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TRANSFERRING L^p MULTIPLIERS

by Anthony H. DOOLEY

1. Introduction.

The homomorphism theorem for L^p multipliers, in its most general form, asserts that if G is a separable locally compact amenable group and if H is a normal subgroup of G then there is a canonical norm-nonincreasing map $\imath^*: \mathrm{CONV}_p(G) \to \mathrm{CONV}_p(G/H)$ (where CONV_p denotes the space of Fourier multipliers of L^p). In this form, it is due to C. S. Herz [11] although of course the theorem is a descendant of de Leeuw's theorem which applies to the case where G is abelian. It is instructive to consider the case $G = \mathbf{R}^n$, $H = \mathbf{Z}^n$. Starting from the quotient map $\pi_\lambda: x \to (e^{ix_1/\lambda}, e^{ix_2/\lambda}, e^{ix_n/\lambda}): \mathbf{R}^n \mapsto \mathbf{T}^n$ one computes that $(\imath_\lambda^*\Phi)^\wedge(k_1,\ldots,k_n) = \frac{1}{\lambda}(k_1,\ldots,k_n)$, for $\Phi \in \mathrm{CONV}_p(\mathbf{R}^n)$.

The genesis of this work was the idea that one could develop an analogue of the homomorphism theorem where the group homomorphism π is replaced by a family of contraction maps ([7]) from one Lie group to another. It appears that this is possible in some generality, but while the foundations of a general theory are presented here, in § 2, we will devote the bulk of the paper to two rather specific cases, the contraction onto a compact Lie group G of its Lie algebra $\mathfrak{g}$, and the contraction of the Cartan motion group $V \rtimes K$ associated to a Riemannian symmetric pair of the compact type (G,K) onto G.

In both these cases, one is able to develop an explicit formula for the periodification maps $\imath_\lambda$ from $\underline{A}_p(G)$ into $\underline{B}_p(\mathfrak{g})$ or $\underline{B}_p(V \rtimes K)$ respectively, associated to the contraction maps. This enables us to compute the Fourier transform of $\imath_\lambda^*\Phi$, also in an explicit manner.

These techniques are useful for producing examples of multipliers on compact groups and homogeneous spaces. Considering first the contraction

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of $\mathfrak{g}$ onto G, one may apply the maps ι_λ^* to a fairly arbitrary multiplier of $L^p(\mathfrak{g})$ and obtain a multiplier of $L^p(G)$. Starting from Ad invariant multiplier one obtains a central multiplier: one can hence prove boundedness of most known central multipliers on G. However, the method applies equally well to non-central multipliers. For example, we will show that for $G = SU(2)$ the Riesz kernels are obtained by applying ι_λ^* to the Riesz kernels on $\mathfrak{g} = \mathbf{R}^3$; hence we obtain new bounds for their norms.

The contraction of $V \rtimes K$ onto G also yields interesting results. Starting from Ad K invariant multipliers of $L^p(V)$ one obtains bi-K invariant multipliers of $L^p(V)$. Thus, for example, from each radial multiplier of the plane, one may obtain a family of multipliers of $L^p(S^2)$. As before, the method applies equally well to non K-invariant multipliers.

The original de Leeuw theorem gains in power by the fact that it has a converse: if Φ is some function on $\mathbf{R}^n$ so that $\iota_\lambda^*\Phi$ is a bounded multiplier of $\iota L^p(\mathbf{T}^n)$ for each λ and if $\limsup_{\lambda\to\infty} |\,\|\iota_\lambda^*\Phi|\,\|_p \leqslant K$, then $\Phi \in M_p(\mathbf{R}^n)$ and $\|\,|\Phi|\,\|_p \leqslant K$. Versions of this theorem have been proved by Clerc [1] for the contraction of $\mathfrak{g}$ to G, by Rubin [5] for the contraction of $M(2)$ onto $SO(3)$ and by Dooley and Gaudry [6] for $V \rtimes K$ onto G. It is interesting to enquire to what extent the present results constitute a converse to these theorems. It is readily seen that the restriction $\Phi^{(\lambda)}$ used by these authors is not the same as $\iota_\lambda^*\Phi$; indeed, $\Phi^{(\lambda)}$ is not in general a multiplier of L^p when Φ is a multiplier of L^p. Rather, it appears that in each degree d_σ, $\iota_\lambda^*\Phi^\wedge(\sigma)$ is an average of $\Phi^{(\lambda)\wedge}(\sigma)$ together with $d_\sigma - 1$ other terms involving representations with lower weights. It seems reasonable to suggest that $\iota_\lambda^*\Phi$, arising as it does from natural structural mappings between the groups and algebras of functions is a better notion of restriction than $\Phi^{(\lambda)}$ (which, after all, was defined using a particular special choice of embeddings for representation spaces).

If this suggestion is to be followed through, one should try to prove converse theorems (à la Clerc, Rubin, etc.) with $\iota_\lambda^*\Phi$ replacing $\Phi^{(\lambda)}$. Looking just at the case of $SO(2)$ invariant multipliers of $SU(2)$, a necessary condition is that $\lim_{\ell\to\infty}\left[\frac{\iota_R^*\Phi}{\ell}\right]^\wedge(\ell)$ should equal $\hat{\Phi}(R)$, or at least a multiple of the latter. Unfortunately this will be seen to be not true for general multipliers, although it does hold for certain special multipliers of interest in particular the Riesz means and the Bochner–Riesz means.

2. The map ι_λ.

2.1. Throughout this section, let G be a connected Lie group and K a reductive subgroup of G. Thus, the Lie algebra $\mathfrak{g}$ of G may be decomposed as

$$\mathfrak{g} = \mathfrak{k} \oplus V$$

where $\mathfrak{k}$ is the Lie algebra of K, and V is an ad(K)-invariant vector subspace of V. We let $V \rtimes K$ denote the semidirect product of V by the adjoint action of K.

In the case where K is amenable, which we will assume henceforth, $V \rtimes K$, a semidirect product of amenable groups, is again amenable.

We introduce, for each $\lambda > 0$, the contraction map

$$\pi_\lambda : \quad V \rtimes K \to G : (v,k) \mapsto \left(\exp_G \frac{v}{\lambda}\right) \cdot k .$$

In the case $(G,K) = (\mathbf{T},\{e\})$, the map π_λ is $x \to e^{ix/\lambda} : \mathbf{R} \to \mathbf{T}$. It is the purpose of this section to use π_λ to construct a periodification map $\underline{\underline{A}}_p(G) \to \underline{\underline{B}}_p(V \rtimes K)$.

We will make the additional assumption, satisfied in all cases of interest to us, that every element of G has the form $(\exp v).k$ for some where $v \in V$, $k \in K$. This implies that the map π_λ onto and hence is a quotient map of locally compact Hausdorff spaces.

2.2. We recall [11] § 3 the notion of $\underline{\underline{V}}_p(Y)$ for a locally compact Hausdorff space Y; it is the algebra of continuous functions $F : Y \times Y \to \mathbf{C}$ which are bounded multipliers of the Banach space tensor product $\ell^p \times \ell^{p'}(Y)$. Such a function has norm

$$\|F\|_{\underline{\underline{V}}_p} = \sup_{u,v} \inf_{(u_n),(v_n)} \sum_{n=1}^{\infty} \|u_n\|_p \|v_n\|_{p'}$$

where the supremum is taken over all pairs u, v of complex valued functions with

$$\left[\sum_{y=Y} |u(y)|^p\right]^{\frac{1}{p}} = \|u\|_p \leqslant 1 \quad \text{and} \quad \left[\sum_{y=Y} |v(y)|^{p'}\right]^{\frac{1}{p'}} = \|v\|_{p'} \leqslant 1$$

and the infimum over all representations

$$F(x,y)u(x)v(y) = \sum_{n=1}^{\infty} u_n(x)v_n(y).$$

According to [11], theorem 5, if $\pi : \Gamma \to G$ is a quotient map of locally compact Hausdorff spaces, then $F \to F \circ \pi \times \pi$ is an isometric Banach algebra inclusion

$$\underline{V}_p(G) \to \underline{V}_p(\Gamma).$$

Applying this to the case where $\pi = \pi_\lambda$, we see that

LEMMA. — *$\psi_\lambda : F \to F \circ \pi_\lambda \times \pi_\lambda$ is an isometric Banach algebra inclusion*

$$\underline{V}_p(G) \to \underline{V}_p(V \rtimes K).$$

2.3. Now $\underline{B}_p(G)$ denotes the closed subspace of elements F of $\underline{V}_p(G)$ for which $F(tx,ty) = F(x,y)$, $\forall x, y, t \in G$. It may be realized as a set of functions on G via $f \longleftrightarrow F$, where

$$F(x,y) = f(y^{-1}x).$$

We will usually consider $\underline{B}_p$ to consist of functions on G.

By theorem 1 of [10], $\underline{A}_p(G) \subseteq \underline{B}_p(G)$ where $\underline{A}_p(G)$ denotes the space of continuous functions on G which are in the image of the projective tensor product $L^p \otimes L^{p'}$ under the map $P : u \otimes v \to u * v^*$. Thus, we have a norm-decreasing map $\imath : \underline{A}_p(G) \to \underline{V}_p(G)$.

On the other hand, $V \rtimes K$ is amenable; let M denote a left invariant mean on $L^\infty(V \rtimes K)$. (We will here and in the sequel, assume that M is normalized so that $M(\underline{1})=1$). We have a map, also denoted M:

$$\underline{V}_p(V \rtimes K) \to \underline{B}_p(V \rtimes K)$$

defined by

$$(MF)(x) = M(t \to F(tx,t)).$$

It is easily checked that M is a norm nonincreasing map $\underline{V}(V \rtimes K) \to \underline{B}_p(V \rtimes K)$.

Finally, let $\imath_\lambda = M \circ \psi_\lambda \circ \imath$ be the concatenation of all the above maps. We have proved, by our construction, PROPOSITION. — $\imath_\lambda : \underline{\underline{A}}_p(G) \to \underline{\underline{B}}_p(V \rtimes K)$ *is a norm-nonincreasing linear map.*

2.4. Herz ([8], théorème 2) has shown that $\underline{\underline{A}}_p^*(G) = CONV_p(G)$, the set of all Fourier multipliers of $L^p(G)$. Further, $V \rtimes K$ being amenable, it results from théorème 2 of [10] that $B_p(V \rtimes K)$ is the dual of PF_p where PF_p is the closure of $L^1(V \rtimes K)$ in $CONV_p$. Thus, we may think of $\underline{\underline{B}}_p^*(V \rtimes K)$ as a kind of ultra-weak closure of $L^1(V \rtimes K)$, $\underline{\underline{B}}_p^* = PF_p^{**}$.

Dualizing proposition (2.3), we obtain :

COROLLARY. — $\imath_\lambda^* : \quad PF_p^{**}(V \rtimes K) \to CONV_p(G)$ *is a norm nonincreasing map.*

In particular, for every $f \in L^1(V \rtimes K)$ and for every $\lambda > 0$ we obtain a Fourier multiplier $\imath_\lambda^*(f)$ of $L^p(G)$ whose norm does not exceed $|\,\|f\|\,|_p$.

2.5. Since $PF_p(V \rtimes K)$ is dense in the weak topology of $CONV_p(V \rtimes K)$ we may, for certain other elements of $CONV_p(V \rtimes K)$ obtain elements of $CONV_p(G)$. This, however, is a rather delicate question which we shall not pursue here.

2.6. *Remark.* — In the case where $(G,K) = (\mathbf{T},\{e\})$ the map $\imath_\lambda$:

$$\underline{\underline{A}}_p(\mathbf{T}) \to \underline{\underline{B}}_p(\mathbf{R})$$

is easily seen to be usual « periodification » map $(\imath_\lambda(f))(x) = f(e^{ix/\lambda})$ and its dual $B_p^*(\mathbf{R}) \to CONV_p(\mathbf{T})$ is the map $\Phi \to \Phi^{(\lambda)}$ defined in the introduction. Thus, corollary (2.4) is a generalization of de Leeuw's theorem.

3. Computation of $\imath_\lambda$.

3.1. In the previous section, we considered a connected Lie group G with a reductive amenable subgroup K and constructed a norm-nonincreasing map $\imath_\lambda : A_p(G) \to B_p(V \rtimes K)$. One can do little more without making specific hypotheses on G and K, which we proceed to do.

3.2. Let us assume that K is compact. The mean M on $L^\infty(V \rtimes K)$ can then be defined as follows. Fix a K-invariant norm $\|.\|$ on V with the property that the Haar measure (in V) of $B_R = \{x \in V : \|x\| \leqslant R\}$ is $R^{\dim V}$. Set

$$Mf = \lim_{R \to \infty} \frac{1}{R^{\dim V}} \int_K \int_{B_R} f(v,k)\, dv\, dk \qquad \text{for} \qquad f \in L^\infty(V \rtimes K).$$

We have

PROPOSITION. — *Let* $f \in \underline{\underline{B}}_p(G)$. *Then*

$$\imath_\lambda f(v,k) = \lim_{R \to \infty} \int_{B_1} f\left(\exp - Rw \exp\left(Rw + \frac{v}{\lambda}\right)k\right) dw.$$

Proof. — This is just a matter of using definitions from § 2. Thus

$$\begin{aligned}
(\imath_\lambda f)(v,k) &= \lim_{R\to\infty} R^{-\dim V} \int_{K \times B_R} f(\pi_\lambda(t^{-1})\pi_\lambda(t.(v,k)))\, dt \\
&= \lim_{R\to\infty} R^{-\dim V} \int_K \int_{B_R} f(\pi_\lambda(-h^{-1}x, h^{-1})\pi_\lambda(w+hv, hk))\, dw\, dh \\
&= \lim_{R\to\infty} R^{-\dim V} \int_K \int_{B_R} f\left(\left(\exp \frac{-h^{-1}w}{\lambda}\right)\cdot h^{-1}\right. \\
&\qquad\qquad \left.\left(\exp \frac{w+hv}{\lambda}\right)\cdot hk\right) dw\, dh \\
&= \lim_{R\to\infty} R^{-\dim V} \int_K \int_{B_R} f\left(\left(\exp \frac{-h^{-1}w}{\lambda}\right)\left(\exp \frac{h^{-1}w+v}{\lambda}\right)k\right) dw\, dh .
\end{aligned}$$

Since B_R is K invariant, the K integral disappears and we are left with

$$\lim_{R\to\infty} R^{-\dim V} \int_{B_R} f\left(\left(\exp \frac{-w}{\lambda} \exp \frac{w+v}{\lambda}\right)k\right) dw .$$

Making the change of variables $w \to R.w$, we obtain

$$\lim_{R\to\infty} \int_{B_1} f\left(\exp \frac{-Rw}{\lambda} \exp\left(\frac{Rw}{\lambda} + \frac{v}{\lambda}\right)k\right) dw$$

which clearly equals the given expression.

In order to eliminate the limit in the above expression, we need to consider the limit as $R \to \infty$ of $\exp - Rw \exp\left(Rw + \frac{v}{\lambda}\right)$. Since R approaches infinity rather than zero, we cannot use the Campbell-Baker-Hausdorff formula. Rather, a method akin to the method of stationary phase seems appropriate.

3.3 Let us specialize the above somewhat, and assume that (G,K) is a Riemannian symmetric pair of the compact type. In this case $\mathfrak{g} = \mathfrak{k} + V$ is the Cartan decomposition and π_λ is onto so that the hypotheses of § 2 are satisfied. We may consider the polar decomposition of the measure dw on B_1, as follows. Let $\mathfrak{a}$ be a maximal abelian subalgebra of V, and let $\mathfrak{a}_1 = \mathfrak{a} \cap B_1$. Then $(Y,k) \to \mathrm{Ad}(k)Y$; $\mathfrak{a} \times K \to V$ is onto with kernel $\{(0,k) : k \in M\}$ where M is the stabilizer of $\mathfrak{a}$ in K, and we have (cf. [9], $\underline{X}$, § 1.5)

$$\int_{B_1} dw = \int_K \int_{\mathfrak{a}_1} \prod_{\alpha \in P_+} \alpha(H)\, dH\, dh.$$

Substituting this in the expression from (3.1) and making the change of variables $h \to h^{-1}$, we obtain

Lemma. — $(\imath_\lambda f)(v,k)$

$$= \int_K \lim_{R \to \infty} \int_{\mathfrak{a}_1} f\left(h^{-1} \exp -RH \exp\left(RH + \frac{h.v}{\lambda}\right) hk\right) \prod_{\alpha \in P_+} \alpha(H)\, dH\, dh.$$

Using this lemma we will prove

3.4. Theorem. — *Let* (G,K) *be a Riemannian symmetric pair of the compact type. Then for* $f \in C(G)$

$$(\imath_\lambda f)(v,k) = \int_K f\left(h^{-1} \exp\left(\frac{h.v}{\lambda}\right)_{\mathfrak{a}} hk\right) dh$$

where $(\ )_{\mathfrak{a}}$ *denotes the projection in* V *onto the maximal abelian subalgebra* $\mathfrak{a}$.

Proof. — By lemma (3.3), it will suffice to show that for every $v \in V$

$$I := \lim_{R \to \infty} \int_{\mathfrak{a}_1} f(\exp -RH, \exp(RH + v)) \prod_{\alpha \in P_+} \alpha(H)\, dH = f(\exp v_{\mathfrak{a}}).$$

The left regular representation of the compact abelian group $A = \exp \mathfrak{a}$ on $C(G)$ may be decomposed as a direct sum of characters of A. Hence, we may assume without loss of generality that $f(t^{-1}g) = \chi(t)f(g)$ for some character χ of A. By approximating f in the uniform norm, if necessary, we may also assume that f is C^{∞}.

In this case, writing $v = A + Q$, $A \in \mathfrak{a}$, $Q \in \mathfrak{a}^{\perp}$, we have

$$f(\exp - RH \exp(RH + v)) = \chi(\exp A) f(\exp - (A + RH) \exp((A + RH) + Q))$$

and $\chi(\exp A) f(e) = f(\exp A)$; thus we obtain

$$\left| \int_{\mathfrak{a}_1} f(\exp - RH \exp RH + v) \prod_{\alpha \in P_+} \alpha(H)\, dH - f(\exp A) \right|$$

$$= \left| \int_{\mathfrak{a}_1 + \frac{A}{R}} (f(\exp - RH \exp(RH + Q)) - f(\exp - RH \exp RH)) \prod_{\alpha \in P_+} \alpha\left(H - \frac{A}{R}\right) dH \right|$$

$$\leqslant \int_{\mathfrak{a}_1 + \frac{A}{R}} |f(\exp - RH \exp(RH + Q)) - f(\exp - RH \exp RH)| \left| \prod_{\alpha \in P_+} \alpha\left(H - \frac{A}{R}\right) \right| dH$$

$$= \int_{\mathfrak{a}_1 + \frac{A}{R}} |f(\exp(RH + Q) - f(\exp RH)| \prod_{\alpha \in P_+} \alpha\left(H - \frac{A}{R}\right) dH.$$

Let $h = f \circ \exp|_V : V \to \mathbf{C}$. Since f is C^{∞}, so is h and by the mean value theorem, there exists for each $R > 0$ a number $S_R \in [0,1]$ with

$$|h(RH + Q) - h(RH)| = |h'(RH + S_R Q)(Q)|.$$

The derivative $h' \circ f$, h is a map $V^* \to V$ which may be computed by the chain rule

$$h(u) = f_{*, \exp u} \circ (L_{\exp u})_{*, e} \circ \left(\frac{1 - e^{-\operatorname{ad} u}}{\operatorname{ad} u} \right)$$

where we have used the expression for the differential of the exponential given in [9], II, Theorem 1.7.

I claim that $|h'(\mathrm{RH}+\mathrm{S_R Q})(\mathrm{Q})|$ is dominated by $\dfrac{\text{const}}{\mathrm{R}\prod_{\alpha\in\mathrm{P}_+}\alpha(\mathrm{H})}$.

Indeed, since G is compact, we have a uniform bound for the operators

$$f_{*,x}:\ \mathfrak{g}\to \mathrm{T}_x(\mathrm{G})\quad (x\in \mathrm{G})$$
$$(\mathrm{L}_x)_{*,e}:\ \mathrm{T}_x(\mathrm{G})\to\mathfrak{g}$$

and $e^{-\operatorname{ad}\mathrm{X}}=\operatorname{ad}(\exp -\mathrm{X}):\ \mathfrak{g}\to\mathfrak{g}$.

Now Q has the form $\sum_{\alpha\in\mathrm{P}_+} c_\alpha(\mathrm{X}^\alpha-\theta\mathrm{X}^\alpha)$.

Writing $\mathrm{Q}=\operatorname{ad}(\mathrm{RH})\sum_{\alpha\in\mathrm{P}_+}\dfrac{c_\alpha}{\alpha(\mathrm{RH})}(\mathrm{X}^\alpha+\theta\mathrm{X}^\alpha)$, we obtain

$$\left\|\frac{1-e^{-\operatorname{ad}(\mathrm{RH}+\mathrm{S_R Q})}}{\operatorname{ad}(\mathrm{RH}+\mathrm{S_R Q})}(\mathrm{Q})\right\|$$
$$\leqslant\left\|\frac{1-e^{-\operatorname{ad}(\mathrm{RH}+\mathrm{S_R Q})}}{\operatorname{ad}(\mathrm{RH}+\mathrm{S_R Q})}\Big((\operatorname{ad}(\mathrm{RH}+\mathrm{S_R Q}))\sum_{\alpha\in\mathrm{P}_+}\frac{c_\alpha}{\alpha(\mathrm{RH})}(\mathrm{X}^\alpha+\theta\mathrm{X}^\alpha)\right.$$
$$\left.-\ \mathrm{S_R}\operatorname{ad}\mathrm{Q}\,\Sigma\frac{c_\alpha}{\alpha(\mathrm{RH})}(\mathrm{X}^\alpha+\theta\mathrm{X}^\alpha)\Big)\right\|.$$

The norm of the first term on the right hand side of this expression is

$$\left\|1-e^{-\operatorname{ad}(\mathrm{RH}+\mathrm{S_R Q})}\sum_{\alpha\in\mathrm{P}_+}\frac{c_\alpha}{\alpha(\mathrm{RH})}(\mathrm{X}^\alpha+\theta\mathrm{X}^\alpha)\right\|\leqslant\frac{\text{const}}{\mathrm{R}\prod_{\alpha\in\mathrm{P}_+}\alpha(\mathrm{H})}.$$

The norm of the second expression is dominated by

$$\frac{\mathrm{S_R}}{\mathrm{R}\prod_{\alpha\in\mathrm{P}_+}|\alpha(\mathrm{H})|}\left\|\frac{1-e^{-\operatorname{ad}(\mathrm{RH}+\mathrm{S_R Q})}}{\operatorname{ad}(\mathrm{RH}+\mathrm{S_R Q})}[\mathrm{Q},c_\alpha^1(\mathrm{X}^\alpha+\theta\mathrm{X}^\alpha)]\right\|$$

where $|c_\alpha^1|=\left|\sum_{\substack{\beta\in\mathrm{P}_+\\ \beta\neq\alpha}}\alpha(\mathrm{H})c_\alpha\right|$ is majorized by a multiple of c_α which is independent of R. Thus, to show that the second term is appropriately bounded it suffices to show that

$$\sup_{\mathrm{R}>0}\left\|\frac{1-e^{-\operatorname{ad}(\mathrm{RH}+\mathrm{S_R Q})}}{\operatorname{ad}(\mathrm{RH}+\mathrm{S_R Q})}(\mathrm{Y})\right\|\leqslant\text{const}\,\|\mathrm{Y}\|$$

or alternatively, that for all $Y \in \mathfrak{g}$

$$\sup_{X \in \mathfrak{g}} \|d \exp_X(V)\| \leqslant \text{const } \|Y\|.$$

This follows from the compactness of G.

$\Big($One notices that for all $g \in G$, $\left\|\frac{d \exp X}{\|Y\|}(Y)\right\| = \left\|\frac{d \exp_{\mathrm{Ad}(g)X}(Y)}{\|Y\|}\right\|$.
Thus, one may take the supremum over $X \in \mathfrak{t}$. Since the X^α span $\mathfrak{g} \ominus \mathfrak{t}$, it suffices to check that $\sup_{H \in \mathfrak{t}} \left\|\frac{1 - e^{-\mathrm{ad}\,H}}{\mathrm{ad}\,(H)}(\alpha)\right\| \leqslant \text{const } \forall \alpha \in \Phi^+$. But $\mathrm{ad}\,(H)X^\alpha = \alpha(H)(X^\alpha)$ where $\alpha(H)$ is purely imaginary; thus we have $\left\|\frac{1 - e^{-\mathrm{ad}\,H}}{\mathrm{ad}\,H}(X^\alpha)\right\| = \left|\frac{1 - e^{\alpha(H)}}{\alpha(H)}\right| \|X^\alpha\| \leqslant \text{const.}\Big)$.

We have thus shown that

$$|f(\exp(RH+Q)) - f(\exp RH)| \leqslant \frac{\text{const}}{R \prod_{\alpha \in P_+} \alpha(H)} \cdot \tag{2}$$

Now let $\varepsilon > 0$ and choose a neighbourhood $\mathfrak{a}_\varepsilon$ in $\mathfrak{a}$ of the set $\{H \in \mathfrak{a}_1 : \alpha(H) = 0 \text{ for some } \alpha \in P_1\}$ such that

$$2\|f\|_\infty \prod_{\alpha \in P_+} \alpha(H) < \varepsilon$$

wherever $H \in \mathfrak{a}_\varepsilon$.

Next, choose R so large that there is $\beta > 0$ for which $\left|\alpha\left(H - \frac{A}{R}\right)\right| > \beta$ for all $\alpha \in P_+$ for all $H \in \mathfrak{a}_\varepsilon$.

We obtain from (2)

$$\begin{aligned}\int_{\mathfrak{a}_1 + \frac{A}{R}} & |f(\exp - RH \exp(RH+Q)) \\ & - f(\exp - RH \exp RH)| \prod_{\alpha \in P_+} \alpha\left(H - \frac{a}{R}\right) dH \\ & \leqslant \varepsilon \int_{\mathfrak{a}_\varepsilon} dH + \frac{\text{const}}{R} \int_{\mathfrak{a}_1 \setminus \mathfrak{a}_\varepsilon} \prod_{\alpha \in P_+} \frac{\alpha(H)}{\alpha\left(H - \frac{A}{R}\right)} dH \\ & \leqslant \varepsilon + \frac{\text{const}}{R} \cdot\end{aligned}$$

Combining this with (1), we obtain, since ε is arbitrary, $I = f(\exp A)$, as claimed.

3.5. One case of interest not covered by the above theorem is the case where G is a compact Lie group and $K = \{e\}$. In that case, $V \rtimes K$ is the Lie algebra $\mathfrak{g}$ of G. Fix a maximal torus T in G. One can prove in a manner similar to the above.

THEOREM. — *Let* $X \in \mathfrak{g}$, *and* $f \in A_p(G)$. *Then*

$$(\imath_\lambda f)(X) = \int_{G/T} f\left(g \exp\left(\frac{g^{-1}.X}{\lambda}\right)_{\mathfrak{t}} g^{-1}\right) d\dot{g}.$$

Proof. — As in the previous theorem, one writes, using the Weyl integration formula

$$\int_{B_1} dw = \int_{G/T} \int_{\mathfrak{t}_1} \prod_{\alpha \in \Phi^+} \alpha(H)\, dH\, d\dot{g}.$$

Essentially the analysis given in (3.4) enables one to deduce that

$$\int_{\mathfrak{t}_1} f\left((\exp - RH) \exp\left(RH + \frac{g^{-1}.X}{\lambda}\right)\right) \prod_{\alpha \in \Phi^+} \alpha(H)\, dH \to f\left(\left(\frac{g^{-1}.X}{\lambda}\right)_{\mathfrak{t}}\right)$$

for all continuous f. □

Presumably there is some more general theorem which contains both (3.4) and (3.5); the author has been unable to find a satisfactory generalization of these two theorems.

Remarks. — These formulae may at first sight seem curious. However, they are in some senses the natural nonabelian analogue of the periodification map which lies at the heart of de Leeuw's theorems (and, indeed, a number of other techniques of abelian harmonic analysis).

If, in theorem (3.5), G were abelian, we would have $\mathfrak{t} = \mathfrak{g}$. Thus, the g's in the integrand would commute with projection onto $\mathfrak{t}$ and cancel, the integral over G/T would disappear and we would obtain $f\left(\exp \frac{X}{\lambda}\right)$ — the periodification of de Leeuw. Similarly in (3.4) if we could commute projection onto $\mathfrak{a}$ with the adjoint action of G, $\imath_\lambda f$ would reduce to $f \circ \pi_\lambda$.

4. The contraction of $\mathfrak{g}$ onto G.

In this section we will study the map $\imath_\lambda^*$ associated to the contraction map of $\mathfrak{g}$ onto G, and obtain formulae for the multipliers of G which arise from multipliers of its Lie algebra $\mathfrak{g}$.

4.1. One may use Theorem (3.5) to obtain an expression for the Fourier transform of $\imath_\lambda^*\varphi$ ($\varphi \in L^1(\mathfrak{g})$, say). For each $\nu \in \Lambda^*(G,T)$, let $\Pi(\nu)$ be the set of weights which occur in $\sigma_{\nu|_T}$. Choose an orthonormal basis

$$\{u^\nu_{\mu,j} : j = 1,\ldots,m_\nu(\mu)\}$$

for

$$\mathscr{H}_\mu = \{f \in \mathscr{H}^\nu : \sigma_\nu(t)f = \chi_\mu(t^{-1})f\}.$$

THEOREM. — *Let* $\varphi \in L^1(\mathfrak{g})$. *Then for* $u, w \in \mathscr{H}^\nu$,

$$\langle (\imath_\lambda * \varphi)^\wedge(\sigma_\nu)u, w\rangle = \sum_{\mu,j} \int_{G/T} \hat\varphi\left(\frac{g.\mu}{\lambda}\right) \langle \sigma_\nu(g)u, w_{\mu,j}\rangle \overline{\langle \sigma_\nu(g)w, w_{\mu,j}\rangle}\, d\dot g.$$

Proof. — By definition,

$$\begin{aligned}\langle \imath_\lambda^*\varphi^\wedge(\sigma_\nu)u,w\rangle &= \langle \varphi, \imath_\lambda(u * w^*)\rangle \\ &= \int_{\mathfrak{g}} \varphi(X)\imath_\lambda(u * w^*)(X)\, dX.\end{aligned}$$

Now by (3.5),

$$\begin{aligned}\imath_\lambda(u * w^*)(X) &= \int_{G/T} (u * w^*)\left(g^{-1}\exp\left(\frac{g^{-1}X}{\lambda}\right)_t g\right) d\dot g \\ &= \int_{G/T}\int_G u\left(g^{-1}\exp\left(\frac{g^{-1}.X}{\lambda}\right)_t gx\right)\overline{w(x)}\, dx\, d\dot g \\ &= \int_{G/T}\int_G (\sigma(g)u)\left(\exp\left(\frac{g^{-1}.X}{\lambda}\right)_t x\right)\overline{\sigma(g)w(x)}\, dx\, d\dot g.\end{aligned}$$

We expand $\sigma(g)u$ as $\sum_{u,j} \langle \sigma(g)u, u^\nu_{\mu,j}\rangle u^\nu_{\mu,j}$ and use the fact that $u^\nu_{\mu,j}(tx) = \chi_\mu(t^{-1})u^\nu_{\mu,j}(x)$ to obtain

$$\imath_\lambda(u * w^*)(x) = \int_{G/T} \sum_{\mu,j} \chi_\mu\left(\exp -\left(\frac{g^{-1}.X}{\lambda}\right)\right)$$
$$\langle \sigma(g)(w), u^\nu_{\mu,j}\rangle \overline{\langle \sigma(g)(w), u^\nu_{\mu,j}\rangle}\, d\dot g.$$

Thus

$$\langle (\imath_\lambda^* * \varphi)^\wedge(\sigma)u,w\rangle = \sum_{\mu,j}\int_{G/T}\int_{\mathfrak{g}} e^{\frac{ig.d\mu(X)}{\lambda}} \varphi(X)\,dX\langle\sigma(g)u,u^\vee_{\mu,j}\rangle\overline{\langle\sigma(g)(w)u^\vee_{\mu,j}\rangle}\,d\dot g$$

$$= \sum_{\mu,j}\int_{G/T}\hat\varphi\left(\frac{g.\mu}{\lambda}\right)\langle\sigma(g)u,u^\vee_{\mu,j}\rangle\overline{\langle\sigma(g)w,u^\vee_{\mu,j}\rangle}\,d\dot g\,.$$

4.2. An important simplification occurs in this formula when φ is an $\mathrm{Ad}(G)$ invariant function. In this case $\hat\varphi\left(\frac{g.\mu}{\lambda}\right) = \hat\varphi\left(\frac{\mu}{\lambda}\right)$ and so we may apply the orthogonality relations to obtain

$$\langle \imath_\lambda^*\varphi^\wedge(\sigma)u,w\rangle = \sum_{\mu\in\Pi(\sigma)}\frac{1}{d_\sigma}\hat\varphi\left(\frac{\mu}{\lambda}\right)m_\sigma(\mu).\langle u,w\rangle\,.$$

COROLLARY. – *If* φ *is an* $\mathrm{Ad}(G)$ *invariant function on* $\mathfrak{g}$, *then* $\imath_\lambda^*\varphi$ *is a central multiplier of* $L^p(G)$ *and*

$$(\imath_\lambda^*\varphi)^\wedge(\sigma) = \frac{1}{d_\sigma}\sum_{\mu\in\Pi(\sigma)}\hat\varphi\left(\frac{\mu}{\lambda}\right)m_\sigma(\mu)I_{d_\sigma}\,.$$

This formula is particularly simple and allows us to obtain from $\mathrm{Ad}(G)$ invariant multipliers of $L^p(\mathfrak{g})$, central multipliers of $L^p(G)$.

4.3. Given a complex valued function ψ on $\hat G = \Lambda^+(G,T)$ we may enquire when there is an $\mathrm{Ad}(G)$ invariant multiplier φ of $L^p(\mathfrak{g})$ with $\imath_\lambda^*\varphi = \psi$, i.e. one may attempt to invert $\imath_\lambda^*$. This can always be done formally by the Weyl character formula ([14], § 20.2). One obtains

$$\hat\varphi\left(\frac{\nu}{\lambda}\right) = \left(\prod_{\alpha\in\Phi^+} D\alpha(d.\psi)\right)(\nu)$$

where $D_\alpha(d.\psi)(\nu) = d_{\nu-\alpha}\psi(\nu-\alpha) - d_\nu\psi(\nu)$ is the difference operator associated to the positive root α.

Hence, we have proved a theorem of Coifman and Weiss [4]; if $\nu \to \prod_{\alpha\in\varphi^+} D\alpha(d.\psi)(\lambda\nu)$ belongs to $PF_p(\mathfrak{g})$ then ψ is a central multiplier of $L^p(G)$. This enables us to prove that a number of well-known multipliers on G are bounded (e.g. the Riesz kernels of Clerc [1]).

4.4 Recall that the central multipliers L^p are in general different from the multipliers of the centre of L^p. We should emphasize that our methods apply to the former but not to the latter. N. J. Weiss [18] has proved some results pertaining to multipliers of the centre of $L^p(G)$, for $G = SO(n)$ for a limited range of p. While it is true that if f is a central invariant function on $A^p(G)$, $\iota_\lambda f \in B_p(\mathfrak{g})$ is $\mathrm{Ad}(G)$ invariant, it is not obvious that $\iota_\lambda f$ belongs to $B_p(\mathfrak{g})_{\mathrm{inv}}$ (i.e. is a pointwise multiplier of $L^p_{\mathrm{inv}} \hat{\otimes} L^{p'}_{\mathrm{inv}}(\mathfrak{g})$). Weiss' restriction, which is particular to $SO(n)$ does not seem to be attainable from ours.

4.5. In this paragraph, we examine the special case $G = SU(2)$, so that $\mathfrak{g} = \mathbf{R}^3$. Choose

$$T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : 0 \leqslant \theta < 2\pi \right\}.$$

We have $\Lambda^+(G,T) = \mathbf{N}$, and for $\ell \in \mathbf{N}$, $\Pi(\ell) = \{-\ell,\ldots,\ell\}$ and,

$$m_\ell(p) = \begin{cases} 1 & \text{if } |p| \leqslant \ell \\ 0 & \text{otherwise} \end{cases}.$$

It follows readily that

$$\langle(\iota_\lambda^* \varphi)^\wedge(\sigma_\ell)u,w\rangle = \sum_{k=-\ell}^{\ell} \int_{G/T} \hat{\varphi}\left(\frac{|k|}{\lambda} g.e_3\right)\langle\sigma(g)u,u_k^\ell\rangle \overline{\langle\sigma(g)w,u_k^\ell\rangle}\, d\dot{g}$$

where e_3 is the element of $\mathbf{R}^3$ corresponding to $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$. (G acts on $\mathbf{R}^3$ by the adjoint action.)

We suppose that φ is a product of a radial function and an associated spherical harmonic. Let $x \in \mathbf{R}^3$; we assume that

$$\varphi(x) = f(|x|)P\left(\frac{x}{|x|}\right)$$

where $f\colon \mathbf{R}^+ \to \mathbf{C}$ and $P = Y_{s,q}$ is a spherical harmonic of degree s, and $-s \leqslant q \leqslant s$.

One knows from the Bochner-Hecke formula that for $g \in SU(2)$

$$\hat{\varphi}(\mu g.e_3) = 2\pi i^{-s}\mu^{-1/2}\tilde{f}(s+1/2,\mu)P(\dot{g})$$

where $\dot{g}$ denotes a coset representative of g in $S^2 = SU(2)/T$ and $\hat{f}(s+1/2,\mu)$ is the Bessel transform defined by

$$\hat{f}(s+1/2,\mu) = \int_0^\infty f(r) J_{s+1/2}(2\pi\mu r) r^{s+3/2}\, dr .$$

Combining this expression with our expression for ι_λ^*, we obtain

$$(\iota_\lambda^*\varphi)^\wedge(\sigma_\ell)_{m,n} = \sum_{k=-\ell}^{\ell} 2\pi i^{-s}\left(\frac{|k|}{\lambda}\right)^{s-1/2} \hat{f}\left(s+1/2, \frac{|k|}{\lambda}\right) \int_{S^2} P(\dot{g}) t^\ell_{mk} t^{-\ell}_{nk}(\dot{g})\, d\dot{g}$$

(it is clear that the product $t^\ell_{mk} t^{-\ell}_{mk}(g)$ depends only on the coset representative of g).

The associated harmonic $P = Y_{sq}$ is in fact the matrix coefficient $t^s_{qo}(\dot{g})$ of SU(2) (see [17], p. 129). The integral over S^2 is than equal to the following integral over SU(2):

$$\int_{SU(2)} t^s_{qo} t^\ell_{mk} t^{-\ell}_{nk}\, d\lambda .$$

The latter may be computed by the Clebsch-Gordan coefficients (see [17], p. 178 ff); it is zero unless $m + q = n$, in which case it is given by

$$\frac{1}{2\ell+1} C(\ell,s,\ell\,;k,o,k)\, C(\ell,s,\ell\,;m,q,n) = A(\ell,s,n,q)\,(\text{say}).$$

Thus we have proved

PROPOSITION. — *$(\iota_\lambda^*\varphi)^\wedge(\sigma_\ell)_{n,m}$ is zero unless $n - m = q$ when it is equal to*

$$\sum_{k=-\ell}^{\ell} 2\pi i^{-s}\left(\frac{|k|}{\lambda}\right)^{s-1/2} \hat{f}\left(s+1/2; \frac{|k|}{\lambda}\right) A(\ell,s,|k|,n,q) .$$

The Clebsh-Gordan coefficients simplify somewhat for the case $P = Y_{1,0}$. One sees easily from the expression on p. 181 of [17] that

$$A(\ell,1,k,m,0) = \frac{2m2k}{(2\ell+2)(2\ell+1)2\ell} .$$

The matrix $(\iota_\lambda^*\varphi)^\wedge(\sigma_\ell)$ is a diagonal matrix: its m,m entry is

$$-\,2\pi i \sum_{k=-\ell}^{\ell} \left(\frac{|k|}{\lambda}\right)^{1/2} \hat{f}\left(3/2; \frac{k}{\lambda}\right) \frac{2m\,2|k|}{(2\ell+2)(2\ell+1)\,2\ell} .$$

This expression becomes particularly simple when $f(r) = r^3$. One computes that $\hat{f}\left(3/2;\frac{k}{\lambda}\right) = \left(\frac{|k|}{\lambda}\right)^{-1/2}\sqrt{\frac{2}{\pi}}$, and hence the m, m entry is

$$-\sqrt{2\pi}\, i\frac{m.}{(\ell+1)(\ell+1/2)\ell}\sum_{k=\ell}^{\ell}|k| = -\sqrt{2\pi}\, i\frac{m}{(\ell+1/2)}.$$

This matrix is closely related to the operator $\bar{\mathrm{B}}_2$ of [3], p. 135 ff; in fact one has $\mathrm{I} - i(2\pi)^{-1/2}i_\lambda^*\varphi = \bar{\mathrm{B}}_2$, so that

$$(2\pi)^{-1/2}\imath_\lambda^*\varphi = \mathrm{I} + \bar{\mathrm{B}}_2.$$

Thus, the operator $\mathrm{I} + \bar{\mathrm{B}}_2$ has multiplier norm on $\mathrm{SU}(2)$ not exceeding $(2\pi)^{-1/2}|||\varphi|||_{p,\mathbf{R}^3}$. This bound is considerably more precise than the bound given in Theorem (1.7) of [3].

It is equally easy to treat the other Riesz kernels B_1, $\bar{\mathrm{B}}_1$ of Coifman and Weiss. In fact, it is obtained by transferring the appropriate Riesz kernel on $\mathbf{R}^3$.

Specifically, we let $\mathrm{P} = \mathrm{Y}_{11}$ and $f(s) = s^{-3}$. In this case,

$$\mathrm{A}(\ell,1,k,m,1) = \frac{k}{\ell(\ell+1)(2\ell+1)}\sqrt{(\ell-i)(\ell+i-1)}.$$

Thus, $\imath_\lambda^*\varphi^\wedge(\sigma_\ell)_{m,m+1} = -2\pi i\dfrac{\sqrt{(\ell-i)(\ell+i-1)}}{2\ell+1}\dfrac{1}{\ell(\ell+1)}$

$$\sum_{k=-\ell}^{\ell}\frac{(|k|)^{3/2}}{\lambda^{1/2}}\hat{f}\left(3/2;\frac{k}{\lambda}\right).$$

As before, this reduces to

$$-\sqrt{2\pi i}\,\frac{\sqrt{(\ell-m)(\ell+m-1)}}{2\ell+1}.$$

Comparing this expression with the expression for $\hat{\mathrm{B}}_1(\ell)$ on page 134 of [3], we have

$$(2\pi)^{-1/2}i(\imath_\lambda^*\varphi)^\wedge(\sigma_\ell). = \mathrm{B}_1(\ell).$$

Thus, the map $\imath_\lambda$ transfers the Riesz kernel on $\mathbf{R}^3$ to the Riesz kernel on $\mathrm{SU}(2)$, and one has immediately

THEOREM. — $|||B_1|||_{p,SU(2)} \leqslant |||\varphi_1|||_{p,\mathbf{R}^3}$

$$|||I - B_2|||_{p,SU(2)} \leqslant |||\varphi_2|||_{p,\mathbf{R}^3}$$

where φ_2 *is the kernel on* $\mathbf{R}^3$ *obtained by taking* $f(s) = s^3$ *and* $P = Y_{11}$, *and* φ_2 *is the kernel on* $\mathbf{R}^3$ *obtained by taking* $f(s) = s^3$ *and* $P = Y_{10}$.

4.6. Coifman and Weiss deduce their estimates from a general theorem giving sufficient conditions for a family $(\mu^{\ell}_{m,n})_\ell = 0, 1/2, 1, \ldots$ to be a multiplier of $L^p(SU(2))$. One can use theorem (4.1) to give a different proof of their result. Details will appear elsewhere.

5. The contraction of $V \rtimes K$ onto G.

We will now return to the case where (G,K) is a Riemannian symmetric space of the compact type. In this case, starting from (3.4) instead of (3.5) we can prove analogues of some of the theorems in § 4, obtaining transference theorems from $V \rtimes K$ to G.

5.1. The representations of $V \rtimes K$.

For each $\psi \in i\mathfrak{a}^*$, we may define a representation ρ_ψ of $V \rtimes K$ on $L^2(K)$ by the formula

$$\rho_\psi(v,k)f(k_0) = e^{-\psi(k_0^{-1}v)}f(k^{-1}k_0)$$

where ψ is extended to an element of V^* by making it zero on $\mathfrak{a}$. This is the representation of $V \rtimes K$ induced from the character $v \to e^{\psi(v)}$ of V. It is not in general irreducible.

In the case where ψ is generic $(\langle\psi,\alpha\rangle \neq 0, \forall\alpha \in P_+)$ we have a canonical procedure for choosing an irreducible subrepresentation of ρ_ψ. The stabilizer of ψ in K is then M (the stabilizer of $\mathfrak{a}$ in K); and we choose a maximal torus T_1 in M. Let $\chi \in \hat{T}_1^+ = \{\chi \in \hat{T} : \langle d_\chi, \alpha\rangle \geqslant 0, \alpha \in P_-\}$ and let β be a character of the finite group $A \cap M$.

Set

$$L^2_{\chi,\beta} = \{f \in L^2(K) : f(kt) = \chi(t^{-1})f(k) \text{ and } f(ka) = \beta(a^{-1})f(k),\ \forall k \in K, t \in T, a \in A\}$$

and

$$\mathcal{H}^{\chi,\beta} = \{f \in L^2_{\chi,\beta} \cap C^\infty(K) : X^{-\alpha} f = 0,\ \forall \alpha \in P_-\}.$$

Then $\rho_{\chi,\beta,\psi} = \rho_\psi|_{\mathcal{H}}\chi, \beta$ is irreducible, and almost all irreducible representations of $V \rtimes K$ are obtained by this construction. (Further details of this construction may be found in [6], Theorem (4.8), but we give a word of explanation here. Evidently, ρ_ψ may be obtained as $e^{i\psi} \times \lambda \uparrow {}^{V \rtimes K}_{V \rtimes M}$, where λ denotes the left regular representation of M. The representation $\rho_{\chi,\beta,\psi}$ corresponds to $e^{i\psi} \times \sigma_{\chi,\beta} \uparrow {}^{V \rtimes K}_{V \rtimes M}$, where $\sigma_{\chi,\beta}$ denotes the irreducible representation of M holomorphically induced from $\chi \times \beta$. We may realise $\sigma_{\chi,\beta}$ as a subrepresentation of $L^2(M)$; by Mackey's restriction theorem, $\rho_{\chi,\beta,\psi}$ is a subrepresentation of ρ_ψ.)

5.2. To describe the irreducible representations of G, we take the maximal torus T corresponding to $t_1 + \mathfrak{a}$, and let

$$\hat{T}^+ = \{\nu \in \hat{T} : \langle d\nu, \alpha\rangle \geq 0, \alpha \in \Phi^+\}.$$

There is a bijection $\nu \longleftrightarrow \sigma_\nu$ of $\hat{T}^+$ onto $\hat{G}$. Given $\nu \in \hat{T}^+$ we let

$$\mathcal{H}^\nu = \{f \in C^\infty(G) : f(gt) = \nu(t^{-1})f(g),\ g \in G,\ t \in T,\ X^{-\alpha}f = 0,\ \alpha \in \Phi^+\}.$$

Then σ_ν is the left regular action of G in H. As in (4.1), one chooses an orthonormal basis

$$\{u^\nu_{\mu,j} : \mu \in \Pi(\nu),\ j = 1, \ldots, m_\nu(\mu)\}$$

for $\mathcal{H}^\nu$, where $\{u^\nu_{\mu,j} : j = 1, \ldots, m_\nu(\mu)\}$ is an orthonormal basis for H^ν_μ. A moment's thought will convince the reader of

LEMMA. — *Let* $\{m_1 = e, m_2, \ldots, m_{|W|}\}$ *be a fixed set of coset representatives for the Weyl group* $W = N_G(T)/T$. *We may choose the basis with the following property. For each* $w_j = m_jT \in W$, *we have*

$$u^\nu_{w\mu,k} = m_j u_{\eta,j} \quad \text{for} \quad 1 \leq k \leq m_\nu(\mu) = m_\nu(w\mu).$$

(Here, $mu(g) = u(mg)$.*)*

We shall henceforth assume the basis $\{u^\nu_{\mu,j}\}$ has this property. Often, when ν is clear from the context, we will write $u_{\mu,j}$ instead of $u^\nu_{\mu,j}$.

5.3. Before stating our first main theorem, a piece of notation. Let $\eta \in \hat{T}^+$. The derivative $d\eta$ then belongs to $it^* = it_1^* \oplus ia^*$. Let

$$d\eta = d\eta_1 + d\eta_2$$

be its decomposition with respect to this direct sum.

5.4. Theorem. — *Let* $\nu \in \hat{T}^+$, *and let* $u, w \in H^\nu$. *For each* $x \in V \rtimes K$ *and for each* $\lambda \in \mathbf{R}^+$, *we have*

$$i_\lambda(u * w^*) = \sum_{\mu \in \Pi(v)} \sum_{j=1}^{m_\nu(\mu)} (\rho_{\frac{d\eta_2}{\lambda}}(x^{-1})u * u^*_{\mu,j}|_K , w * u^*_{\mu,j}|_K)_{L^2(K)} .$$

Proof. — The proof of this theorem is sufficiently similar to the proof of Theorem (4.1) to be left to the reader.

Remark. — It is obvious that $u * u^*_{\mu n' j|K} \in L^2(K)$. The question of when it belongs to one of the $\mathscr{H}^{\nu,\beta}$'s and hence of when the matrix coefficients occuring on the right hand side belong to our standard family of irreducible representations will be considered below.

5.5 We shall consider the Fourier transform of $f \in L^1(V \rtimes K)$ at the not necessarily irreducible representation ρ_ψ to be defined by

$$\hat{f}(\rho_\psi) = \int f(x)\rho_\psi(x^{-1})\, dx$$

(Bochner integral). We obtain immediately from (5.4)

Corollary. — *Let* $f \in L^1(V \rtimes K)$, $\nu \in \hat{T}^+$, $u, w \in \mathscr{H}^\nu$. *Then*

$$(i_\lambda * f)^\wedge (u * w^*) = \sum_{\mu \in \Pi(\nu)} \sum_{j=1}^{m_\nu(\mu)} (\hat{f}(\rho_{\frac{d\eta_2}{\lambda}})\, u * u^*_{\mu,j}|_K , w * u^*_{\mu,j}|_K)_{L^2(K)} .$$

Notice that $(i_\lambda^* f)(u * w^*) = ((i_\lambda^* f)^\wedge(\sigma_\nu)u,w)$. We see that this gives us an explicit formula for the matrix of the Fourier transform of $i_\lambda^* f$ in terms of certain non-irreducible matrix coefficients of f. This is an analogue of Theorem (4.1) (Although there, because of the simplicity of the vector group we did not have to worry about non-irreducible matrix entries.) Again, we note that we have on the right an average of a number of terms equal to the dimension of ν.

The next theorem eliminates the non irreducible matrix coefficients — at the expense of introducing some differential operators.

5.6. THEOREM. — *Let* $f \in L^1(V \rtimes K)$, $\nu \in \hat{T}^+$, $u, w \in \mathscr{H}^\nu$. Let $\nu_1 = \nu|_{T_1}$, $\beta_1 = \nu|_{M \cap A}$. *Then for every* $\mu \in \Lambda^+$ *with* $\mu < \nu$ *and for every* $j = 1, \ldots, m_\nu(\mu)$, *there are differential operators* $Z(\nu,\mu,j) \in u(m^C)$ *and* $Y(\nu,\mu,j) \in u(k^C)$ *such that*

$$((i_\lambda^* f)^\wedge(\sigma_\nu)u,w)_{L^2(G)} = \frac{1}{d_v} \sum_{\substack{\mu<\nu \\ \mu \in \Lambda^+(G,T)}} |W\mu| \sum_{j=1}^{m_\nu(\mu)} (Z(\nu,\mu,j)\hat{f}(\rho_{\nu,\beta_1,\frac{d\mu_2}{\lambda}})Y(\nu,\mu,j)u|_K, \; Z(\nu,\mu,j)Y(\nu,\mu,j)w|_K)_{L^2(K)}$$

where $Y(\nu,\mu,j)(u|_K) \in \mathscr{H}_{\nu_1,\beta_1}$.

An explicit description of the operators Y and Z is given below, although their determination in practice is difficult.

Proof. — First, notice that by our choice of orthonormal basis — lemma (5.3) — we have for all $w \in W$, for all $\eta \in i\mathfrak{a}^*$ we have

$$\rho_{w\eta}(x)(u * u^*_{w\eta,j})|_K, (w * u^*_{w\mu,j})|_K = (\rho_\eta(x)(u * u^*_{\mu,j})|_K, w * u^*_{\mu,j}|_K)$$

hence the sum over $\Pi(\nu)$ in corollary (5.4) may be changed to $\sum_{\substack{\mu<\nu \\ \mu\in\Lambda^+}}$, each term occuring with multiplicity $|W\mu|$.

For $\nu \in \hat{T}^+$, let u^ν denote the unique element of $\mathscr{H}^\nu_\nu$ such that $\|u^\nu\| = 1$ and $u^\nu(e) > 0$. It is well known (cf. [14], § 20.2) that u^ν is a cyclic vector for $\mathscr{H}^\nu$ considered as a module over the universal enveloping algebra $u(g^C)$. Thus, for each μ and for each $j \in \{1, \ldots, m_\nu(\mu)\}$ there is an element $X(\nu,\mu,j) \in u(g^C)$ such that

$$\sigma_\nu(X(\nu,\mu,j))u^\nu = u^\nu_{\mu,j}.$$

In fact, it is not hard to see that action of the elements of $u(k^C)$ generate the whole of $\mathscr{H}^\nu$ (see [14] for a proof), and hence, the operators $X(\nu,\mu,j)$ may be taken to belong to $u(\mathfrak{k}^C)$.

Now according to the orthogonality relations, for any $u \in \mathscr{H}^\nu$, $u * u^\nu = \frac{1}{d_\nu^{1/2}} u$.

It follows readily that, for all $u \in \mathscr{H}^\nu$,

$$u * u^*_{\mu,j} = \overline{X(\nu,\mu,j)u * u^\nu} = \frac{1}{d_\nu^{1/2}} \overline{X(\nu,\mu,j)u}$$

where $-$ denotes the complex conjugation in $\mathfrak{u}(k^{\mathbf{C}})$ induced by

$$X + iY \longleftrightarrow X - iY(X,Y \in \mathfrak{k}),$$

and $X \in \mathfrak{k}$ acts on u by the right derivative

$$Xu(G) = \frac{d}{dt}\bigg|_{t=0} u(g \exp tX).$$

By the Poincaré-Birkhoff-Witt theorem ([14], p. 92), we may suppose that

$$X(\nu,\mu,j) = Z(\nu,\mu,j)Y(\nu,\mu,j)$$

where $Z(\nu,\mu,j) \in \mathfrak{u}(\mathfrak{m}^{\mathbf{C}})$ and Y is in the algebra generated by $\{X+\theta X : X \in \mathfrak{g}_A, \alpha \in P_+\}$.

Indeed, Z may be taken to have the form

$$\prod_{\alpha \in P_-} (X^\alpha)^{n_\alpha}$$

where $\nu_1 - \mu_1 = \sum_{\alpha \in P_-} n_\alpha \alpha$.

Now $\sigma_\nu(Y(\nu,\mu,j))u^\nu$ transforms according to ν_1 under T_1 and according to β_1 under $M \cap A$; hence we have

$$Y(\nu,\mu,j)h|_K \in \mathscr{H}^{\nu_1,\beta_1}.$$

Since $Z \in \mathfrak{u}(\mathfrak{m}^{\mathbf{C}})$ and since $[\mathfrak{m},\mathfrak{a}] = 0$, it is easy to check that

$$Z\rho_\psi(x) = \rho_\psi(x)Z.$$

We have proved the theorem. □

5.7. *Remarks.* — (i) Theorem (5.5) gives a description of $((i_\chi^* f)^\wedge(\sigma_\nu)u,w)$ in terms of the coefficients of the Fourier transform of f at our standard irreducible representations.

The particular matrix entry chosen is obtained as a sum of $u|_K$ acted upon by certain differential operators. While the latter are hard to compute, the restriction of u to K is well-behaved; in fact ([6], Theorem (6.2)) assures us that the map $u \longleftrightarrow u|_K : \mathscr{H}^\nu \to \mathscr{H}^{\nu_1,\beta_1}$ is an injection, and the images, as $\nu_2 \to \infty$ fill out a dense subspace of $\mathscr{H}^{\nu_1,\beta_1}$.

(ii) We used in the above proof, that for

$$Z \in u(\mathfrak{m}^{C}), \quad Z\rho_{\psi}(x) = \rho_{\psi}(x)Z.$$

One may check in general that for $X \in \mathfrak{k}^{C}$ and for $x = (v,k) \in V \rtimes K$, $h \in C^{\infty}(K)$,

$$\begin{aligned}(X\rho_{\psi}(x)h)(k_0) &= \frac{d}{ds}\bigg|_{s=0} e^{i\langle\psi,(k_0 \exp sX)^{-1} v\rangle/\lambda} h(k^{-1} \exp sX)\\ &= (\rho_{\psi}(x)Xh)(k_0) - i\frac{\langle\psi,[X,v]\rangle}{\lambda}\rho_{\psi}(x)h(k_0)\\ &= (\rho_{\psi}(x)Xf)(k_0) - O\left(\frac{1}{\lambda}\right).\end{aligned}$$

Thus, one obtains $\langle i_{\lambda}^{*}fu,w\rangle$ as a sum of terms of the form

$$\hat{f}(\rho_{\nu_1,\beta_1,\frac{d\eta_2}{\lambda}}(u|_K), Xw|_k)$$

plus an « error term » which is $\frac{1}{\lambda}$ times a sum of

$$\langle Y_1\hat{f}(\rho)Y_2(u|_K), Y_1Y_2(w|_K)\rangle.$$

Without an explicit knowledge of the operators $X(\nu,\mu,j)$ it seems difficult effectively to estimate this error. It is desirable to estimate it, because, the leading term is simpler to evaluate.

(iii) We noted in the above proof that $u(\mathfrak{g}^{C})u^{\nu} = u(\mathfrak{k}^{C})u^{\nu}$. In fact, this can be seen by a simple inductive procedure. Let $X = X_1X_2 \ldots X_n$ be a monomial in $u(\mathfrak{g}^{C})$ and suppose that $X_2, \ldots, X_n \in u(k^{C})$. Writing $X_1 = 1/2(X_1 + \theta X_1)$, we notice that $X_1 - \theta X_1 \in V$, and since $[\mathfrak{k},V] \subseteq V$, $(X_1 - \theta X_1)X_2 \ldots X_n u^{\nu}$ is a sum of terms of the form $Y_1 \ldots Y_m W u^{\nu}$ where $Y_1 \ldots Y_m \in \mathfrak{k}^{C}$ and $W \in V$. But V is spanned by $(X_\alpha - \theta X_\alpha)$, $\alpha \in P_+$: since $\theta X_\alpha u^{\nu} = 0$, $(X_\alpha - \theta X_\alpha)u^{\nu} = (X_\alpha + \theta X_\alpha)u^{\nu}$. Hence $Wu^{\nu} = Y_{m+1}u^{\nu}$ for $Y_{m+1} \in \mathfrak{k}^{C}$. This procedure can be turned into combinatorial procedure for calculating the operators X, Y and Z of theorem (5.5).

5.8. In the next section, we will compute the operators Y and Z occurring in (5.6) for the case $(G,K) = (SU(2),SO(2))$. Before making this computation, we shall specialize Theorem (5.6) to the case of K-bi-invariant multipliers.

If f is a bi-K-invariant function in $L^1(V \rtimes K)$, then $\varphi : v \to f(e,v)$ is a radial function on V. The Fourier transform of f is

$$\hat{f}(\rho_{\nu,\beta,\psi}) = \hat{\varphi}(\psi)P_{(\nu,\beta)}$$

where $P_{\nu,\beta}$ is the projection onto the zero or one dimensional subspace of K invariant vectors of $\mathscr{H}_{\nu,\beta,\psi}$; in fact, $P_{(\nu,\beta)} = 0$ unless $(\nu,\beta) = (0,0)$).

Since φ is radial, $\hat{\varphi}(\psi) = \hat{\varphi}(k.\psi)$. Thus ψ may be supposed to belong to $\mathfrak{a}^{*+}$. The radial Fourier transform of f is then given for $\mu \in \mathfrak{a}^{*+}$ by

$$\hat{f}(\mu) = \hat{\varphi}(u).$$

Similarly, (G,K) is a Gelfand pair, so for each $\nu \in \Lambda^+(G,T)$, the space of K-fixed vectors in $\mathscr{H}^\nu$ is zero or one dimensional – the Cartan-Helgason theorem states that it is one dimensional iff $\nu_1 = 0$ and $\nu(X) = 0$ whenever $\exp(X) \in K \cap A$, i.e. iff $\nu_1 = 0$ and $\beta_1 = 0$. Thus ν can be considered to belong to a certain lattice Λ_1^+ in $\hat{A}$.

The Fourier transform of a bi-K-invariant function f on G is then $\nu \longleftrightarrow \Psi(\nu_2)P_\nu$ where $\nu_2 \in \Lambda_1^+$; P_ν is the projection onto the 1 dimensional subspace and $\psi(\nu_2) \in \mathbf{C}$.

The complex valued mapping ψ is the radial Fourier transform of f.

THEOREM. – *Let $f \in L^1(V \times K)$ be bi-K-invariant. Then $\iota_\lambda^* f$ is a bi-K-invariant multiplier of G and*

$$(\iota_\lambda^* f)^\wedge(\nu) = \frac{1}{d_\nu} \sum_{\substack{\mu<\nu_+ \\ \mu \in \Lambda_1}} \sum_{j=1}^{m_\nu(\mu)} |c(\nu,\mu,j)|^2 \varphi\left(\frac{\mu}{\lambda}\right)$$

where $c(\nu,\mu,j)$ is the constant term of the operator $X(\nu,\mu,j) \in \mathfrak{u}(\mathfrak{k}^{\mathbf{C}})$ of theorem (5.6) and φ is the radial Fourier transform of f.

This theorem enables us to transfer K-invariant multipliers on V to bi-K-invariant multipliers on G thus, for example, taking $(G,K) = (SO(3), SO(2))$, we can transfer radial multipliers of $L^p(\mathbf{R}^2)$ to give bi-SO(2)-invariant multipliers of $L^p(S^3)$.

The catch is that the constants $c(\nu,\mu,j)$ are rather hard to compute, even for this simple case. (Nevertheless, there is an algorithm, based on (5.6) for computing them, in general).

6. Transferring from M(2) to SU(2).

6.1. The aim of this section is to explicitly calculate the expressions for $\iota_\lambda^* f$ given in the previous section in the case where $(G,K) = (SU(2), SO(2))$. Here, the associated motion group is $\widetilde{M(2)}$, the twofold cover of the Euclidean motion group. (It is the semidirect product of **C** by **T** where **T** acts on **C** by $w.z = w^2 z$). Much of the apparent complexity of the formulae of the previous section disappears. Indeed,

$$A = T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \; 0 \leqslant \theta < 2\pi \right\}.$$

Thus $P_- = \varnothing$. The elements of $\Lambda^+(G,T)$ are indexed by the nonnegative half integers $\{0, 1/2, 1, 3/2, \ldots\}$ and for $\ell \in \Lambda^+$, we have $\pi(\ell) = \{-\ell, -\ell+1, \ldots, \ell\}$. The integers $m_\ell(t)$ for $|t| \leqslant \ell$ are all 1.

Furthermore, the representations $\rho_{R,\pm 1}$ of $\widetilde{M(2)}$, acting in $L^2(K) = L^2(SO(2))$ are all irreducible. Hence, corollary (5.5) takes the form, for $u, w \in \mathscr{H}^\ell$,

$$(\iota_\lambda^* f)(u * w^*) = \sum_{|t| \leqslant \ell} \left(\hat{f}(\rho_{\frac{|t|}{\lambda}, (-1)^{2\ell}}) u * u_t^{\ell^*}|_{SO(2)}, w * u_t^{\ell^*}|_{SO(2)} \right)$$

where $\{u_j^\ell : -\ell \leqslant j \leqslant \ell\}$ is an orthonormal basis for $\mathscr{H}^\ell$. In particular, choosing $u_j^\ell = \psi_j$ as in [17], III, § 2, (4) we may see that matrix of the Fourier transform of $\iota_\lambda^* f$ is

$$(\iota_\lambda^* f)^\wedge (\sigma_\ell)_{j,k} = \frac{1}{2\ell+1} \sum_{|t| \leqslant \ell} \left(\hat{f}(\sigma_{\frac{|t|}{\lambda}, (-1)^{2\ell}}) t_{jt}^\ell|_{SO(2)}, t_{kt}^\ell|_{SO(2)} \right).$$

Now, using the notation of [17], we have, for $\theta \in SO(2)$, $t_{jt}^\ell(\theta) = P_{jt}^\ell(\cos\theta)$. The remarks of the preceding section, together with the fact that $SO(2)$ is commutative enables us to see that $P_{jt}^\ell(\cos\theta)$ is a linear combination of the functions $P_{s\ell}^\ell(\cos\theta)$, $-\ell \leqslant s \leqslant \ell$. While the following formula may be known to the experts, we have not been able to find a reference for it, so we give a proof.

6.2. Lemma. — *Let* $\ell \in \Lambda^+$, *and let* $-\ell \leqslant j, \; t \leqslant \ell$. *Then*

$$P_{j,t}^\ell(\cos\theta) = \sum_{s=-\ell}^{\ell} a(\ell, j, t, s) P_{s,\ell}^\ell(\cos\theta)$$

where

$$a(\ell,j,t,s) = \binom{\ell-t}{\frac{\ell-t+j+s}{2}}\binom{\ell+t}{\frac{\ell+t+j-s}{2}}\left(\frac{(\ell+s)!\,(\ell-s)!}{(2\ell)!}\,\frac{(\ell-j)!\,(\ell+j)!}{(\ell-t)!\,(\ell+t)!}\right)^{\frac{1}{2}}.$$

Proof. – We start from the generating function for $P^\ell_{j,t}$, [17], III, § 3 (3):

$$\left(w\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^{\ell-t}\left(iw\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^{\ell+t}$$
$$= \sum_{j=-\ell}^{\ell}\left(\frac{(\ell-t)!\,(\ell+t)!}{(\ell-j)!\,(\ell+j)!}\right)^{\frac{1}{2}} P^\ell_{jt}(\cos\theta)w^{\ell-j}.$$

We may expand the left hand side of this expression by using the binomial theorem, obtaining

$$\sum_{m=0}^{\ell+t}\sum_{n=0}^{\ell-t}\binom{\ell+t}{m}\binom{\ell-t}{n}\left(\cos\frac{\theta}{2}\right)^{\ell+t-m+n}\left(i\sin\frac{\theta}{2}\right)^{\ell-t+m-n} w^{2\ell-m-n}.$$

Changing variables in these sums, to $j=m+n-\ell$ and $s=t-m+n$, we obtain

$$\sum_{j=-\ell}^{\ell}\sum_{s=-\ell}^{\ell}\binom{\ell+t}{\frac{\ell+t+j-s}{2}}\binom{\ell-t}{\frac{\ell-t+j+s}{2}}\left(\cos\frac{\theta}{2}\right)^{\ell+s}\left(i\sin\frac{\theta}{2}\right)^{\ell-s} w^{\ell-j}.$$

Finally, consulting III, § 3 (6), we have

$$\left(\cos\frac{\theta}{2}\right)^{\ell+s}\left(i\sin\frac{\theta}{2}\right)^{\ell-s} = \left(\frac{(\ell-s)!\,(\ell+s)!}{(2\ell)!}\right)^{\frac{1}{2}} P^\ell_{s\ell}.$$

Substituting this expression in the previous one and comparing coefficients of $w^{\ell-j}$, we obtain the lemma. □

We are now in a position to compare our restriction $\iota^*_\lambda f$ with that used by Rubin [15] and later in [6] in proving the other direction of de Leeuw's theorem. We use the notation of [6]; thus the elements

$$\{t^\ell_{s\ell} : -\ell \leqslant s \leqslant \ell\}$$

span $\mathscr{H}^\ell \subseteq L^2(SO(2))$. The images $\mathscr{R}_\ell\mathscr{H}^\ell$, $\ell = 0, 1/2, 1, \ldots$, form an increasing family of finite dimensional subspaces whose union is dense.

Rubin's restriction is defined, for φ a multiplier of $L^p(M(2))$, by

$$\hat{\varphi}_\lambda(\ell) = \mathcal{R}_\ell^{-1} P_\ell \varphi\left(\frac{\ell}{\lambda}\right)\mathcal{R}_\ell$$

where P_ℓ denotes projection onto $\mathcal{R}_\ell \mathcal{H}^\ell$. We have shown that

$$(\iota^* f)^\wedge(\sigma_\ell)_{jk} = \frac{1}{2\ell+1}\sum_{t=-\ell}^{\ell}\sum_{s=-\ell}^{\ell}\sum_{p=-\ell}^{\ell} a(\ell,j,t,s)a(\ell,k,t,p)(f_\lambda(|t|)_{s,p}.$$

Thus, $\iota_\lambda^* f^\wedge$ is a kind of average of $\hat{f}_\lambda(|t|)$. In the abelian case $\iota_\lambda^* \hat{f}$ is the same as $\hat{f}_\lambda$. It seems that for some purposes, $\iota_\lambda^* f$ is to be preferred as a restriction to f_λ.

6.3. It is interesting to enquire whether an analogue of Theorem B holds for the restriction $\iota_\lambda^* f$. A condition which is certainly necessary is for each i, j,

$$\lim_{\ell\to\infty}\left(\iota^*_{\frac{\ell}{R}}\hat{f}\right)(\sigma_\ell)_{ij} = \hat{f}(\rho_R)_{ij}.$$

This is certainly satisfied by the restriction f_λ, and in fact it is not hard to trace through the arguments of [1] to see that any restriction for which this holds will satisfy the appropriate version of Theorem B. (Indeed, this remark holds in the generality of [6].)

It seems that no such theorem holds in general. To see this, we suppose that $\hat{f}(\rho_R)_{ij} = 0$ unless $i = j = 0$ and we compute $\left(\iota^*_{\frac{\ell}{R}} f\right)^\wedge(\sigma_\ell)_{00}$.

Now

$$\begin{aligned} a(\ell,0,t,0)^2 &= \binom{\ell-t}{\frac{\ell-t}{2}}^2\binom{\ell+t}{\frac{\ell+t}{2}}^2\frac{(\ell!)}{2\ell!}\frac{1}{(\ell-t)!\,(\ell+t)!} \\ &= \frac{(\ell-t)!\,(\ell+t)!\,(\ell!)^4}{(2\ell)!\left(\frac{\ell-t}{2}\right)!^4\left(\frac{\ell+t}{2}\right)!^4} \\ &= \binom{\ell}{\frac{\ell-t}{2}}^4\binom{2\ell}{\ell-t}^{-1}. \end{aligned}$$

Even the $t = 0$ term of this sum blows up; by using Stirling's approximation, we have

$$\begin{pmatrix} \ell \\ \frac{\ell}{2} \end{pmatrix}^4 \begin{pmatrix} 2\ell \\ \ell \end{pmatrix}^{-1} \sim 2^{2\ell-7/2}\ell^{-3/2}.$$

6.4. The above argument means that we cannot hope to recover any type of reasonable average of f from its restrictions $\iota_\lambda^* f$ unless we place some extra conditions on f.

We now specialize to the case where f is bi-K-invariant. We start from corollary 5.5, substituting $u = w = a^\ell$, where a^ℓ is a K-invariant vector in $\mathscr{H}^\ell$ of norm 1. Then

$$\begin{aligned} \iota_\lambda^* f(a^\ell * a^{\ell*}) &= (\iota_\lambda^* f^\wedge(\sigma_\ell)a^\ell, a^\ell) = (\iota_\lambda^* f)^\wedge(\sigma_\ell) \\ &= \sum_{|t|\leq\ell} \left\langle \hat{f}\left(\frac{|t|}{\lambda}\right) a^{\ell*}u_t^\ell|_{SO(2)}, a^{\ell*}u_t^\ell|_{SO(2)} \right\rangle \\ &= \sum_{|t|\leq\ell} \hat{f}\left(\frac{|t|}{\lambda}\right) |a^{\ell*}u_t^\ell(e)|^2 \\ &= \sum_{|t|\leq\ell} \hat{f}\left(\frac{|t|}{\lambda}\right) |\langle a^\ell, u_t^\ell\rangle|^2. \end{aligned}$$

Now a^ℓ may be computed as follows. The vector $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \longmapsto (\alpha^2+\beta^2)^\ell$ is a K-invariant vector in $\mathscr{H}_\ell$, for ℓ integral.

By the binomial theorem

$$a^\ell = c \sum_{\substack{t=-\ell \\ \ell-t \text{ even}}}^{\ell} \begin{pmatrix} \ell \\ \frac{\ell-t}{2} \end{pmatrix} \alpha^{\ell-t}\beta^{\ell+t}$$

where c is a constant which we will determine presently.

Thus

$$\begin{aligned} a^\ell &= c \sum_{\substack{t=-\ell \\ \ell-t \text{ even}}}^{\ell} \begin{pmatrix} \ell \\ \frac{\ell-t}{2} \end{pmatrix} \begin{pmatrix} 2\ell \\ \ell-t \end{pmatrix}^{-1/2} t_{t\ell}^\ell(\alpha,\beta) \\ &= \frac{c}{\sqrt{2\ell+1}} \sum_{\substack{t=-\ell \\ \ell-t \text{ even}}}^{p} \begin{pmatrix} \ell \\ \frac{\ell-t}{2} \end{pmatrix} \begin{pmatrix} 2\ell \\ \ell-t \end{pmatrix}^{-1/2} u_t^\ell. \end{aligned}$$

Thus, since the u_t^ℓ are an orthonormal basis,

$$1 = \|a^\ell\|^2_{\mathscr{H}^\ell} = \frac{c^2}{2\ell+1} \sum_{\substack{t=-\ell \\ (\ell-t)\text{ even}}}^{\ell} \binom{\ell}{\frac{\ell-t}{2}}^2 \binom{2\ell}{\ell-t}^{-1}.$$

Furthermore

$$\begin{aligned} |\langle a^\ell, u_p^\ell\rangle|^2 &= \frac{c^2}{2\ell+1} \binom{\ell}{\frac{\ell-p}{2}}^2 \binom{2\ell}{\ell-p}^{-1} \\ &= \binom{\ell}{\frac{\ell-p}{2}}^2 \binom{2\ell}{\ell-p}^{-1} \left(\sum_{\substack{t=-\ell \\ \ell-t\text{ even}}}^{\ell} \binom{\ell}{\frac{\ell-t}{2}}^2 \binom{2\ell}{\ell-t}^{-1} \right)^{-1}. \end{aligned}$$

We have proved

THEOREM. — *If f is bi-K-invariant on $\mathbf{R}^2$, then*

$$\iota_\lambda^* f^\wedge(\sigma_\ell) = \frac{\displaystyle\sum_{\substack{|p|\leqslant\ell \\ \ell-p\text{ even}}} \hat{f}\left(\frac{|p|}{\lambda}\right) \binom{\ell}{\frac{\ell-p}{2}}^2 \binom{2\ell}{\ell-p}^{-1}}{\displaystyle\sum_{\substack{|t|\leqslant\ell \\ \ell-t\text{ even}}} \binom{\ell}{\frac{\ell-t}{2}}^2 \binom{2\ell}{\ell-t}^{-1}}.$$

From this it follows that

$$\left(\iota_{\frac{\ell}{R}}^* f\right)^\wedge(\sigma_\ell) = \frac{\displaystyle\sum_{p=0}^{\ell} \hat{f}\left(R\left|1-\frac{2p}{\ell}\right|\right) \binom{\ell}{p}^2 \binom{2\ell}{2p}^{-1}}{\displaystyle\sum_{p=0}^{\ell} \binom{\ell}{p}^2 \binom{2\ell}{2p}^{-1}}.$$

By Stirling's approximation, $\binom{\ell}{p}^2 \binom{2\ell}{2p}^{-1} \sim \frac{1}{2\sqrt{\pi}} \sqrt{\frac{p}{\ell}(\ell-p)}$. Hence, as $\ell \to \infty$, the above quotient approaches

$$\frac{\displaystyle\int_0^1 \hat{f}(R(|1-2x|)\sqrt{x(1-x)}\,dx}{\displaystyle\int_0^1 \sqrt{x(1-x)}\,dx}.$$

This is an average value of f. In certain cases, this expression is a multiple of $\hat{f}(\mathrm{R})$.

For example, taking $f(x) = |x|^{i-2}$, one computes $\hat{f}(\mathrm{R}) = c.\mathrm{R}^{-i}$ where $c = \int_0^\infty t^{i-1} \mathrm{J}_0(t)\,dt$

$$\lim_{\ell \to \infty} \left(\iota^*_{\frac{\ell}{\mathrm{R}}} f \right)^\wedge (\sigma_\ell) = c.\mathrm{R}^{-i} \frac{\int_0^1 (1-2x)^i \sqrt{x(1-x)}\,dx}{\int_0^1 \sqrt{x(1-x)}\,dx}$$

which is a nonzero multiple of $\hat{f}(\mathrm{R})$.

A similar calculation enables one to conclude that the Bochner-Riesz means are bounded on $\mathrm{L}^p(\mathrm{S}^2)$.

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