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APPLICATIONS OF CONVEX INTEGRATION TO SYMPLECTIC AND CONTACT GEOMETRY

by Dusa McDUFF (*)

1. Introduction.

Gromov's method of convex integration [G2, 3] gives a very general way of constructing differential forms on closed manifolds. It is not powerful enough actually to construct symplectic or contact forms. However it does construct such forms "in codimension 1": for example, it constructs closed 2-forms of maximal rank on odd-dimensioned manifolds. Geometrically, such forms correspond to 1-dimensional foliations with transverse symplectic or contact structures. In this paper we discuss the implications of such constructions. In particular, we give a geometric meaning to the concept of formal equivalence of symplectic or contact structures. We also calculate the first non-trivial homotopy group of the classifying space for transversally symplectic (or contact) foliations.

We begin by describing the problems which concern us. A symplectic form σ on a manifold X^{2m} gives rise to a cohomology class $[\sigma] \in H^2(X, \mathbf{R})$ and to a homotopy class of almost complex structures on X, or, equivalently, to a homotopy class of reductions of the structural group of the tangent bundle of X to the unitary

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group $U(m)$. Similarly, a transversally oriented contact structure on X^{2m+1} gives rise to a homotopy class of reductions of this structural group to $U(m) \oplus 1$. This will be called the formal data underlying the symplectic or contact structure. We are interested in the existence problem: is there always a structure corresponding to given formal data ? If such a structure exists, we would also like to know how unique it is.

The techniques of Gromov's thesis [G1, 3] solve the existence problem in the affirmative on open manifolds. No general method is known for closed manifolds, although Lutz and Martinet have solved it in the 3-dimensional case by very geometric methods. See [M]. Techniques for constructing contact and symplectic structures may be found, for example, in [G3], [MD4], [Me], [T], [TW]. See also the references in [B]. However Gromov's most recent results in [G4] suggest that the existence problem may have a negative answer, in general.

It is now known that formally equivalent structures need not be diffeomorphic. Bennequin [B] constructed explicit counter examples in the case of contact manifolds of dimension 3. See also [E], [G4], [MD5]. In § 3 below, we discuss the connection between different notions of equivalence for symplectic structures. We also define some further isotopy invariants of cohomological type which arise for symplectic forms on open manifolds.

Convex integration allows one to construct from appropriate formal data closed non degenerate 2-forms τ (i.e. τ^m never vanishes) on manifolds of dimension $2m + 1$, as well as non-degenerate 1-forms α (i.e. $\alpha \wedge d\alpha^{m-1}$ never vanishes) on manifolds of dimension $2m$. Our first application of this result is given in § 4, where we interpret geometrically what it means for two symplectic structures to have the same formal data. The analogous result in the contact case appears in § 7.

In § 5 we use Sullivan's language of structural cycles to pinpoint the difference between a contact structure and a non-vanishing flow with transverse exact symplectic structure. Since the formal data for a contact structure gives rise to such a flow, this shows just how far the method of convex integration goes towards solving the existence problem in the contact case. This result has no symplectic analogue since symplectic forms do not give rise to flows.

Our final application of convex integration is a calculation of $\pi_n(\mathrm{B}\,\Gamma^n_{\mathrm{str}})$, where for even (resp. odd) n, $\mathrm{B}\,\Gamma^n_{\mathrm{str}}$ denotes the classifying space for codimension n foliations with transverse symplectic (resp. contact) structure. See § 6, § 7.

In order to make this paper reasonably self-contained, we begin by describing the method of convex integration. Here is a list of the contents.

Throughout we will write $\gamma \neq 0$ to mean that the form γ never vanishes. Also, a 2-form (resp. 1-form) γ on a manifold of dimension n will be called *non-degenerate* if $\gamma^k \neq 0$ where $k = \left[\frac{n}{2}\right]$ $\left(\text{resp. } \gamma \wedge d\gamma^k \neq 0 \text{ where } k = \left[\frac{n-1}{2}\right]\right)$.

2. Convex Integration.

In this section we state the version of Gromov's theorem which we shall need. Then we discuss the two main examples, the symplectic case (2.2) and the contact case (2.6).

Let $p : \mathrm{X} \longrightarrow \mathrm{V}$ be a smooth fibration, X^r be the manifold of r-jets of germs of sections of p and $p^r : \mathrm{X}^r \longrightarrow \mathrm{V}$ be the natural projection. A *differential relation* is a set $\Omega \subset \mathrm{X}^r$. A *solution* of the relation Ω is a C^r-section $f : \mathrm{V} \longrightarrow \mathrm{X}$ whose r-jet $\mathrm{J}^r(f_0)$ takes V into Ω. Let f_0 be a solution of Ω defined in an open neighbourhood of the closed subset W of V. Then we write $\Gamma(\Omega, f_0, \mathrm{W})$ for the space of all continuous sections $\mathrm{V} \longrightarrow \Omega$ which equal $\mathrm{J}^r(f_0)$ on W, with the compact-open topology. Similarly, let $\mathrm{Sol}(\Omega, f_0, \mathrm{W})$ be the space of all C^r-solutions of

Ω over V which equal f_0 near W, with the appropriate direct limit C^r-topology. Then we say that Ω satisfies the *relative weak homotopy equivalence principle* (or relative *w.h.e.* principle) if the map $f \longmapsto J^r(f)$ induces a weak homotopy equivalence $\mathrm{Sol}(\Omega, f_0, W) \longrightarrow \Gamma(\Omega, f_0, W)$ for all closed submanifolds $W \subset V$ and all f_0. The method of convex integration gives a very general condition under which a relation $\Omega \subset X^1$ satisfies this principle. To state it, we need the following definitions.

A subset Q of an affine space L is said to be *ample* if, for every connected component Q_0 of Q, the convex hull $\mathrm{Conv}(Q_0)$ of Q_0 equals L. For example, if Q is the complement $L - W$ of an affine subspace W of L, then Q is ample unless W has codimension 1. In particular, the empty set is considered to be ample. Here we follow the terminology of [G2]. The definitions in [G3] are more general.

A *q-dimensional direction* in an affine space L is a decomposition of L into a disjoint union of parallel q-dimensional affine subspaces. A subset $Q \subset L$ is said to be ample in a given direction if, for every subspace S in this direction, the set $Q \cap S$ is ample in S.

Now let us go back to the fibration $p : X \longrightarrow V$ and the associated fibration $p^1 : X^1 \longrightarrow X$, where X^1 is the space of 1-jets of section of p. If $u_1, \ldots, u_n$ are local coordinates about $v \in V$ and $y_1, \ldots, y_q$ are local coordinates in the fiber $p^{-1}(v)$ about $x \in X$, then the fiber $(p^1)^{-1}(x)$ has coordinates $a_{jk}, 1 \leqslant j \leqslant n, 1 \leqslant k \leqslant q$, where (a_{jk}) represents the 1-jet $\left(x, \Sigma\, a_{jk} \dfrac{\partial y_k}{\partial u_j}\right)$. One can easily check that these coordinates provide the map $p^1 : X^1 \longrightarrow X$ with the structure of an nq-dimensional affine bundle over X. Also, each coordinate u_i determines a q-dimensional direction in the fiber $(p^1)^{-1}(x)$ as follows: the affine subspace containing the point (b_{jk}) is just $\{(a_{jk}) : a_{jk} = b_{jk} \text{ if } j \neq i\}$. Note that these directions do not depend on the choice of the fiber coordinates y_k. In fact, the *ith.* direction in $(p^1)^{-1}(x)$ may be defined as follows: two jets are in the same affine subspace if they may be represented by germs of sections whose restrictions to the submanifold $u_i = \text{const.}$ through $p(x)$ are equal. These directions in the fibers of p^1 are called *coordinate directions.* The version of Gromov's theorem which we shall use is the following:

THEOREM 2.1. – *Let* $p : X \longrightarrow V$ *be a smooth fibration and* $\Omega \subset X^1$ *an open set. If, for every* $x \in X$, *there are local coordinates about* $p(x)$ *such that* Ω *is ample in the coordinate directions for all the fibers of* p^1 *sufficiently close to that over* x, *then* Ω *satisfies the relative w.h.e. principle.*

This is proved in [G2] § 2,3. See also [G3] 2.4.1-2.4.3 and [*Sp*]. The essential idea is contained in the proof below.

Proof of the simplest case. – Let $(V, W) = (I, \partial I), X = I \times \mathbf{R}^q$ and $\Omega = X \times A \subset X \times \mathbf{R}^q = X^1$. We will show that

$$\pi_0(\mathrm{Sol}(\Omega, f_0, \partial I)) = \pi_0(\Gamma(\Omega, f_0, \partial I))$$

for any solution f_0 of Ω which is defined near ∂I.

A section $\phi_0 \in \Gamma(\Omega, f_0, \partial I)$ is a pair (θ_0, ψ_0) where $\theta_0 : I \longrightarrow R^q$ and $\psi_0 : I \longrightarrow A$ and where

$$(\theta_0(t), \psi_0(t)) = \left(f_0(t), \frac{df_0}{dt}(t)\right)$$

for $t = 0, 1$. We must show that ϕ_0 can be homotoped relative to ∂I to a section (θ, ψ) such that $\psi(t) = \dfrac{d\theta}{dt}(t)$ for all t. Let A_0 be the connected component of A which contains $\dfrac{df_0}{dt}(1)$. Because A is ample, the element $f_0(1) - f_0(0)$ is a convex linear combination of elements of A_0. Therefore, there is a step function $\sigma : I \longrightarrow A_0$ whose integral $\int_0^1 \sigma(s)\, ds$ equals $f_0(1) - f_0(0)$. Because $\dfrac{df_0}{dt}(1)$ belongs to the open, connected set A_0, this function σ may be approximated by a continuous loop $(I, \partial I) \longrightarrow \left(A_0, \dfrac{df_0}{dt}(1)\right)$ whose integral is arbitrarily close to $f_0(1) - f_0(0)$. Let $\hat{\psi} : I \longrightarrow A_0$ be a continuous path formed by going quickly along ψ_0 and then round such a loop. By parametrizing $\hat{\psi}$ suitably, we may assume that the integral $\int_0^1 \hat{\psi}(s)\, ds$ is arbitrarily close to $f_0(1) - f_0(0)$. Because A_0 is open, $\hat{\psi}$ may be adjusted to a path ψ whose integral is exactly $f_0(1) - f_0(0)$. (For example, we may assume that $\hat{\psi}$ equals σ

on some fixed interval J, and has integral arbitrarily close to $f_0(1) - f_0(0)$. Then we may adjust $\hat{\psi}$ on J). Now define

$$\theta(t) = f_0(0) + \int_0^t \psi(s)\, ds.$$

Then, by construction, $\theta(t) = f_0(t)$ and $\psi(t) = \frac{df_0}{dt}(t)$ for $t = 0, 1$. Also, since we may assume that the loop which approximates σ is contractible, the initial path ψ_0 is homotopic rel ∂I to $\hat{\psi}$, and hence to ψ, through paths in A_0. Since $\theta : I \longrightarrow \mathbf{R}^q$ is obviously homotopic rel ∂I to $\theta_0 : I \longrightarrow \mathbf{R}^q$, the proof is complete.

□

In [G2, 3] Gromov gives many applications of this theorem, and also discusses variants which apply when Ω is closed or when $r > 1$. In particular he points out that the theorem applies when Ω is the complement of a generic singularity $\Sigma \subset X^1$ of codimension $\geqslant 2$. The examples which follow are of this type.

Example 2.2. – Let $X = T^*V$, so that sections of $p : X \longrightarrow V$ are 1-forms on V. Because p has a canonical section (the zero section), the bundle $p^1 : X^1 \longrightarrow X$ has a linear rather than just an affine structure. In fact, it is the pull-back over p of a vector bundle $q : J \longrightarrow V$, and there is a commutative diagram

$$\begin{tikzcd} X^1 \arrow[r, "g"] \arrow[d, "p^1"'] & J \arrow[r, "A"] \arrow[d, "q"] & \Lambda^2(T^*V) \arrow[d] \\ X \arrow[r, "p"] & V \arrow[r, "id"] & V \end{tikzcd} \qquad (2.3)$$

where A is the unique vector bundle homomorphism such that

$$d\alpha = A \circ g(J^1(\alpha)),$$

for all C^1 1-forms α on V. Given any 2-form λ on V, define

$$\Omega_\lambda = \{\phi \in X^1 : \lambda + A \circ g(\phi) \text{ is non-degenerate}\}.$$

When dim V is even, Ω_λ is the complement of a set of codimension 1, and it is easy to see that it is not ample in any coordinate direction.

LEMMA 2.4. – *When* dim V *is odd,* Ω_λ *is ample in every coordinate direction.*

Proof. – Given $x \in X$, let $u_1, \ldots, u_n$ be local coordinates near $p(x)$ and coordinatize the fibers of p by the map

$$(y_1, \ldots, y_n) \longrightarrow \Sigma\, y_j\, du_j .$$

Then the coordinates (a_{jk}), $1 \leqslant j, k \leqslant n$, represent the 1-jet

$$\left(x\,,\, \Sigma\, a_{jk} \frac{\partial y_k}{\partial u_j}\right) \qquad \text{in} \qquad \mathrm{L} = (p^1)^{-1}(x).$$

Clearly $\mathrm{A} \circ g(a_{jk}) = \sum\limits_{j<k} (a_{jk} - a_{kj})\, du_j \wedge du_k$. It will suffice to check that $\Omega_\lambda \cap \mathrm{L}$ is ample in the first coordinate direction. If S is a subspace in this direction, the coordinates a_{jk}, $j > 1$, are constant on S. Let $\lambda = \sum\limits_{j<k} \lambda_{jk}\, du_j \wedge du_k$, and let us write $z_1, \ldots, z_n$ for the unknowns $a_{11}, a_{12} - a_{21} + \lambda_{12}, \ldots, a_{1n} - a_{n1} + \lambda_{1n}$ and b_{jk}, $2 \leqslant j < k \leqslant n$, for the known quantities $a_{jk} - a_{kj} + \lambda_{jk}$. Then, we must show that for every choice of the b_{jk} the set

$$\mathrm{Q} = \{(z_1, \ldots, z_n) \in \mathbf{R}^n : \beta(z)^m \neq 0\}$$

is ample in $\mathrm{S} = \mathbf{R}^n$, where $\beta(z) = \Sigma\, z_k\, du_1 \wedge du_k + \Sigma\, b_{jk}\, du_j \wedge du_k$ and $n = 2m + 1$. Since the coefficient of $du_2 \wedge \ldots \wedge du_n$ in $\beta(z)^m$ is independent of z, the set Q will equal S when this coefficient is non-zero. If this coefficient is zero, it is easy to check that Q is the complement in S of the set of solutions to the equations

$$\sum_{k=2}^{n} \epsilon_{ik}\, \mathrm{B}_{ik}\, z_k = 0\,, \qquad i = 2, \ldots, n,$$

where $\mathrm{B}_{ik} = \mathrm{B}_{ki}$ is the coefficient of

$$du_2 \wedge \ldots \wedge \widehat{du_i} \wedge \ldots \wedge \widehat{du_k} \wedge \ldots \wedge du_n$$

in $\beta(z)^m$, and where $\epsilon_{ik} = \pm 1$. If $\mathrm{B}_{ik} \neq 0$, then the ith. and the

kth. rows are linearly independent since $B_{ii} = B_{kk} = 0$. Thus this system of equations does not have rank 1. It follows that Q never has codimension 1 in S, and so is ample. □

Therefore Theorem 2.1 applies to the relations Ω_λ. To illustrate what this means, let us take λ to be a closed 2-form which represents some cohomology class $a \in H^2(V, \mathbf{R})$, and let ρ_0 be a closed non-degenerate 2-form which is defined near W and is such that

$$[\rho_0 \mid W] = a \mid W.$$

Further, let us suppose given a non-degenerate 2-form μ_0 which equals ρ_0 near W. Because the fibers of $A \circ g$ are contractible, $\mu_0 - \lambda$ may be lifted to an element ϕ_0 of $\Gamma(\Omega_\lambda, \rho_0 - \lambda, W)$. Theorem 2.1 then implies that ϕ_0 is homotopic to a solution ϕ_1 of Ω_λ. Hence μ_0 is homotopic, through non-degenerate forms μ_t which equal ρ_0 near W, to a closed non-degenerate 2-form

$\mu_1 = \lambda + A \circ g(\phi_1) = \lambda +$ (exact). Therefore, we have shown:

THEOREM 2.5. – *If* dim V *is odd, any non-degenerate 2-form* μ_0 *on* V *is homotopic to a non-degenerate closed 2-form* μ_1. *We may specify in advance the cohomology class* a *of* μ_1. *Further, if* μ_0 *is closed in a neighbourhood of the submanifold* W, *we may perform the homotopy relative to* W *provided that* $a \mid W = [\mu_0 \mid W]$.

The geometric implications of this result are discussed in § 4.

Example 2.6. – Let $p : X \longrightarrow V$ be as in the previous example, but suppose now that $\dim V = 2m > 2$. Using diagram (2.3), we will think of elements of X^1 as pairs

$$(\theta, \psi) \in T^*V \times J \quad \text{with} \quad p(\theta) = q(\psi).$$

Then, given any 2-form λ on V1, define

$$\Omega'_\lambda = \{(\theta, \psi) : \theta \wedge (\lambda + A\psi)^{m-1} \neq 0\} \subset X^1.$$

LEMMA 2.7. – Ω'_λ *is ample in all coordinate directions.*

Proof. – We will show as in Lemma 2.4 that Ω'_λ is ample at $x_0 = (\theta_0, \psi_0)$ in the first coordinate direction. Let us write $\theta_0 = \Sigma\, \theta_i\, du_i$ and

$$\lambda + A\psi = \beta(z) = \sum_k z_k\, du_1 \wedge du_k + \sum_{j<k} b_{jk}\, du_j \wedge du_k$$

as before, where the θ_i, b_{jk} are known and the z_k are unknown. If the component of $\theta_0 \wedge \beta(z)^{m-1}$ which does not contain du_1 is non-zero, then $Q = S$. Otherwise, Q is the complement in S of the set of solutions to a system of equations of the form

$$\sum_{i,j} \epsilon_{ijk}\, \theta_i\, z_j\, D_{ijk} = c_k\,, k = 2, \ldots, 2m\,.$$

Here $c_k = c_k(\theta_i, b_{jk})$ is known, D_{ijk} is the coefficient of the term in $\beta(z)^{m-2}$ which does not involve du_1, du_i, du_j, or du_k and so is independent of the ordering of i, j, k, while ϵ_{ijk} is the sign of the permutation $(i, j, 2, 3, \ldots, \hat{\imath} \ldots \hat{\jmath} \ldots \hat{k} \ldots, 2m)$. One can check that the ratio $\epsilon_{ijk}/\epsilon_{ikj}$ depends only on j and k. It follows that the coefficient $\sum_i \epsilon_{ijk}\, \theta_i\, D_{ijk}$ of z_j in the kth. equation is $\pm$ the coefficient of z_k in the jth. equation. As before, this implies that Q is ample.

□

A typical conclusion would be the following. Let us call a pair (α, β), consisting of a 1-form α together with a 2-form β, non-degenerate if $\alpha \wedge \beta^{m-1} \neq 0$. Then any such pair is homotopic through non-degenerate pairs to a non-degenerate pair of the form $(\alpha, d\alpha)$. A geometric interpretation of this result is given in § 7.

3. Equivalences of symplectic forms.

In § 3 and § 4 we discuss various equivalence relations on the set of symplectic forms on a $2m$-dimensional manifold X. One of our main aims is to show that the relation of formal equivalence which was mentioned in § 1 has some geometric content. In particular, we will see that two formally equivalent forms are *concordant*, that is, there is a non-vanishing flow on $X \times I$ with a transverse symplectic form τ, which restricts to the given forms on $X \times i$, $i = 0, 1$. To put this result in perspective, we will begin by discussing some other equivalence relations for symplectic forms.

Note that a reduction of the structural group of TX to $Sp(2m, \mathbf{R})$ determines and is determined by a non-degenerate 2-form β on X. Thus a homotopy class of reduction of the structure group to $U(m)$ is equivalent to a homotopy class of non-degenerate 2-forms β. Therefore, we may recast the definition of formal equivalence as follows:

DEFINITION 3.1. – *Two symplectic forms* σ_0, σ_1 *are:*

(i) *formally equivalent iff* $[\sigma_0] = [\sigma_1]$ *and there is a smooth family of non-degenerate 2-forms* $\beta_t, 0 \leqslant t \leqslant 1$, *such that*

$$\beta_i = \sigma_i, \quad i = 0, 1$$

(ii) *homotopic iff there is a smooth family* $\rho_t, 0 \leqslant t \leqslant 1$, *of cohomologous symplectic forms with* $\rho_i = \sigma_i, i = 0, 1$, *and*

(iii) *isotopic iff there is a smooth family* $g_t, 0 \leqslant t \leqslant 1$, *of diffeomorphisms of* X *with* $g_0 = \mathrm{id}$ *and* $g_1{}^* \sigma_1 = \sigma_0$.

Two further equivalence relations, concordance and strong concordance, are defined in (4.1). As we shall see, the following implications hold:

(if X is closed)

isotopy $\xLeftrightarrow{\text{(Moser)}}$ homotopy $\Longrightarrow$ formal equivalence $\Longleftrightarrow$ strong concordance $\Longrightarrow$ concordance.

(if X is open)

isotopy $\Longrightarrow$ homotopy $\xLeftrightarrow{\text{(Gromov)}}$ formal equivalence $\Longleftrightarrow$ strong concordance $\Longrightarrow$ concordance.

Indeed it is immediate that isotopy implies homotopy, and that homotopy implies formal equivalence. For the other direction, we must distinguish between open and closed manifolds: Moser [Mo] proved that homotopy implies isotopy on open manifolds, and, by [G1,3], formal equivalence implies homotopy on open manifolds.

However, formal equivalence does not imply isotopy in either case. Some counter examples for closed manifolds are given in [MD5], and for open manifolds in [G4] 0.3.A. These examples involve quite subtle invariants. Indeed Gromov's examples are simply the products $D(R_1) \times D(R_2)$ in $\mathbf{R}^4$ with the induced structure, where $D(R) \subset \mathbf{R}^2$ is an open disc of radius R. Here there is no obvious invariant apart from the volume, and yet he shows that these structures for $R_1 \leqslant R_2$ are all distinct.

As the last sentence makes clear, there are further isotopy invariants of formally equivalent forms on open manifolds which are elementary, in the sense that they are of cohomological type and do not involve the almost-complex geometry developed in [G4]. We will now discuss some obstructions to isotopy which arise from the volume and other related forms on X.

Evidently the total σ-volume of X is an isotopy invariant, as is the set of ends of infinite volume. (An end E has infinite volume if, for every compact set $K \subset X$ the component of $X - K$ which contains E has infinite volume.) Moser's Theorem has been extended in [GS] to show that these are the only invariants which arise solely from the volume form. However, in the symplectic case, one may assign a numerical volume $\nu(E)$ to certain isolated ends E of finite σ-volume as follows:

If S is a compact oriented hypersurface in X such that $\sigma^k \mid S$ is exact, where $k = \left[\dfrac{m}{2}\right]$, we define $I(S)$ by the formula:

$$I(S) = \int_S \alpha \wedge \sigma^{\left[\frac{m+1}{2}\right]},$$

where α is any form on S such that $d\alpha = \sigma^k \mid S$. It is easy to check that $I(S)$ is independent of the choice of α. An isolated end E has a neighbourhood N which is a connected manifold with boundary ∂N and has E as its only end. If, in addition, $\sigma^k \mid N$ is exact then $I(\partial N)$ is defined and we put

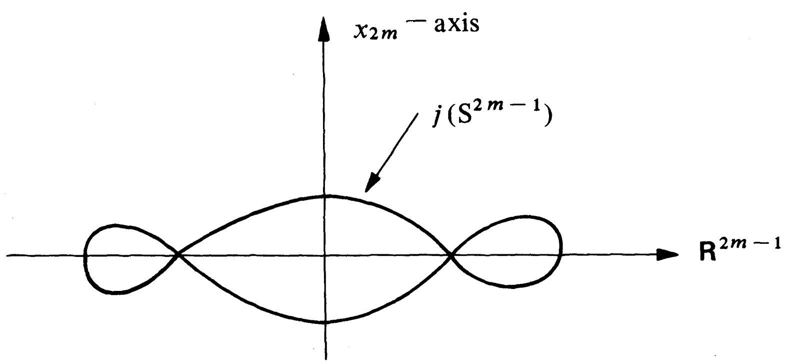
$$\nu(E) = \mathrm{vol}_\sigma N - I(\partial N),$$

where ∂N is oriented as the boundary of N. It is easy to check that $\nu(E)$ is independent of the choice of N satisfying the above conditions. Note, also, that if X has isolated ends and if σ^k is exact then $\nu(\mathrm{E})$ is defined for them all, and $\mathrm{vol}_\sigma \mathrm{X} = \sum_{\mathrm{E}} \nu(\mathrm{E})$.

Example 3.2. – Let $\sigma_{r,s}$ be the symplectic form on the annulus $\mathrm{A} = \mathrm{S}^{2m-1} \times \mathbf{R}$, $m \geqslant 2$, which is induced from the standard structure σ_c on $\mathbf{R}^{2m}$ by radially identifying A with the annulus $\{x \in \mathbf{R}^{2m} : r < \| x \| < s\}$. Then

$$\nu(\mathrm{S}^{2m-1} \times \infty) = \mathrm{vol}\, \{\| x \| < s\}$$

and $\nu(\mathrm{S}^{2m-1} \times -\infty) = -\mathrm{vol}\, \{\| x \| < r\}$. Therefore, none of these forms are isotopic, although they are homotopic, and in some cases have the same total volume. However, they are all essentially the same: for instance, their germs at $\mathrm{S}^{2m-1} \times 0$ are isotopic when multiplied by a suitable constant. This is no longer the case if we consider forms induced on A by immersions $j : \mathrm{A} \longrightarrow \mathbf{R}^{2m}$. For example, it is easy to construct an immersion j such that $\int_{\mathrm{S}^{2m-1}\times 0} j^*\lambda = 0$, where $d\lambda = \sigma_c^m$. (See figure.) It follows easily that both ends of A have $\nu(\mathrm{E}) > 0$. Also, the germ of $j^*\sigma_c$ at $\mathrm{S}^{2m-1} \times 0$ does not extend over any disc D with boundary $\partial\mathrm{D} = \mathrm{S}^{2m-1} \times 0$, since if it did the integral $\int_{\mathrm{S}^{2m-1}\times 0} j^*\lambda$ would be positive. Hence this germ does not embed in $\mathbf{R}^{2m}$.



Note 3.3. – Suppose that S is a compact oriented hypersurface in (X, σ) such that $\mathrm{I}(\mathrm{S})$ is defined and non-zero. Then a small

collar neighbourhood of S will have one end with $\nu(\mathrm{E}) > 0$ and the other with $\nu(\mathrm{E}) < 0$. Note that this designation is independent of the choice of orientation for S. It follows that S has a "positive" and a "negative" side which cannot be interchanged by any symplectic diffeomorphism. This generalizes a remark of Calabi-Weinstein in [W] § 3.

4. Transversally symplectic foliation and concordance.

The main aim of this section is to prove Theorem 4.2 which gives a geometric interpretation of the relation of formal equivalence.

Let $\mathfrak{F}$ be a codimension $2m$ foliation on a manifold Y^{p+2m}. It may be described by an atlas $(\mathrm{U}_i, (\psi_i, \phi_i))_{i \in \mathrm{A}}$, where (ψ_i, ϕ_i) is a diffeomorphism of $\mathrm{U}_i \subset \mathrm{Y}$ onto an open subset of $\mathbf{R}^p \times \mathbf{R}^{2m}$, and where the leaves of $\mathfrak{F} \mid \mathrm{U}_i$ are $\phi_i^{-1}(pt)$. A *transverse symplectic structure* on $\mathfrak{F}$ is a closed 2-form τ on Y such that, for some such atlas, $\tau \mid \mathrm{U}_i = \phi_i^* (\sigma_c)$ for all i, where σ_c is the canonical form $dx_1 \wedge dx_2 + \ldots + dx_{2m-1} \wedge dx_{2m}$ on $\mathbf{R}^{2m}$. Such τ will exist if and only if the transition functions $\phi_j \cdot \phi_i^{-1}$ all preserve σ_c and so are local symplectic diffeomorphisms of $\mathbf{R}^{2m}$. Darboux's theorem implies that a closed 2-form τ is a transverse symplectic form for $\mathfrak{F}$ if and only if $\tau^m \neq 0$ and the 1-form $\xi \lrcorner \tau$ vanishes for every vector field ξ tangent to the leaves of $\mathfrak{F}$. Using the identity $\mathcal{L}_\xi \tau = d(\xi \lrcorner \tau) + \xi \lrcorner d\tau$, one easily sees that the latter condition is equivalent to requiring that τ is invariant under every vector field tangent to $\mathfrak{F}$. In particular, if τ is a non-degenerate closed 2-form on a $(2m+1)$-dimensional manifold Y, its kernel $\{v \in \mathrm{TY} : v \lrcorner \tau = 0\}$ is everywhere 1-dimensional. Hence τ defines a 1-dimensional foliation $\mathfrak{F}_\tau$ for which τ is the traverse symplectic form. If Y is orientable, then $\mathfrak{F}_\tau$ will be too. If ξ is any non-vanishing vector field along $\mathfrak{F}_\tau$, the pair (ξ, τ) will be called a *flow with transverse symplectic structure.*

DEFINITION 4.1. – Two symplectic forms σ_0 and σ_1 on X are said to be *concordant* if there is a closed non-degenerate 2-form τ

on $X \times I$ which restricts to σ_i on $X \times i$ for $i = 0,1$. Further, the forms will be called *strongly concordant* if in addition, the foliation $\mathfrak{F}_\tau$ is homotopic rel $X \times \partial I$ to the trivial foliation $\mathfrak{F}_0$ with leaves $pt \times I$. More precisely, this means that there is a family $\xi_t, 0 \leqslant t \leqslant 1$, of non-vanishing vector fields on $X \times I$, which are transverse to $X \times \partial I$ and are such that ξ_0 is tangent to $\mathfrak{F}_0$ and ξ_1 is tangent to $\mathfrak{F}_\tau$. (Note that X and $X \times I$ are oriented.)

Thus two concordant symplectic forms are connected by a non-vanishing transversally symplectic flow on $X \times I$. Of course, this flow will not in general be everywhere transverse to the slices $X \times s$, $s \in I$. If it is, then σ_0 and σ_1 are homotopic, and hence also isotopic in the case of compact X.

Our main result is:

THEOREM 4.2. – *Two symplectic forms σ_0, σ_1 are formally equivalent if and only if they are strongly concordant.*

Before proving this, we will reformulate the extra condition which occurs in the definition of strong concordance in terms of differential forms.

LEMMA 4.3. – *Two symplectic forms σ_0, σ_1 are strongly concordant if and only if there is a continuous family μ_t, $0 \leqslant t \leqslant 1$, of non-degenerate 2-forms on $X \times I$ such that :*

(i) $\mu_t \mid X \times i = \sigma_i$ *for all* t *and* $i = 0,1$,

(ii) $\xi_0 \lrcorner \mu_0 = 0$, *where* ξ_0 *is the vector field along the foliation* $pt \times I$, *and*

(iii) μ_1 *is closed.*

Proof. – ($\Longrightarrow$) The requisite family of vector fields may be taken to lie in ker μ_t.

($\Longleftarrow$) Put $G = GL^+(2m+1, \mathbf{R})$. Let P be the principal G-bundle over $X \times \mathbf{R}$ associated to $T(X \times \mathbf{R})$, and let E_H, E_K be the associated bundles with fibers G/H, G/K, where

$$H = \left\{ \begin{pmatrix} S & O \\ * & * \end{pmatrix} \in G : S \in Sp(2m, \mathbf{R}) \right\} \subset K$$

and

$$K = \left\{ \begin{pmatrix} A & O \\ * & * \end{pmatrix} \in G : A \in GL^+(2m, \mathbf{R}) \right\}.$$

Sections of E_H (resp. E_K) correspond to non-degenerate 2-forms on $X \times I$ (resp. to non-zero oriented line fields on $X \times I$). Clearly E_H fibers over E_K with fiber K/H. Moreover, this map takes a non-degenerate 2-form to its (oriented) kernel. Since vector fields determine oriented line fields and since ξ_1 lifts to τ, the homotopy $\xi_t, 0 \leqslant t \leqslant 1$, lifts to a family μ_t of non-degenerate 2-forms on $X \times I$ with $\mu_1 = \tau$ and $\xi_0 \subset \ker \mu_0$. □

Proof of Theorem 4.2. – Suppose first that σ_0 and σ_1 are strongly concordant, and let μ_0 be the non-degenerate 2-form with kernel along ξ_0 which is mentioned in Lemma 4.3. Then for each $t \in [0, 1]$ the form $\mu_0 \mid X \times t$ may be identified with a non-degenerate form β_t on X. Clearly, this family $\beta_t, 0 \leqslant t \leqslant 1$, satisfies the conditions of Definition 3.1.

Now suppose that σ_0 and σ_1 are formally equivalent. If X is open, [H2] II Thm 3 implies that they are strongly concordant. In the general case, we must use the method of convex integration. Let β_t be given by (3.1), and let μ_0 be the unique form on $X \times I$ such that $\mu_0 \mid X \times t = \beta_t$ for all t and $\zeta_0 \lrcorner \mu_0 = 0$. Clearly, we may assume that μ_0 is closed near $X \times \partial I$. (One just has to reparametrize the β_t so that they are constant for t near 0 and 1.) Let a be the cohomology class $\pi^*[\sigma_0] = \pi^*[\sigma_1]$, where $\pi : X \times I \longrightarrow X$ is the projection. Then it follows from Theorem 2.5 that μ_0 may be homotoped rel $X \times \partial I$ to a closed non-degenerate form τ which represents a. □

5. Transversally exact symplectic flows and contact structures.

Let X be an oriented compact manifold of dimension $2m + 1$ with a non-degenerate exact 2-form $d\alpha$. Following [W], we will say that $(X, d\alpha)$ is of *contact type* if there is a contact form α' on X with $d\alpha' = d\alpha$. In this section we give a necessary and sufficient condition for $(X, d\alpha)$ to be of contact type.

It is of considerable interest to know when $(X, d\alpha)$ is of contact type. In [W] Weinstein considers the following situation. Suppose that X is embedded as a hypersurface in some $(2m+2)$-dimensional symplectic manifold (P, Ω). Then the 2-form $\Omega \mid X$ is non-degenerate on X and so, as in § 4, gives rise to a flow (which is called the Hamiltonian flow) on X. If $\Omega \mid X$ is exact, we have a pair $(X, d\alpha)$ as above. Weinstein conjectures that if $(X, d\alpha)$ is of contact type and if $H^1(X; \mathbf{R}) = 0$ then this flow has a closed orbit. He presents certain evidence for this, though the general question seems very hard. (See [V].) The question of when $(X, d\alpha)$ is of contact type also arises when one is trying to construct contact structures. For, the formal data for a transversally oriented contact structure is a homotopy class of reductions of the structural group of TX to $U(m)$, or, equivalently, a homotopy class of non-degenerate 2-forms on X. Theorem 2.5 shows that such a homotopy class always contains exact 2-forms $d\alpha$. Thus the formal data always gives rise to pairs $(X, d\alpha)$, and one would like to know which, if any, are of contact type.

We will formulate the condition for $(X, d\alpha)$ to be of contact type in terms of Sullivan's theory of cone structures [S]. (Sullivan discusses the relevence of cone structures to symplectic forms, but does not treat the contact case). Throughout, we will assume that X and α are as above. Further, we will choose once and for all a non-vanishing vector field ξ in $\ker \alpha$. (Our criterion will not depend on this choice). It is easy to see that a 1-form $\alpha' = \alpha + (\text{closed})$ is a contact form on X if and only if the function $\alpha'_x(\xi_x)$ never vanishes. Our main result says roughly that such a form α' will exist if and only if certain integrals $\int_b \alpha$ do not vanish, where b runs over the set of all "structural boundaries" associated to ξ. In particular, any closed null-homologous orbit of ξ is such a boundary.

We will begin by recalling some notation and definitions from [S]. Let $\mathcal{D}'_p$ be the space of p-currents on X, topologized as the dual of the space $\mathcal{D}_p$ of C^∞ p-forms on X. The boundary

$$\partial : \mathcal{D}'_p \longrightarrow \mathcal{D}'_{p-1}$$

is dual to the exterior derivative. Its range $\mathcal{B}_{p-1}$ is closed, and is a direct summand in $\mathcal{D}'_{p-1}$. Further, the dual $\mathcal{D}''_p$ of $\mathcal{D}'_p$ equals $\mathcal{D}_p$. These facts imply:

LEMMA 5.1. – (i) *A p-form* β *is closed if and only if its kernel*

$$H_\beta = \{c \in \mathcal{D}'_p : \langle c , \beta \rangle = 0\}$$

contains the subspace $\mathcal{B}_p$ *of boundaries*

(ii) $c \in \mathcal{D}'_p$ *is a boundary if and only if* $\langle c , k \rangle = 0$ *for all closed p-forms* k.

The vector field ξ on X determines a *cone* $\mathcal{C} \subset \mathcal{D}'_1$ *of structure currents* as follows: $\mathcal{C}$ is the closed convex cone in $\mathcal{D}'_1$ which is generated by the Dirac currents $\delta(\xi_x)$ along ξ ([S] I.4). Here $\langle \delta(\xi_x), \beta \rangle = \beta_x(\xi_x)$. It follows that elements of $\mathcal{C}$ have the form

$$\beta \longmapsto \int_x \beta(v)\, d\mu$$

for $\beta \in \mathcal{D}_1$, where μ is a positive Borel measure on X and $v = \lambda\xi$ is a continuous vector field such that the function λ is $\geqslant 0$ ([S] I.8). Note that $\mathcal{C}$ depends only on the direction of ξ and hence only on the form $d\alpha$ and the given orientation of X. Because X is compact, $\mathcal{C}$ is compactly supported ([S] I.5). Indeed, the closure $\bar{\mathcal{C}}$ of the set $\{\Sigma\, \lambda_i\, \delta(\xi_{x_i}) : \lambda_i > 0 , \Sigma\, \lambda_i = 1\}$ is compact and convex, and is a base for $\mathcal{C}$. The *cone of structural boundaries* is the intersection of $\mathcal{C}$ with the subspace of boundaries in $\mathcal{D}'_1$. Observe that the cone $\mathcal{C}$ always contains some non-zero boundaries. For, otherwise, [S] II.27 implies that there would be a closed manifold T of codimension 1 which is everywhere transverse to ξ. Then $d\alpha$ would restrict to a symplectic form on T, which is impossible since $d\alpha$ is exact.

Here is our main result.

THEOREM 5.2. – $(X, d\alpha)$ *has contact type if and only if*

$$\langle b , \alpha \rangle \neq 0$$

for every non-zero structural boundary b *in* $\mathcal{C}$.

Proof. – ($\Longrightarrow$) Suppose that $\alpha' = \alpha + \kappa$ is a contact form with κ closed. Then $\alpha'(\xi)$ never vanishes, so that $\langle c , \alpha' \rangle \neq 0$ for all non-zero $c \in \mathcal{C}$. But $\langle b , \kappa \rangle = 0$ for all boundaries b, by Lemma 5.1. (i). Therefore $\langle b , \alpha \rangle = \langle b , \alpha' \rangle \neq 0$ for all non-zero boundaries $b \in \mathcal{C}$.

($\Longleftarrow$) Let H_α be the hyperplane $\{a \in \mathcal{D}_1' : \langle a, \alpha \rangle = 0\}$, and let $\overline{\mathcal{C}}$ be the compact convex base of $\mathcal{C}$ defined above. Our hypothesis implies that the compact convex set $\overline{\mathcal{C}} \cap H_\alpha$ does not meet the closed subspace $\mathcal{B}_1$ of boundaries. Therefore, by the Hahn-Banach theorem, there is a hyperplane H in $\mathcal{D}_1'$ which contains $\mathcal{B}_1$ and is disjoint from $\overline{\mathcal{C}} \cap H_\alpha$. Thus

$$(H \cap H_\alpha) \cap \overline{\mathcal{C}} = \phi .$$

Let κ' be a 1-form with kernel H. Because $\mathcal{B}_1 \subset H$, Lemma 5.1 (i) implies that κ' is closed. Now consider the quotient $\mathcal{D}_1 / H \cap H_\alpha$. This is a 2-dimensional vector space V whose points are separated by the functionals α and κ'. Moreover the image of $\overline{\mathcal{C}}$ in V is a compact convex set which does not meet $\{0\}$ because $H \cap H_\alpha \cap \overline{\mathcal{C}} = \phi$. Therefore, there is some linear combination, $\lambda_1 \alpha + \lambda_2 \kappa'$ say, of α and κ' which is always positive on $\overline{\mathcal{C}}$. Since $\overline{\mathcal{C}}$ is compact, we may assume that $\lambda_1 \neq 0$. Then, setting $\kappa = \dfrac{\lambda_2}{\lambda_1} \kappa'$, we see that $\langle c, \alpha + \kappa \rangle \neq 0$ for all $c \in \overline{\mathcal{C}}$. Hence $(\alpha + \kappa)(\xi) \neq 0$, and so $(\alpha + \kappa) \wedge (d\alpha)^m \neq 0$ as required.

□

Note 5.3. – In [W] Lemma 4, Weinstein proves that if $(X, d\alpha)$ has contact type and if $H^1(X; \mathbf{R}) = 0$, then there cannot be recurrent orbits $\mathcal{O}, \mathcal{O}'$ of ξ such that $\alpha_x(\xi_x)$ is positive for all $x \in \mathcal{O}$ and negative for all $x \in \mathcal{O}'$. This result (without the hypothesis of recurrence) follows from Theorem 5.2. For, by [S] II.8, the closure of every orbit $\mathcal{O}$ of ξ supports at least one non-zero structural cycle $c_{\mathcal{O}}$. Because $H^1(X; \mathbf{R}) = 0$, this cycle is a boundary. Clearly, if $\alpha_x(\xi_x)$ is positive on $\mathcal{O}$, then $\langle c_{\mathcal{O}}, \alpha \rangle \geqslant 0$. Therefore if $\alpha_x(\xi_x)$ is positive on $\mathcal{O}$ and negative on $\mathcal{O}'$, there is some convex linear combination of $c_{\mathcal{O}}$ and $c_{\mathcal{O}'}$, on which α vanishes.

6. The classifying space for transversally symplectic foliations.

Let σ_c be the canonical symplectic form

$$dx_1 \wedge dx_2 + \ldots + dx_{2m-1} \wedge dx_{2m}$$

on $\mathbf{R}^{2m}$, and denote by Γ_{sp}^{2m} the groupoid of germs of symplectic diffeomorphisms of $(\mathbf{R}^{2m}, \sigma_c)$. Then the space $B\Gamma^{2m}_{sp}$ classifies foliations with transverse symplectic structure. (See § 4 above, and [H2].) The differential $\Gamma^{2m}_{sp} \longrightarrow Sp(2m, \mathbf{R})$, which takes the germ g at x to its derivative dg_x, is a homomorphism of groupoids, and so induces a map $\nu : B\Gamma^{2m}_{sp} \longrightarrow BSp(2m, \mathbf{R}) \simeq BU(m)$. There is also a map $\epsilon : B\Gamma^{2m}_{sp} \longrightarrow K(\mathbf{R}^\delta, 2)$ which classifies the cohomology class of the "universal symplectic form" on $B\Gamma^{2m}_{sp}$. (Here $\mathbf{R}^\delta$ denotes the additive group of reals with the discrete topology.) In other words, if $(\mathfrak{F}, \tau)$ is a foliation on Y with transverse symplectic structure τ, and if $f : Y \longrightarrow B\Gamma^{2m}_{sp}$ is its classifying map, then $(\epsilon \circ f)^*(e) = [\tau] \in H^2(Y; \mathbf{R})$, where $e \in H^2(K(\mathbf{R}^\delta, 2); \mathbf{R})$ is the fundamental class. Similarly, the map $\nu \circ f$ classifies the normal bundle of $\mathfrak{F}$.

Let $\overline{\overline{B}}\Gamma^{2m}_{sp}$ denote the homotopy fiber of the product map

$$\nu \times \epsilon : B\Gamma^{2m}_{sp} \longrightarrow BU(m) \times K(\mathbf{R}^\delta, 2).$$

Haefliger showed in [H2] II.6 that $\overline{\overline{B}}\Gamma^{2m}_{sp}$ is $(2m-1)$-connected. In this section we prove:

THEOREM 6.1. – $\pi_{2m}(\overline{\overline{B}}\Gamma^{2m}_{sp}) = 0$.

Proof. – We adapt the proof of [H3] Theorem 3(b). An element of $\pi_{2m}(\overline{\overline{B}}\Gamma^{2m}_{sp})$ may be represented by a Γ^{2m}_{sp}-structure $(\mathfrak{F}_0, \tau_0)$ on S^{2m}, where τ_0 is exact, together with a trivialization T_0 of its normal bundle as a $Sp(2m, \mathbf{R})$-bundle. (For short, we will call such a trivialization a framing of $(\mathfrak{F}_0, \tau_0)$.) This element of $\pi_{2m}(\overline{\overline{B}}\Gamma^{2m}_{sp})$ will be zero if and only if both the Γ^{2m}_{sp}-structure and its framing extend over D^{2m+1}. By [H2] II.3, we may suppose that $(\mathfrak{F}_0, \tau_0)$ is a transversally symplectic foliation defined in a neighbourhood V of S^{2m} in $\mathbf{R}^{2m+1}$. The framing T_0 of $\nu(\mathfrak{F}_0)$ then defines a bundle epimorphism ϕ_0 which covers the constant

$$\begin{array}{ccc} TV & \xrightarrow{\phi_0} & T\mathbf{R}^{2m} \\ \downarrow & & \downarrow \\ V & \longrightarrow & \mathbf{R}^{2m}, \end{array}$$

map $V \longrightarrow pt$. and is such that $\phi_0{}^*(\sigma_c) = \tau_0$. By [H2] II.2, ϕ_0 is homotopic through bundle epimorphisms ϕ_t to an epimorphism ϕ_1 which is the derivative dg of a smooth map $g : V \longrightarrow \mathbf{R}^{2m}$. Thus g is a submersion.

Let $(\mathfrak{F}_1, \tau_1, T_1)$ be the transversally symplectic framed foliation which is defined near the sphere $\{x \in \mathbf{R}^{2m+1} : \|x\| = 1/2\}$ by the single projection $x \longmapsto g(2x)$. Since g extends to a map of the disc $\{\|x\| \leqslant 1/2\}$ into $\mathbf{R}^{2m}$, this foliation extends to a framed Γ_{sp}^{2m}-structure over this disc. Therefore, it remains to show that there is a transversally symplectic framed foliation on the annulus $A = \{x : 1/2 \leqslant \|x\| \leqslant 1\}$ which extends the foliation given near ∂A.

It is easy to see that there is a non-degenerate 2-form μ_0 on a neighbourhood W of A which equals τ_1 if $\|x\| \leqslant 1/2$ and τ_0 if $\|x\| \geqslant 1$, and is given on the sphere $\|x\| = \lambda, 1/2 < \lambda < 1$, by the form $\phi_t{}^*(\sigma_c) \mid S^{2m}$, where $t = t(\lambda)$. The "normal bundle" $TW/\ker \mu_0$ of μ_0 has a canonical $Sp(2m, \mathbf{R})$-structure coming from μ_0. Moreover, the framing which is already defined near ∂A extends to a framing T' of the whole normal bundle. Indeed, on the sphere $\|x\| = \lambda, T'$ will project via $(\phi_t)_*$ to the standard framing on $T\mathbf{R}^{2m}$.

Since $[\tau_1] = [\tau_0] = 0$, we may use Theorem 2.5 to homotop the non-degenerate 2-form μ_0 to a non-degenerate closed 2-form μ_1 on W which equals μ_0 near ∂A. Further, it is not hard to check that the homotopy μ_t lifts to a family of framings T'_t of the normal bundle of μ_t, such that $T'_0 = T'$ and $T'_t = T'$ near ∂A. (Compare the proof of Lemma 4.3.) Hence the pair (μ_1, T'_1) defines a transversally symplectic framed foliation $\mathfrak{F}$ on A which extends the foliations $\mathfrak{F}_0, \mathfrak{F}_1$ given near ∂A. The result follows.

□

Note 6.2. – A similar result for foliations with transverse volume form may be found in [MD1] Lemma 1. The proof there is not quite correct since it was not shown that the given framing of $\mathfrak{F}_0$ extends over the disc. However, this gap may be filled in by arguments similar to those above.

The next result was pointed out to me by Steven Hurder. Here, $\mathbf{R}^\delta$ denotes $\mathbf{R}$ with the discrete topology.

PROPOSITION 6.3. – $\pi_{2m+1}(\overline{\overline{\mathrm{B}}}\Gamma_{sp}^{2m})$ *surjects onto* $\mathbf{R}^{\delta}$.

Proof. – Let $\hat{\mathrm{B}}\Gamma_{sp}^{2m}$ denote the homotopy fiber of the map $\epsilon : \mathrm{B}\Gamma_{sp}^{2m} \longrightarrow \mathrm{K}(\mathbf{R}^{\delta}, 2)$. Thus $\hat{\mathrm{B}}\Gamma_{sp}^{2m}$ classifies transversally exact symplectic foliations, and there is a fibration

$$\overline{\overline{\mathrm{B}}}\Gamma_{sp}^{2m} \xrightarrow{j} \hat{\mathrm{B}}\Gamma_{sp}^{2m} \xrightarrow{\nu} \mathrm{BU}(m).$$

Observe that if $(\mathfrak{F}, \tau)$ is a transversally symplectic foliation of codimension $2m$ on a manifold Y, then the form τ^{m+1} vanishes identically because τ is locally pulled back from $\mathbf{R}^{2m}$. Moreover, if τ is exact, it is easy to check that the cohomology class of the $(2m+1)$-form $\beta \wedge \tau^m$ does not depend on the choice of β satisfying $d\beta = \tau$. Applying this to the universal Γ_{sp}^{2m}-structures over $\hat{\mathrm{B}}\Gamma_{sp}^{2m}$ and $\overline{\overline{\mathrm{B}}}\Gamma_{sp}^{2m}$, one gets classes $\hat{a} \in \mathrm{H}^2(\hat{\mathrm{B}}\Gamma_{sp}^{2m}; \mathbf{R})$ and $a \in \mathrm{H}^2(\overline{\overline{\mathrm{B}}}\Gamma_{sp}^{2m}; \mathbf{R})$ such that $j^* \hat{a} = a$.

It is easy to see that $\hat{a}$ induces a surjection $\hat{\mathrm{A}}$ from $\pi_{2m+1}(\hat{\mathrm{B}}\Gamma_{sp}^{2m})$ to $\mathbf{R}^{\delta}$. For example, given a contact form α on S^{2m+1} and $\lambda \in \mathrm{R}$, let χ_λ be the element of $\pi_{2m+1}(\hat{\mathrm{B}}\Gamma_{sp}^{2m})$ which is represented by the classifying map of the transversally symplectic 1-dimensional foliation defined by the 2-form $\lambda d\alpha$. If $\mathcal{H}$ denotes the Hurewicz map, then

$$\hat{a}(\mathcal{H}(\chi_\lambda)) = \lambda^{m+1} \hat{a}(\mathcal{H}(\chi_1)) \neq 0,$$

which implies that $\hat{\mathrm{A}}$ is surjective. The result now follows easily, since $\nu_*(\chi_\lambda - \chi_1) = 0$ for all λ.

□

A similar result for the groupoid Γ_{sl}^{n} of germs of volume-preserving diffeomorphisms is discussed in [MD3]. One can also calculate $\pi_{n+1}(\overline{\mathrm{B}}\Gamma_{sl}^{n})$: see [MD2].

Note 6.4. – Let α_c be the standard contact form

$$dx_0 + x_1 dx_2 + \ldots + x_{2m-1} dx_{2m}$$

on $\mathbf{R}^{2m+1}$, and let Γ_{α}^{2m+1} denote the groupoid of germs of diffeomorphisms of $\mathbf{R}^{2m+1}$ which preserve α_c. (These are sometimes called strict contact diffeomorphisms.) The projection of $\mathbf{R}^{2m+1}$ onto $\mathbf{R}^{2m}$ along the coordinate x_0 induces a surjection of groupoids $\Gamma_{\alpha}^{2m+1} \longrightarrow \Gamma_{sp}^{2m}$ whose kernel is the discrete group $\mathbf{R}^{\delta}$.

As Haefliger observed, this implies that $B\Gamma_\alpha^{2m+1} \cong \hat{B}\Gamma_{sp}^{2m}$. Thus, if $\bar{B}\Gamma_\alpha^{2m+1}$ is the homotopy fiber of the map $B\Gamma_\alpha^{2m+1} \longrightarrow BU(m)$ we have $\bar{B}\Gamma_\alpha^{2m+1} \simeq \bar{\bar{B}}\Gamma_{sp}^{2m}$. We will see in (7.4) below that convex integration allows us to calculate $\pi_{2m+1}(\bar{B}\Gamma_{ct}^{2m+1})$, where Γ_{ct}^{2m+1} is the groupoid of germs of contact diffeomorphisms of $\mathbf{R}^{2m+1}$. Therefore, one might hope that the same method would work for $\pi_{2m+1}(\bar{B}\Gamma_\alpha^{2m+1})$. However, this is not the case, since there is no analogue of Proposition 7.1. Indeed, the best way to calculate $\pi_i(\bar{B}\Gamma_\alpha^{2m+1})$ appears to be to exploit its relationship with $\bar{\bar{B}}\Gamma_{sp}^{2m}$, rather than the other way round.

7. The contact case.

Recall that a 1-form α on a manifold of dimension $2m+2$ is said to be non-degenerate if $\alpha \wedge (d\alpha)^m \neq 0$. We saw in example 2.6 that the method of convex integration allows the construction of such forms from suitable formal data. The first result of § 7 is that non-degenerate 1-forms correspond to 1-dimensional foliations with (transversely oriented) contact structure. After establishing this, we will briefly discuss the contact analogues of Theorems 4.2 and 6.1. For simplicity, we will consider only transversally oriented contact structures. Such a structure is given by an equivalence class of non-degenerate 1-forms on an odd-dimensional manifold, where α and α' are called equivalent if $\alpha = \lambda\alpha'$ for some positive function λ.

DEFINITION 7.1. — A foliation $\mathfrak{F}$ of codimension $2m+1$ on a manifold Y^{p+2m+1} is said to have *transverse contact structure* if there is a transversely oriented field $\mathcal{H}$ of hyperplanes on Y which contains the tangent planes to $\mathfrak{F}$ and restricts to a contact structure on any $(2m+1)$-dimensional manifold which is transverse to $\mathfrak{F}$. Further, $\mathcal{H}$ must be invariant under any flow along the leaves of $\mathfrak{F}$.

It follows from Darboux's theorem that near every point one can choose local coordinates $(y, x) = (y_1, \dots, y_p, x_0, \dots, x_{2m})$ so that the foliation is given by $x =$ const. and $\mathcal{H}$ is the kernel of the 1-form $\alpha_c = dx_0 + x_1\, dx_2 + \dots + x_{2m-1}\, dx_{2m}$. As in Definition 4.1, this implies that the transition functions of $\mathfrak{F}$ may be assumed to be local contact diffeomorphisms of $\mathbf{R}^{2m+1}$. Observe, also, that because $\mathcal{H}$ is transversally oriented there is a global 1-form α on Y such that $\mathcal{H} = \ker \alpha$. Clearly, α is determined up to multiplication by a positive function.

PROPOSITION 7.2. – *There is a 1-1 correspondence between equivalence classes of non-degenerate 1-forms on* Y^{2m+2} *and 1-dimensional foliations* $\mathfrak{F}$ *with transverse contact structure.*

Proof. – We must show that if α is non-degenerate, the hyperplane field $\mathcal{H} = \ker \alpha$ is a transverse contact structure for a unique 1-dimensional foliation $\mathfrak{F}$. Let Ω be a volume form on Y, and define the vector field ξ by $\xi \lrcorner \Omega = \alpha \wedge (d\alpha)^m$. Then $\xi \neq 0$, and so it defines a foliation $\mathfrak{F}$ on Y. We will show that the field

$$\mathcal{H} = \ker \alpha$$

is a transverse contact structure for $\mathfrak{F}$. Observe that

$$0 = \xi \lrcorner (\xi \lrcorner \Omega) = (\xi \lrcorner \alpha) \wedge (d\alpha)^m - m\alpha \wedge (\xi \lrcorner d\alpha) \wedge (d\alpha)^{m-1} .$$

Multiplying this equation by α, we see that $\xi \lrcorner \alpha = 0$. Thus $T\mathfrak{F} \subset \mathcal{H}$. It remains to check that $\mathcal{H}$ is invariant under any flow along $\mathfrak{F}$. This will follow if $\mathcal{L}_\xi \alpha = \rho\alpha$ for some function ρ. (To see this, note that, if $\xi \lrcorner f = \rho$ where $\mathcal{L}_\xi \alpha = \rho\alpha$, then $\mathcal{L}_{\lambda\xi}(e^{-f}\alpha) = 0$, which implies that the field $\mathcal{H} = \ker(e^{-f}\alpha)$ is invariant under $\lambda\xi$).

We claim that $\mathcal{L}_\xi \alpha = \rho\alpha$, where ρ is defined by the equation $(d\alpha)^{m+1} = (m+1)\rho\Omega$. This is proved as follows. Since $\xi \lrcorner \alpha = 0$, we have $\mathcal{L}_\xi \alpha = \xi \lrcorner d\alpha$. Also,

$$\begin{aligned}(m+1)(\xi \lrcorner d\alpha) \wedge (d\alpha)^m &= \xi \lrcorner (d\alpha)^{m+1} = \xi \lrcorner (m+1)\rho\Omega \\ &= (m+1)\rho\alpha \wedge (d\alpha)^m .\end{aligned}$$

If $(d\alpha)^{m+1} \neq 0$ near y, one may use Darboux's theorem for the symplectic form $d\alpha$ to conclude that $\xi \lrcorner d\alpha = \rho\alpha$ near y. Thus the equation holds on the support of $(d\alpha)^{m+1}$. On the other hand, if $(d\alpha)^{m+1} \equiv 0$ near y, then one may choose coordinates

$$(y_0, \ldots, y_{2m+1})$$

near y so that $\alpha = dy_0 + y_1 dy_2 + \ldots + y_{2m-1} dy_{2m}$. (See [C] Ch III.47). Then $\xi \lrcorner \Omega$ is a multiple of $dy_0 \wedge \ldots \wedge dy_{2m}$, so that ξ is a multiple of $\partial/\partial y_{2m+1}$. Hence $\xi \lrcorner d\alpha = 0 = \rho\alpha$ near y, as required.

It remains to show that there is at most one foliation $\mathfrak{F}$ with a given transverse contact structure $\mathcal{H}$. So let us suppose that ξ is a vector field tangent to $\mathfrak{F}$, and that $\mathcal{H} = \ker \alpha$ is a transverse contact structure for $\mathfrak{F}$. Then $\xi \lrcorner \alpha = 0$. Also, because $\mathcal{H}$ is invariant, we can express $\mathcal{H}$ locally as $\ker \lambda\alpha$ where $\mathcal{L}_\xi(\lambda\alpha) = \xi \lrcorner d(\lambda\alpha) = 0$. It follows that $\xi \lrcorner d\alpha = -\xi(\lambda)\,\alpha$, and hence that $\xi \lrcorner(\alpha \wedge (d\alpha)^m) = 0$. Since $\alpha \wedge (d\alpha)^m \neq 0$, this implies that $\alpha \wedge (d\alpha)^m = \xi \lrcorner \Omega$ for some volume form Ω on Y. Hence the direction of ξ is determined by α, and $\mathfrak{F}$ is unique.

□

One can now repeat the discussion of § 3,4, making the obvious changes. Thus we say that two transversally oriented contact structures $\mathcal{H}_0$ and $\mathcal{H}_1$ are:

(i) *Isotopic* iff there is an isotopy g_t with $g_0 = \mathrm{id}.$ and $g_{1*}\ \mathcal{H}_0 = \mathcal{H}_1$;

(ii) *Homotopic* iff they may be joined by a smooth family of contact structures;

(iii) *Formally equivalent* iff they define homotopic reductions of the structural group of TX to $\mathrm{U}(m)$;

(iv) *Concordant* iff there is a 1-dimensional foliation $\mathfrak{F}$ on $\mathrm{X} \times \mathrm{I}$ with transverse contact structure which restricts to $\mathcal{H}_i$ on the end $\mathrm{X} \times i$; and

(v) *Strongly concordant* iff they are concordant and if in addition, $\mathfrak{F}$ is homotopic rel $\mathrm{X} \times \partial\mathrm{I}$ to the foliation with leaves $pt \times \mathrm{I}$.

The relations between these concepts are just the same as in the symplectic case. Note that there are no isotopy invariants for open contact manifolds corresponding to the volume of ends. On the other hand, Bennequin's examples show that formal equivalence does not imply isotopy for either open or closed manifolds(*).

(*) Other possible equivalence relations on contact manifolds are discussed in [To]. This paper should be read with a certain amount of caution since it contains several mistakes. The author confuses the groupoid Γ_α^{2m+1} defined in Note 6.4 above with the groupoid of germs of contact diffeomorphisms Γ_{ct}^{2m+1}. Theorems 6 and 7 are wrong. Also Theorem 2 is unproven with his (rather than Haefliger's) definition of integrable homotopy.

PROPOSITION 7.3. – *Two contact structures are formally equivalent if and only if they are strongly concordant.*

Proof. – One first shows that $\mathcal{H}_0 = \ker \alpha_0$ is formally equivalent to $\mathcal{H}_1 = \ker \alpha_1$ if and only if there are 1-forms α_t, $0 \leqslant t \leqslant 1$, and 2-forms β_t, $0 \leqslant t \leqslant 1$, with $\alpha_t \wedge (\beta_t)^m \neq 0$ and such that $\beta_i = d\alpha_t$, $i = 0, 1$. Then one argues as in the proof of Theorem 4.2, using example 2.6 instead of example 2.2.

□

Finally consider the groupoid Γ_{ct}^{2m+1} of germs of contact diffeomorphisms of $\mathbf{R}^{2m+1}$. These diffeomorphisms preserve the standard contact form α_c up to multiplication by a non-vanishing function which could be negative. Therefore $\overline{\mathrm{B}}\Gamma_{ct}^{2m+1}$ classifies foliations with a transverse contact structure which need not be transversally oriented. Thus, there is a map

$$\nu : \mathrm{B}\Gamma_{ct}^{2m+1} \longrightarrow \mathrm{BC}_{2m+1},$$

where C_{2m+1} is the subgroup of $\mathrm{O}(2m+1)$ which is generated by $\mathrm{U}(m) \oplus 1$ together with the element (conjugation) $\oplus -1$. Let $\overline{\mathrm{B}}\Gamma_{ct}^{2m+1}$ be the homotopy fiber of ν. It follows as in the symplectic case (see [9] II.6) that $\overline{\mathrm{B}}\Gamma_{ct}^{2m+1}$ is $2m$-connected. We can now go one step further.

PROPOSITION 7.4. – $\pi_{2m+1}(\overline{\mathrm{B}}\Gamma_{ct}^{2m+1}) = 0$.

Proof. – This is entirely analogous to the proof of Theorem 6.1 and will be left to the reader.

□

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