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Deformations of coherent foliations on a compact normal space


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DEFORMATIONS OF COHERENT FOLIATIONS
ON A COMPACT NORMAL SPACE

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Introduction.

Let $X$ be a normal reduced compact analytic space with countable
topology. Let $\Omega^1_X$ be the coherent sheaf of holomorphic 1-forms on $X$ and
$\Theta_X = \text{Hom}_{O_X}(\Omega^1_X, O_X)$ its dual sheaf. The bracket of holomorphic vector
fields on the smooth part of $X$ induces a $C$-bilinear morphism
$m : \Theta_X \times \Theta_X \to \Theta_X$ (section 1); therefore, for any open subset $U$ of $X$, $m$
defines a map $m_U : \Theta_X(U) \times \Theta_X(U) \to \Theta_X(U)$ which is continuous for the
usual topology on $\Theta_X(U)$.

We shall study coherent foliations on $X$ (section 1 definition 2), using
the definition given in [2], this notion generalizes the notion of analytic
foliations on manifolds introduced by P. Baum ([1]) (see also [8]). A
coherent foliation on $X$ defines a quotient $O_X$-module of $\Theta_X$ by a $m$-stable
submodule (condition (i) of definition 2), this quotient being a non zero
locally free $O_X$-module outside a rare analytic subset of $X$ (condition (ii) of
definition (ii)).

Then the set of the coherent foliations on $X$ is a subset of the universal
space $H$ of all the quotient $O_X$-modules of $\Theta_X$; the analytic structure of $H$
has been constructed by A. Douady in [4].

The aim of this paper is to prove that the set of the quotient $O_X$-modules
of $\Theta_X$ which satisfy conditions (i) and (ii) of definition 2 is an analytic
subspace $\mathcal{H}$ of an open set of $H$ and that $\mathcal{H}$ satisfies a universal property
(Theorem 2). Any coherent foliation gives a point of $\mathcal{H}$, any point of $\mathcal{H}$
defines a coherent foliation but two different points of $\mathcal{H}$ can define the
same foliation (cf. section 1, remark 3).

Key-words: Singular holomorphic foliations - Deformations.
In section 2 one proves that, in the local situation, $m$-stability is an analytic condition on a suitable Banach analytic space (of infinite dimension).

In section 3 we follow the construction of the universal space of A. Douady and we get the analytic structure of $\mathcal{H}$.

**Notations:**

- For any analytic space $Y$ and any analytic space not necessarily of finite dimension $Z$ let us denote $p_{Z}: Z \times Y \to Y$ the projection.

- For any $O_{Z \times Y}$-module $\mathcal{F}$ and any $z \in Z$ let us denote $\mathcal{F}(z)$ the $O_{Y}$-module which is the restriction to $\{z\} \times Y$ of $\mathcal{F}$, by definition we have for any $y \in Y$

$$\mathcal{F}(z)_{y} = \mathcal{F}(z,y) \otimes_{O_{Z \times Y}} O_{Z,z}/m_{z}.$$

1. **Coherent foliations.**

Let $X$ be a reduced connected normal analytic space with countable topology; let $\Omega_{X}^{1}$ be the coherent sheaf of holomorphic differential 1-forms on $X$ and

$$\Theta_{X} = \text{Hom}_{O_{X}}(\Omega_{X}^{1}, O_{X})$$

$\Theta_{X}$ is called the tangent sheaf on $X$. Let $S$ be the singular locus of $X$, then $S$ is at least of codimension two and the restriction of $\Theta_{X}$ to $X - S$ is the sheaf of holomorphic vector fields on the manifold $X - S$.

**Bracket of two sections of $\Theta_{X}$.**

The bracket of two holomorphic vector fields on the manifold $X - S$ is well-defined; recall that, if $z = (z_{1}, \ldots, z_{p})$ denotes the coordinates on $C^{p}$, if $U$ is an open set in $C^{p}$ and if $a$ and $b$ are two holomorphic vector fields on $U$, with

$$a = \sum_{i=1}^{p} a_{i}(z) \frac{\partial}{\partial z_{i}}, \quad b = \sum_{i=1}^{p} b_{i}(z) \frac{\partial}{\partial z_{i}}$$

then we have $[a,b] = c$ with

$$c = \sum_{i=1}^{p} c_{i} \frac{\partial}{\partial z_{i}} \text{ where } c_{i} = \sum_{j=1}^{p} \left( a_{j} \frac{\partial b_{i}}{\partial z_{j}} - b_{j} \frac{\partial a_{i}}{\partial z_{j}} \right).$$
Let $m_U : O(U)^p \times O(U)^p \to O(U)^p$ be the $\mathbb{C}$-bilinear map which sends $((a_1, \ldots, a_p), (b_1, \ldots, b_p))$ onto $(c_1, \ldots, c_p)$; the Cauchy majorations imply the continuity of $m_u$ for the Frechet topology of uniform convergence on compacts of $U$.

**Proposition 1.** — For every open subset $U$ of $X$ the restriction homomorphism

$$\rho : H^0(U, \Theta_X) \to H^0(U - U \cap S, \Theta_X)$$

is an isomorphism of Frechet spaces.

**Proof.** — One knows that $\rho$ is continuous; by the open mapping theorem it is sufficient to prove that $\rho$ is bijective.

Now we may suppose that $X$ is an analytic subspace of an open set $V$ in $\mathbb{C}^n$; let $I$ be the coherent ideal sheaf defining $X$ in $V$; one has an exact sequence

$$0 \to \Theta_X \to O_X^e \xrightarrow{\alpha} \text{Hom}_{O_U}(I/I^2, O_X)$$

where the map $\alpha$ is defined by

$$\alpha(a_1, \ldots, a_n)(f) = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial z_i}|_X$$

$z_1, \ldots, z_n$ being the coordinates in $\mathbb{C}^n$.

Because the complex space $X$ is reduced and normal it follows from the second removable singularities theorem two isomorphisms

$$O_X(V) \cong O_X(V - S)$$
$$I(V) \cong I(V - S).$$

Then the proposition 1 follows from (1) and (2). As an immediate consequence of proposition 1 we obtain the following corollary:

**Corollary and Definition.** — It exists a unique homomorphism of sheaves of $\mathbb{C}$-vector spaces

$$m : \Theta_X \times \Theta_X \to \Theta_X$$

extending the bracket defined on $X - S$. Therefore, for every open subset $U$
of $X$, the induced map

$$m_U : H^0(U, \Theta_X) \times H^0(U, \Theta_X) \to H^0(U, \Theta_X)$$

is $C$-bilinear and continuous for the Frechet topology on $H^0(U, \Theta_X)$. We call bracket-map the sheaf morphism $m : \Theta_X \times \Theta_X \to \Theta_X$.

Coherent foliations.

**Definition 1.** A coherent $O_X$-submodule $T$ of $\Theta_X$ is said to be maximal if for any open $U \subset X$, any section $s \in \Theta_X(U)$ and any nowhere dense analytic set $A$ in $U$

$$s \in T(U - A) \Rightarrow s \in T(U)$$

holds.

Because $X$ is reduced and normal, then locally irreducible, $T$ is maximal if and only if $\Theta_X/T$ has no $O_X$-torsion.

**Definition 2** [2]. A coherent foliation on $X$ is a coherent $O_X$-submodule $T$ of $\Theta_X$ such that:

(i) $\Theta_X/T$ is non zero locally free outside a nowhere dense analytic subset of $X$;

(ii) $T$ is a subsheaf of $\Theta_X$ stable by the bracket-map;

(iii) $T$ is maximal.

Remarks. 1) A coherent foliation induces a classical smooth holomorphic foliation outside a nowhere dense analytic subset of $X - S$.

2) If $T$ is maximal the stability of $T$ by the bracket-map on $X$ is equivalent to the stability of $T$ on $X - A$, for any rare analytic subset $A$.

3) A coherent foliation on a connected reduced normal complex space $X$ is characterized by a quotient module $F$ of $\Theta_X$, without $O_X$-torsion, such that ker $[\Theta_X \to F]$ is stable by the bracket-map and which is a non zero locally free $O_X$-module outside a rare analytic subset of $X$.

4) Let $T$ be a coherent $O_X$-submodule of $\Theta_X$ satisfying conditions (i) and (ii) of definition 2; then $T$ is included in a maximal coherent sheaf $\hat{T}$ which is equal to $T$ outside a rare analytic subset of $X$ ([7] 2.7); the conditions (i) and (ii) are also fulfilled for $\hat{T}$, hence one can associate to $T$ a maximal foliation on $X$. But two different $T$ for which (i) and (ii) hold may give the same maximal sheaf $\hat{T}$. 
We suppose $X$ compact.

The purpose of this paper is to put an analytic structure on the set of all subsheaves of $\mathcal{O}_X$ satisfying conditions (i) and (ii) of Definition 2 (Theorem 2 below), that gives a versal family of holomorphic singular foliations for which a coherent extension exists.

First we have the following proposition:

**Proposition 2.** — Let $X$ be an irreducible complex space; let $Z$ be a complex space and $F$ a coherent $\mathcal{O}_{Z \times X}$-module. Let $F$ be $Z$-flat.

Let $Z_1$ be the set of points $z \in Z$ such that $F(z)$ is a non-zero locally free $\mathcal{O}_X$-module outside a rare analytic subset of $X$.

Then $Z_1$ is an open subset of $Z$.

**Proof.** — For every $z \in Z$ let $\sigma_z$ be the analytic subset of points $x \in X$ where $F(z)$ is not locally free ([3]). Put $z_0 \in Z_1$. The irreducibility of $X$ implies that $G_{z_0}$ is nowhere dense; fix $x_0 \in X - S \cap \sigma_{z_0}$ and denote $r > 0$ the rank of the $\mathcal{O}_{X,x_0}$-module $F(z_0)$. The $Z$-flatness of $F$ implies that $F$ is $\mathcal{O}_{Z \times X}$-free of rank $r$ in an open neighborhood $V$ of $(z_0, x_0)$. Let $U$ be the projection of $V$ on $Z$. For any point $z$ of the open set $U$ the $Z$-flatness of $F$ implies that $F(z)_{x_0}$ is $\mathcal{O}_{x_0}$-free of rank $r$; then the support of the sheaf $F(z)$ contains a neighborhood of $x_0$; hence the irreducibility of $X$ implies

$$\text{support } F(z) = X$$

and the proposition.

For any analytic space $S$ $m_S : p_S^* \mathcal{O}_X \times p_S^* \mathcal{O}_X \to p_S^* \mathcal{O}_X$ denotes the pull back of $m$ by the projection $p_S : S \times X \to X$ (i.e. the bracket map in the direction of the fibers of the projection $S \times X \to S$). Our aim is the proof of the following theorem:

**Theorem 1.** — Let $X$ be a compact connected normal space. There exist an analytic space $\bar{A}$ and a coherent $\mathcal{O}_{\bar{A} \times X}$-submodule $\mathcal{T}$ of $p_{\bar{A}}^* \mathcal{O}_X$ such that:

(i) $p_{\bar{A}}^* \mathcal{O}_X \big/ \mathcal{T}$ is $\bar{A}$-flat;

(ii) $\mathcal{T}$ is a $m_{\bar{A}}$-stable submodule of $p_{\bar{A}}^* \mathcal{O}_X$;

(iii) $(\bar{A}, \mathcal{T})$ is universal for properties (i) and (ii).

As a corollary of proposition 2 and theorem 1 we obtain:
THEOREM 2. — Let X be a compact connected normal space and r a positive integer. There exist an analytic space \( \mathcal{H} \) and a coherent \( O_{X \times X} \)-submodule \( \mathcal{E} \) of \( p^*_X \Theta_X \) such that:

(i) \( p^*_X \Theta_X/\mathcal{E} \) is \( \mathcal{H} \)-flat;

(ii) \( \mathcal{E} \) is \( m_X \)-stable and for any \( h \in \mathcal{H} \Theta_X/\mathcal{E}(h) \) is a locally free \( O_X \)-module of rank \( r \) outside a rare analytic subset of \( X \);

(iii) \( (\mathcal{H}, \mathcal{E}) \) is universal, i.e. for any analytic space \( S \) and any coherent \( O_{S \times X} \)-submodule \( \mathcal{F} \) of \( p^*_S \Theta_X \) such that

- \( p^*_S \Theta_X/\mathcal{F} \) is \( S \)-flat;
- \( \mathcal{F} \) is \( m_S \)-stable and for any \( s \in S \Theta_X/\mathcal{F}(s) \) is a locally free \( O_X \)-module of rank \( r \) outside a rare analytic subset of \( X \) then it exists a unique morphism \( f: S \rightarrow \mathcal{H} \) satisfying

\[
(f \times 1_X)^*(p^*_X \Theta_X/\mathcal{E}) = p^*_S \Theta_X/\mathcal{F}.
\]

We shall use the following theorem and Douady ([4]):

THEOREM. — Let \( X \) be a compact analytic space and \( \mathcal{E} \) a coherent \( O_X \)-module; there exist an analytic space \( \mathcal{H} \) and a quotient \( O_{\mathcal{H} \times X} \)-module \( \mathcal{R} \) of \( p^*_\mathcal{H} \mathcal{E} \) such that:

(i) \( \mathcal{R} \) is \( \mathcal{H} \)-flat;

(ii) for any analytic space \( S \) and any quotient \( O_{S \times \mathcal{H}} \)-module \( \mathcal{F} \) of \( p^*_S \mathcal{E} \) which is \( S \)-flat, it exists a unique morphism \( f: S \rightarrow \mathcal{H} \) satisfying

\[
(f \times 1_\mathcal{H})^* \mathcal{R} = \mathcal{F}.
\]

2. Local deformations.

One uses notations and results of [4]; the notions of infinite dimensional analytic spaces, called Banach analytic spaces, and of anaflatness are defined respectively in ([4] § 3) and in ([4] § 8).

In this section we fix an open subset \( U \) of \( C^n \), two compact polycylinders of non-empty interior \( K \) and \( K' \) satisfying

\[
K' \subset \hat{K} \subset K \subset U
\]

and a reduced normal analytic subspace \( X \) of \( U \). Let \( B(K) \) be the Banach algebra of those continuous functions on \( K \) which are analytic on the interior \( \hat{K} \) of \( K \); one defines \( B(K') \) in an analogous way.
For every coherent sheaf $\mathcal{F}$ on $U$, one knows that it exists finite free resolutions of $\mathcal{F}$ in a neighborhood of $K$; for such a resolution

$$(L.) \quad 0 \to L_n \to L_{n-1} \to \cdots \to L_0$$

let us consider the complex of Banach spaces

$$B(K,L.) = B(K) \otimes_{O(K)} H^0(K,L.)$$

and the vector space

$$B(K,\mathcal{F}) = \text{coker } [B(K;L_1) \to B(K,L_0)].$$

**Definition 1** ([4] §7, [5]). $K$ is $\mathcal{F}$-privileged if and only if it exists a finite free resolution $L.$ of $\mathcal{F}$ on a neighborhood of $K$ such that the complex $B(K,L.)$ is direct exact.

Then this is true for every finite free resolution; therefore $B(K,\mathcal{F})$ is a Banach space which does not depend of the resolution; $\mathcal{F}$-privileged polycylinders give fundamental systems of neighborhoods at every point of $U$. For a more geometric definition of privilege, the reader can refer to ([6]).

In the following, we always suppose that the two polycylinders $K$ and $K'$ are $\Theta_x$-privileged, $\Theta_x$ being the tangent sheaf defined by $1 - (*)$.

Let $G_K$ be the Banach analytic space of those $B(K)$-submodules $Y$ of $B(K,\Theta_x)$ (or equivalently of quotient modules) for which it exists an exact sequence of $B(K)$-modules

$$0 \to B(K)^n \to \cdots \to B(K)^0 \to B(K,\Theta_x) \to B(K,\Theta_x)/Y \to 0$$

which is a direct sequence of Banach vector spaces.

A universal sheaf $R_K$ on $G_K \times K$ is constructed in [4]; $R_K$ satisfies the following proposition:

**Proposition 1** ([4] §8 no 5). (i) $R_K$ is $G_K$-anaflat.

(ii) For every Banach analytic space $Z$ and for every $Z$-anaflat quotient $\mathcal{F}$ of $p_2^*\Theta_x$ it exists a natural morphism $\varphi : Z \to G_K$ such that

$$(\varphi \times 1_K)^* R_K = \mathcal{F}_{S \times K}.$$

Recall that the $Z$-anaflatness generalizes to the infinite dimensional space $Z$ the notion of flatness; pull back preserves anaflatness.
Let $G_{K,K}$ be the set of the $B(K)$-submodules $E$ of $B(K,\Theta_X)$, element of $G_K$, such that $E \otimes_{B(K)} B(K')$ gives an element of $G_{K'}$.

**Proposition 2.** — (i) $G_{K,K}$ is an open subset of $G_K$.

(ii) Let $\mathcal{R}$ be the pull back of $R_K$ by the inclusion $G_{K,K} \hookrightarrow G_K$. Then the map from $G_{K,K}$ to $G_K$ which maps every $B(K)$-module $E$ element of $G_{K,K}$ onto the $B(K')$-module $E \otimes_{B(K)} B(K')$ is given by a unique morphism

$$\rho_{K,K'} : G_{K,K} \rightarrow G_K$$

satisfying

$$\rho_{K,K'}^* R_{K'} = \mathcal{R}.$$ 

**Proof.** — Proposition 2 follows from ([4] 14 prop. 4).

Let $\rho_1 : B(K,\Theta_X) \times B(K,\Theta_X) \rightarrow \Theta_X(\hat{K}) \times \Theta_X(\hat{K})$ and $\rho_2 : \Theta_X(\hat{K}) \rightarrow B(K',\Theta_X)$ be the restriction homomorphisms and

$$m : \Theta_X(\hat{K}) \times \Theta_X(\hat{K}) \rightarrow \Theta_X(\hat{K})$$

the bracket map.

Let

$$m_{K,K'} : B(K,\Theta_X) \times B(K,\Theta_X) \rightarrow B(K',\Theta_X)$$

be the continuous $C$-bilinear map defined by

$$m_{K,K'} = \rho_2 \circ m \circ \rho_1.$$ 

**Definition 2.** — A $B(K)$-submodule $Y$ of $B(K,\Theta_X)$ is said to be $m_{K,K'}$-stable if it verifies:

(i) $Y$ is an element of $G_{K,K'}$,

(ii) for every $f$ and $g$ in $Y$ one has

$$m_{K,K'}(f,g) \in \rho_{K,K}(Y).$$

Then, if $\bar{\Theta}$ is a $m$-stable $O_X$-submodule of $\Theta_X$ such that $K$ and $K'$ are $\bar{\Theta}$-privileged, $B(K,\bar{\Theta})$ is $m_{K,K'}$-stable; the converse is not necessarily true; however we have the following proposition:
Proposition 3. — Let \( Y \) be a \( m_{K,K} \)-stable \( B(K) \)-submodule of \( B(K,\Theta_X) \); then \( Y \) defines in a natural way a coherent \( O_X \)-submodule of \( \Theta_X \) on \( \hat{K} \), the restriction to \( \hat{K}' \) of which is \( m \)-stable (i.e. stable by the bracket-map).

Proof. — Let \( B_Y \) be the privileged \( B_K \)-module given by \( Y \) ([6]); the restriction to \( \hat{K} \) of \( B_Y \) is a coherent sheaf; therefore one has ([6] th. 2.3 (ii) and prop. 2.11)

\[
Y = \hat{H}(K,B_Y)
\]

and the restriction homomorphism

\[
i: Y = H^0(K,B_Y) \to H^0(\hat{K},B_Y)
\]

is injective and has dense image; therefore the restriction \( B_{Y_K} \) is a submodule of \( \Theta_X \) ([4] § 8 lemme 1(b)), hence \( H^0(\hat{K}',B_Y) \) is a closed subspace of the Frechet space \( H^0(\hat{K}',\Theta_X) \).

Let us show that \( m_{K,K} \) induces a \( C \)-bilinear continuous map

\[
m: H^0(\hat{K},B_Y) \times H^0(\hat{K},B_Y) \to H^0(\hat{K}',B_Y).
\]

Take \( t_1, t_2 \) two elements of \( H^0(\hat{K},B_Y) \) and \( (t_1^n) \) and \( (t_2^n) \) two sequences of elements of \( Y \) with

\[
\lim_{n \to \infty} t_i^n = t_i, \quad i = 1, 2.
\]

Because the bracket-map \( m: H^0(\hat{K},\Theta_X) \times H^0(\hat{K},\Theta_X) \to H^0(\hat{K},\Theta_X) \) is continuous one has

\[
\lim_{n \to \infty} m(t_1^n, t_2^n) = m(t_1, t_2) \in H^0(\hat{K},\Theta_X).
\]

Therefore the \( m_{K,K} \)-stability of \( Y \) implies for every \( m \)

\[
m_{K,K}(t_1^n, t_2^n) \in B(\hat{K}',B_Y) \subseteq H^0(\hat{K}',B_Y)
\]

then \( m(t_1, t_2)|_{\hat{K}'} \in H^0(\hat{K}',B_Y) \) follows.

In order to prove the proposition it is sufficient to remark that, for every polycylinder \( K'' \subset \hat{K}' \), the restriction homomorphism

\[
H^0(\hat{K}',B_Y) \to H^0(\hat{K}'',B_Y)
\]

has a dense image. Q.E.D.
Recall some properties of infinite dimensional spaces: let $V$ be an open subset of a Banach $C$-vector space; let $F$ be a Banach vector space and $f: V \to F$ an analytic map. Let $\mathcal{X}$ the Banach analytic space defined by the equation $f = 0$; $\mathcal{X}$ is a local model of general Banach analytic space; the morphisms from $\mathcal{X}$ into a Banach vector space $G$ extend locally in analytic maps on open subsets of $V$; for such a morphism $\varphi: \mathcal{X} \to G$ the equation $\varphi = 0$ defines in a natural way a Banach analytic subspace of $\mathcal{X}$; the morphisms from a Banach analytic space $\mathcal{Y}$ into $\mathcal{X}$ are exactly the morphisms $\psi: \mathcal{Y} \to V$ such that $f \circ \psi = 0$.

**Proposition 4.** — Let $S_{K,K'}$ be the subset of elements of $G_{K,K'}$ which are $m_{K,K'}$-stable. Then $S_{K,K'}$ is a Banach analytic subspace of $G_{K,K'}$.

**Proof.** — Let $Y_0 \in S_{K,K'}$ and $Y_0' = \rho_{K,K}(Y_0)$; let $G_0$ (resp. $G_0'$) a closed $C$-vector subspace of $B(K,\Theta_{\chi})$ (resp. $B(K',\Theta_{\chi'})$) which is a topological supplementary of $Y_0$ (resp. $Y_0'$). Let $U_0$ (resp. $U_0'$) the set of closed $C$-vector subspaces of $B(K,\Theta_{\chi})$ (resp. $B(K',\Theta_{\chi'})$) which are topological supplementaries of $G_0$ (resp. $G_0'$); we identify $U_0$ and $L(Y_0,G_0)$, hence $U_0 \cap G_K$ is a Banach analytic subspace of $U_0(\text{[4] § 4})$.

For every $Y$ in $U_0$ one denotes $p_Y : B(K,\Theta_{\chi}) = Y \oplus G_0 \to G_0$ the projection and $j_Y : Y_0 \to Y \subset B(K,\Theta_{\chi})$ the reciprocal map of the restriction to $Y$ of the projection $B(K,\Theta_{\chi}) = Y_0 \oplus G_0 \to Y_0$.

Then the two maps

$$
p^K : G_K \to L(B(K,\Theta_{\chi}),G_0)
$$

$$
j^K : G_K \to L(Y_0,B(K,\Theta_{\chi}))
$$

defined by $p^K(Y) = p_Y$ and $j^K(Y) = j_Y$ are induced by morphisms ([4] § 4, n° 1); associated to the polycylinder $K'$ we have in the same way morphisms $p^K$ and $j^K$. Put $W_0 = G_{K,K'} \cap U_0 \cap \rho_{K,K}^{-1}(U_0')$; $W_0$ is an open subset of $G_{K,K'}$. Let be

$$
\varphi_1 = p^K \circ \rho_{K,K'} : W_0 \to L(B(K',\Theta_{\chi}),G_0')
$$

and $\Delta : G_K \to L(Y_0 \otimes Y_0,B(K',\Theta_{\chi}))$ the morphism defined by

$$
\Delta(Y) = m_{K,K'} \circ (j_Y \times j_Y).
$$

Let be $\varphi_2 = \Delta \circ j^K : W_0 \to L(Y_0 \otimes Y_0,B(K',\Theta_{\chi}))$; $\varphi_1$ and $\varphi_2$ are
morphism; let

\[ \phi : W_0 \to L(Y_0 \otimes Y_0, G'_0) \]

be the morphism defined by

\[ \phi(Y) = \phi_2(Y) \circ \phi_1(Y). \]

We have \( W_0 \cap S_{K,K'} = \phi^{-1}(0) \), hence \( S_{K,K'} \cap W_0 \) is a Banach analytic subspace of \( W_0 \); following ([4] § 4, n° 1 (i) and (ii)) one easily proves that the analytic structures obtained in the different charts of \( G_K \) and \( G_{K'} \) patch together in an analytic structure on \( S_{K,K'} \); that proves proposition 4.

Remark 1. — With the previous notations the morphisms of Banach analytic spaces \( g : Z \to S_{K,K'} \cap W_0 \) are the morphisms \( g : Z \to W_0 \) satisfying \( \phi \circ g = 0 \).

Let \( \iota : S_{K,K'} \to G_K \) be the inclusion and \( R_{K,K'} \) the pullback of \( R_K \) by \( \iota \); \( R_{K,K'} \) is \( S_{K,K'} \)-anaflat; by construction \( R_{K,K'} \) is a quotient of \( p_{S_{K,K}}^* \Theta_X \), then put

\[ R_{K,K'} = p_{S_{K,K}}^* \Theta_X / T_{K,K'}. \]

By anaflatness one obtains for every \( s \in S_{K,K'} \) exact sequence of coherent sheaves on \( \mathfrak{K} : \)

\[ 0 \to T_{K,K'}(s) \to \Theta_X \to R_{K,K'}(s) \to 0. \]

From the definition of the analytic structure of \( S_{K,K'} \) and from proposition 3 one deduces the following theorem:

Theorem 3. — (i) For every \( s \in S_{K,K'} \) the restriction to \( \mathfrak{K}' \) of the coherent subsheaf \( T_{K,K'}(s) \) of \( \Theta_X \) is stable by the bracket-map.

(ii) For every Banach analytic space \( Z \) and every quotient \( \mathcal{F} = p_{S_{K,K}}^* \Theta_X / T \) of \( p_{S_{K,K}}^* \Theta_X \) by a \( O_{Z \times X} \)-submodule \( T \) such that

- \( \mathcal{F} \) is \( Z \)-anaflat.
- \( T \) is \( m_Z \)-stable and for any \( z \in Z \) the polycylinders \( K \) et \( K' \) are \( \mathcal{F}(z) \)-privileged;

then the unique morphism \( g : Z \to G_K \) satisfying

\[ (g \times I_K)^* R_K = \mathcal{F} \]

factorizes through \( S_{K,K'} \) (i.e. it exists a unique morphism \( f : Z \to S_{K,K'} \) with \( r \circ f = g \)).
Remark 2. — We don’t know if the restriction of \( R_{K,K} \) to \( S_{K,K} \times \hat{K} \) is \( m_{S,K,K} \)-stable; but if \( S \) is a finite dimensional analytic space then the pull back of \( R_{K,K} \) by any morphism \( S \to S_{K,K} \) is \( m_S \)-stable.


In this section \( X \) denotes a compact reduced normal space and \( \Theta_X \) its tangent sheaf. Let \( H \) be the universal space of quotient \( O_X \)-modules of \( \Theta_X \) and \( \mathcal{R} \) the \( H \)-flat universal sheaf on \( H \times X \) ([4]). Put \( \mathcal{R} = p^\#_X \Theta_X/E \), \( E \) being a coherent submodule of \( p^\#_X \Theta_X \); for any \( h \in H \) \( \mathcal{R}(h) \) is a coherent submodule of \( \Theta_X \). We shall construct the space \( \hat{H} \) as an analytic subspace of an open subset of \( H \).

1. Refining of a privileged « cuirasse ».

Let \( M \) be a \( \Theta_X \)-privileged « cuirasse » ([4] § 9, n° 2); \( M \) is given by,

(i) a finite family \((\phi_i)_{i \in I}\) of charts of \( X \), i.e. for every \( i \in I \) \( \phi_i \) is an isomorphism from an open set \( X_i \subset X \) onto a closed analytic subspace of an open set \( U_i \) in \( C^n \),

(ii) for every \( i \in I \) a \( \Theta_X \)-privileged polycylinder \( K_i \subset U_i \) (i.e. a \( \phi_i \*-\Theta_X \)-privileged polycylinder), and an open set \( V_i \subset X_i \) satisfying

\[
V_i = \phi_i^{-1}(\hat{K}_i) \subset X_i
\]

\[
X = \bigcup_{i \in I} V_i
\]

(iii) for every \((i,j) \in I \times J\) a chart \( \phi_{ij} \) defined on \( X_i \cap X_j \) with values in an open \( U_{ij} \subset C^{n_j} \) and a finite family \((K_{i\alpha})\) of \( \Theta_X \)-privileged polycylinders in \( U_{ij} \) such that conditions

\[
V_i \cap V_j = \bigcup_{\alpha} \psi_{ij}^{-1}(K_{i\alpha})
\]

\[
\phi_{ij}^{-1}(K_{i\alpha}) \subset \phi_i^{-1}(\hat{K}_i) \cap \phi_j^{-1}(\hat{K}_j)
\]

are fulfilled.

As in ([4]) let us denote \( H_M \) the open subset of the elements \( F \) of \( H \) for which \( M \) is \( F \)-privileged (i.e. all the polycylinders \( K_i, K_{i\alpha} \) are \( F \)-privileged); we shall construct \( \hat{H} \) as union of open subsets \( \hat{H} \cap H_M \).

— For any \( \Theta_X \)-privileged polycylinder \( K \) let us denote \( G_K \) (§ 2) the Banach analytic space of quotients of \( B(K,\Theta_X) \) with finite direct resolution.
For every $i \in I$ let $G_i$ be the open subset of $G_{K_i}$ on which, for any $\alpha$, the restriction homomorphisms $B(K_i) \rightarrow B(K_{ij\alpha})$ induce morphisms $G_i \rightarrow G_{K_{ij\alpha}}$. The Douady construction of $H_M$ gives a natural injective morphism

$$i : H_M \rightarrow \prod_{i \in I} G_i.$$ 

**Definition 5.** — A refining of the « cuirasse » $M$ is given by a family $(K_i)_{i \in I}$ of poly-cylinders satisfying:

(i) for every $i \varphi_i(V_i) \subseteq \hat{K}_i \subseteq K_i \subseteq \hat{K}_i$,

(ii) for every $i, j, \alpha \varphi_{ij}^{-1}(K_{ij\alpha}) \subseteq \varphi_i^{-1}(\hat{K}_i) \cap \varphi_j^{-1}(\hat{K}_j)$,

(iii) for every $i K_i$ is $\Theta_X$-privileged.

We denote by $M((K_i))$ such a refining; for any coherent sheaf $\mathcal{F}$ on $X$ we shall say that $M((K_i))$ is $\mathcal{F}$-privileged if $M$ is $\mathcal{F}$-privileged and if, for every $i$, $K_i$ is $\mathcal{F}$-privileged.

**Lemma 1.** — (i) Let $\mathcal{F}$ be a coherent sheaf such that $M$ is $\mathcal{F}$-privileged; then it exists a $\mathcal{F}$-privileged refining of $M$.

(ii) Let $M((K_i))$ a refining of $M$; then the set of quotient $\mathcal{F}$ of $\Theta_X$ such that $M((K_i))$ is $\mathcal{F}$-privileged is open in $H_M$.

**Proof.** — (i) follows from ([4] § 7, n° 3 corollary of prop. 6) and (ii) is an immediate consequence of flatness and privilege.

2. Now we fix a $\Theta_X$-privileged « cuirasse » $M = M(I, (K_i), (V_i), (K_{ij\alpha}))$ and a $\Theta_X$-privileged refining $M((K_i))$ of $M$.

**Lemma 2.** — Let $H'_M$ be the subset of $H_M$ the points of which are quotients $\Theta_X/T$ satisfying:

(i) $M((K_i))$ is $\Theta_X/T$-privileged,

(ii) $T$ is a subsheaf of $\Theta_X$ stable by the bracket-map.

Then $H'_M$ is an analytic subspace of an open subset of $H_M$.

**Proof.** — Using notations of section 2 one puts for every $i \in I$

$$G'_i = G_{K_i,K_i} \cap G_i$$

$G'_i$ is an open subset of $G_i$ and $G_{K_i}$; put $S_i = S_{K_i,K_i} \cap G'_i$. 
One knows that the category of Banach analytic spaces has finite products, kernel of double arrows and hence fiber products (for all this notions the reader can refer to ([4] § 3, n° 3). Then \( \prod_{i \in I} S_i \) is a Banach analytic subspace of \( \prod_{i \in I} G_i \); since \( \prod_{i \in I} G_i \) is an open subset of \( \prod_{i \in I} G_i \) it follows from (§II Theorem 3)

\[
H'_M = H_M \times \prod_{i \in I} S_i
\]

and the lemma is proved.

— Now let \( R'_M \) (resp. \( T'_M \)) be the pull back of \( \mathcal{R} \) (resp. \( \mathcal{E} \)) by the inclusion morphism \( H'_M \times X \to H \times X \); \( R'_M \) is the quotient of \( p^*_H \Theta_X \) by \( T'_M \) (the sheaves \( T'_M \) and \( \ker[p^*_H \Theta_X \to R'_M] \) are \( H'_M \)-flat and equal on the fibers \( \{h\} \times X \).

**Lemma 3.** — \( T'_M \) is a \( m_{H'_M} \)-stable submodule of \( p^*_H \Theta_X \).

The proof follows immediately of the remark 2 of paragraph 2 and of

\[
X = \bigcup_{i \in I} V_i = \bigcup_{i \in I} \varphi_i^{-1}(K'_i).
\]

— Using the universal property of \( H_M \), Theorem 3 §2 and the commutative diagram

\[
\begin{array}{ccc}
H'_M \times X & \to & H_M \times X \\
\downarrow & & \downarrow \\
(\prod_{i \in I} G'_i) \times X & \to & (\prod_{i \in I} G_i) \times X
\end{array}
\]

we obtain the following proposition:

**Proposition 1.** — Let \( Z \) be an analytic space and \( T_Z \) a coherent subsheaf of \( p^*_Z \Theta_X \) satisfying:

(i) \( p^*_Z \Theta_X/T_Z \) is \( Z \)-flat.

(ii) For every \( z \in Z \) the cuirasse \( M((K'_i)) \) is \( \Theta_X/T_Z(z) \)-privileged.

(iii) \( T_Z \) is a \( m_Z \)-stable submodule of \( p^*_Z \Theta_X \).
Then the unique morphism \( g : Z \to H \) such that

\[
(g \times I_X)^* \mathcal{R} = p_2^* \Theta_X/T_z
\]

factorizes through \( H'_M \) and verifies

\[
(g \times I_X)^* T'_M = T_z.
\]

3. End of the proof of Theorem 1.

Notations are those of the previous proposition; the unicity of \( g \) implies the unicity of its factorization through the subspace \( H'_M \) of \( H \). Hence, when the refinings of a given \( M \) are varying, one obtains analytic spaces \( H'_M \) which patch together in an analytic subspace of an open subset of \( H_M \).

When the « cuirasse » \( M \) varies in the family of all the \( \Theta_X \)-privileged « cuirasse » the spaces \( H_M \) form an open covering of \( H \); then the universal property of the \( H_M \)'s implies that \( \mathcal{H} = \bigcup_M H'_M \) is an analytic subspace of an open subset of \( H \). Theorem 4 is proved.

Remark. — More generally if \( X \) is not compact, let \( \Theta \) be a coherent sheaf on \( X \) and \( m : \Theta \times \Theta \to \Theta \) a sheaf morphism inducing for each open set \( U \) a continuous \( C \)-bilinear map \( m_U : \Theta(U) \times \Theta(U) \to \Theta(U) \); let \( H \) be the Douady space of the coherent quotients of \( \Theta \) with compact support ([4]). We get a universal analytic structure on the subset of those quotients which are \( m \)-stable.

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