ANNALES DE L'INSTITUT FOURIER

MARKO TADIC

Spherical unitary dual of general linear group over non-Archimidean local field

Annales de l'institut Fourier, tome 36, n° 2 (1986), p. 47-55

<http://www.numdam.org/item?id=AIF_1986__36_2_47_0>

Ann. Inst. Fourier, Grenoble
36, 2 (1986), 47-55.

SPHERICAL UNITARY DUAL OF GENERAL LINEAR GROUP OVER NON-ARCHIMEDEAN LOCAL FIELD

by **Marko TADIĆ**

1. Introduction.

Let G be a connected reductive group over a local non-archimedean field F. Fix a special good maximal compact subgroup K of G (in the sense of Bruhat and Tits). Let $\tilde{G}$ be the set of all equivalence classes of irreducible smooth representations of G, and let $\hat{G}$ be the subset of all unitarizable classes in $\tilde{G}$. The subset of all classes π in $\tilde{G}$ such that π restricted to K contains the trivial representation of K as a composition factor, is denoted by $\tilde{G}^s$. Let

$$\hat{G}^s = \tilde{G}^s \cap \hat{G}.$$

The set $\tilde{G}$ (resp. $\tilde{G}^s$) is called a non-unitary dual of G (resp. non-unitary spherical dual of G) and $\hat{G}$ (resp. $\hat{G}^s$) is called the unitary dual of G (resp. unitary spherical dual of G).

A basic problem of the harmonic analysis of the Gelfand pair (G,K) is to describe $\hat{G}^s$.

Notice that $\hat{G}^s$ is not a very big part of the whole unitary dual $\hat{G}$, but it plays a very important role in the description of the unitary duals of adelic reductive groups.

The set $\tilde{G}^s$ is in bijection with the set of all zonal spherical functions on G with the respect to K, and $\hat{G}^s$ is in bijection with the subset of all positive definite zonal spherical functions.

The spherical functions on a p-adic group and related problems were studied for the first time in 1958 by F. I. Mautner on the group PGL (2)

Key-words : Unitary representations - Spherical representations - General linear group - Non archimedean local field.

([7]). The basic ideas of the theory of spherical representations in the general setting, interesting for our point of view, seems to belong to I. M. Gelfand (for general ideas one can consult [4] and also [6]). Papers [9], [6] and [3] deals with formulas for spherical functions on reductive p-adic groups, while [5] contains among other things a description of the spherical unitary dual of $\mathrm{SL}(2,\mathrm{F})$.

In this paper we classify all unitary spherical representations of the groups $\mathrm{GL}(n,\mathrm{F})$. The notion of spherical representation does not depend on the choice of K, because in $\mathrm{GL}(n,\mathrm{F})$ all maximal compact subgroups are conjugated. In this way, one obtains also classification of positive definite zonal spherical functions on $\mathrm{GL}(n,\mathrm{F})$.

The biggest part of the present paper is devoted to the identification of $(\mathrm{GL}(n,\mathrm{F})^{\sim})^{s}$ in the Zelevinsky parametrization of $\mathrm{GL}(n,\mathrm{F})^{\sim}$ (Proposition 2.1). In particular, we prove the following result.

PROPOSITION. — *Let* π *be a spherical representation of* $\mathrm{GL}(n,\mathrm{F})$. *Then there exists a parabolic subgroup* P *of* $\mathrm{GL}(n,\mathrm{F})$ *with Levi factorisation* $\mathrm{P} = \mathrm{MN}$ *and a spherical one-dimensional representation* χ *of* M, *such that the representation of* $\mathrm{GL}(n,\mathrm{F})$ *induced from* P *by* χ *is irreducible and equal to* π.

The description of the spherical unitary dual is obtained after identification of non-unitary spherical dual, as a direct consequence of the Bernstein result which states that the induced representation of $\mathrm{GL}(n,\mathrm{F})$ by an irreducible unitarizable representation of a parabolic subgroup, is irreducible.

We present now our result. Let $\mathrm{U}(\mathrm{F}^{\times})$ be the set of all unramified unitary characters of the multiplicative group of F. The normalized absolute value of F is denoted by $|\ |_{\mathrm{F}}$.

First of all, we have very simple unitary spherical representations

$$\chi(\det_n): \ g \mapsto \chi(\det_n g), \qquad g \in \mathrm{GL}(n,\mathrm{F}),$$

when $\chi \in \mathrm{U}(\mathrm{F}^{\times})$. Let $\pi(\chi(\det_n),\alpha)$ be the representation of $\mathrm{GL}(2n,\mathrm{F})$ induced by

$$\chi(\det_n)\,|\det_n|_{\mathrm{F}}^{\alpha} \otimes \chi(\det_n)\,|\det_n|_{\mathrm{F}}^{-\alpha}.$$

If $0 < \alpha < 1/2$, then $\pi(\chi(\det_n),\alpha)$ is unitarizable. That was shown by G. I. Olshansky in [8] (see also [1]).

THEOREM. — *Fix a positive integer* t.

(i) *Let* $n_1, \dots, n_p, m_1, \dots, m_q$ *be positive integers such that*

$$n_1 + \cdots + n_p + 2(m_1 + \cdots + m_q) = t .$$

Let $\chi_1, \dots, \chi_p, \mu_1, \dots, \mu_q \in \mathrm{U}(\mathrm{F}^\times)$ *and let* $0 < \alpha_1, \dots, \alpha_q < 1/2$. *Then the representation induced by*

$$\chi_1(\det_{n_1}) \otimes \cdots \otimes \chi_p(\det_{n_p}) \otimes \pi(\mu_1(\det_{m_1}), \alpha_1) \otimes \cdots \otimes \pi(\mu_q(\det_{m_q}), \alpha_q)$$

from a suitable parabolic subgroup of $\mathrm{GL}(t,\mathrm{F})$, *is in* $(\mathrm{GL}(t,\mathrm{F})^\wedge)^s$.

(ii) *Each unitary spherical representation can be obtained as it is described in* (i), *and the parameters*

$$(\chi_1, n_1), \dots, (\chi_p, n_p), (\mu_1, m_1, \alpha_1), \dots, (\mu_q, m_q, \alpha_q)$$

are uniquely determined, up to a permutation.

One can obtain the preceding theorem from the description of the whole unitary dual (a part of the solution of unitarizability problem for p-adic GL(n) is contained in [10]). One can also obtain the preceding theorem using Lemma 8.8 of [1], after one has Proposition 2.1. The proof we present here is based on very simple ideas (*).

We shall now introduce some basic notation. The field of complex numbers is denoted by $\mathbf{C}$, the subfield of reals is denoted by $\mathbf{R}$, the subring of integers is denoted by $\mathbf{Z}$. The subset of positive integers is denoted by $\mathbf{N}$, and the subset of non-negative integers is denoted by $\mathbf{Z}_+$.

At the end, I would like to thank the Max-Planck-Institut für Mathematik for hospitality and excellent working conditions in which this paper was written.

2. Zelevinsky classification and identification of the non-unitary spherical dual in this classification.

Set $\mathrm{G}_n = \mathrm{GL}(n,\mathrm{F})$. The category of all smooth representations of G_n of finite length is denoted by $\mathrm{Alg}\,\mathrm{G}_n$. The induction functor defines the mapping

$$\mathrm{Alg}\,\mathrm{G}_n \times \mathrm{Alg}\,\mathrm{G}_m \to \mathrm{Alg}\,\mathrm{G}_{n+m},$$
$$(\tau,\sigma) \mapsto \tau \times \sigma$$

(*) *Added in proof:* S. Sahi obtained also the same result in his 1985 Yale Ph. D. thesis.

(see § 1 of [11]). Let R_n be the Grothendieck group of the category $\text{Alg } G_n$. Let

$$R = \bigoplus_{n \geqslant 0} R_n .$$

The mapping $(\tau,\sigma) \to \tau \times \sigma$ induces a structure of a commutative graded algebra on R. We identify $\tilde{G}_n$ with a subset of R_n. Let $C(G_n)$ be the set of all cuspidal representations in $\tilde{G}_n$. Set

$$\text{Irr} = \bigcup_{n=0}^{\infty} \tilde{G}_n .$$

$$\text{Irr}^u = \bigcup_{n=0}^{\infty} \hat{G}_n$$

$$C = \bigcup_{n=0}^{\infty} C(G_n) .$$

If X is a set then a function f from X into the non-negative integers with finite support will be called a multiset in X. If $\{x_1,\ldots,x_n\}$ is the support of f we shall write f also as

$$f = (\underbrace{x_1,\ldots,x_1}_{f(x_1)\text{-times}},\ \underbrace{x_2,\ldots,x_2}_{f(x_2)\text{-times}},\ldots,\ \underbrace{x_n,\ldots,x_n}_{f(x_n)\text{-times}}) .$$

The set of all multisets in X is denoted by $M(X)$. The number $|f| = \sum_{x \in X} f(x)$ is called the cardinal number of f.

The representation $g \mapsto |\det g|_F$ of G_n is denoted ν. For $\rho \in C$ and a non-negative integer n we set

$$[\rho,\nu^n\rho] = \{\rho,\nu\rho,\ldots,\nu^{n-1}\rho,\nu^n\rho\} .$$

Then $\Delta = [\rho,\nu^n\rho]$ is called a segment in C. The set of all segments in C is denoted by $S(C)$. We shall identify C with a subset of $S(C)$ in a natural way, and also $M(C)$ with a subset of $M(S(C))$.

If $\Delta = [\rho,\nu^n\rho] \in S(C)$, then the representation $\rho \times \nu\rho \times \cdots \times \nu^n\rho$ contains a unique irreducible subrepresentation which is denoted by $\langle \Delta \rangle$. For $a = (\Delta_1,\ldots,\Delta_n) \in M(S(C))$ set

$$\pi(a) = \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_n \rangle \in R .$$

We construct $\langle a \rangle \in \text{Irr}$ like in 6.5 of [11]. Then $\langle a \rangle$ is a composition factor of $\pi(a)$.

The mapping

$$a \mapsto \langle a \rangle$$

is a bijection of M(S(C)) onto Irr. This is *Zelevinsky classification.*

Let $\Delta_i \in S(C)$, $i = 1, 2$. Suppose that $\Delta_1 \cup \Delta_2$ is a segment and

$$\Delta_1 \cup \Delta_2 \notin \{\Delta_1, \Delta_2\}.$$

Then we say that Δ_1 and Δ_2 are linked.

We introduce an ordering on the set M(S(C)) like in 7.1. of [11]. By Theorem 7.1. of [11], $\langle a \rangle$ is a composition factor of $\pi(b)$ if and only if $a \leqslant b$.

Let $\mathcal{O}_F$ be the maximal compact subring of F. Let $K_n = GL(n, \mathcal{O}_F)$. Then K_n is a maximal compact subgroup of G_n. A representation $\pi \in \tilde{G}_n$ is called *spherical* if it contains a non-trivial vector invariant under the action of K_n. Denote by $\tilde{G}_n^s$ the subset of all spherical representations in $\tilde{G}_n$. Let

$$\hat{G}_n^s = \tilde{G}_n^s \cap \hat{G}_n$$

$$\mathrm{Irr}^s = \bigcup_{n=0}^{\infty} \tilde{G}_n^s$$

$$\mathrm{Irr}^{su} = \bigcup_{n=0}^{\infty} \hat{G}_u^s.$$

Note that G_1 is isomorphic to the multiplicative group $F^\times$ of F. We shall identify G_1 with $F^\times$. Then $\tilde{G}_1^s$ is identified also with the group of all unramified quasicharacters of $F^\times$ and $\hat{G}_1^s$ is identified with the group of all unramified unitary characters of $F^\times$. Let ω be a generator of the maximal ideal in $\mathcal{O}_F$. Then the mapping

$$\chi \mapsto \chi(\omega)$$

is a bijection of $\tilde{G}_1^s$ onto $C^\times$, and a bijection of $\hat{G}_1^s$ onto $\{z \in C : |z| = 1\}$. Notice that $\tilde{G}_1 = C(G_1)$.

Let us remind of some well known facts about spherical representations ([2],4.4). For each $a \in M(\tilde{G}_1^s)$ the representation $\pi(a)$ contains as a composition factor exactly one spherical representation which we shall denote by $s(a)$. Now

$$a \mapsto s(a), \qquad M(\tilde{G}_1^s) \to \mathrm{Irr}^s$$

is a bijection. The restriction

$$\{a \in M(\tilde{G}_1^s) : |a| = n\} \rightarrow \tilde{G}_n^s$$

is also a bijection. We want to describe $s(a)$ in terms of $M(S(C))$.

2.1. PROPOSITION. — *For* $a \in M(G_1^s)$ *let* $m(a)$ *be a minimal element, with respect to the ordering of* $M(S(C))$, *in the set*

$$(*) \qquad \{b \in M(S(C)); b \leq a\}.$$

Then $m(a)$ *is unique and*

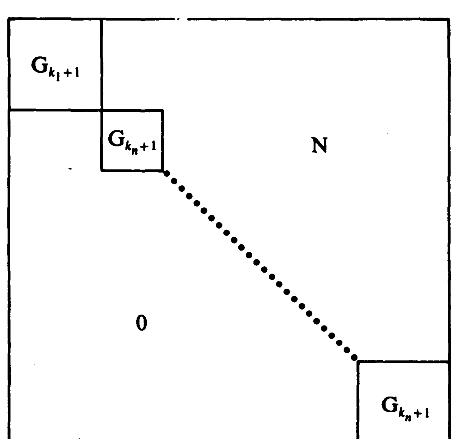
$$s(a) = \langle m(a) \rangle = \pi(m(a)).$$

Proof. — Let $m(a) = (\Delta_1, \ldots, \Delta_n\}$ be a minimal element of (*). Then b is minimal in (*) if and only if we have no linked segments in b. Now $\pi(m(a))$ is irreducible by Theorem 4.2 of [11]. Since $\langle m(a) \rangle$ is a composition factor of $\pi(m(a))$ we have $\langle m(a) \rangle = \pi(m(a))$. Now we shall prove that $\pi(m(a))$ is spherical.

Let $\Delta_i = [\chi_i, \nu^{k_i}\chi_i]$, where k_i are non-negative integers. Then $\langle \Delta_i \rangle \in \tilde{G}_{k_i+1}$,

$$\langle \Delta_i \rangle(g) = \nu^{k_i/2}\chi_i(\det g)$$

(see 3.2 of [11]). Now the groups $G_{k_1+1}, \ldots, G_{k_n+1}$ determine a parabolic subgroup P of G_p where $p = \sum_{i=1}^{n} (k_i+1)$. Let $P = MN$ be a Levi decomposition of P such that $M = G_{k_1+1} \times \cdots \times G_{k_n+1}$ (see the following illustration of P).



We consider G_{k_i+1} as a subgroup of G_p in a natural way. Let δ_p be the modular character of P. Clearly $\delta_P(M\cap K_p) = \{1\}$. We have the Iwasawa decomposition

$$G_p = PK_p.$$

Using this decomposition we can construct the function defined by

$$f(u\cdot g_1 g_2 \cdots g_n\cdot k) = \prod_{i=1}^{n} \delta_p^{\frac{1}{2}}(g_i)(\nu^{k_i/2}\chi_i)(\det g_i),$$

$$u \in N, \qquad g_i \in G_{k_i+1}, \qquad k \in K_p.$$

In a standard way we prove that f is well defined. Now $f \neq 0$, $f \in \langle \Delta_1 \rangle \times \cdots \times \langle \Delta_n \rangle$ and f is fixed for the action of K_p. Thus $\pi(m(a))$ is spherical.

Since $m(a) \leqslant a$ and $\langle m(a) \rangle$ is spherical, uniqueness of $s(a)$ in $\pi(a)$ implies $s(a) = \langle m(a) \rangle$. Now the fact that $b \mapsto \langle b \rangle$ is a bijection implies uniqueness of $m(a)$.

3. Spherical unitary dual.

We consider $R^* = R\backslash\{0\}$ as a commutative multiplicative semigroup with identity. Corollary 8.2 *a*) of [1] implies that Irr^u is a subsemigroup of R^*. From the proof of Proposition 2.1 one can obtain directly the following proposition. We give another proof.

3.1. PROPOSITION. — Irr^{su} *is a subsemigroup of* Irr^u.

Proof. — Let $\pi_1, \pi_2 \in \mathrm{Irr}^{su}$. Choose $a_1, a_2 \in M(\tilde{G}_1^s)$ such that

$$\pi_i = s(a_i) = \langle m(a_i) \rangle = \pi(m(a_i)), \qquad i = 1, 2.$$

Clearly $m(a_1+a_2) \leqslant m(a_1) + m(a_2)$. Now

$$\pi_1 \times \pi_2 = \pi(m(a_1)) \times \pi(m(a_2))$$

is irreducible by Corollary 8.2 of [1].

Thus $\pi(m(a_1)) \times \pi(m(a_2)) = \pi(m(a_1)+m(a_2))$. The representation $\langle m(a_1+a_2) \rangle$ is a composition factor of $\pi(m(a_1)+m(a_2))$ since

$m(a_1+a_1) \leqslant m(a_1) + m(a_2)$. The irreducibility of $\pi(m(a_1)+m(a_2))$ implies

$$\pi_1 \times \pi_2 = \langle m(a_1+a_2)\rangle = s(a_1+a_2).$$

Thus $\pi_1 \times \pi_2 \in \mathrm{Irr}^{su}$.

Let n be a positive integer and $\chi \in \tilde{G}_1^s$. Set

$$\begin{aligned} \Delta[n] &= \{-(n-1)/2,\ 1-(n-1)/2,\dots,(n-1)/2\}, \\ \Delta[n]^{(\chi)} &= \{\nu^\alpha\chi\,;\alpha\in\Delta[n]\}. \end{aligned}$$

Note that the representation $\langle\Delta[n]^{(\chi)}\rangle$ is just

$$g \to \chi(\det_n g).$$

Therefore $\langle\Delta[n]^{(\chi)}\rangle \in \mathrm{Irr}^{su}$, $\chi \in \hat{G}_1^s$.

Also

$$\pi(\langle\Delta[n]^{(\chi)}\rangle,\alpha) = (\nu^\alpha\langle\Delta[n]^{(\chi)}\rangle) \times (\nu^{-\alpha}\langle\Delta[n]^{(\chi)}\rangle), \qquad \alpha \in (0, 1/2)$$

is irreducible by Theorem 4.2. of [11]. It is unitarizable by Theorem 2 of [8]. One can obtain this also from [1].

Let S be the subsemigroup of Irr^u generated by all $\langle\Delta[n]^{(\chi)}\rangle$, $\pi(\langle\Delta[n]^{(\chi)}\rangle,\alpha)$ where n is a positive integer, $\chi \in \hat{G}_1^s$ and $0 < \alpha < 1/2$. Since $\langle\Delta[n]^{(\chi)}\rangle$, $\pi(\langle\Delta[n]^{(\chi)}\rangle,\alpha)$ are unitarizable spherical representations, we have $S \subseteq \mathrm{Irr}^{su}$.

3.2. THEOREM. — *We have* $S = \mathrm{Irr}^{su}$.

Proof. — We need to prove that $\mathrm{Irr}^{su} \subseteq S$. Let $\pi \in \mathrm{Irr}^{su}$. Then $\pi = \pi(m(a))$ for some $a \in M(\tilde{G}_1^s)$. Proposition 2.1 and the fact that π is a Hermitian representation imply that

$$(*) \quad \pi = \pi(m(a)) = (\nu^{\alpha_1}\Delta[n_1]^{(\chi_1)} \times \nu^{-\alpha_1}\Delta[n_1]^{(\chi_1)}) \times \cdots \\ \cdots \times (\nu^{\alpha_k}\Delta[n_k]^{(\chi_k)} \times \nu^{-\alpha_k}\Delta[n_k]^{(\chi_k)}) \\ \times \Delta[m_1]^{(\mu_1)} \times \cdots \times \Delta[m_i]^{(\mu_i)}$$

where $\alpha_j > 0$, n_j, $m_j \in \mathbf{N}$, χ_j, $\mu_j \in \hat{G}_1^s$.

The theorem will be proved if we show that all $\alpha_j < 1/2$.

Let $\alpha_j \geqslant 1/2$ for some j. Now $\alpha_j \neq 1/2$ since (*) is irreducible. Let

$\alpha_j = t + \beta$ for some $t \in (1/2)\mathbf{Z}$ and $0 \leqslant \beta < 1/2$. Now

$$(**) \quad \pi \times \nu^{\beta}\Delta[2t+n_j-2]^{(\alpha_j)} \times \nu^{-\beta}\Delta[2t+n_j-2]^{(\alpha_j)} \in \mathrm{Irr}^{su}$$

by Proposition 3.1. But (**) reduces, since $\nu^{\beta}\Delta[2t+n_j-2]^{(\alpha_j)}$ and $\nu^{\alpha_j}\Delta[n_j]^{(\alpha_j)}$ are linked segments. We have obtained a contradiction and this proves the theorem.

BIBLIOGRAPHY

[1] I. N. Bernstein, P-invariant distributions on GL(N) and the classification of unitary representations of GL(N) (non-archimedean case), in Lie Group Representations II, Proceedings, University of Maryland 1982-1983, *Lecture Notes in Math.*, vol. 1041, Springer-Verlag, Berlin, (1983), 50-102.

[2] P. Cartier, Representations of p-adic groups : a survey, in *Proc. Sympos. Pure Math.* Vol. XXXIII, part 1, *Amer. Math. Soc.*, Providence, R.I., 1979, 111-155.

[3] W. Casselman, The unramified principal series of p-adic groups I, The spherical functions, *Comp. Math.*, vol. 41 (1980), 387-406.

[4] J. Dieudonné, *Treatise on analysis*, vol. VI, Academic Press, New York, 1978.

[5] I. M. Gelfand, M. I. Graev, Representations of a group of the second order with elements from a locally compact field, *Russian Math. Surveys*, 18 (1963), 29-100.

[6] I. G. Macdonald, Spherical functions on a group of p-adic type, Rammanjan Institute, *Univ. of Madras Publ.* (1971).

[7] F. I. Mautner, Spherical functions over p-adic fields I, II, *Amer. J. Math.*, vol. 80 (1958), 441-457 and vol. 86 (1964), 171-200.

[8] G. I. Olshansky, Intertwining operators and complementary series in the class of representations of the general group of matrices over a locally compact division algebra, induced from parabolic subgroups, *Math. Sb.*, vol. 93, no. 2 (1974), 218-253.

[9] I. Satake, Theory of spherical functions on reductive algebraic groups over p-adic fields, *Inst. Hautes Études Sci. Publ. Math.*, 18 (1963), 1-69.

[10] M. Tadić, Classification of unitary representations in irreducible representations of general linear group (non-archimedean case), to appear in *Ann. Scient. École Norm. Sup.*

[11] A. V. Zelevinsky, Induced representations of reductive p-adic groups II, *Ann. Scient. École Norm. Sup.*, 13 (1980), 165-210.

Manuscrit reçu le 22 avril 1985.

Marko Tadic,
Department of Mathematics
University of Zagreb
41001 Zagreb (Yugoslavia).