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ON FINITELY GENERATED CLOSED IDEALS IN $H^\infty(D)$

by Jean BOURGAIN

1. Introduction.

Let $H^\infty = H^\infty(D)$ be the Banach algebra of bounded analytic functions on the open disc $D = \{z \in \mathbf{C}; |z| < 1\}$, endowed with the sup norm $\|f\| = \sup_{z \in D} |f(z)|$. The result presented in this paper may as well be formulated for H^∞ on the upper half plane $\mathbf{R}^2_+ = \{x+iy;\ x \in \mathbf{R}, y \in \mathbf{R}_+\}$.

If $f_1, \ldots, f_N$ is a finite sequence in H^∞, we denote $I(f_1, \ldots, f_N)$ the ideal generated by $\{f_1, \ldots, f_N\}$, thus

$$I(f_1, \ldots, f_N\} = \left\{ \sum_{1 \leqslant j \leqslant N} f_j g_j \,;\, g_j \in H^\infty \right\}.$$

The solution to the corona problem for $H^\infty(D)$ (see [1], [4]) states that the 1-function 1 belongs to $I(f_1, \ldots, f_N)$ if

$$\inf_{z \in D} \max_{1 \leqslant j \leqslant N} |f_j(z)| > 0 .$$

There are by now several methods to prove this result. The reader may consult [4] for a systematic exposition. The simple approach due to T. Wolff permits to obtain certain extensions. For instance, if $\{f_j\}_{1 \leqslant j \leqslant N}$ and f in H^∞ satisfy

$$(1) \qquad |f(z)| \leqslant C \max_{1 \leqslant j \leqslant N} |f_j(z)|$$

for all $z \in D$ and a constant C, then f^3 belongs to $I(f_1, \ldots, f_N)$. An

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example of Rao, which we improve here, shows that (1) does not necessarily implies $f \in I(f_1, \ldots, f_N)$ while for f^2 the question seems unsolved at the time of this writing (see again [4] for details).

In the spirit of Rao's example, consider Blaschke product B_1, B_2 with disjoint zero sets such that however

$$\inf_{z \in B} (|B_1(z)| + |B_2(z)|) = 0. \tag{2}$$

Fix an integer $r \geqslant 2$ and let $f_1 = B_1^r$, $f_2 = B_2^r$ and $f = B_1^{r-1} B_2^{r-1}$. Then

$$|f(z)| \leqslant \alpha(|f_1(z)| + |f_2(z)|) \quad \text{for} \quad z \in B, \qquad \text{defining} \quad \alpha(t) = t^{2\left(1-\frac{1}{r}\right)}.$$

We claim that f does not belong to $I(f_1, f_2)$. Otherwise indeed, one should have

$$(B_1 B_2)^{r-1} = g_1 B_1^r + g_2 B_2^r; \qquad g_1, g_2 \in H^\infty$$

hence, since B_1 and B_2 do not have common zero's

$$g_1 = h_1 B_2^{r-1}, \qquad g_2 = h_2 B_1^{r-1}; \qquad h_1, h_2 \in H^\infty$$
$$1 \equiv h_1 B_1 + h_2 B_2$$

which is contradicted by (2).

Our purpose is to prove following property, implying in particular that in previous example f is in the closure of $I(f_1, f_2)$ for $r \geqslant 3$ (actually already for $r = 2$ if we assume that the zero-set of either B_1 or B_2 forms an interpolating sequence).

THEOREM. — *Let* $\{f_j\}_{1 \leqslant j \leqslant N}$ *and* f *be* $H^\infty(D)$-*functions. Assume the existence of a function* α *on* $\mathbf{R}_+$, $\lim\limits_{t \to 0} \dfrac{\alpha(t)}{t} = 0$ *such that*

$$|f(z)| \leqslant \alpha(|f_1(z)| + \cdots |f_N(z)|) \qquad \textit{for} \qquad z \in D. \tag{3}$$

Then f *belongs to the norm closure of the ideal* $I(f_1, \ldots, f_N)$.

Notice that the construction described above gives an example of a non-closed ideal $I(f_1, f_2)$.

It will be improved later in order to show that condition (3) taking $\alpha(t) = C.t$ does not imply the conclusion of the theorem. This second example will require some new ideas, such as the use of the Douglas-Rudin approximation result.

Recall that a positive measure μ on D is a Carleson-measure provided there is a constant C satisfying

$$\mu(R(I)) \leqslant C|I| \tag{4}$$

whenever I is an interval in $\Pi = \partial D = \{z \in \mathbf{C};|z|=1\}$, denoting $|I|$ the length of I and R(I) the region

$$\left\{z \in D; \frac{z}{|z|} \in I \text{ and } |z| \geqslant 1-|I|\right\}.$$

Let $\|\mu\|_C$ stand for the smallest constant verifying (4).

Carleson measures will be exploited here in connection with following result due to L. Carleson on existence of bounded solution to the $\bar\partial$-equation.

PROPOSITION 1. — *If* μ *is a Carleson-measure on* D, *then the equation*

$$\bar\partial F = \mu, \qquad \bar\partial = \partial/\partial x + i(\partial/\partial y)$$

has a solution on D *satisfying* $\|F\|_{L^\infty(\Pi)} \leqslant C_1\|\mu\|_C$, *where* C_1 *is numerical.*

Of course, in proving the theorem, we will make use of the Koszul-complex and $\bar\partial$-correction. The following two facts explain how Carleson measures may arise.

The first is due to J. Garnett [4] (in a weaker form) and B. Dahlberg [3]. The second, I think, is new and improves on the « contour » construction in Carleson original solution to the corona-problem.

PROPOSITION 2. — *Let* u *be a* 1*-bounded harmonic function on* D. *For each* $\varepsilon > 0$, *there exists a* $\mathscr{C}^\infty$*-function* $v = v_\varepsilon$ *on* D *satisfying*

$$|v-u| < \varepsilon \quad \text{on D} \tag{5}$$

$$\|\,|\nabla u|\,dx\,dy\|_C < \frac{C_2}{\varepsilon}. \tag{6}$$

PROPOSITION 3. — *Assume* B *a Blaschke product on* D. *Given* $\varepsilon > 0$, *there exists an open set* R *in* D *such that* ∂R *is a rectifiable curve and*

$$|B(z)| < \varepsilon \quad \text{if} \quad z \in R$$

$$|B(z)| > \delta(\varepsilon) \quad \text{if} \quad z \in D\backslash R$$

$$\|\lambda_{\partial R}\|_C < C_3$$

where $\delta(\varepsilon)$ *is a fixed function of* ε *(independent of* B) *and* $\lambda_{\partial R}$ *refers to the arc-length-measure of the boundary of* R.

This result should be compared with [4], p. 342. The new feature is the estimate on $\|\lambda_{\partial R}\|_C$ independent of ε.

In the next section, the theorem will be deduced from Props. 1, 2 and 3. In section 3 we elaborate the example which was announced earlier. Section 4 of the paper is devoted to the proof of Prop. 3.

The author is indebted to T. Wolff and J. Garnett for some valuable discussions.

2. Proof of the theorem.

Assume $\{f_j\}_{j=1}^N$ and f in H^∞ satisfying $|f(z)| \leqslant \alpha\left(\sum_{1\leqslant j\leqslant N} |f_j(z)|\right)$ for $z \in D$, where $\alpha : \mathbf{R}_+ \to \mathbf{R}_+$ fulfils $\lim_{t\to 0} \frac{\alpha(t)}{t} = 0$. To be more explicit, we give the argument for $N = 2$, the general case being completely analogue. We use the letter C to denote numerical constants. It follows from our hypothesis that in particular

$$|f(z)| \leqslant C(|f_1(z)| + |f_2(z)|), \qquad z \in D$$

for some constant C. Without restriction, we may take $\|f\|_\infty$, $\|f_1\|_\infty$ and $\|f_2\|_\infty < 1$.

Writing $f \in H^\infty$ as product $f = B.F$ where B is a Blaschke product and F a zero-free function on $D(|F| \leqslant 1)$, the approximation argument given in [2] (p. 204, Lemma 3) permits to formulate Prop. 3 for functions in the unit-ball of H^∞ as well.

Fix $\varepsilon > 0$ and apply Prop. 3 to our function f, providing the region R with boundary $\Gamma = \partial R$ satisfying the estimate on arc-length.

Let $\tau > 0$ to be defined later and apply Prop. 2 to each of the functions f_1, f_2, giving C^∞-functions v_1, v_2 on D such that

$$(7) \quad |f_1(z) - v_1(z)| < \tau, \qquad |f_2(z) - v_2(z)| < \tau, \qquad (z \in D)$$

$$(8) \quad \| |\nabla v_1|\, dx\, dy \|_C < C\tau^{-1}; \qquad \| |\nabla v_2|\, dx\, dy \|_C < C\tau^{-1}.$$

If we define for $j = 1,2$

$$g_j = \bar{v}_j (f_1 \bar{v}_1 + f_2 \bar{v}_2)^{-1} \chi_{D\setminus R}$$

then

(9) $1-(g_1f_1+g_2f_2)=\chi_R$ hence $f_1\,\bar\partial g_1+f_2\,\bar\partial g_2=-\bar\partial\chi_R$.

Consider solutions a_{12}, a_{21} and b_1, b_2 of the respective $\bar\partial$-equations

$$\begin{cases}\bar\partial a_{12}=fg_1\,\bar\partial g_2\\ \bar\partial a_{21}=fg_2\,\bar\partial g_1\end{cases}\tag{10}$$

and

$$\bar\partial b_j=f\bar v_j(f_1\bar v_1+f_2\bar v_2)^{-1}\,\bar\partial\chi_R\qquad (j=1,2)\tag{11}$$

with L^∞-norm control on ∂D. This will be realized with Prop. 1. Postponing the estimations on Carleson-norm, put

$$\begin{cases}h_1=fg_1+(a_{12}-a_{21})f_2+b_1\\ h_2=fg_2+(a_{21}-a_{12})f_1+b_2.\end{cases}\tag{12}$$

By construction and (9)

$$f-(h_1f_1+h_2f_2)=f\chi_R-b_1f_1-b_2f_2$$

$$\|f-(h_1f_1+h_2f_2)\|_{L^\infty(\Pi)}\leqslant\varepsilon+\|b_1\|_{L^\infty(\Pi)}+\|b_2\|_{L^\infty(\Pi)}.\tag{13}$$

Next, we verify that h_j $(j=1,2)$ are analytic. From (13) will then follow an estimation on $\|f-(h_1f_1+h_2f_2)\|_{L^\infty(D)}$. Consider h_1 for instance. By (9), (10), (11)

$$\begin{aligned}\bar\partial h_1&=f\,\bar\partial g_1+f(g_1\,\bar\partial g_2-g_2\,\bar\partial g_1)f_2+\bar\partial b_1\\&=f(1-g_2f_2)\,\bar\partial g_1-fg_1(f_1\,\bar\partial g_1+\bar\partial\chi_R)+\bar\partial b_1\\&=-f\bar v_1(f_1\bar v_1+f_2\bar v_2)^{-1}\,\bar\partial\chi_R+\bar\partial b_1=0.\end{aligned}$$

We now turn our attention to eqs. (10), (11). Choose $\tau<\frac{1}{10}\delta(\varepsilon)^2$ where $\delta(\varepsilon)$ refers to Prop. 3. Since by (7)

$$|fg_j|\leqslant C(|f_1|+|f_2|)\,(|f_j|+\tau)(|f_1|^2+|f_2|^2-2\tau)^{-1}\,\chi_{D\setminus R}$$

and

$$\delta(\varepsilon)\leqslant|f|\leqslant C(|f_1|+|f_2|)\quad\text{outside the region } R$$

it follows thus

$$\|fg_j\|_{L^\infty(D)}\leqslant C,\qquad (j=1,2).\tag{14}$$

We have, letting $\partial = \partial/\partial x - i(\partial/\partial y)$

$$\bar\partial g_j = \overline{\partial v_j}(f_1\bar v_1+f_2\bar v_2)^{-1}\chi_{D\setminus R} + \bar v_j(f_1\bar v_1+f_2\bar v_2)^{-2}(f_1\,\overline{\partial v_1}+f_2\,\overline{\partial v_2})\chi_{D\setminus R} - \bar v_j(f_1\bar v_1+f_2\bar v_2)^{-1}\,\bar\partial\chi_R$$

implying by (14) for $j = 1,2$

$$\|fg_j\,\bar\partial g_{3-j}\|_C \leqslant \delta^{-2}\|\{|\nabla v_1|+|\nabla v_2|\}\,dx\,dy\|_C + \delta^{-1}\|\lambda_\Gamma\|_C\,.$$

From (8) and Prop. 1, we may conclude the existence of solutions a_{12}, a_{21} to (10) satisfying

$$(15)\qquad \|a_{j,\,3-j}\|_{L^\infty(\partial D)} \leqslant C\tau^{-1}\,\delta^{-2}\,.$$

Remains to analyze the right members in eqs. (11). Since $\bar\partial\chi_R$ is supported by Γ and on Γ

$$f\bar v_j(f_1\bar v_1+f_2\bar v_2)^{-1} \leqslant 2\{\varepsilon\wedge\alpha(|f_1|+|f_2|)\}(|f_j|+\tau)(|f_1|^2+|f_2|^2)^{-1} < 2\sup\left\{\frac{\alpha(t)}{t}\wedge\varepsilon\right\} + C\tau$$

the hypothesis $\lim\limits_{t\to 0}\frac{\alpha(t)}{t} = 0$ and numerical estimate $\|\lambda_\Gamma\|_C \leqslant$ const., imply

$$\|f\bar v_j(f_1\bar v_1+f_2\bar v_2)^{-1}\,\bar\partial\chi_R\|_C \to 0 \qquad \text{taking} \qquad \varepsilon\to 0\,.$$

Again by Prop. 1, this allows us to obtain b_j satisfying (11) and $\|b_j\|_{L^\infty(\partial D)}$ as small as desired. In particular, we proved, see (14), (15),

$$h_j\in H^\infty(D)\,;\qquad \|h_j\|_\infty < C\,\delta(\varepsilon)^{-4}\,,\qquad (j=1,2)\,.$$

As a consequence of (13), $\|f-h_1f_1-h_2f_2\|_\infty$ can be made arbitrarily small. Hence $\operatorname{dist}_{H^\infty}(f, I(f_1,f_2)) = 0$, completing the proof.

Remark. — Estimations appearing above are not best possible. We did not attempt to do so in order to avoid unnecessary complications.

3. An example.

The purpose of this section is to improve Rao's example of Blaschke products B_1, B_2 on D such that $B_1.B_2$ does not belong to the ideal

$I(B_1^2, B_2^2)$. In this construction, $f = B_1 . B_2$ will not be in the uniform closure of $I(B_1^2, B_2^2)$. The argument is particularly simple. Let us recall the classical Douglas-Rudin theorem asserting that any unimodular function σ, $|\sigma| = 1$, on Π can be uniformly approximated by a quotient of Blaschke products [4], thus there are Blaschke products B_1, B_2 s.t.

$$|\sigma(\theta) - B_1^\varepsilon(e^{i\theta})\overline{B_2^\varepsilon(e^{i\theta})}| < \varepsilon . \tag{16}$$

Apply this property to the function $\sigma(\theta) = \operatorname{sign} \sin \theta$. Assume g_1, $g_2 \in H^\infty(D)$ such that for some $\gamma > 0$

$$|B_1 B_2 - g_1 B_1^2 - g_2 B_2^2| < 1 - \gamma \tag{17}$$

assuming $\varepsilon \ll \gamma$. Let for $0 < r < 1$ the conjugate Poisson kernel $2r \sin \theta[1 - 2r \cos \theta + r^2]^{-1}$ be denoted by $Q_r(\theta)$. Recall that if $g \in H^\infty$, then

$$\frac{1}{2\pi} \int_0^{2\pi} g(e^{i\theta}) Q_r(\theta)\, d\theta = (\mathscr{H} g)(r) = -\, i[g(r) - g(0)]$$

hence

$$\left| \int_\Pi g . Q_r \right| < 2\|g\|_\infty .$$

Restricting (17) to Π implies by (16)

$$|\sigma - g_1 - g_2| < 1 - \gamma + \varepsilon + 2\|g_1\|_\infty \varepsilon .$$

Therefore, clearly

$$\|Q_r\|_1 = \left| \int_\Pi Q_r \sigma \right| \leqslant (1 - \gamma + \varepsilon + 2\|g_1\|_\infty \varepsilon)\|Q_r\|_1 + 2(\|g_1\|_\infty + \|g_2\|_\infty)$$

and consequently

$$\|g_1\|_\infty + \|g_2\|_\infty \geqslant \frac{\gamma}{10} \min\{\varepsilon^{-1}, \log(1 - r)^{-1}\} .$$

Letting $r \underset{<}{\to} 1$, this implies the lower bound

$$\|g_1\|_\infty + \|g_2\|_\infty \geqslant \frac{\gamma}{10\varepsilon}$$

on any pair of H^∞-functions satisfying (17).

From this observation and a compactness-argument, whose details are left to the reader, one can produce a pair B_1, B_2 of Blaschke products such that $\operatorname{dist}_{H^\infty}(B_1B_2, I(B_1^2, B_2^2)) = 1$, In fact, the proof of the Douglas-Rudin theorem given by P. Jones (see [4], p. 428) permits to construct B_1, B_2 satisfying

$$\lim_{|\theta|\to 0} |B_1(e^{i\theta})\overline{B_2(e^{i\theta})} - \operatorname{sign}\sin\theta| = 0$$

which will have the desired property as a consequence of previous reasoning.

4. Proof of proposition 3.

The argument does not require preliminaries from potential theory, such as harmonic measures and Hall's lemma. Define $\rho(a,b) = \left|\frac{a-b}{1-\bar{a}b}\right|$ for $a,b \in D$, i.e. the pseudo-hyperbolic distance. Following lemma is well-known.

LEMMA 1. – *Let* $B(z) = \Pi \frac{z-a_n}{1-\bar{a}_n z}$ *be a (finite) Blaschke product and* $\varepsilon > 0$. *Then*

$$\frac{1}{2}\Sigma \frac{(1-|z|^2)(1-|a_n|^2)}{|1-\bar{a}_n z|^2} \leqslant -\log|B(z)| \leqslant 2\log\frac{1}{\varepsilon}\Sigma\frac{(1-|z|^2)(1-|a_n|^2)}{|1-\bar{a}_n z|^2}$$

provided $\inf_n \rho(z,a_n) > \varepsilon$.

Proof. – $\left|\frac{z-a_n}{1-\bar{a}_n z}\right|^2 = 1 - \frac{(1-|z|^2)(1-|a_n|^2)}{|1-\bar{a}_n z|^2}$ and

$$t \leqslant -\log(1-t) \leqslant 2\left(\log\frac{1}{\varepsilon}\right)t \quad \text{if} \quad 0 \leqslant t \leqslant 1-\varepsilon. \qquad \square$$

If I is an interval in Π, denote $\tilde{I}$ the interval with same midpoint as I and $|\tilde{I}| = 2|I|$.

LEMME 2. – *Let* $a \in D$ *and* $|B(a)| > \varepsilon > 0$. *Let* I *be an interval in* Π *such that* $\frac{1}{2}|I| < 1 - |a| < |I|$ *and* $\frac{a}{|a|} \in I$, *i.e.* a *belongs to the upperhalf of* $R(I)$.

Given $M < \infty$ *there exists a collection* $\{I_k\}$ *of disjoint subintervals of* $\tilde{I}$ *satisfying*

(18) $$\Sigma|I_k| \leqslant C_4\left(\log\frac{1}{\varepsilon}\right)M^{-1}|I|.$$

(19) If $z \in R(I)\setminus \bigcup R(I_k)$ *and* $\inf_n \rho(z,a_n) > \gamma$,

then

$$\log|B(z)|^{-1} \leqslant C_4\left(\log\frac{1}{\gamma}\right)\left(M+\log\frac{1}{\varepsilon}\right).$$

Proof. – Let $\{I_k\}$ be the collection of maximal subintervals of $\tilde{I}$, obtained by diadic splitting of $\tilde{I}$, satisfying

$$\sum_{a_n \in R(I_k)} (1-|a_n|) > M|I_k|.$$

By lemma 1

$$M\Sigma|I_k| \leqslant \sum_{a_n\in R(I)}(1-|a_n|) \leqslant C|I|\Sigma\frac{(1-|a|^2)(1-|a_n|^2)}{|1-\bar{a}.a_n|^2} \leqslant C|I|\log\frac{1}{|B(a)|}$$

implying (18).

Let now $z \in R(I)\setminus\bigcup R(I_k)$. Let $\{J_s\}_{1\leqslant s\leqslant t}$ be the increasing sequence of diadic subintervals of $\tilde{I}$ containing z, where

$$1-|z| < |J_1| < 2(1-|z|) \qquad \text{and} \qquad J_t = \tilde{I}.$$

By construction, for $s = 1, \dots, t$

$$\sum_{a_n\in R(J_s)}(1-|a_n|) < M|J_s|$$

and hence, as easily verified

(20) $$\sum_{a_n\in R(\tilde{I})}\frac{(1-|z|^2)(1-|a_n|^2)}{|1-\bar{a}_n z|^2} \leqslant \sum_{a_n\in R(J_2)} + \sum_{s=2}^{t}\ \sum_{a_n\in R(J_{s+1})\setminus R(J_s)} \leqslant CM + \sum_{s=2}^{t} CM\,2^{-s} = CM.$$

If $a_n \notin R(\tilde{I})$, then clearly

$$\frac{(1-|z|^2)(1-|a_n|^2)}{|1-\bar{a}_n z|^2} \leqslant C\frac{(1-|a|^2)(1-|a_n|^2)}{|1-\bar{a}_n z|^2}$$

and so, by lemma 1 and hypothesis

$$(21)\qquad \sum_{a_n \notin R(\tilde{I})} \frac{(1-|z|^2)(1-|a_n|^2)}{|1-\bar{a}_n z|^2} \leqslant C \log \frac{1}{\varepsilon}.$$

Adding (20), (21) and using again Lemma 1 gives (19).

LEMMA 3. — *In the situation of Lemma 2 and given $\gamma > 0$, there is a set $O(I)$ in $R(I)$ with rectifiable boundary $\Gamma(I)$ such that*

$$(22)\qquad \|\lambda_{\Gamma(I)}\|_C \leqslant C_5 \gamma \left[M + \log \frac{1}{\varepsilon} \right]$$

$$(23)\qquad |B(z)| < \gamma \quad \text{if} \quad z \in O(I)$$

$$(24)\qquad |B(z)| > \exp\left(- C_5\left(\log \frac{1}{\gamma}\right)\left(M + \log \frac{1}{\varepsilon}\right)\right)$$

if

$$z \in R(I) \backslash \{O(I) \cup \bigcup R(I_k)\}.$$

Proof. — Define $O(I) = [R(I) \setminus \bigcup R(I_k)] \cap \bigcup_{a_n \in R(\tilde{I})} [\rho(z,a_n) \leqslant \gamma]$. Then (23) is trivial and (24) follows from (19). Also, again

$$(25)\qquad |\Gamma| \leqslant C\gamma \sum_{a_n \in R(\tilde{I})} (1-|a_n|) \leqslant C\gamma |I| \Sigma \frac{(1-|a|^2)(1-|a_n|^2)}{|1-\bar{a}_n a|^2}$$
$$\leqslant C\gamma\left(\log \frac{1}{\varepsilon}\right)|I|.$$

Suppose now J is a diadic subinterval of $\tilde{I}$ contained in I. If J is contained in $\bigcup I_k$, then $\Gamma \cap R(J) = \varnothing$. Otherwise, by construction of the I_k-intervals

$$\sum_{a_n \in R(\tilde{J})} (1-|a_n|) \leqslant 2M|J|.$$

Therefore

$$(26)\qquad |\Gamma \cap R(J)| \leqslant \sum_{[\rho(z,a_n)\leqslant\gamma]\cap R(J) \neq \varnothing} C(1-|a_n|) \leqslant C\gamma \sum_{a_n \in R(\tilde{J})} (1-|a_n|)$$
$$\leqslant C\gamma M|J|$$

and (22) as consequence of (25), (26).

The proof of Proposition 3 follows at this point by a standard reasoning of constructing successive generations of intervals in Π. Take $M = 100\, C_4 \log \frac{1}{\varepsilon}$, $\gamma^{-1} = 200\, C_4 C_5 \log \frac{1}{\varepsilon}$ and

$$\delta(\varepsilon) = \exp\left(-2C_5 M \log \frac{1}{\gamma}\right) = \exp\left(-C \log \frac{1}{\varepsilon}\left(\log \log \frac{1}{\varepsilon}\right)\right).$$

Start with $\mathscr{F}^{(1)} = \{\Pi\}$. Assume now the generation $\mathscr{F}^{(s)}$ obtained and J an interval in $\mathscr{F}^{(s)}$. Let $\mathscr{D}(J)$ be the maximal diadic subintervals I of J such that the upperhalf of R(I) contains a point a where $|B(a)| > \varepsilon$. Hence

$$|B| \leqslant \varepsilon \qquad \text{on} \qquad R(J) \setminus \bigcup_{I \in \mathscr{D}(J)} R(I).$$

To each $I \in \mathscr{D}(J)$, we apply lemma's 2 and 3, providing intervals $\{I_k\}$ and a region O(I) of R(I). Define

$$\mathscr{F}^{(s+1)} = \bigcup_{J \in \mathscr{F}^{(s)}} \bigcup_{I \in \mathscr{D}(J)} \{I_k ; k=1,2,\ldots\}$$

where for each $I \in \mathscr{D}(J)$, by construction

$$\Sigma |I_k| < \frac{1}{100} |I| . \tag{27}$$

Let $R_s(J)$ be the union of $R(J) \setminus \bigcup_{I \in \mathscr{D}(J)} R(I)$ and $\bigcup_{I \in \mathscr{D}(J)} O(I)$. Take $R_s = \bigcup_{J \in \mathscr{F}^{(s)}} R_s(J)$. Thus, by (23) of Lemma 3, $|B| \leqslant \varepsilon$ on $R_s(J)$, hence on R_s.

Also, by (24) of Lemma 3

$$|B(z)| > \delta(\varepsilon) \quad \text{if} \quad z \in \bigcup_{J \in \mathscr{F}^{(s)}} R(J) \setminus \left(R_s \cup \bigcup_{J \in \mathscr{F}^{(s+1)}} R(J)\right).$$

Define $R = \bigcup R_s$. Our construction yields $|B| \leqslant \varepsilon$ on R and $|B| \geqslant \delta(\varepsilon)$ on $D \setminus R$.

As a consequence of (22) and (27) and the choice of γ,

$$\|\lambda_{\partial R}\|_C \leqslant 10$$

as the reader will easily check. This completes the proof.

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