E(M), \mathcal{D}(M))$ *is in the second final situation at* P(M).

Proof. — Let us show first that *a)* implies *b)*. Without loss of generality we can assume that for each $i \geqslant 0$, $P(i) \in Y(i)$. Moreover, by (4.10.1) we can also assume that P(0) is a simple point or a simple corner (hence so is each $P(i)$, $i \geqslant 0$). Since 6.3 *b)* and 4.8 *b)* are the same, (4.10.12) is also true and we can assume without loss of generality that E(0) and $S_1(\mathcal{D}(0))$ have normal crossings at P(0). In order to prove *b)* it is enough to find $M \geqslant 0$ such that $(X(M), E(M), D(M))$ is in the second final situation. Let us define

(6.4.1) $I(i) = \{s\,;$ if $F = E_s(i)$, then $F \not\subset \operatorname{Sing}(\mathcal{D}(i),\varnothing)$, $P(i)$ is a singular point of $(F, E(i)/F, \alpha_{E(i)/F}(\mathcal{D}(i)/F))$ which is not pre-simple$\}$

for each $i \geqslant 0$. If $\# I(0) = 0$, we are done, since $(X(0), E(0), \mathcal{D}(0)$ is in the second final situation at P(0). If $\# I(0) \geqslant 1$, then $\pi(1)$ is quadratic in view of 6.3 *c)*. By the proof of (4.10.12), $S_1(\mathcal{D}(1))$ and E(1) have normal crossings at P(1). If $\# I(1) = 0$ we are done, otherwise the situation repeats. We shall prove first that $I(0) \supset I(1)$ and second that there is an index i such that $\# I(0) > \# I(i)$ (strictly). Hence we finish by induction on $\# I(0)$.

Let $s \leqslant 0$ be an index such that $s \notin I(0)$. If $E_s(0) \subset \operatorname{Sing}(\mathcal{D}(0),\varnothing)$, then $E_s(1) \subset \operatorname{Sing}(\mathcal{D}(1),\varnothing)$, hence $s \notin I(1)$. Assume $E_s(0) \not\subset \operatorname{Sing}(\mathcal{D}(0),\varnothing)$. We can also assume that $P(0) \in E_s(0)$ and $P(1) \in E_s(1)$. Denote $F = E_s(0)$, $F' = E_s(1)$. Then P(0) is either a regular point or a pre-simple point of $(F, E(0)/F, \alpha_{E(0)/F}(\mathcal{D}(0)/F))$. Let $\pi_F : F' \to F$ be the restriction of π to F. Then

$$(6.4.2) \qquad (F', E(1)/F', \alpha_{E(1)/F'}(\mathcal{D}(1)/F'))$$

is the strict transform of $(F, E(0)/F, \alpha_{E(0)/F}(\mathcal{D}(0)/F))$ by π_F. Hence P(1) is either a regular point or a pre-simple point of (6.4.2), by the results for the two-dimensional case. Then $s \notin I(1)$. It remains to prove that $1 \notin I(1)$. Take a regular system of parameters (x,y,z) of $\mathcal{O}_{X(0),P(0)}$ such that E(0) is given locally at P(0) by $x = 0$, $xy = 0$ or $xyz = 0$ and there is a generator D of $\mathscr{D}(0)_{P(0)}$ satisfying

$$(6.4.3)\quad \begin{aligned} D(x)/x &= 1 \\ D(y) &= \alpha x + \beta y + \text{terms of order} \geqslant 2 \\ D(z) &= \delta x + \varepsilon y + \psi z + \text{terms of order} \geqslant 2 \end{aligned}$$

where $\beta \notin \mathbf{Q}_+$. (Recall that P(0) is a simple point or a simple corner). If $P(1) = [(1,\zeta,\xi)] \in E_1(1)$, applying $T(1,\zeta,\xi)$ (see (2.3.5)), a generator D′ of $\mathscr{D}(1)_{P(1)}$ satisfies

$$(6.4.4)\quad \begin{aligned} D'(x')/x' &= 1 \\ D'(y') &= (\beta-1)y' + x'(\ldots) \\ D'(z') &= \varepsilon y' + (\psi-1)z' + x'(\ldots) \end{aligned}$$

(note that $P(1) \in \operatorname{Sing}(\mathscr{D}(1),\varnothing))$. Hence $D(1)/E_1(1)$ is generated at P(1) by

$$(6.4.5)\quad D'/E_1(1) = (\beta-1)\; y'\, \partial/\partial y' + [\varepsilon y' + (\psi-1)z']\, \partial/\partial z'.$$

Since $\beta \notin \mathbf{Q}_+$, $\beta - 1 \neq 0$ and P(1) is a pre-simple point of (6.4.2) for $F' = E_1(1)$. Hence $1 \notin I(1)$ in this case. If $P(1) = [(0,1,\xi)]$, a generator D′ of $\mathscr{D}(1)_{P(1)}$ satisfies

$$(6.4.6)\quad \begin{aligned} D'(x')/x' &= 1 - \beta - \alpha x' - y'(\ldots) \\ D'(y')/y' &= \alpha x' + \beta + y'(\ldots) \\ D'(z') &= (\delta-\alpha\xi)x' + (\psi-\beta)\; z' - \alpha x'z' + y'(\ldots). \end{aligned}$$

Hence $\mathscr{D}(1)/E_1(1)$ is generated at P(1) by

$$(6.4.7)\quad \begin{aligned} D'/E_1(1) = {} & (1-\beta-\alpha x')x'\, \partial/\partial x' \\ & + ((\delta-\alpha\xi)x' + (\psi-\beta)z' - \alpha x'z')\, \partial/\partial z'. \end{aligned}$$

Thus P(1) is a pre-simple point of (6.4.2) for $E' = E_1(1)$. Hence $1 \notin I(1)$. If $P(1) = [(0,0,1)]$, a generator D′ of $\mathscr{D}(1)_{P(1)}$ satisfies

$$(6.4.8)\quad \begin{aligned} D'(x')/x' &= 1 - \psi - \delta x' - \varepsilon y' + z'(\ldots) \\ D'(y') &= \alpha x' + (\beta-\psi)y' - \delta x'y' - \varepsilon y'^2 + z'(\ldots) \\ D'(z')/z' &= \psi + \delta x' + \varepsilon y' + z'(\ldots). \end{aligned}$$

Hence $\mathscr{D}(1)/E_1(1)$ is generated at P(1) by

$$(6.4.9)\quad \begin{aligned} D'/E_1(1) = {} & (1-\psi-\delta x'-\varepsilon y')x'\, \partial/\partial x' \\ & + (\alpha x' + (\beta-\psi)y' - \delta x'y' - \varepsilon y'^2)\, \partial/\partial y'. \end{aligned}$$

Since $\beta \notin \mathbf{Q}_+$, $(1-\psi, \beta-\psi) \neq (0,0)$ and P(1) is a pre-simple point of (6.4.2) for $F' = E_1(1)$. Hence $1 \notin I(1)$. This ends the proof of $I(0) \supset I(1)$.

In order to prove that $\# I(0) > \# I(i)$ for some i, let us fix $s \in I(0)$. By 5.2, there is an index i such P(i) is either a regular point or a pre-simple point of

$$(6.4.10) \qquad (E_s(i), E(i)/E_s(i), \alpha_{E(i)/E_s(i)}(\mathscr{D}(i)/E_s(i)))$$

and hence $s \notin I(i)$.

Let us prove that Σ is non-degenerate. Assume that the following statement is true :

(6.4.11) « Let $\mathscr{S}$ be a sequence which respects the second procedure of reduction. Then there is a step N such that E(N) and $S_1(\mathscr{D}(N))$ have normal crossings and such that for each irreducible component F of E(N), such that $F \not\subset \mathrm{Sing}(\mathscr{D}(N), \emptyset)$, all the points of F are either regular points or pre-simple points for $(F, E(N)/F, \alpha_{E(N)/F}(\mathscr{D}(N)/F))$ ».

Then we can reason as in the proof of 4.10 *a*) and Σ is non-degenerate. In order to prove (6.4.11) we can reason as in the proof of (4.10.12) together with the above argument to prove $\# I(0) > \# I(i)$. This ends the proof of the theorem.

6.5. Proposition. — *If* $(X, E, \mathscr{D})$ *is in the second final situation at* $P \in X$ *and* $\pi : X' \to X$ *is a quadratic blowing-up, then* $(X', E', \mathscr{D}')$ *is also in the second final situation at each* P′ *such that* $\pi(P') = P$. *If* π *is monoidal with permissible center, then a*) *and c*) *of* (6.2) *hold for each* P′ *such that* $\pi(P') = P$.

Proof. — If π is quadratic, then the result follows easily from the proof of (4.10.12) and 6.4. Assume that π is monoidal. By 3.7, we have 6.2 *a*) for P′. Let F be an irreducible component of E at P, such that $F \not\subset \mathrm{Sing}(\mathscr{D}, \emptyset)$. Let Y be the center of π and let us denote by F′ the strict transform of F by π.

Assume first that $Y \subset F$, hence $\pi : F' \to F$ is the identity morphism and since $Y \subset \mathrm{Sing}(\mathscr{D}, \emptyset)$, we deduce easily that $\alpha_{E'/F'}(\mathscr{D}'/F') = \alpha_{E/F}(\mathscr{D}/F)$ at the point $P = P'$. If $Y \not\subset F$, then $\pi : F' \to F$ is a quadratic blowing-up and by (6.4.2) and 5.2, P′ is either a regular point or a pre-simple point for $(F', E'/F', \alpha_{E'/F'}(\mathscr{D}'/F'))$. Assume now that $F' = \pi^{-1}(Y)$. If $e(E,P) = 1$, the result follows from (3.6.17) and (3.6.19). If $e(E,P) \geqslant 2$, the result follows from (3.6.20), (3.6.21) and also (3.6.17) and (3.6.19).

6.6. *Remark.* — 1) The second final situation is not stable under permissible blowing-ups, as we see from the example (2.9.1).

2) In general, we cannot expect to obtain that $(F,E/F,\alpha_{E/F}(\mathscr{D}/F))$ be in the final situation described in (5.3.2) and (5.3.3), even for the simple corners. For instance, let $X = \mathbf{A}^3(k)$, $E = (xyz)=0)$ and let $\mathscr{D}$ be globally generated by

$$\text{(6.6.1)} \qquad D = x\partial/\partial x - ny\partial/\partial y - nz\partial/\partial z$$

where $n \in \mathbf{N}$. Then origin is the only point in Sing $(\mathscr{D},\emptyset)$. If $F = (x = 0)$ then $\alpha_{E/F}(\mathscr{D}/F)$ is generated by

$$\text{(6.6.2)} \qquad y\partial/\partial y + z\partial/\partial z$$

which does not satisfy (5.3.3). If $\pi : X' \to X$ is the quadratic blowing-up centered at the origin, for the point $P' = [(1,0,0)] \in \pi^{-1}$ (origin) we obtain that $\mathscr{D}'_{P'}$ is generated by

$$\text{(6.6.3)} \quad D' = x'\partial/\partial x' - (n+1)y'\partial/\partial y' - (n+1)z'\partial/\partial z'$$

and this situation is repeated indefinitely.

7. Integral branches.

7.1. In this paragraph we shall describe the integral branches of $(X,E,\mathscr{D})$ which are not contained in E in terms of the first (or second) final situation which we can obtain by means of permissible blowing-ups. This gives a description of the leaves of a foliation $\mathscr{F}_{\mathscr{D}}$ with isolated singularities in the case when $(X,\emptyset,\mathscr{D})$ may be transformed into $(X',E',\mathscr{D}')$ as in (1.5) by means of blowing-ups at the points of Sing $(\mathscr{D},\emptyset)$ or points (possibly not closed) which project (under the successive transformations) into Sing $(\mathscr{D},\emptyset)$.

7.2. Definition ([4], [8],...). — *Let* $P \in X$ *be a closed point, a « branch » at* P *will be any integral irreducible closed subscheme of dimension* 1 *of* Spec $(\hat{\mathcal{O}}_{X,P})$. *Given a branch* Γ *at* P *let us denote by* $I(\Gamma)$ *the ideal of* Γ, $I(\Gamma) \subset \hat{O}_{X,P}$.

Let $\mathscr{D}$ be any unidimensional distribution over X. Then $\mathscr{D}$ defines a submodule $\hat{\mathscr{D}}_P$ of $\mathrm{Der}_k(\hat{\mathcal{O}}_{X,P})$. The branch Γ is said to be an « integral branch » of $\mathscr{D}$ at P iff

$$\text{(7.2.1)} \qquad \hat{\mathscr{D}}_P(I(\Gamma)) \subset I(\Gamma).$$

7.3. LEMMA. — *Let* $\mathscr{D}$ *be a unidimensional multiplicatively irreducible distribution over* X (*no assumption on* $\dim X$), *adapted to a normal crossings divisor* E *of* X. *Let* $Y \subset X$ *be a regular closed subscheme having normal crossings with* E, *such that* $Y \subset \mathrm{Sing}(\mathscr{D},\varnothing)$. *Let* $\pi : X' \to X$ *be the blowing-up of* X *centered at* Y *and let* $(X',E',\mathscr{D}')$ *be the strict transform of* $(X,E,\mathscr{D})$. *Let us assume that* $Y \subset E$ *or that* $Y = \{P\}$ *is a closed point. Then, there is a bijection*

$$\text{(7.3.1)} \quad \psi : \{\textit{integral branches of } \mathscr{D} \textit{ not contained in } E\} \to \\ \to \{\textit{integral branches of } \mathscr{D}' \textit{ not contained in } E'\}$$

given by $\psi(\Gamma) = \Gamma' =$ *strict transform of* Γ *by* π.

Proof. — $\Gamma \mapsto \Gamma'$ is a bijection between the branches of X not contained in E and the branches of X′ not contained in E′. Now, it is enough to show that Γ is an integral branch of $\mathscr{D}$ iff Γ' is an integral branch of $\mathscr{D}'$. Let Γ be a branch at $P \in Y$ (otherwise the result is easy) and assume Γ' is a branch at $P' \in \pi^{-1}(P)$. Let f be a local equation of $\pi^{-1}(Y)$ at P′. Let μ be as in (1.3.3) and let ν be the multiplicity of Γ at P. Since $\Gamma' \not\subset \pi^{-1}(P)$, we have

$$\text{(7.3.2)} \quad \hat{\mathscr{D}_P}(I(\Gamma).\hat{\mathscr{O}}_{X,P}) \subset I(\Gamma).\hat{\mathscr{O}}_{X,P} \Leftrightarrow f^{\mu}\hat{\mathscr{D}'_{P'}}(f^{\nu}I(\Gamma').\hat{\mathscr{O}}_{X',P'}) \\ \subset f^{\nu}.I(\Gamma').\hat{\mathscr{O}}_{X',P'}.$$

Since $\hat{\mathscr{D}'_{P'}}(f) \subset (f)$, (7.3.2) is also equivalent to

$$\text{(7.3.3)} \qquad f^{\mu}\hat{\mathscr{D}'_{P'}}(I(\Gamma')\hat{\mathscr{O}}_{X'P'}) \subset I(\Gamma').\hat{\mathscr{O}}_{X'P'}.$$

Since $\mu \geqslant 0$, (7.3.3) is equivalent to

$$\text{(7.3.4)} \qquad \hat{\mathscr{D}'_{P'}}(I(\Gamma').\hat{\mathscr{O}}_{X',P'}) \subset I(\Gamma).\hat{\mathscr{O}}_{X',P'}$$

and the result is proved.

7.4. PROPOSITION. — *Let* $\dim X = 3$. *Let* $(X,E,\mathscr{D})$ *be in the first* (*or second*) *final situation at a closed point* $P \in E$. *Then*

a) *If* $\nu(\mathscr{D},\varnothing,P) = 0$, *then there is exactly one integral branch through* P *and it is contained in* E. *Moreover, it is a regular branch.*

b) *If* P *is a simple point, then there is exactly one integral branch* Γ *through* P *which is not contained in* E, *the others are contained in* E. *Moreover,* Γ *is a regular branch.*

c) If P *is a simple corner, then every integral branch through* P *is contained in* E.

Proof. – *a*) P is a regular point of $\mathscr{D}$ and hence there is only one integral branch Γ through P and, in addition, Γ is a regular branch. Since $\mathscr{D}(\mathrm{I(E)}) \subset \mathrm{I(E)}$, it is easily seen that $\Gamma \subset \mathrm{E}$.

c) Assume that $\Gamma \not\subset \mathrm{E}$ is an integral branch. By *a*) and 7.3 the sequence of infinitely near points of Γ cannot touch a regular point. Hence, by 3.6 *c*) all of them must be simple corners. But this is not possible, since if $\Gamma \not\subset \mathrm{E}$, then after finitely many blowing-ups Γ is regular and touches exactly one component of the divisor.

b) First, let us prove the uniqueness. Assume that $\Gamma \neq \Gamma'$ are integral branches through P. By 7.3 and by 3.6 *a*), *b*), the sequences of infinitely near points of Γ and Γ' agree, hence $\Gamma = \Gamma'$: in fact, the above sequence cannot touch either a regular point (by *a*)) or a simple corner (by *c*)), hence each time we have the simple point given by 3.6 *a*). This also implies that Γ is regular. In order to prove the existence, let us fix a coordinate system (x,y,z) in $\hat{\mathcal{O}}_{\mathrm{X,P}}$, such that the point P′ of 3.6 *a*) is successively given by [(1,0,0)]. Assume that $\mathscr{D}_{\mathrm{P}}$ is generated by

$$\mathrm{D} = x\partial/\partial x + a\partial/\partial y + b\partial/\partial z\,. \tag{7.4.1}$$

Then $\nu_{(y,z)}(a)$, $\nu_{(y,z)}(b) \geqslant 1$ and hence (y,z) gives an integral curve of $\mathscr{D}$ at P.

7.5. Let us fix a sequence $\mathscr{S}$ which respects the procedure of reduction (or the second procedure of reduction). Let $\Sigma = \{\mathrm{P}(i)\}_{i \geqslant 0}$ be a sequence of infinitely near singular points in $\mathscr{S}$. By 3.6 and the reduction theorems 4.10 and 6.4, one of the following possibilities holds :

a) There is an index N such that for each $i \geqslant \mathrm{N}$, then P(i) is a simple point. In this case we shall say that Σ « stabilizes at simple points ».

7.6. Corollary. – *Let* $\mathscr{S}$ *be a sequence which respects the procedure of reduction (or the second procedure of reduction). Let us fix a point* $\mathrm{P} \in \mathrm{Sing}\,(\mathscr{D},\varnothing)$. *Then there is a bijection between the integral branches of* $\mathscr{D}$ *not in* E *and passing through* P *and the sequences of infinitely near singular points in* $\mathscr{S}$, *starting at* P *and which stabilize at simple points.*

Proof. – Follows from 7.3 and 7.4.

7.7. *Remark.* – In this paper we do not consider the convergence problems for the integral branches in the case $k = \mathbf{C}$ (see, e.g. [3]).

Appendix.

A.1. Assume that $\dim X = 3$. Let E be a normal crossings divisor of X and let $\mathscr{D}$ be a unidimensional distribution, multiplicatively irreducible and adapted to E over X. The main result of [6] (th. I (4.2.9)) asserts that after finitely many permissible blowing-ups we can have $\nu(\mathscr{D},E,P) \leqslant 1$, at least « punctually ». Moreover, the techniques of [6] work without essential obstruction for the case $\nu(\mathscr{D},E,P) = 1$ and P being not of « type zero » (see [6]). In the case $\nu(D,E,P) = 1$, « type zero » is equivalent to the fact that $L(D;\{P\};P)$ is not nilpotent (hence it has a nonzero eigenvalue). In this appendix we shall consider exactly this situation, i.e. $\nu(\mathscr{D},E,P) = 1$ and type zero. Moreover, we shall assume that $\operatorname{Sing}(\mathscr{D},\varnothing) \subset E$, which can be achieved by a sequence of permissible blowing-ups (see [6] and [7]).

A.2. Definition (See also [7]). – *Let us consider the ideal*

$$\text{(A.2.1)} \qquad J(\mathscr{D},P) = [\bigcap_{n\geqslant 1} \operatorname{Im}[L(D;\{P\};P)]^n].\operatorname{Gr}\mathscr{O}_{X,P}$$

(note that $J(\mathscr{D},P) \neq 0$, *since* $L(D;\{P\};P)$ *is not nilpotent). The directrix* $\operatorname{Dir}(\mathscr{D},P)$ *is defined to be the linear subvariety of* T_PX *given by the zeroes of* $J(\mathscr{D},P)$.

A.3. Theorem ([7], II.2.4) – *Assume that* $\pi : X' \to X$ *is the quadratic blowing-up centered at* P *and let* $(X',E',\mathscr{D}')$ *be the strict transform of* $(X,E,\mathscr{D})$. *Let* $P' \in X'$ *be such that* $\pi(P') = P$ *and* $\nu(\mathscr{D}',E',P') = 1$. *Then*

a) $P' \in \operatorname{Proj}(\operatorname{Dir}(\mathscr{D},P)) \subset \operatorname{Proj}(T_PX) = \pi^{-1}(P)$.

b) $\dim \operatorname{Dir}(\mathscr{D}',P') \leqslant \dim \operatorname{Dir}(\mathscr{D},P)$.

A.4. Theorem. – *Assume that* $\dim \operatorname{Dir}(\mathscr{D},P) = 1$. *Then one of the following two possibilities holds :*

a) *After finitely many quadratic blowing-ups centered at the point given by* A.3 *a*), *each point over* P *has adapted order equal to zero.*

b) There is a regular system of parameters (x,y,z) *of* $\mathcal{O}_{X,P}$ *such that* $E = (x = 0)$ *and a generator* D *of* $\mathcal{D}_P$ *satisfies :*

$b - 1)$ $\nu_P(D(x)/x) \geqslant 1$,

$b - 2)$ $\nu_{(y,z)}(D(y)) \geqslant 1$, $\nu_{(y,z)}(D(z)) \geqslant 1$,

$b - 3)$ $cl^1(D(y)) = \alpha \underline{y} + \beta \underline{z}$; $cl^1(D(z)) = \gamma \underline{y} + \delta \underline{z}$, *with* $\alpha\delta - \beta\gamma \neq 0$.

Moreover, assume that we have the situation b) and let $\pi : (X',E',\mathcal{D}') \to (X,E,\mathcal{D})$ *be the quadratic blowing-up centered at* P. *Then there is exactly one point* $P' \in X'$ *with* $\pi(P') = P$ *satisfying* $b - 1)$, $b - 2)$ *and* $b - 3)$ *above. The other points over* P *are regular points or simple corners. Hence the formal branch* $y = z = 0$ *is the only integral branch of* $\mathcal{D}$ *at* P *which is not contained in* E.

Proof. — Assume that *a*) is not satisfied. Note that dim Dir $(\mathcal{D},P) = 1$ implies $e(E,P) \leqslant 1$, hence $e(E,P) = 1$. We can take a regular system of parameters (x,y,z) such that $E = (x=0)$ and a generator D of $\mathcal{D}_P$ satisfies

$$\text{(A.4.1)} \qquad \nu(D(x)/x) \geqslant 1; \quad cl^1(D(y)) = \alpha \underline{y} + \beta \underline{z}; \quad cl^1(D(z)) = \gamma \underline{y} + \delta \underline{z}, \text{ with } \alpha\delta - \beta\gamma \neq 0.$$

Thus, the points P′ of A.3 *a*) is $P' = [(1,0,0)] \in \pi^{-1}(P)$. Applying T(1,0,0) a generator D′ of $\mathcal{D}'_{P'}$ satisfies

$$\text{(A.4.2)} \quad \nu(D(x')/x') \geqslant 1 \ ; \ cl^1(D(y')) = \alpha \underline{y}' + \beta \underline{z}' + \varepsilon \underline{x}' \ ; \quad cl^1(D(z)) = \gamma \underline{y}' + \delta \underline{z}' + \psi \underline{x}'.$$

After a coordinate change of the type $y_1' = y' + \lambda x'$, $z_1' = z' + \mu x'$, we have the situation of (A.4.1). But this change corresponds to a coordinate change $y_1 = y + \lambda x^2$, $z_1 = z + \mu x^2$ before blowing-up. Then, after a formal change of coordinates $y \mapsto y + \sum_{i \geqslant 2} \lambda_i x^i$, $z \mapsto z + \sum_{i \geqslant 2} \mu_i x^i$, we can assume that the point P′ of A.3 *a*) is given at each step by $P' = [(1,0,0)]$ (hence this points correspond to the infinitely near points of $y = z = 0$) and at each step we make the transformation T(1,0,0). Let us consider the invariant

$$\text{(A.4.3)} \quad \Delta(D,P;(x,y,z)) = \min \{\nu(D(y) \bmod (y,z)),\ \nu(D(z) \bmod (y,z))\}.$$

If $\Delta = \infty$, we have $b - 2)$. If $\Delta < \infty$, it decreases strictly by one at each step and we obtain a contradiction. This proves the first part of the theorem. For the second part it is enough to make the computations after $T(1,\zeta,\xi)$, $T(0,1,\xi)$ and $T(0,0,1)$. The final part comes from 7.4 *a*) and *c*).

A.5. THEOREM. – *Assume that* dim Dir $(\mathcal{D},P) = 2$. *Then there is a regular system of parameters* (x,y,z) *of* $\hat{\mathcal{O}}_{X,P}$, *suited for* (E,P) *such that*

a) $E = (x=0)$ or $E = (xy=0)$,

b) *Let* $F = (z=0)$, *then* $\mathcal{D} \subset \Xi_X[F]$ (*after the formal completion*).

c) $\nu(D, E\cup F; P) = 0$ *and* P *is a simple corner for* $(X, E\cup F, D)$.

Proof. – Let (x,y,z) be a regular system of parameters of $\mathcal{O}_{X,P}$ satisfying *a*) and such that Dir $(\mathcal{D},P) = (\underline{z}=0)$. Moreover, we can choose (x,y,z) and a generator D of $\mathcal{D}_P$ such that

$$\text{(A.5.1)} \qquad cl^1(D(y)) = \alpha\underline{x}\ ;\ cl^1(D(z)) = \underline{z}\,.$$

Let us write $D(z) = f(x,y) + zg(x,y,z)$, where $g(0,0,0) = 1$. Let $f = \sum_{(i,j)} f_{ij}x^iy^j$ and let us denote

$$\text{(A.5.2)} \qquad \operatorname{Exp}(f;(x,y)) = \{(i,j)\,;f_{ij}\neq 0\} \subset \mathbf{Z}^2_+\,.$$

If $\operatorname{Exp}(f;(x,y)) = \varnothing$, we are done since z divides $D(z)$ and since $\nu(D(x)/x) \geqslant 1$ and then the eigenvalues of $L(D;Y;P)$ are $(0,1)$, where $Y = (x=z=0)$. Assume the contrary. Let $(\beta,\gamma) = \min \operatorname{Exp}(f;(x,y))$ for the ordering

$$\text{(A.5.3)} \quad (i,j) \leqslant (i',j') \Leftrightarrow (i+j<i'+j') \text{ or } (i+j = i'+j' \quad\text{and}\quad i \leqslant i')\,.$$

Now, let us make the coordinate change $z_1 = z + f_{\beta\gamma}x^\beta y^\gamma$. Clearly, we have *a*) and (A.5.1) for the regular system of parameters (x,y,z_1). Let us write

$$\text{(A.5.4)} \quad D(z_1) = D(z) + f_{\beta\gamma}x^\beta y^\gamma(\beta D(x)/x+\gamma D(y)/y) = f_1(x,y) + z_1g_1(x,y,z_1)\,.$$

Since $\nu(D(x)/x) \geqslant 1$ and $cl^1(D(y)) = \alpha x$, we deduce easily that $(\beta_1,\gamma_1) < (\beta,\gamma)$ (strictly), where $(\beta_1,\gamma_1) = \min \operatorname{Exp}(f_1;(x,y))$. In this way, we make a formal coordinate change $z \mapsto z + \Sigma\lambda_{ij}x^i y^j$ such that $\operatorname{Exp}(f;(x,y)) = \emptyset$. This ends the proof of the theorem.

A.6. *Remarks.* — We can add the situations of A.4 *b*) and A.5 as new final forms. In the case of A.5 we have to control the integral branches contained in F, but this is a two dimensional problem. Finally, note that $\dim \operatorname{Dir}(\mathscr{D},P) = 0$ is not possible, since $e(E,P) \geqslant 1$.

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