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A SIMPLEX WITH DENSE EXTREME POINTS

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1. — Introduction.

Let L be a locally convex linear topological space, and let C be a compact convex subset of L. The Krein-Milman theorem [3] asserts that C is the closed convex hull of the set E(C) of extreme points of C. It follows that for every $x \in C$ there exists a positive measure μ_x of mass 1 on $\overline{E(C)}$ such that

$$x = \int_{\overline{E(C)}} y \, d\mu_x(y).$$

This representation is of little interest in the case where $C = \overline{E(C)}$, and according to a result due to Klee [2] this is the rule rather than the exception.

Recently Choquet [1] has shown that if C is metrizable the measures μ_x may be chosen so as to be supported by E(C) itself, and furthermore that these measures are uniquely determined if and only if C is a simplex (i.e. such that the intersection of any two positive homothetic images of C is either empty, a single point or a positive homothetic image of C).

The question is raised by Choquet whether the situation $C = \overline{E(C)}$ can arise when C is a simplex. It is the object of this note to construct an example which shows that the answer is affirmative. The ideas governing the construction are closely related to the ideas of [4] where a simple example of a convex set with dense extreme points is exhibited. In § 2 we perform the actual construction of the simplex S and observe that $S = \overline{E(S)}$, and in § 3 we prove that S really is a simplex.

2. — Construction of the example.

In the Hilbert space l^2 of sequences

$$x = (\xi_1, \xi_2, \ldots, \xi_n, \ldots)$$

we denote by e_j the unit vector having the coordinates $\xi_i = \delta_{ij}$. Further, we denote by E_n the subspace spanned by $e_1, e_2, \ldots, e_n$ and by P_n the projection on E_n.

We first construct a sequence of simplexes S_n with the following properties:

(i) $S_n \subset E_n$ for every n.

(ii) $S_n \subset S_m$ and $E(S_n) \subset E(S_m)$ for $n < m$.

(iii) $P_n S_m = S_n$ for $n < m$.

(iv) for every $\varepsilon > 0$ there exists an n such that every point of S_n has distance at most ε from $E(S_n)$.

The construction of the simplexes S_n falls in groups as follows:

a) The first group consists of one simplex

$$S_1 = \{x | 0 \leqq \xi_1 \leqq 2^{-1};\ x \in E_1\}.$$

b) Assume that $S_1, S_2, \ldots, S_{n_p}$ have been constructed, S_{n_p} being the last simplex in the p'th group. Choose points $y_1, y_2, \ldots, y_{q_p}$ in S_{n_p} such that every point of S_{n_p} has distance at most 2^{-p} from the set $\{y_1, y_2, \ldots, y_{q_p}\}$.

$$\text{For } n_p < k \leqq n_p + q_p = n_{p+1} \text{ we define}$$
$$z_k = y_{k-n_p} + 2^{-k} e_{k'},$$

whereupon we define S_k as the convex hull of the set

$$S_{n_p} \cup \{z_{n_p+1}, \ldots, z_k\}.$$

With this construction it is clear that the sets S_n are simplexes satisfying (i), (ii), (iii) and (iv).

Now define

$$T_n = P_n^{-1}(S_n) = \{x | P_n x \in S_n\}$$

and

$$S = \bigcap_{n=1}^{\infty} T_n$$

It then follows that

(ii′) $T_n \supset T_m$ for $n < m$.
(iii′) $P_n T_m = S_n$ for $n < m$.
(iii″) $P_n S = S_n$ for all n.

(iv′) The set $\bigcup_{n=1}^{\infty} E(S_n)$ is dense in S.

Thus, to prove that $S = \overline{E(S)}$ it suffices to prove that $E(S_n) \subset E(S)$ for all n. The proof of this is exactly the same as in [4], but it is so short that we may as well repeat it here: Let $z \in E(S_n)$ and let $y \neq 0$. Then there exists $m \geq n$ so that $P_m y \neq 0$, and by (ii) $z \in E(S_m)$. Therefore, the segment

$$\{x | x = z + tP_m y ; -1 \leq t \leq 1\} \not\subset S_m,$$

and consequently

$$\{x | x = z + ty ; -1 \leq t \leq 1\} \not\subset S.$$

Hence, $z \in E(S)$.

Finally, let us note for completeness that S is compact and convex.

3. — Proof that S is a simplex.

We must prove that every set of the form

$$A = S \cap (qS + a) \qquad \text{with} \qquad q > 0$$

containing at least two points is itself of the form

$$A = rS + b \qquad \text{with} \qquad r > 0.$$

Now since

$$A = \bigcap_{n=1}^{\infty} T_n \cap (q \bigcap_{n=1}^{\infty} T_n + a)$$
$$= \bigcap_{n=1}^{\infty} (T_n \cap (qT_n + a))$$

each of the sets $T_n \cap (qT_n + a)$ contains at least two points, and therefore

$$P_n(T_n \cap (qT_n + a)) = S_n \cap (qS_n + a_n),$$

where $a_n = P_n a$, is non-empty for every n and contains at least two points for sufficiently large n.

Since S_n is a simplex, we have

$$S_n \cap (qS_n + a_n) = r_n S_n + b_n \qquad \text{with} \qquad r_n \geqq 0$$

for every n and $r_n > 0$ for sufficiently large n. Now, for $m > n$ we have

$$P_n(S_m \cap (qS_m + a_m)) \subset P_n S_m \cap P_n(qS_m + a_m),$$

i.e.

$$P_n(r_m S_m + b_m) \subset S_n \cap (qS_n + a_n)$$

or

$$r_m S_n + P_n b_m \subset r_n S_n + b_n$$

from where it follows that

1) $r_m \leqq r_n$.

2) $P_n b_m \in r_n S_n + b_n$ (since $0 \in S_n$).

By the construction all points of S_n have all their coordinates non-negative, and hence, writing

$$b_n = (\beta_{n1}, \beta_{n2}, \ldots, \beta_{nn}, 0, \ldots)$$

we get

3) $\beta_{mi} \geqq \beta_{ni}$ for all i.

From 1) it follows that

$$r_n \to r \ (\geqq 0) \qquad \text{for} \qquad n \to \infty$$

and from 3) that

$$\beta_{ni} \to \beta_i \text{ (for } n \to \infty) \text{ for all i.}$$

It is easily seen that the sequence

$$b = \{\beta_1, \beta_2, \ldots\}$$

belongs to l^2 and that

$$b_n \to b \qquad \text{for} \qquad n \to \infty$$

whence $b \in A$.

We shall complete our proof by showing that

$$A = rS + b.$$

First, since $r \leqq r_m$ for every m, we have

$$rS + b_m \subset rT_m + b_m \subset r_m T_m + b_m = T_m \cap (qT_m + a_m) = T_m \cap (qT_m + a)$$

for every m, and since

$$T_m \cap (qT_m + a) \subset T_n \cap (qT_n + a) \qquad \text{for} \qquad m > n$$

we have

$$rS + b_m \subset T_n \cap (qT_n + a) \qquad \text{for} \qquad m > n.$$

Since T_n is closed, it follows that

$$rS + b \subset T_n \cap (qT_n + a) \qquad \text{for every } n,$$

whence $rS + b \subset A$.

Secondly, since

$$r_n \geq r_m \qquad \text{for} \qquad m > n,$$

we have
$$\begin{aligned} r_nT_n + b_m &\supset r_nT_m + b_m \\ &\supset r_mT_m + b_m \\ &= T_m \cap (qT_m + a) \\ &\supset A \qquad \text{for every} \qquad m > n. \end{aligned}$$

It follows that

$$r_nT_n + b \supset A \qquad \text{for every } n,$$

hence also that

$$r_nT_m + b \supset r_mT_m + b \supset A \qquad \text{for} \qquad m > n,$$

whence
$$r_nS + b \supset A \qquad \text{for all } n.$$

From here, finally, it follows that

$$rS + b \supset A,$$

and the proof is completed.

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