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# DEFORMATIONS OF COHERENT FOLIATIONS ON A COMPACT NORMAL SPACE

by Geneviève POURCIN

## Introduction.

Let X be a normal reduced compact analytic space with countable topology. Let $\Omega_X^1$ be the coherent sheaf of holomorphic 1-forms on X and $\Theta_X = \mathrm{Hom}_{O_X}(\Omega_X^1, O_X)$ its dual sheaf. The bracket of holomorphic vector fields on the smooth part of X induces a C-bilinear morphism $m : \Theta_X \times \Theta_X \to \Theta_X$ (section 1); therefore, for any open subset U of X, $m$ defines a map $m_U : \Theta_X(U) \times \Theta_X(U) \to \Theta_X(U)$ which is continuous for the usual topology on $\Theta_X(U)$.

We shall study coherent foliations on X (section 1 definition 2), using the definition given in [2], this notion generalizes the notion of analytic foliations on manifolds introduced by P. Baum ([1]) (see also [8]). A coherent foliation on X defines a quotient $O_X$-module of $\Theta_X$ by a $m$-stable submodule (condition (i) of definition 2), this quotient being a non zero locally free $O_X$-module outside a rare analytic subset of X (condition (ii) of definition (ii)).

Then the set of the coherent foliations on X is a subset of the universal space H of all the quotient $O_X$-modules of $\Theta_X$; the analytic structure of H has been constructed by A. Douady in [4].

The aim of this paper is to prove that the set of the quotient $O_X$-modules of $\Theta_X$ which satisfy conditions (i) and (ii) of definition 2 is an analytic subspace $\mathscr{H}$ of an open set of H and that $\mathscr{H}$ satisfies a universal property (Theorem 2). Any coherent foliation gives a point of $\mathscr{H}$, any point of $\mathscr{H}$ defines a coherent foliation but two different points of $\mathscr{H}$ can define the same foliation (cf. section 1, remark 3).

*Key-words:* Singular holomorphic foliations - Deformations.

In section 2 one proves that, in the local situation, $m$-stability is an analytic condition on a suitable Banach analytic space (of infinite dimension).

In section 3 we follow the construction of the universal space of A. Douady and we get the analytic structure of $\mathscr{H}$.

*Notations :*

– For any analytic space Y and any analytic space not necessarly of finite dimension Z let us denote $p_Z : Z \times Y \to Y$ the projection.

– For any $O_{Z\times Y}$-module $\mathscr{F}$ and any $z \in Z$ let us denote $\mathscr{F}(z)$ the $O_Y$-module which is the restriction to $\{z\} \times Y$ of $\mathscr{F}$, by definition we have for any $y \in Y$

$$\mathscr{F}(z)_y = \mathscr{F}_{(z,y)} \otimes_{O_{Z,z}} O_{Z,z}/m_z .$$

## 1. Coherent foliations.

Let X be a reduced connected normal analytic space with countable topology; let $\Omega^1_X$ be the coherent sheaf of holomorphic differential 1-forms on X and

$$(*) \qquad \Theta_X = \mathrm{Hom}_{O_X}(\Omega^1_X, O_X)$$

$\Theta_X$ is called the tangent sheaf on X. Let S be the singular locus of X, then S is at least of codimension two and the restriction of $\Theta_X$ to $X - S$ is the sheaf of holomorphic vector fields on the manifold $X - S$.

*Bracket of two sections of* $\Theta_X$.

The bracket of two holomorphic vector fields on the manifold $X - S$ is well-defined; recall that, if $z = (z_1, \ldots, z_p)$ denotes the coordinates on $\mathbf{C}^p$, if U is an open set in $\mathbf{C}^p$ and if $a$ and $b$ are two holomorphic vector fields on U, with

$$a = \sum_{i=1}^{p} a_i(z) \frac{\partial}{\partial z_i}, \qquad b = \sum_{i=1}^{p} b_i(z) \frac{\partial}{\partial z_i}$$

then we have $[a,b] = c$ with

$$c = \sum_{i=1}^{p} c_i \frac{\partial}{\partial z_i} \text{ where } c_i = \sum_{j=1}^{p} \left( a_j \frac{\partial b_i}{\partial z_j} - b_j \frac{\partial a_i}{\partial z_j} \right).$$

Let $m_U : O(U)^p \times O(U)^p \to O(U)^p$ be the **C**-bilinear map which sends $((a_1,\ldots,a_p), (b_1,\ldots,b_p))$ onto $(c_1,\ldots,c_p)$; the Cauchy majorations imply the continuity of $m_u$ for the Frechet topology of uniform convergence on compacts of U.

PROPOSITION 1. — *For every open subset* U *of* X *the restriction homomorphism*

$$\rho : H^0(U,\Theta_X) \to H^0(U-U_\cap S,\Theta_X)$$

*is an isomorphism of Frechet spaces.*

*Proof.* — One knows that $p$ is continuous; by the open mapping theorem it is sufficient to prove that $p$ is bijective.

Now we may suppose that X is an analytic subspace of an open set V in $\mathbf{C}^n$; let I be the coherent ideal sheaf defining X in V; one has an exact sequence

$$(1) \qquad O \to \Theta_X \to O_X^n \xrightarrow{\alpha} \mathrm{Hom}_{O_U}(I/I^2,O_X)$$

where the map $\alpha$ is defined by

$$\alpha(a_1,\ldots,a_n)(f) = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial z_i}\bigg|_X$$

$z_1, \ldots, z_n$ being the coordinates in $\mathbf{C}^n$.

Because the complex space X is reduced and normal it follows from the second removable singularities theorem two isomorphisms

$$(2) \qquad \begin{aligned} O_X(V) &\approx O_X(V-S) \\ I(V) &\approx I(V-S). \end{aligned}$$

Then the proposition 1 follows from (1) and (2). As an immediate consequence of proposition 1 we obtain the following corollary:

COROLLARY AND DEFINITION. — *It exists a unique homomorphism of sheaves of* **C**-*vector spaces*

$$m : \Theta_X \times \Theta_X \to \Theta_X$$

*extending the bracket defined on* X — S. *Therefore, for every open subset* U *of* X, *the induced map*

$$m_U : H^0(U,\Theta_X) \times H^0(U,\Theta_X) \to H^0(U,\Theta_X)$$

*is* **C**-*bilinear and continuous for the Frechet topology on* $H^0(U,\Theta_X)$. *We call bracket-map the sheaf morphism* $m : \Theta_X \times \Theta_X \to \Theta_X$.

*Coherent foliations.*

DEFINITION 1. — *A coherent* $O_X$-*submodule* T *of* $\Theta_X$ *is said to be maximal if for any open* $U \subset X$, *any section* $s \in \Theta_X(U)$ *and any nowhere dense analytic set* A *in* U

$$s \in T(U-A) \Rightarrow s \in T(U)$$

*holds.*

Because X is reduced and normal, then locally irreducible, T is maximal if and only if $\Theta_X/T$ has no $O_X$-torsion.

DEFINITION 2 [2]. — *A coherent foliation on* X *is a coherent* $O_X$-*submodule* T *of* $\Theta_X$ *such that:*

(i) $\Theta_X/T$ *is non zero locally free outside a nowhere dense analytic subset of* X;

(ii) T *is a subsheaf of* $\Theta_X$ *stable by the bracket-map;*

(iii) T *is maximal.*

*Remarks.* — 1) A coherent foliation induces a classical smooth holomorphic foliation outside a nowhere dense analytic subset of $X - S$.

2) If T is maximal the stability of T by the bracket-map on X is equivalent to the stability of T on $X - A$, for any rare analytic subset A.

3) A coherent foliation on a connected reduced normal complex space X is characterized by a quotient module F of $\Theta_X$, without $O_X$-torsion, such that ker $[\Theta_X \to F]$ is stable by the bracket-map and which is a non zero locally free $O_X$-module outside a rare analytic subset of X.

4) Let T be a coherent $O_X$-submodule of $\Theta_X$ satisfying conditions (i) and (ii) of definition 2; then T is included in a maximal coherent sheaf $\hat{T}$ which is equal to T outside a rare analytic subset of X ([7] 2.7); the conditions (i) and (ii) are also fullfilled for $\hat{T}$, hence one can associate to T a maximal foliation on X. But two different T for which (i) and (ii) hold may give the same maximal sheaf $\hat{T}$.

We suppose X compact.

The purpose of this paper is to put an analytic structure on the set of all subsheaves of $\Theta_X$ satisfying conditions (i) and (ii) of Definition 2 (Theorem 2 below), that gives a versal family of holomorphic singular foliations for which a coherent extension exists.

First we have the following proposition :

PROPOSITION 2. — *Let X be an irreducible complex space; let Z be a complex space and F a coherent $O_{Z\times X}$-module. Let F be Z-flat.*

*Let $Z_1$ be the set of points $z \in Z$ such that F(z) is a non-zero locally free $O_X$-module outside a rare analytic subset of X.*

*Then $Z_1$ is an open subset of Z.*

*Proof.* — For every $z \in Z$ let $\sigma_z$ be the analytic subset of points $x \in X$ where F(z) is not locally free ([3]). Put $z_0 \in Z_1$. The irreducibility of X implies that $G_{z_0}$ is nowhere dense; fix $x_0 \in X - S \cap \sigma_{z_0}$ and denote $r > 0$ the rank of the $O_{X,x_0}$-module $F(z_0)$. The Z-flatness of F implies that F is $O_{Z\times X}$-free of rank $r$ in an open neighborhood V of $(z_0, x_0)$. Let U be the projection of V on Z. For any point $z$ of the open set U the Z-flatness of F implies that $F(z)_{x_0}$ is $O_{X,x_0}$-free of rank $r$; then the support of the sheaf F(z) contains a neighborhood of $x_0$; hence the irreducibility of X implies

$$\text{support } F(z) = X$$

and the proposition.

For any analytic space S $m_S : p_S^*\Theta_X \times p_S^*\Theta_X \to p_S^*\Theta_X$ denotes the pull back of $m$ by the projection $p_S : S \times X \to X$ (i.e. the bracket map in the direction of the fibers of the projection $S \times X \to S$). Our aim is the proof of the following theorem :

THEOREM 1. — *Let X be a compact connected normal space. There exist an analytic space $\tilde{H}$ and a coherent $O_{\tilde{H}\times X}$-submodule $\tilde{T}$ of $p_{\tilde{H}}^*\Theta_X$ such that :*

(i) $p_{\tilde{H}}^*\Theta_X/\tilde{T}$ *is* $\tilde{H}$*-flat;*

(iii) $\tilde{T}$ *is a* $m_{\tilde{H}}$*-stable submodule of* $p_{\tilde{H}}^*\Theta_X$;

(iii) $(\tilde{H},\tilde{T})$ *is universal for properties* (i) *and* (ii).

As a corollary of proposition 2 and theorem 1 we obtain :

THEOREM 2. — *Let* X *be a compact connected normal space and* $r$ *a positive integer. There exist an analytic space* $\mathscr{H}$ *and a coherent* $O_{\mathscr{H}\times X}$-*submodule* $\mathscr{C}$ *of* $p_{\mathscr{H}}^{*}\Theta_X$ *such that :*

(i) $p_{\mathscr{H}}^{*}\Theta_X/\mathscr{C}$ *is* $\mathscr{H}$-*flat;*

(ii) $\mathscr{C}$ *is* $m_{\mathscr{H}}$-*stable and for any* $h \in \mathscr{H}\ \Theta_X/\mathscr{C}(h)$ *is a locally free* $O_X$-*module of rank* $r$ *outside a rare analytic subset of* X;

(iii) $(\mathscr{H},\mathscr{C})$ *is universal, i.e. for any analytic space* S *and any coherent* $O_{S\times X}$-*submodule* $\mathscr{F}$ *of* $p_S^{*}\Theta_X$ *such that*

— $p_S^{*}\Theta_X/\mathscr{F}$ *is* S-*flat;*

— $\mathscr{F}$ *is* $m_S$-*stable and for any* $s \in S\ \Theta_X/\mathscr{F}(s)$ *is a locally free* $O_X$-*module of rank* $r$ *outside a rare analytic subset of* X *then it exists a unique morphism* $f: S \to \mathscr{H}$ *satisfying*

$$(f\times I_X)^{*}(p_{\mathscr{H}}^{*}\Theta_X/\mathscr{C}) = p_S^{*}\Theta_X/\mathscr{F}.$$

We shall use the following theorem and Douady ([4]) :

THEOREM. — *Let* X *be a compact analytic space and* $\mathscr{E}$ *a coherent* $O_X$-*module; there exist an analytic space* H *and a quotient* $O_{H\times X}$-*module* $\mathscr{R}$ *of* $p_H^{*}\mathscr{E}$ *such that :*

(i) $\mathscr{R}$ *is* H-*flat;*

(ii) *for any analytic space* S *and any quotient* $O_{S\times H}$-*module* $\mathscr{F}$ *of* $p_S^{*}\mathscr{E}$ *which is* S-*flat, it exists a unique morphism* $f: S \to H$ *satisfying*

$$(f\times I_X)^{*}\mathscr{R} = \mathscr{F}.$$

## 2. Local deformations.

One uses notations and results of [4]; the notions of infinite dimensional analytic spaces, called Banach analytic spaces, and of anaflatness are defined respectively in ([4] § 3) and in ([4] § 8).

In this section we fix an open subset U of $\mathbf{C}^n$, two compact polycylinders of non-empty interior K and K′ satisfying

$$K' \subset \mathring{K} \subset K \subset U$$

and a reduced normal analytic subspace X of U. Let B(K) be the Banach algebra of those continuous functions on K which are analytic on the interior $\mathring{K}$ of K; one defines B(K′) in an analogous way.

For every coherent sheaf $\mathscr{F}$ on U, one knows that it exists finite free resolutions of $\mathscr{F}$ in a neighborhood of K; for such a resolution

$$(\mathrm{L.})\quad \mathrm{O} \to \mathrm{L}_n \to \mathrm{L}_{n-1} \to \cdots \to \mathrm{L}_0$$

let us consider the complex of Banach spaces

$$\mathrm{B}(\mathrm{K},\mathrm{L.}) = \mathrm{B}(\mathrm{K}) \otimes_{\mathrm{O}(\mathrm{K})} \mathrm{H}^0(\mathrm{K},\mathrm{L.})$$

and the vector space

$$\mathrm{B}(\mathrm{K},\mathscr{F}) = \operatorname{coker}\,[\mathrm{B}(\mathrm{K};\mathrm{L}_1)\to\mathrm{B}(\mathrm{K},\mathrm{L}_0)]\,.$$

DEFINITION 1([4] §7, [5]). — K *is $\mathscr{F}$-privileged if and only if it exists a finite free resolution* L. *of $\mathscr{F}$ on a neighborhood of* K *such that the complex* B(K,L.) *is direct exact.*

Then this is true for every finite free resolution; therefore B(K,$\mathscr{F}$) is a Banach space which does not depend of the resolution; $\mathscr{F}$-privileged polycylinders give fundamental systems of neighborhoods at every point of U. For a more geometric definition of privilege, the reader can refer to ([6]).

In the following, we always suppose that the two polycylinders K and K′ are $\Theta_X$-privileged, $\Theta_X$ being the tangent sheaf defined by 1 – (*).

Let $G_K$ be the Banach analytic space of those B(K)-submodules Y of B(K,$\Theta_X$) (or equivalently of quotient modules) for which it exists an exact sequence of B(K)-modules

$$\mathrm{O} \to \mathrm{B}(\mathrm{K})^{r_n} \to \cdots \to \mathrm{B}(\mathrm{K})^{r_0} \to \mathrm{B}(\mathrm{K},\Theta_X) \to \mathrm{B}(\mathrm{K},\Theta_X)/\mathrm{Y} \to \mathrm{O}$$

which is a direct sequence of Banach vector spaces.

A universal sheaf $R_X$ on $G_X \times \mathring{K}$ is constructed in [4]; $R_K$ satisfies the following proposition:

PROPOSITION 1 ([4] § 8 n° 5). — (i) $R_K$ *is* $G_K$*-anaflat.*

(ii) *For every Banach analytic space* Z *and for every* Z*-anaflat quotient $\mathscr{F}$ of $p_Z^*\Theta_X$ it exists a natural morphism* $\varphi : Z \to G_K$ *such that*

$$(\varphi\times I_{\mathring{K}})^*R_K = \mathscr{F}_{S\times\mathring{K}}\,.$$

Recall that the Z-anaflatness generalizes to the infinite dimensional space Z the notion of flatness; pull back preserves anaflatness.

Let $G_{K,K'}$ be the set of the B(K)-submodules E of $B(K,\Theta_X)$, element of $G_K$, such that $E \otimes_{B(K)} B(K')$ gives an element of $G_{K'}$.

PROPOSITION 2. — (i) *$G_{K,K'}$ is an open subset of $G_K$.*

(ii) *Let $\mathscr{R}$ be the pull back of $R_K$ by the inclusion $G_{K,K'} \hookrightarrow G_K$. Then the map from $G_{K,K'}$ to $G_{K'}$ which maps every B(K)-module E element of $G_{K,K'}$ onto the B(K′)-module $E \otimes_{B(K)} B(K')$ is given by a unique morphism*

$$\rho_{K,K'} : G_{K,K'} \to G_{K'}$$

*satisfying*

$$\rho^{*}_{K,K'} R_{K'} = \mathscr{R}.$$

*Proof.* — Proposition 2 follows from ([4] 14 prop. 4).

Let $\rho_1 : B(K,\Theta_X) \times B(K,\Theta_X) \to \Theta_X(\mathring{K}) \times \Theta_X(\mathring{K})$ and

$$\rho_2 : \Theta_X(\mathring{K}) \to B(K',\Theta_X)$$

be the restriction homomorphisms and

$$m : \Theta_X(\mathring{K}) \times \Theta_X(\mathring{K}) \to \Theta_X(\mathring{K})$$

the bracket map.

Let

$$m_{K,K'} : B(K,\Theta_X) \times B(K,\Theta_X) \to B(K',\Theta_X)$$

be the continuous C-bilinear map defined by

$$m_{K,K'} = \rho_2 \circ m \circ \rho_1 .$$

DEFINITION 2. — *A B(K)-submodule Y of $B(K,\Theta_X)$ is said to be $m_{K,K'}$-stable if it verifies :*

(i) *Y is an element of $G_{K,K'}$,*

(ii) *for every $f$ and $g$ in Y one has*

$$m_{K,K'}(f,g) \in \rho_{K,K'}(Y).$$

Then, if $\mathscr{E}$ is a $m$-stable $O_X$-submodule of $\Theta_X$ such that K and K′ are $\mathscr{E}$-privileged, $B(K,\mathscr{E})$ is $m_{K,K'}$-stable ; the converse is not necessarily true ; however we have the following proposition :

PROPOSITION 3. — *Let* Y *be a* $m_{K,K'}$*-stable* B(K)*-submodule of* $B(K,\Theta_X)$; *then* Y *defines in a natural way a coherent* $O_X$*-submodule of* $\Theta_X$ *on* $\mathring{K}$, *the restriction to* $\mathring{K}'$ *of which is m-stable (i.e. stable by the bracket-map).*

*Proof.* — Let $B_Y$ be the privileged $B_K$-module given by Y ([6]); the restriction to $\mathring{K}$ of $B_Y$ is a coherent sheaf; therefore one has ([6] th. 2.3 (ii) and prop. 2.11)

$$Y = \mathring{H}(K,B_Y)$$

and the restriction homomorphism

$$i : Y = H^0(K,B_Y) \to H^0(\mathring{K},B_Y)$$

is injective and has dense image; therefore the restriction $B_{Y|\mathring{K}}$ is a submodule of $\Theta_X$ ([4] § 8 lemme 1 (*b*)), hence $H^0(\mathring{K}',B_Y)$ is a closed subspace of the Frechet space $H^0(\mathring{K}',\Theta_X)$.

Let us show that $m_{K,K'}$ induces a C-bilinear continuous map

$$\mathring{m} : H^0(\mathring{K},B_Y) \times H^0(\mathring{K},B_Y) \to H^0(\mathring{K}',B_Y).$$

Take $t_1$, $t_2$ two elements of $H^0(\mathring{K},B_Y)$ and $(t_1^n)$ and $(t_2^n)$ two sequences of elements of Y with

$$\lim_{n\to\infty} t_i^n = t_i, \qquad i = 1, 2.$$

Because the bracket-map $m : H^0(\mathring{K},\Theta_X) \times H^0(\mathring{K},\Theta_X) \to H^0(\mathring{K},\Theta_X)$ is continuous one has

$$\lim_{n\to\infty} m(t^n_{1|\mathring{K}},t^n_{2|\mathring{K}}) = m(t_1,t_2) \in H^0(\mathring{K},\Theta_X).$$

Therefore the $m_{K,K'}$-stability of Y implies for every $m$

$$m_{K,K'}(t_1^n,t_2^n) \in B(K',B_Y) \subset H^0(\mathring{K}',B_Y)$$

then $m(t_1,t_2)|_{\mathring{K}'} \in H^0(\mathring{K}',B_Y)$ follows.

In order to prove the proposition it is sufficient to remark that, for every polycylinder $K'' \subset \mathring{K}'$, the restriction homomorphism

$$H^0(\mathring{K}',B_Y) \to H^0(\mathring{K}'',B_Y)$$

has a dense image. Q.E.D.

Recall some properties of infinite dimensional spaces : let V be an open subset of a Banach **C**-vector space; let F be a Banach vector space and $f: V \to F$ an analytic map. Let $\mathscr{X}$ the Banach analytic space defined by the equation $f = 0$; $\mathscr{X}$ is a local model of general Banach analytic space; the morphisms from $\mathscr{X}$ into a Banach vector space G extend locally in analytic maps on open subsets of V; for such a morphism $\varphi : \mathscr{X} \to G$ the equation $\varphi = 0$ defines in a natural way a Banach analytic subspace of $\mathscr{X}$; the morphisms from a Banach analytic space $\mathscr{Y}$ into $\mathscr{X}$ are exactly the morphisms $\psi : \mathscr{Y} \to V$ such that $f \circ \psi = 0$.

PROPOSITION 4. — *Let* $S_{K,K'}$ *be the subset of elements of* $G_{K,K'}$ *which are* $m_{K,K'}$*-stable. Then* $S_{K,K'}$ *is a Banach analytic subspace of* $G_{K,K'}$.

*Proof.* — Let $Y_0 \in S_{K,K'}$ and $Y'_0 = \rho_{K,K'}(Y_0)$; let $G_0$ (resp. $G'_0$) a closed **C**-vector subspace of $B(K,\Theta_X)$ (resp. $B(K',\Theta_X)$) which is a topological supplementary of $Y_0$ (resp. $Y'_0$). Let $U_0$ (resp. $U'_0$) the set of closed **C**-vector subspaces of $B(K,\Theta_X)$ (resp. $B(K',\Theta_X)$) which are topological supplementaries of $G_0$ (resp. $G'_0$); we identify $U_0$ and $L(Y_0,G_0)$, hence $U_0 \cap G_K$ is a Banach analytic subspace of $U_0$([4] § 4).

For every Y in $U_0$ one denotes $p_Y : B(K,\Theta_Y) = Y \oplus G_0 \to G_0$ the projection and $j_Y : Y_0 \to Y \subset B(K,\Theta_X)$ the reciprocal map of the restriction to Y of the projection $B(K,\Theta_X) = Y_0 \oplus G_0 \to Y_0$.

Then the two maps

$$p^K : G_K \to L(B(K,\Theta_X),G_0)$$
$$j^K : G_K \to L(Y_0,B(K,\Theta_X))$$

defined by $p^K(Y) = p_Y$ and $j^K(Y) = j_Y$ are induced by morphisms ([4] § 4, n° 1); associated to the polycylinder K′ we have in the same way morphisms $p^{K'}$ and $j^{K'}$. Put $W_0 = G_{K,K'} \cap U_0 \cap \rho_{K,K'}^{-1}(U'_0)$; $W_0$ is an open subset of $G_{K,K'}$. Let be

$$\varphi_1 = p^{K'} \circ \rho_{K,K'} : W_0 \to L(B(K',\Theta_X), G'_0)$$

and $\Delta : G_K \to L(Y_0 \underset{\pi}{\otimes} Y_0, B(K',\Theta_X))$ the morphism defined by

$$\Delta(Y) = m_{K,K'} \circ (j_Y \times j_Y).$$

Let be $\varphi_2 = \Delta \circ j^K : W_0 \to L(Y_0 \underset{\pi}{\otimes} Y_0, B(K',\Theta_X))$; $\varphi_1$ and $\varphi_2$ are morphisms; let

$$\varphi : W_0 \to L(Y_0 \underset{\pi}{\otimes} Y_0, G'_0)$$

be the morphism defined by

$$\varphi(Y) = \varphi_2(Y) \circ \varphi_1(Y).$$

We have $W_0 \cap S_{K,K'} = \varphi^{-1}(0)$, hence $S_{K,K'} \cap W_0$ is a Banach analytic subspace of $W_0$; following ([4] § 4, n° 1 (i) and (ii)) one easily proves that the analytic structures obtained in the different charts of $G_K$ and $G_{K'}$ patch together in an analytic structure on $S_{K,K'}$; that proves proposition 4.

*Remark* 1. — With the previous notations the morphisms of Banach analytic spaces $g : Z \to S_{K,K'} \cap W_0$ are the morphisms $g : Z \to W_0$ satisfying $\varphi \circ g = 0$.

Let $\imath : S_{K,K'} \to G_K$ be the inclusion and $R_{K,K'}$ the pullback of $R_K$ by $\imath$; $R_{K,K'}$ is $S_{K,K'}$-anaflat; by construction $R_{K,K'}$ is a quotient of $p^*_{S_{K,K'}}\Theta_X$, then put

$$R_{K,K'} = p^*_{S_{K,K'}}\Theta_X/T_{K,K'}.$$

By anaflatness one obtains for every $s \in S_{K,K'}$ an exact sequence of coherent sheaves on $\mathring{K}$:

$$O \to T_{K,K'}(s) \to \Theta_X \to R_{K,K'}(s) \to 0.$$

From the definition of the analytic structure of $S_{K,K'}$ and from proposition 3 one deduces the following theorem:

THEOREM 3. — (i) *For every* $s \in S_{K,K'}$ *the restriction to* $\mathring{K}'$ *of the coherent subsheaf* $T_{K,K'}(s)$ *of* $\Theta_X$ *is stable by the bracket-map.*

(ii) *For every Banach analytic space* Z *and every quotient* $\mathscr{F} = p^*_Z\Theta_X/T$ *of* $p^*_Z\Theta_X$ *by a* $O_{Z\times X}$*-submodule* T *such that*

— $\mathscr{F}$ *is* Z*-anaflat.*

— T *is* $m_Z$*-stable and for any* $z \in Z$ *the polycylinders* K *et* K′ *are* $\mathscr{F}(z)$*-privileged;*

*then the unique morphism* $g : Z \to G_K$ *satisfying*

$$(g \times I_{\mathring{K}})^* R_K = \mathscr{F}$$

*factorizes through* $S_{K,K'}$ *(i.e. it exists a unique morphism* $f : Z \to S_{K,K'}$ *with* $r \circ f = g$*).*

*Remark* 2. — We don't know if the restriction of $R_{K,K'}$ to $S_{K,K'} \times \mathring{K}'$ is $m_{S_{K,K'}}$-stable; but if S is a finite dimensional analytic space then the pull back of $R_{K,K'}$ by any morphism $S \to S_{K,K'}$ is $m_S$-stable.

## 3. Proof of theorem 1.

In this section X denotes a compact reduced normal space and $\Theta_X$ its tangent sheaf. Let H be the universal space of quotient $O_X$-modules of $\Theta_X$ and $\mathcal{R}$ the H-flat universal sheaf on $H \times X$ ([4]). Put $\mathcal{R} = p_H^*\Theta_X/\mathcal{T}$, $\mathcal{T}$ being a coherent submodule of $p_H^*\Theta_X$; for any $h \in H$ $\mathcal{T}(h)$ is a coherent submodule of $\Theta_X$. We shall construct the space $\tilde{H}$ as an analytic subspace of an open subset of H.

1. *Refining of a privileged « cuirasse ».*

Let M be a $\Theta_X$-privileged « cuirasse »» ([4] § 9, n° 2); M is given by,

(i) a finite family $(\varphi_i)_{i \in I}$ of charts of X, i.e. for every $i \in I$ $\varphi_i$ is an isomorphism from an open set $X_i \subset X$ onto a closed analytic subspace of an open set $U_i$ in $\mathbf{C}^{n_i}$,

(ii) for every $i \in I$ a $\Theta_X$-privileged polycylinder $K_i \subset U_i$ (i.e. a $\varphi_{i*}\Theta_X$-privileged polycylinder). and an open set $V_i \subset X_i$ satisfying

$$\bar{V}_i \subset \varphi_i^{-1}(\mathring{K}_i) \subset X_i$$

$$X = \bigcup_{i \in I} V_i$$

(iii) for every $(i,j) \in I \times J$ a chart $\varphi_{ij}$ defined on $X_i \cap X_j$ with values in an open $U_{ij} \subset \mathbf{C}^{n_{ij}}$ and a finite family $(K_{ij_\alpha})$ of $\Theta_X$-privileged polycylinders in $U_{ij}$ such that conditions

$$\bar{V}_1 \cap \bar{V}_j \subset \bigcup_{\alpha} \psi_{ij}^{-1}(K_{ij\alpha})$$

$$\varphi_{ij}^{-1}(K_{ij\alpha}) \subset \varphi_i^{-1}(\mathring{K}_i) \cap \varphi_j^{-1}(\mathring{K}_j)$$

are fullfilled.

As in ([4]) let us denote $H_M$ the open subset of the elements F of H for which M is F-privileged (i.e. all the polycylinders $K_i$, $K_{ij\alpha}$ are F-privileged); we shall construct $\tilde{H}$ as union of open subsets $\tilde{H} \cap H_M$.

— For any $\Theta_X$-privileged polycylinder K let us denote $G_K$ (§ 2) the Banach analytic space of quotients of $B(K,\Theta_X)$ with finite direct resolution.

For every $i \in I$ let $G_i$ be the open subset of $G_{K_i}$ on which, for any $\alpha$, the restriction homomorphisms $B(K_i) \to B(K_{ij\alpha})$ induce morphisms $G_i \to G_{K_{ij\alpha}}$. The Douady construction of $H_M$ gives a natural injective morphism

$$i : H_M \to \prod_{i \in I} G_i .$$

DEFINITION 5. — *A refining of the « cuirasse » M is given by a family $(K'_i)_{i \in I}$ of polycylinders satisfying :*

(i) *for every i* $\varphi_i(V_i) \subset \mathring{K}'_i \subset K'_i \subset \mathring{K}_i$,

(ii) *for every* $i, j, \alpha$ $\varphi_{ij}^{-1}(K_{ij\alpha}) \subset \varphi_i^{-1}(\mathring{K}'_i) \cap \varphi_j^{-1}(\mathring{K}'_j)$,

(iii) *for every i* $K'_i$ *is* $\Theta_X$*-privileged.*

We denote by $M((K'_i))$ such a refining; for any coherent sheaf $\mathscr{F}$ on X we shall say that $M((K'_i))$ is $\mathscr{F}$-privileged if M is $\mathscr{F}$-privileged and if, for every $i$, $K'_i$ is $\mathscr{F}$-privileged.

LEMMA 1. — (i) *Let $\mathscr{F}$ be a coherent sheaf such that* M *is $\mathscr{F}$-privileged; then it exists a $\mathscr{F}$-privileged refining of* M.

(ii) *Let* $M((K'_i))$ *a refining of* M; *then the set of quotient $\mathscr{F}$ of $\Theta_X$ such that* $M((K'_i))$ *is $\mathscr{F}$-privileged is open in* $H_M$.

*Proof.* — (i) follows from ([4] § 7, n° 3 corollary of prop. 6) and (ii) is an immediate consequence of flatness and privilege.

**2.** Now we fix a $\Theta_X$-privileged « cuirasse » $M = M(I,(K_i),(V_i),(K_{ij\alpha}))$ and a $\Theta_X$-privileged refining $M((K'_i))$ of M.

LEMMA 2. — *Let $H'_M$ be the subset of $H_M$ the points of which are quotients $\Theta_X/T$ satisfying :*

(i) $M((K'_i))$ *is* $\Theta_K/T$*-privileged,*

(ii) T *is a subsheaf of* $\Theta_X$ *stable by the bracket-map.*

*Then $H'_M$ is an analytic subspace of an open subset of $H_M$.*

*Proof.* — Using notations of section 2 one puts for every $i \in I$

$$G'_i = G_{K_i,K'_i} \cap G_i$$

$G'_i$ is an open subset of $G_i$ and $G_{K_i}$; put $S_i = S_{K_i,K'_i} \cap G'_i$.

One knows that the category of Banach analytic spaces has finite products, kernel of double arrows and hence fiber products (for all this notions the reader can refer to ([4] § 3, n° 3). Then $\prod_{i \in I} S_i$ is a Banach analytic subspace of $\prod_{i \in I} G'_i$; since $\prod_{i \in I} G'_i$ is an open subset of $\prod_{i \in I} G_i$ it follows from (§ II Theorem 3)

$$H'_M = H_M \times_{\prod_{i \in I} G_i} \prod_{i \in I} S_i$$

and the lemma is proved.

– Now let $R'_M$ (resp. $T'_M$) be the pull back of $\mathcal{R}$ (resp. $\mathcal{T}$) by the inclusion morphism $H'_M \times X \to H \times X$; $R'_M$ is the quotient of $p^*_{H'_M}\Theta_X$ by $T'_M$ (the sheaves $T'_M$ and $\ker [p^*_{H'_M}\Theta_X \to R'_M]$ are $H'_M$-flat and equal on the fibers $\{h\} \times X$).

LEMMA 3. – *$T'_M$ is a $m_{H'_M}$-stable submodule of $p^*_{H'_M}\Theta_X$.*

The proof follows immediatly of the remark 2 of paragraph 2 and of

$$X = \bigcup_{i \in I} V_i = \bigcup_{i \in I} \varphi_i^{-1}(\mathring{K}'_i).$$

– Using the universal property of $H_M$, Theorem 3 § 2 and the commutative diagram

$$\begin{array}{ccc} H'_M \times X & \to & H_M \times X \\ \downarrow & & \downarrow \\ \left(\prod_{i \in I} G'_i\right) \times X & \to & \left(\prod_{i \in I} G_i\right) \times X \end{array}$$

we obtain the following proposition:

PROPOSITION 1. – *Let Z be an analytic space and $T_Z$ a coherent subsheaf of $p^*_Z\Theta_X$ satisfying:*

(i) *$p^*_Z\Theta_X/T_Z$ is Z-flat.*

(ii) *For every $z \in Z$ the cuirasse $M((K'_i))$ is $\Theta_X/T_Z(z)$-privileged.*

(iii) *$T_Z$ is a $m_Z$-stable submodule of $p^*_Z\Theta_X$.*

*Then the unique morphism* $g : Z \to H$ *such that*

$$(g \times I_X)^* \mathscr{R} = p_Z^* \Theta_X / T_Z$$

*factorizes through* $H'_M$ *and verifies*

$$(g \times I_X)^* T'_M = T_Z .$$

**3.** *End of the proof of Theorem* 1.

Notations are those of the previous proposition; the unicity of $g$ implies the unicity of its factorization through the subspace $H'_M$ of H. Hence, when the refinings of a given M are varying, one obtains analytic spaces $H'_M$ which patch together in an analytic subspace of an open subset of $H_M$.

When the « cuirasse » M varies in the family of all the $\Theta_X$-privileged « cuirasse » the spaces $H_M$ form an open covering of H; then the universal property of the $H_M$ 's implies that $\tilde{H} = \bigcup_M H'_M$ is an analytic subspace of an open subset of H. Theorem 4 is proved.

*Remark.* — More generally if X is not compact, let $\Theta$ be a coherent sheaf on X and $m : \Theta \times \Theta \to \Theta$ a sheaf morphism inducing for each open set U a continuous **C**-bilinear map $m_U : \Theta(U) \times \Theta(U) \to \Theta(U)$; let H be the Douady space of the coherent quotients of $\Theta$ with compact support ([4]). We get a universal analytic structure on the subset of those quotients which are $m$-stable.

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