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Vanishing theorems for compact hessian manifolds


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Let $M$ be a flat affine manifold with a locally flat affine connection $D$. Among the Riemannian metrics on $M$ there is an important class of Riemannian metrics which are compatible with the flat affine structure on $M$. A Riemannian metric $g$ on $M$ is said to be Hessian if $g$ has an expression $g = D^2 u$ where $u$ is a local $C^\infty$-function. A flat affine manifold provided with a Hessian metric is called a Hessian manifold. A certain geometry of Hessian manifolds has been studied in Shima [10]-[14]. See also Cheng and Yau [2] and Yagi [15].

Hessian manifolds have in a certain sense some analogy with Kählerian manifolds. In this paper, being motivated by the theory of cohomology for Kählerian manifolds we study cohomology groups for Hessian manifolds.

Let $F$ be a locally constant vector bundle over $M$. We denote by $\Omega^{p\cdot\cdot\cdot, q}(F)$ the space of all sections of $(\wedge^p T^*) \otimes (\wedge^q T^*) \otimes F$, where $T^*$ is the cotangent bundle over $M$. Since the vector bundle $(\wedge^q T^*) \otimes F$ is locally constant, we can naturally define a complex

$$\cdots \to \Omega^{p-1\cdot\cdot\cdot, q}(F) \overset{\delta}{\to} \Omega^{p\cdot\cdot\cdot, q}(F) \overset{\delta}{\to} \Omega^{p+1\cdot\cdot\cdot, q}(F) \overset{\delta}{\to} \cdots$$

We denote by $H^{p\cdot\cdot\cdot, q}(F)$ the $p$-th cohomology group of the complex. Then we have the following duality theorem analogous to that of Serre [9].

**Theorem.** — Let $M$ be a compact oriented flat affine manifold of dimension $n$. Then we have

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\[ H^{p,q}(F) \cong H^{n-p,n-q}((K \otimes F)^*) , \]

where \( K \) is the canonical line bundle over \( M \) and \((K \otimes F)^*\) is the dual bundle of \( K \otimes F \).

Let \( F \) be a locally constant line bundle over \( M \). Choose an open covering \( \{U_\lambda\} \) of \( M \) such that the local triviality holds on each \( U_\lambda \). Denote by \( \{f_{\lambda\mu}\} \) the constant transition functions with respect to \( \{U_\lambda\} \). A fiber metric \( a = \{a_\lambda\} \) on \( F \) is a collection of positive \( C^\infty \)-functions \( a_\lambda \) on \( U_\lambda \) such that
\[ a_\mu = f_{\lambda\mu}^2 a_\lambda . \]

Using this we can define a globally defined closed 1-form \( A \) and a symmetric bilinear form \( B \) by
\[ A = -D \log a_\lambda , \]
\[ B = -D^2 \log a_\lambda , \]
and we call them the first Koszul form and the second Koszul form of \( F \) with respect to the fiber metric \( a = \{a_\lambda\} \) respectively.

A locally constant line bundle \( F \) is said to be positive (resp. negative) if the second Koszul form is positive (resp. negative) definite with respect to a certain fiber metric. It should be remarked that if a compact connected flat affine manifold \( M \) admits a locally constant positive (resp. negative) line bundle, then by a theorem of Koszul [6] \( M \) is a hyperbolic affine manifold, that is, the universal covering of \( M \) is an open convex cone not containing any full straight line.

Kodaira-Nakano’s vanishing theorem for compact Kählerian manifolds plays an essential role in the theory of compact Kählerian manifolds. In this paper we prove the following vanishing theorem for a compact Hessian manifold analogous to that of Kodaira-Nakano.

**Theorem.**—Let \( M \) be a compact connected oriented Hessian manifold. Denote by \( K \) the canonical line bundle over \( M \). Let \( F \) be a locally constant line bundle over \( M \).

(i) If \( 2F + K \) is positive, then
\[ H^{p,q}(F) = 0 \quad \text{for} \quad p + q > n . \]
(ii) If $2F + K$ is negative, then

$$H^{p\cdot q}(F) = 0 \quad \text{for} \quad p + q < n.$$ 

As to vanishing theorem for compact hyperbolic affine manifolds we should mention the following theorem due to Koszul [7].

Theorem. — Let $M$ be a compact oriented hyperbolic affine manifold. Then we have

$$H^{p\cdot q}(1) = 0 \quad \text{for} \quad p, q > 0,$$

where 1 is the trivial line bundle over $M$.

In § 1 and § 2 a Riemannian metric $g$ is not assumed to be Hessian. We define in § 1 fundamental operators $e(g), i(g), \Pi, \ast, \partial, \delta$ and $\Box$. In § 2 we define the Laplacian $\Box_{g}$ on $\Omega^{p\cdot q}(F)$, and prove the duality theorem $H^{p\cdot q}(F) \cong H^{n-p\cdot n-q}((K \otimes F)^*)$ and the cohomology isomorphisms $\mathcal{H}^{p\cdot q}(F) \cong H^{p\cdot q}(F) \cong H^{p}(P^q(F))$. In § 3 we give the local expressions for geometric concepts on Hessian manifolds. In § 4 and § 5 the formulae of Weitzenböck type for $\Box$ and $\Box_a$ are obtained. In § 6 we prove a vanishing theorem analogous to that of Kodaira-Nakano. In § 7 we mention a vanishing theorem of Koszul type.

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1. The Laplacian $\Box$ on $\Omega^{p\cdot q}$.

Let $M$ be a flat affine manifold with a locally flat affine connection $D$. Then there exist local coordinate systems $\{x^1, \ldots, x^n\}$ such that $Dx^i = 0$, which will be called affine local coordinate systems. Throughout this paper the local expressions for geometric concepts on $M$ will be given in terms of affine local coordinate system. From now on we assume further that $M$ is compact, connected and oriented.

Choose an arbitrary Riemannian metric $g$ on $M$. Let $\Omega^{p\cdot q}$ be the space of all sections of $(\wedge T^*) \otimes (\wedge T^*)$. We denote the local
expression of $\phi \in \Omega^{p,q}$ by

$$\phi = \frac{1}{p! \, q!} \sum \phi_{i_1 \ldots i_p \bar{i}_1 \ldots \bar{i}_q} (dx^{i_1} \wedge \ldots \wedge dx^{i_p}) \otimes (dx^{\bar{i}_1} \wedge \ldots \wedge dx^{\bar{i}_q}).$$

For simplicity let us fix some notation. We denote as follows:

$$I_p = (i_1, \ldots, i_p), \quad i_1 < i_2 < \ldots < i_p, \quad 1 \leq i_a \leq n,$$

$$I_{n-p} = (i_{p+1}, \ldots, i_n), \quad i_{p+1} < \ldots < i_n, \quad 1 \leq i_r \leq n,$$

and $(i_1, \ldots, i_p, i_{p+1}, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$. Then with this notation we write

$$\phi = \sum_{i_p, i_q} \phi_{i_p \bar{i}_q} dx^{i_p} \otimes dx^{\bar{i}_q},$$

where $dx^{i_p} = dx^{i_1} \wedge \ldots \wedge dx^{i_p}$.

For $\phi, \psi \in \Omega^{p,q}$ we set

$$h(\phi, \psi) = \frac{1}{p! \, q!} \phi_{i_1 \ldots i_p \bar{i}_1 \ldots \bar{i}_q} \psi^{i_1 \ldots i_p \bar{i}_1 \ldots \bar{i}_q} \quad (\ast)$$

$$= \phi_{i_p \bar{i}_q} \psi^{i_p \bar{i}_q} \quad (\ast\ast)$$

**DEFINITION 1.1.** — The inner product of $\phi, \psi \in \Omega^{p,q}$ is

$$(\phi, \psi) = \int_M h(\phi, \psi) \, v,$$

where $v$ is the volume element determined by $g$.

**DEFINITION 1.2.** — We define $\ast$-operation

$$\ast : \Omega^{p,q} \rightarrow \Omega^{n-p,n-q}$$

by $(\ast \phi)_{i_{n-p} \bar{i}_{n-q}} = (-1)^{pq} \operatorname{sgn}(I_p I_{n-p}) \operatorname{sgn}(I_q \bar{I}_{n-q}) G \phi^{i_p \bar{i}_q}$, where $\operatorname{sgn}(I_p I_{n-p})$ is the signature of the permutation $(I_p I_{n-p})$ of $(1, \ldots, n)$ and $G = \det(g_{ij})$.

$(\ast)$ Throughout this paper we use Einstein’s convention on indices.

$(\ast\ast)$ $\phi_{i_p \bar{i}_q} \psi^{i_p \bar{i}_q}$ means $\sum_{i_p, i_q} \phi_{i_p \bar{i}_q} \psi_{i_p \bar{i}_q}$. 
**Definition 1.3.** Let \( \phi = \sum \phi_p \tau_q \, dx^p \otimes dx^q \) and
\[
\psi = \sum \psi_{k, r} \, dx^k \otimes dx^r.
\]
We set \( \phi \wedge \psi = \sum \phi_p \tau_q \psi_{k, r} \, (dx^p \wedge dx^k) \otimes (dx^q \wedge dx^r) \).

A straightforward calculation shows

**Proposition 1.1.** Let \( \phi, \psi \in \Omega^{p,q} \). Then

(i) \( \star \phi = (-1)^{p+q} \phi \),

(ii) \( \phi \wedge \star \psi = (-1)^{p+q} \phi \wedge \psi \).

**Definition 1.4.** Considering the Riemannian metric \( g \) as an element in \( \Omega^{1,1} \) we define
\[
e(g) : \Omega^{p,q} \longrightarrow \Omega^{p+1,q+1},
i(g) : \Omega^{p,q} \longrightarrow \Omega^{p-1,q-1},
\]
by \( e(g) \phi = g \wedge \phi \) for \( \phi \in \Omega^{p,q} \) and \( i(g) = (-1)^{p+q+1} \star e(g) \star \).

Then \( i(g) \) is the adjoint operator of \( e(g) \) with respect to the inner product given in Definition 1.1:
\[
(i(g) \phi, \psi) = (\phi, e(g) \psi) \quad \text{for} \quad \phi \in \Omega^{p,q}, \psi \in \Omega^{p-1,q-1}.
\]

**Definition 1.5.** We set
\[
\Pi = \sum_{p,q} (n-p-q) \pi_{p,q},
\]
where \( \pi_{p,q} \) is the projection from \( \sum_{r,s} \Omega^{r,s} \) onto \( \Omega^{p,q} \).

**Proposition 1.2.** We have
\[
[\Pi, e(g)] = -2e(g), \quad [\Pi, i(g)] = 2i(g), \quad [i(g), e(g)] = \Pi.
\]

The proof is carried out by a direct calculation and so it is omitted.
DEFINITION 1.6. — Define
\[ \partial : \Omega^p,q \rightarrow \Omega^{p+1},q \]
by \[ \partial = \sum_k (e(dx^k) \otimes \text{id}) D_k, \]
where \( e(dx^k) \) is a linear map from \( ^p \Lambda T^* \) to \( ^{p+1} \Lambda T^* \) given by \( e(dx^k) \omega = dx^k \wedge \omega \), \( \text{id} \) is the identity map on \( ^p \Lambda T^* \) and \( D_k \) is the covariant derivation with respect to \( \partial/\partial x^k \) for the locally flat affine connection \( D \).

Then we have
\[ \partial^2 = 0. \quad (1.2) \]

DEFINITION 1.7. — Define
\[ \delta : \Omega^p,q \rightarrow \Omega^{p-1},q \]
by \[ \delta = (-1)^{n+1} \sqrt{G} \star \partial \left( \frac{1}{\sqrt{G}} \star \right) . \]

PROPOSITION 1.3. — \( \delta \) is the adjoint operator of \( \partial \) with respect to the inner product given in Definition 1.1;
\[ (\partial \phi, \psi) = (\phi, \delta \psi) \quad \text{for} \quad \phi \in \Omega^p,q, \ \psi \in \Omega^{p+1},q. \]
In Proposition 2.1 we prove the above fact in more general situation and so we omit the proof.

DEFINITION 1.8. — We define
\[ \Box : \Omega^p,q \rightarrow \Omega^p,q \]
by \( \Box = \partial^2 + \delta^2 \), and call it the Laplacian. \( \phi \in \Omega^p,q \) is said to be \( \Box \)-harmonic if \( \Box \phi = 0 \).

2. The Laplacian \( \Box \) on \( \Omega^p,q \) (F).

Let \( F \) be a locally constant vector bundle over \( M \). Choose an open covering \( \{ U_{\lambda} \} \) of \( M \) such that the local triviality holds
on each \( U_\lambda \). Let \( \{ \xi^1_\lambda, \ldots, \xi^m_\lambda \} \) be fiber coordinate systems such that the transition functions \( \{ f_{\lambda\mu} \} \) defined by

\[
x^i_\lambda = \sum_j f_{\lambda\mu}^i f_{\mu j}^i \xi^j_\mu
\]

are constants. A fiber metric \( a = \{ a_\lambda \} \) on \( F \) is a collection of \( m \times m \) positive definite symmetric matrices \( a = (a_{ij}) \) such that each \( a_{ij} \) is a \( C^\infty \)-function on \( U_\lambda \) and

\[
a_\lambda = \sum f_{\mu\lambda} f_{\mu j} a_{ij}
\]

holds.

Let \( \Omega^{p,q}(F) \) denote the space of all sections of \( (\Lambda^p T^*) \otimes (\Lambda^q T^*) \otimes F \).

Using fiber coordinate systems \( \{ \xi^i_\lambda \} \) we express an element \( \phi_\lambda \in \Omega^{p,q}(F) \) as \( \phi = \{ \phi^i_\lambda \} \).

**Definition 2.1.** — Define

\[
\partial : \Omega^{p,q}(F) \rightarrow \Omega^{p+1,q}(F)
\]

by

\[
\partial \{ \phi^i \} = \{ \partial \phi^i \} . (*)
\]

We have then

\[
\partial \partial = 0. \quad (2.1)
\]

**Definition 2.2.** — The inner product of \( \phi, \psi \in \Omega^{p,q}(F) \) is

\[
(\phi, \psi) = \int_M \sum a_{ij} h(\phi^i, \psi^j) \nu.
\]

**Definition 2.3** — Define

\[
\delta_a : \Omega^{p,q}(F) \rightarrow \Omega^{p-1,q}(F)
\]

by

\[
\delta_a \{ \phi^i \} = \left\{ (-1)^{n+1} \sum_{l,k} \sqrt{G} a^{ij} \ast \partial \left( \frac{a_{jk}}{\sqrt{G}} \phi^k \right) \right\}, \quad \text{where} \quad a^{ij} \quad \text{is the} \quad (i,j)\text{-component of} \quad (a_{ij})^{-1}.
\]

(*) For brevity the subscripts \( \lambda, \mu, \ldots \) are dropped where no confusion will arise.
PROPOSITION 2.1. \( \delta_a \) is the adjoint operator of \( \partial \) with respect to the inner product given in Definition 2.2:
\[
(\partial \phi, \psi) = (\phi, \delta_a \psi) \quad \text{for} \quad \phi \in \Omega^{p-1,q}(F), \quad \psi \in \Omega^{p,q}(F).
\]

Proof. – Since \( \sum_{i,j} a_{ij} \phi^i \wedge * \psi^j \) is globally defined on \( M \), there exists \((n-1)\)-form \( \omega \) on \( M \) such that \( \omega \otimes v = \Sigma a_{ij} \phi^i \wedge * \psi^j \).
Then
\[
\partial (\omega \otimes v) = (\alpha \wedge \omega + d\omega) \otimes v,
\]
where \( \alpha = d \log \sqrt{G} \), and
\[
\partial (\Sigma a_{ij} \phi^i \wedge * \psi^j)
= (-1)^p q \Sigma a_{ij} h(\partial \phi^i, \psi^j) v \otimes v + (-1)^{n-q} \Sigma \phi^i \wedge ** \partial (a_{ij} * \psi^j).
\]
Since
\[
\delta_a \psi^i = -(-1)^{n+1} (\alpha \wedge * \psi^i) + (-1)^{n+1} \Sigma a^q_k \partial (a_{jk} * \psi^k),
\]
we have
\[
(\alpha \wedge \omega + d\omega) \otimes v
= (-1)^p q \Sigma a_{ij} h(\partial \phi^i, \psi^j) v \otimes v + (-1)^{n-q} \Sigma a_{ij} \phi^i \wedge ** (\alpha \wedge * \psi^j)
+ (-1)^{q+1} \Sigma a_{ij} \phi^i \wedge * \delta_a \psi^j
= (-1)^p q \Sigma a_{ij} h(\partial \phi^i, \psi^j) v \otimes v + (\alpha \wedge \omega) \otimes v
+ (-1)^{p-q-1} \Sigma a_{ij} h(\phi^i, \delta_a \psi^j) v \otimes v,
\]
and so
\[
d\omega = (-1)^p q (\Sigma a_{ij} h(\partial \phi^i, \psi^j) - \Sigma a_{ij} h(\phi^i, \delta_a \psi^j)) v.
\]
Therefore
\[
0 = \int_M d\omega = (-1)^p q ((\partial \phi, \psi) - (\phi, \delta_a \psi)).
\]
Q.E.D.

DEFINITION 2.4. – We define
\[
\Box_a : \Omega^{p,q}(F) \rightarrow \Omega^{p,q}(F)
\]
by $\Box_a = \partial \delta_a + \delta_a \partial$, and call it the Laplacian. $\phi \in \Omega^{p,q}(F)$ is said to be $\Box_a$-harmonic if $\Box_a \phi = 0$.

**Definition 2.5.** We set

$$\mathcal{H}^{p,q}(F) = \{ \phi \in \Omega^{p,q}(F) \mid \Box_a \phi = 0 \}.$$

**Theorem 2.2.** We have the following duality:

$$\mathcal{H}^{p,q}(F) \cong \mathcal{H}^{n-p,n-q}((K \otimes F)^*),$$

where $K$ is the canonical line bundle over $M$ and $(K \otimes F)^*$ is the dual bundle of $K \otimes F$.

**Proof.** For $\psi = \{ \psi^i \} \in \Omega^{p,q}(F)$ we set

$$\psi^*_i = \sum_j \frac{a_{ij}}{\sqrt{G}} \ast \psi^j. \quad (2.2)$$

Then we have $\psi^* = \{ \psi^*_i \} \in \Omega^{n-p,n-q}((K \otimes F)^*)$. It follows from Proposition 1.1 (i)

$$\psi^j = (-1)^{n+p+q} \sum_i \sqrt{G} d^i \ast \psi^*_i. \quad (2.3)$$

Thus the map $\psi \mapsto \psi^*$ is a linear isomorphism from $\Omega^{p,q}(F)$ onto $\Omega^{n-p,n-q}((K \otimes F)^*)$.

Let $\phi \in \Omega^{p,q}(F)$ and $\psi^* \in \Omega^{n-p,n-q}((K \otimes F)^*)$. Then

$$\sum_i \sqrt{G} \phi^i \wedge \psi^*_i$$

is globally defined on $M$. Hence there exists a $C^\infty$-function $k(\phi, \psi^*)$ on $M$ such that

$$\sum_i \sqrt{G} \phi^i \wedge \psi^*_i = k(\phi, \psi^*) v \otimes v.$$

We set

$$\langle \phi, \psi^* \rangle = (-1)^{pq} \int_M k(\phi, \psi^*) v.$$

Since

$$k(\phi, \psi^*) v \otimes v = \sum_{i,j} a_{ij} \phi^i \Lambda \ast \psi^j = (-1)^{pq} \sum_{i,j} a_{ij} h(\phi^i, \psi^j) v \otimes v,$$

we have

$$\langle \phi, \psi^* \rangle = \langle \phi, \psi \rangle \quad \text{for} \quad \phi, \psi \in \Omega^{p,q}(F).$$
Define the inner product of $\psi^*, \phi^* \in \Omega^{n-p, n-q} ((K \otimes F)^*)$ by

$$(\psi^*, \phi^*) = \int_M \sum G a^{ij} h(\psi^*_i, \phi^*_j) v.$$ 

Since

$$\sum_{i,j} G a^{ij} h(\psi^*_i, \phi^*_j) v \otimes v = \sum_{i,j} a_{ij} h(\psi^i, \phi^j) v \otimes v$$

$$= (-1)^p \sum_{i,j} a_{ij} \phi^j \wedge \psi^i = \sum_{i,j} a_{ij} h(\phi^j, \psi^i) v \otimes v,$$

we obtain

$$(\psi^*, \phi^*) = (\phi, \psi) \quad \text{for} \quad \phi, \psi \in \Omega^{p,q} (F).$$

Let $\phi \in \Omega^{p-1, q} (F)$ and $\psi^* \in \Omega^{n-p, n-q} ((K \otimes F)^*)$. Then $\sum_i \sqrt{\nabla} \phi^i \wedge \psi_i^*$ is globally defined on $M$ and hence there exists $(n-1)$-form $\omega$ on $M$ such that

$$\sum_i \sqrt{\nabla} \phi^i \wedge \psi_i^* = \omega \otimes v.$$

Since

$$\partial \left( \sum_i \sqrt{\nabla} \phi^i \wedge \psi_i^* \right)$$

$$= \sum_i \{ \alpha \wedge \sqrt{\nabla} \phi^i \wedge \psi_i^* + \sqrt{\nabla} \partial \phi^i \wedge \psi_i^* + (-1)^{p-1} \sqrt{\nabla} \phi^i \wedge \partial \psi_i^* \}$$

$$= (\alpha \wedge \omega) \otimes v + \sum_i \{ k(\partial \phi^i, \psi_i^*) + (-1)^{p-1} k(\phi^i, \partial \psi_i^*) \} v \otimes v,$$

and

$$\partial (\omega \otimes v) = (\alpha \wedge \omega + d\omega) \otimes v,$$

we obtain

$$d\omega = \sum_i \{ k(\partial \phi^i, \psi_i^*) + (-1)^{p-1} k(\phi^i, \partial \psi_i^*) \} v.$$ 

Therefore

$$0 = \int_M d\omega$$

$$= (-1)^p \langle \partial \phi, \psi^* \rangle + (-1)^{p-1+(p-1)q} \langle \phi, \partial \psi^* \rangle.$$
This implies
\[ \langle \partial \phi, \psi^* \rangle = (-1)^{p+q} \langle \phi, \partial \psi^* \rangle. \]

Using these facts we obtain
\[
(\phi^*, \partial \psi^*) = (\phi, \partial \psi^*) = (-1)^{p+q} (\partial \phi, \psi^*) = (-1)^{p+q} (\partial \phi, \psi)
= (-1)^{p+q} (\phi, \delta_a \psi) = (-1)^{p+q} (\phi^*(\delta_a \psi)^*),
\]

hence
\[
\partial \psi^* = (-1)^{p+q} (\delta_a \psi^*) \quad \text{for} \quad \psi \in \Omega^{p,q}(F). \tag{2.4}
\]

By the same way we have
\[
(\psi^*, \delta_a \phi^*) = (\partial \psi^*, \phi^*) = (\phi, \partial \psi^*) = (-1)^{p+q} (\partial \phi, \psi^*)
= (-1)^{p+q} (\partial \phi, \psi) = (-1)^{p+q} (\delta(\phi)^*, \psi^*),
\]

hence
\[
\delta_a \phi^* = (-1)^{p+q} (\partial \phi)^*.
\]

Thus
\[
\delta_a \psi^* = (-1)^{p+q+1} (\partial \psi)^* \quad \text{for} \quad \psi \in \Omega^{p,q}(F). \tag{2.5}
\]

(2.4) and (2.5) imply that \( \psi^* \) is harmonic if and only if \( \psi \) is harmonic.

Q.E.D.

**DEFINITION 2.6.** — We set
\[
H^{p,q}(F) = \{ \phi \in \Omega^{p,q}(F) \mid \partial \phi = 0 \} / \{ \partial \psi \mid \psi \in \Omega^{p-1,q}(F) \}.
\]

A \( q \)-form \( \omega \) on \( M \) is said to be \textit{D-parallel} if \( D\omega = 0 \). Let us denote by \( \mathcal{P}^d(F) \) the sheaf over \( M \) of germs of \( F \)-valued \( D \)-parallel \( q \)-forms.

**DEFINITION 2.7.** — We denote by \( H^p(\mathcal{P}^d(F)) \) the \( p \)-th cohomology group of \( M \) with coefficients on \( \mathcal{P}^d(F) \).

**THEOREM 2.3.** — We have the following isomorphisms:
\[
\mathcal{E}^{p,q}(F) \cong H^{p,q}(F) \cong H^p(\mathcal{P}^d(F)).
\]
Proof. — By the theory of harmonic integral we have
\[ \mathcal{H} P^q(F) \cong H^p\cdot q(F). \]
Let \( A^p\cdot q(F) \) denote the sheaf over \( M \) of germs of sections of
\( (\wedge^p T^*) \otimes (\wedge^q T^*) \otimes F \). Then
\[ 0 \rightarrow P^q(F) \rightarrow A^0\cdot q(F) \xrightarrow{\partial} A^1\cdot q(F) \xrightarrow{\partial} A^2\cdot q(F) \xrightarrow{\partial} \ldots \]
is a fine resolution of \( P^q(F) \). Thus we have \( H^p\cdot q(F) \cong H^p(P^q(F)) \).
Q.E.D.

3. Hessian metrics on affine local coordinate systems.

Let \( M \) be a Hessian manifold with a locally flat affine connection \( D \) and a Hessian metric \( g \). We denote by \( \nabla \) the Riemannian connection for \( g \). In this section we shall express various geometric concepts on the Hessian manifold \( M \) in terms of affine local coordinate systems. Let us denote by \( D_k \) and \( \nabla_k \) the covariant derivations with respect to \( \partial/\partial x^k \) for \( D \) and \( \nabla \) respectively. Since the Christoffel symbol \( \Gamma^l_{jk} \) for \( g \) is the difference between the components of affine connections \( \nabla \) and \( D \), we may consider that \( \Gamma^l_{jk} \) is a tensor field. We have then
\[ \Gamma^l_{jk} = \frac{1}{2} g^{ls} D_k g_{lj}, \quad (3.1) \]
\[ D_k g_{ij} = 2 \Gamma_{ijk}, \quad D_k g^{ij} = -2 \Gamma^{ij}_k, \]
\[ \Gamma_{jk} = \Gamma_{jk} = \Gamma_{kj}. \]

Definition 3.1. — We define a 1-form \( \alpha \) and a symmetric bilinear form \( \beta \) by
\[ \alpha = D \log \sqrt{G}, \]
\[ \beta = D^2 \log \sqrt{G}, \]
where \( G = \det(g_{ij}) \), and call them the first Koszul form and the second Koszul form of \( M \) respectively.
Then we have

\[ \alpha_i = \Gamma^r_{ir}, \quad (3.2) \]
\[ \beta_q = D_j \Gamma^r_{ir}. \]

**Definition 3.2.** Let \( \gamma_k \) be the derivation of the algebra of tensor fields defined by

\[ \gamma_k = \nabla_k - D_k. \]

Let \( T^p \) be the space of tensor fields of type \((p, q)\) defined on \( M \).

**Definition 3.3.** We define certain covariant derivations \( \nabla^l_k \), \( \bar{\nabla}^l_k \) on \( T^p_q \otimes T^r_s \) by

\[ \nabla^l_k = (2\gamma_k) \otimes \text{id} + D_k, \]
\[ \bar{\nabla}^l_k = \text{id} \otimes (2\gamma_k) + D_k, \]
where \( \text{id} \) are the identity transformations.

Notice that

\[ \nabla_k = \frac{1}{2}(\nabla^l_k + \bar{\nabla}^l_k), \quad \text{where} \quad k = \bar{k}. \]

**Lemma 3.1.** For the Hessian metric \( g \) we have

\[ \nabla^l_k g_{ij} = 0, \quad \bar{\nabla}^l_k g_{ij} = 0, \]
\[ \nabla^l_k g_{ij} = 0, \quad \bar{\nabla}^l_k g_{ij} = 0. \]

**Proof.** By (3.1) we obtain

\[ \nabla^l_k g_{ij} = D_k g_{ij} - 2\gamma^m_{ki} g_{mj} = 2\Gamma^r_{ij} - 2\Gamma^r_{jki} = 0. \]

Similarly we can prove the other equalities.

Q.E.D.

**Definition 3.4.** Considering \( \gamma_i \) as tensor fields of type \((1, 1)\) we define tensor fields \( \gamma \) and \( S \) by

\[ \gamma = \sum_i \gamma_i \otimes dx^i, \]
\[ S = D\gamma. \]
The component of $S$ is given by

$$S^i_{jkl} = D_k \Gamma^i_{jl}.$$ 

**Lemma 3.2.** $S^i_{jkl} = S^k_{jil} = S^k_{klij} = S^i_{lkl}.$

**Proof.** Let $g_{ij} = D_i D_j u$. By (3.1) we have

$$S^i_{jkl} = g_{ip} D_k \Gamma^p_{jl} = g_{ip} D_k (g^{pq} \Gamma_{qjl}) = g_{ip} (D_k g^{pq}) \Gamma_{qjl} + g_{ip} g^{pq} D_k \Gamma_{qjl}$$

$$= -2 \Gamma^q_{i k} \Gamma_{qjl} + D_k \Gamma_{jl} = -2g^{qr} \Gamma_{ir k} \Gamma_{qjl} + D_k \Gamma_{jl}$$

$$= \frac{1}{2} D_i D_j D_k D_l u - \frac{1}{2} g^{qr} (D_r D_i D_k u) D_q D_j D_l u.$$

This proves the Lemma. Q.E.D.

**Lemma 3.3.** $g^i_{jkl} = S^i_{jkl} = S_{ijkl}$.

**Proof.** $g^i_{jkl} = D_i \alpha_j = D_i \alpha_j = D_i \Gamma^r_{rij} = S^r_{rij}$. By Lemma 3.2 we have $g^i_{jkl} = g^{ip} S_{prij} = g^{ip} S_{ijpr} = S_{ijkl}$.

Q.E.D.

4. The local expression for $\square$.

From now on we always assume that $M$ is a compact connected oriented Hessian manifold.

**Proposition 4.1.** Let $\phi \in \Omega^{p,q}$. Then we have

$$(\partial \phi)_{l_1 \ldots l_p + 1} \beta_q = \sum_{\alpha} (-1)^{\alpha - 1} \nabla_{i_\alpha}^l \phi_{l_1 \ldots l_\alpha \ldots l_p + 1} \beta_q,$$

where $i_\alpha$ means "omit $i_\alpha"."

**Proof.** By Definition 1.6 we have

$$(\partial \phi)_{l_1 \ldots l_p + 1} \beta_q = \sum_{\alpha = 1}^{p + 1} (-1)^{\alpha - 1} \nabla_{i_\alpha}^l \phi_{l_1 \ldots l_\alpha \ldots l_p + 1} \beta_q.$$  \hspace{1cm} (4.1)

Using this and (3.1) we obtain the proposition. Q.E.D.
PROPOSITION 4.2. - Let $\phi \in \Omega^{p, q}$. Then we have

$$(\delta \phi)_{p-1 \mathbf{1}} = - g^{\sigma^p} \nabla^i_{\sigma^p} \phi_{\mathbf{1} p-1 \mathbf{1}} + \alpha^q \phi_{\mathbf{1} p-1 \mathbf{1}}.$$ 

Proof. - Let $\psi \in \Omega^{p-1, q}$. By (4.1) and Green's theorem we have

$$(\phi, \partial \psi) = - \int_M D_r(\phi^r_{p-1 \mathbf{1}} \sqrt{G}) \frac{1}{\sqrt{G}} \psi_{p-1 \mathbf{1}} v.$$ 

Thus we obtain

$$(\delta \phi)_{p-1 \mathbf{1}} = - D_r \phi^r_{p-1 \mathbf{1}} - \alpha_r \phi^r_{p-1 \mathbf{1}} = - \nabla_r \phi^r_{p-1 \mathbf{1}} + \alpha_r \phi^r_{p-1 \mathbf{1}}.$$ 

This completes the proof.

Q.E.D.

THEOREM 4.1. - Let $\phi \in \Omega^{p, q}$. Then we have

$$(\square \phi)_{p \mathbf{1}} = - g^{\sigma^p} \nabla^i_{\sigma^p} \phi_{\mathbf{1} p} - \sum_a \beta_{\mathbf{1} a} \phi_{\mathbf{1} a^p}.$$ 

Proof. - Using Proposition 4.1, Proposition 4.2 and $\nabla^i \alpha^j = \beta^j_i$, we obtain

$$(\delta \delta \phi)_{p \mathbf{1}} = - g^{\sigma^p} \sum_a \nabla^i_{\sigma^p} \phi_{\mathbf{1} a^p} + \sum_a \beta_{\mathbf{1} a} \phi_{\mathbf{1} a^p}.$$ 

$$(\delta \delta \phi)_{p \mathbf{1}} = - g^{\sigma^p} \nabla^i_{\sigma^p}(\nabla^i \phi_{\mathbf{1} a^p} - \sum_a \nabla^i \phi_{\mathbf{1} a^p}).$$ 

$$(\delta \delta \phi)_{p \mathbf{1}} = - g^{\sigma^p} \nabla^i_{\sigma^p}(\nabla^i \phi_{\mathbf{1} a^p} - \sum_a \nabla^i \phi_{\mathbf{1} a^p}),$$

where $(s)_a$ means "substitute $s$ for $a$-th place".

Q.E.D.
and so
\[(\Box \phi)_{l_p i_q} = - g^{\alpha \beta} \nabla_\alpha^l \nabla_\beta^i \phi_{l_p i_q} + \alpha \nabla_\alpha^l \phi_{l_p i_q}
\]
\[= - g^{\alpha \beta} \sum_{\sigma} [\nabla_\alpha^l, \nabla_\beta^i] \phi_{l_\sigma \ldots (\sigma) \ldots i_p i_q}
\]
\[+ \sum_{\sigma} \beta_{l_\sigma} \phi_{l_\sigma \ldots (\sigma) \ldots i_p i_q}.
\]

Let us calculate the third term on the right-hand of the above formula. Since \([\nabla_\alpha^l, \nabla_\beta^i] \) is a derivation of the algebra of tensor fields which maps every function to 0 and since
\[[\nabla_\alpha^l, \nabla_\beta^i] = 2 S^m_{l_\sigma i_\tau} \phi_{l_\sigma \ldots (\sigma) \ldots i_\tau i_q},
\]
\[[\nabla_\alpha^l, \nabla_\beta^i] = - 2 S^m_{l_\sigma i_\tau} \phi_{l_\sigma \ldots (\sigma) \ldots i_\tau i_q},
\]
we have
\[[\nabla_\alpha^l, \nabla_\beta^i] \phi_{l_\sigma \ldots (\sigma) \ldots i_p i_q} = \sum_{\tau} 2 S^m_{l_\sigma i_\tau} \phi_{l_\sigma \ldots (\sigma) \ldots (\tau)} + 2 S^m_{l_\sigma i_\tau} \phi_{l_\sigma \ldots (\sigma) \ldots i_\tau i_q}
\]
\[+ 2 S^m_{l_\sigma i_\tau} \phi_{l_\sigma \ldots (\sigma) \ldots i_\tau i_q}.
\]

Thus, by Lemma 3.2 and 3.3 we obtain
\[g^{\alpha \beta} \sum_{\sigma} [\nabla_\alpha^l, \nabla_\beta^i] \phi_{l_\sigma \ldots (\sigma) \ldots i_p i_q} = 2 \sum_{\sigma} \beta_{l_\sigma} \phi_{l_\sigma \ldots (\sigma) \ldots i_p i_q}
\]
\[+ 2 \sum_{\sigma, \tau} S^m_{l_\sigma i_\tau} \phi_{l_\sigma \ldots (\sigma) \ldots i_\tau i_q}.
\]

This completes the proof.

Q.E.D.

Example. — For the Hessian metric \(g\) we have
\[\Box g \nabla^i = - \beta_{i \nabla} \]
Thus the Hessian metric $g$ is $\Box$-harmonic if and only if the second Koszul form $\beta = 0$. Therefore, by [12] the following conditions are equivalent:

(i) $g$ is $\Box$-harmonic.

(ii) The first Koszul form $\alpha = 0$.

(iii) The second Koszul form $\beta = 0$.

(iv) $g$ is locally flat.

5. The local expression for $\square_a$.

Let $F$ be a locally constant line bundle over a compact connected oriented Hessian manifold $M$, and let $a$ be a fiber metric on $F$.

**Proposition 5.1.** — We have

$$\delta_a = \delta + i(A),$$

where $A = - D \log a$ and $i(A) \phi_{l_{p-1} \vec{l}_q} = \Phi_{r l_{p-1} \vec{l}_q}$ for $\phi \in \Omega^{p,q}(F)$.

**Proof.** — By Definition 1.2, 1.7 and 2.3 we have

$$\delta_a = (-1)^{n+1} \frac{\sqrt{G}}{a} \partial \left( \frac{a}{\sqrt{G}} \right)$$

$$= (-1)^n \ast e(A) \ast + (-1)^{n+1} \sqrt{G} \ast \partial \left( \frac{1}{\sqrt{G}} \right)$$

$$= i(A) + \delta,$$

where

$$(e(A) \phi)_{l_1 \ldots l_p \vec{l}_q} = \sum_{\sigma} (-1)^{\sigma-1} A_{l_{\sigma}} \phi_{l_1 \ldots \vec{l}_\sigma \ldots l_p \vec{l}_q}$$

for $\phi \in \Omega^{p,q}(F)$. Q.E.D.
DEFINITION 5.1. — For $\phi \in \Omega^{p,q}(F)$ we set
\[ \nabla^t(a) \phi = \frac{1}{a} \nabla^t(a \phi). \]

THEOREM 5.1. — Let $\phi \in \Omega^{p,q}(F)$. Then we have
\[
\Box \phi_{1p} \overline{\phi} = - g^F \nabla^t(a) \nabla' \phi_{1p} \overline{\phi} + \alpha^F \nabla' \phi_{1p} \overline{\phi} + \sum_{\sigma} (- \beta_{i\sigma} + B_{i\sigma} \phi_{1\ldots(\sigma)\ldots p} \overline{\phi} + 2 \sum_{\sigma, \tau} S_{i\sigma \overline{\tau}} \phi_{1\ldots(\tau)\ldots p} \overline{\phi} \overline{\phi} \ldots \phi_{1\ldots(\tau)\ldots p}. \]

Proof. — By Proposition 5.1 we have
\[
\Box_a = \Box + i(A) \partial + \partial i(A). \]

A straightforward calculation shows
\[
(i(A) \partial \phi)_{1p} \overline{\phi} + (\partial i(A) \phi)_{1p} \overline{\phi} = g^F A_{i} \nabla' \phi_{1p} \overline{\phi} + \sum_{\sigma = 1}^{p} B_{i\sigma} \phi_{1\ldots(\sigma)\ldots p} \overline{\phi} \overline{\phi} \ldots \phi_{1\ldots(\tau)\ldots p}. \]

Thus our assertion follows from the above facts and Theorem 4.1.

Q.E.D.

6. A vanishing theorem of Kodaira-Nakano type.

Let $\theta$ be a symmetric covariant tensor field of degree 2. Considering $\theta$ as an element in $\Omega^{1,1}$ we define
\[
e(\theta) : \Omega^{p,q} \longrightarrow \Omega^{p+1,q+1}, \]
\[
i(\theta) : \Omega^{p,q} \longrightarrow \Omega^{p-1,q-1}, \]
by $e(\theta) \phi = \theta \wedge \phi$ for $\phi \in \Omega^{p,q}$ and $i(\theta) = (-1)^{n+p+q+1} * e(\theta) *$.

Then $i(\theta)$ is the adjoint operator of $e(\theta)$ with respect to the inner product in Definition 1.1 and 2.2.
In this section we always assume that $F$ is a locally constant line bundle over $M$.

**Proposition 6.1.** — We have

(i) $\Box_a e(g) = e(B + \beta)$,

(ii) $\Box_a i(g) = -i(B + \beta)$.

The proof follows from a straightforward calculation and so it is omitted.

**Proposition 6.2.** — Suppose $\Box_a \phi = 0$. Then we have

(i) $(e(B + \beta) i(g) \phi, \phi) \leq 0$.

(ii) $(i(g) e(B + \beta) \phi, \phi) \geq 0$.

(iii) $(i(g), e(B + \beta) \phi, \phi) \geq 0$.

**Proof.** — By Proposition 6.1 (i) we have $\Box_a e(g) = e(B + \beta) \phi$. Thus we have

$0 \leq (\Box_a e(g) \phi, e(g) \phi) = (e(B + \beta) \phi, e(g) \phi) = (i(g) e(B + \beta) \phi, \phi)$,

which implies (ii). By the same way, since $\Box_a i(g) \phi = -i(B + \beta) \phi$ we obtain

$0 \leq (\Box_a i(g) \phi, i(g) \phi) = (-i(B + \beta) \phi, i(g) \phi)

= (\phi, -e(B + \beta) i(g) \phi),

which shows (i). (iii) follows from (i) and (ii).

Q.E.D.

**Theorem 6.1.** — Let $M$ be a compact connected oriented Hessian manifold. Denote by $K$ the canonical line bundle over $M$. Let $F$ be a locally constant line bundle over $M$.

(i) If $2F + K$ is positive, then

$H^{p,q}(F) = 0$ for $p + q > n$.

(ii) If $2F + K$ is negative, then

$H^{p,q}(F) = 0$ for $p + q < n$. 
Proof. – Suppose $2F + K$ is negative. Then $B + \beta$ is negative definite. Therefore $g' = -(B + \beta)$ gives a Hessian metric on $M$. If we denote by $\beta'$ the Koszul form on $M$ with respect to $g'$, then there exists a positive $C^\infty$-function $f$ on $M$ such that

$$\beta' = \beta + D^2 \log f.$$ 

If $B$ is a Koszul form of $F$ with respect to a fiber metric $a = \{a^i\}$, then the Koszul form $B'$ of $F$ with respect to the fiber metric $a' = \{fa^i\}$ satisfies

$$B' + \beta' = B + \beta = -g'.$$

Therefore if we use $-(B + \beta)$ as a Hessian metric, the formula in Proposition 6.2 (iii) is reduced to

$$([i(g) - e(g)], \phi, \phi) \geq 0 \quad \text{for} \quad \phi \in \mathcal{H}^{p,q} (F).$$

Thus by Proposition 1.2 we have

$$(n - p - q) (\phi, \phi) \leq 0 \quad \text{for} \quad \phi \in \mathcal{H}^{p,q} (F).$$

Therefore, if $n - p - q > 0$ then $\phi = 0$. Hence (ii) is proved. (i) follows from (ii) and Theorem 2.2

Q.E.D.

7. A vanishing theorem of Koszul type.

In this section we mention a vanishing theorem of Koszul type. Let $M$ be a compact oriented hyperbolic affine manifold. Then there exists a canonical Hessian metric $g$ and a unique Killing vector field $H$ on $M$ such that

$$D_X H = X,$$

(7.1)

for all vector field $X$ on $M[7]$. The following theorem is essentially due to Koszul.

**Theorem 7.1.** – Let $F$ be a locally constant vector bundle over a compact hyperbolic affine manifold. If there exist a fiber metric $a = \{(a_i)\}$ and a constant $c (\neq -2q)$ such that

$$Ha_y = ca_y,$$
then we have

\[ H^{p,q}(F) = 0, \quad \text{for} \quad p > 0 \quad \text{and} \quad q \geq 0. \]

The proof of this theorem is nearly the same as Koszul [7], and so we omit the proof.

**COROLLARY 7.1.** — Let \( M \) be a compact oriented hyperbolic affine manifold. Then we have

\[ H^{p,q}(1) = 0, \quad \text{for} \quad p, q > 0, \]

where 1 is the trivial vector bundle over \( M \).

The tensor bundle \( \otimes T \otimes T^* \) satisfies the condition of Theorem 7.1 if \( q - r + s \neq 0 \).

We give another example of locally constant vector bundle over \( M \) which satisfies the conditions of Theorem 7.1. Let \( \Omega \) be an open convex cone in \( \mathbb{R}^n \) with vertex 0 not containing any full straight line. Suppose that a discrete subgroup \( \Gamma \) of \( GL(n, \mathbb{R}) \) acts properly discontinuously and freely on \( \Omega \) such that \( M = \Gamma \backslash \Omega \) is compact. Assume further that there exist a linear mapping from \( \Omega \) to the space of all \( m \times m \) positive definite real symmetric matrices and a homomorphism from \( \Gamma \) to \( GL(m, \mathbb{R}) \), which are denoted by the same letter \( \rho \), such that

\[ \rho(\gamma x) = \rho(\gamma) \rho(x) \gamma^t \rho(\gamma) \quad \text{for} \quad \gamma \in \Gamma, \ x \in \Omega. \]

We denote by \( F_{\rho} \) the vector bundle over \( M \) associated with the universal covering \( \Omega \to M \) and \( \rho \). Let \( U \) be an evenly covered open set in \( M \). Choosing a section \( \sigma \) on \( U \) we set

\[ a = (\rho \circ \sigma)^{-1}. \]

Then \( a \) is a fiber metric on \( F_{\rho} \) and we have

\[ Ha = -a. \]

Therefore

**COROLLARY 7.2.** — We have

\[ H^{p,q}(F_{\rho}) = 0 \quad \text{for} \quad p > 0 \quad \text{and} \quad q \geq 0. \]
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