ANNALES DE L'INSTITUT FOURIER

MICHEL TALAGRAND

Choquet simplexes whose set of extreme points is K-analytic

Annales de l'institut Fourier, tome 35, nº 3 (1985), p. 195-206

<http://www.numdam.org/item?id=AIF_1985__35_3_195_0>

Ann. Inst. Fourier, Grenoble
35, 3 (1985), 195-206

CHOQUET SIMPLEXES WHOSE SET OF EXTREME POINTS IS $\mathcal{K}$-ANALYTIC

Dedicated to Professor G. Choquet, on his 70th birthday

by Michel TALAGRAND

Introduction.

When the author started research in mathematics, he asked his advisor Professor Choquet a list of problems. This list consisted of ten problems. On nine of them the author could make no progress. The tenth was : If the set of extreme points of a convex compact set is $\mathcal{K}$-analytic, must it be a $K_{\sigma\delta}$ set (or more generally, a $\mathcal{K}$-Borel set) ? Let us recall that a subset T of a compact set K is called $\mathcal{K}$-analytic if it is the image of the irrationals under an upper continuous compact-valued map [1], [2]. The classes $\mathcal{K}_\alpha$ of $\mathcal{K}$-Borel sets of K are defined by induction over the ordinal α in the following way. $\mathcal{K}_0(K)$ is the class of compact sets. When α is even (resp. odd) $\mathcal{K}_{\alpha+1}(K)$ consists of the countable intersections (resp. unions) of sets of $\mathcal{K}_\alpha(K)$. Finally, if α is limit, $\mathcal{K}_\alpha(K)$ is the union of the classes $\mathcal{K}_\beta(K)$ for $\beta < \alpha$. A subset of K is called $\mathcal{K}$-Borel if it belongs to some class $\mathcal{K}_\alpha$. A $\mathcal{K}$-Borel set is $\mathcal{K}$-analytic.

It has been known for some time that the set of extreme points $\mathcal{E}$ of a convex compact set K has a lot of structure. It is known that $\mathcal{E}$ can be topologically very irregular [5], [6]. However, if one assumes some regularity for $\mathcal{E}$, then $\mathcal{E}$ often turns out to be very regular. Along this line R. Haydon showed that if E is a continuous image of a separable metric space, then K is metrizable, so E is actually a G_δ set [3]. See also [4]. The hypothesis that $\mathcal{E}$ is a continuous image of a separable metric space is an hypothesis of smallness as well as of regularity, so it is of a fairly different nature than the hypothesis that $\mathcal{E}$ is $\mathcal{K}$-analytic.

Mots-clés : Choquet simplex – $\mathcal{K}$-analytic – Extreme point.

In [8], the author showed that when $\mathcal{E}$ is $\mathcal{K}$-analytic, it can be written as $\mathcal{E} = \bigcap_n (U_n \cup F_n)$, where U_n is open in $\mathcal{E}$ and F_n is closed. So when $\mathcal{E}$ is $\mathcal{K}$-analytic, it must be Borel of a very special type. So, the problem of Choquet is connected to the following question asked by Goullet de Rugy : If a subset X of a compact set is at the same time $\mathcal{K}$-analytic and Borel, must it be $\mathcal{K}$-Borel ? The answer is yes when X is open, since then it is a K_σ set.

A seemingly unrelated question is the following question ([2], 10-7, 10-8). If a topological space X is a G_δ set in its Stone-Cech compactification it is a G_δ set in every compactification. But if X is a $K_{\sigma\delta}$ set in some compactification, is it a $K_{\sigma\delta}$ set in every compactification ? (If X is a $K_{\sigma\delta}$ set in each compactification, it is called an absolute $K_{\sigma\delta}$ set). Our main construction will answer these questions.

THEOREM A. – *There exists a Choquet simplex* K *with the following properties :*

1) *The set of extreme points* $\mathcal{E}$ *of* K *is* $\mathcal{K}$*-analytic.*

2) $\mathcal{E}$ *is not* $\mathcal{K}$*-Borel in* $\bar{\mathcal{E}}$*.*

3) $\mathcal{E}$ *is a* $K_{\sigma\delta}$ *set in its Stone-Cech compactification.*

4) *There is an open set* U *of* $\bar{\mathcal{E}}$ *and a point* ω *of* $\bar{\mathcal{E}}$ *such that* $\mathcal{E} = \{\omega\} \cup U$.

5) $\bar{\mathcal{E}} \setminus \mathcal{E}$ *is discrete.*

So our construction provides a negative answer to the problems of Choquet and Goullet de Rugy, as well as an example of a $K_{\sigma\delta}$ set that is not absolute.

2. Construction.

The construction will use ideas from [7]. Let $\mathcal{A}$ be a family of subsets of $\mathbf{N}^{\mathbf{N}}$ that are closed and discrete for the usual topology. Let ω be a point which does not belong to $\mathbf{N}^{\mathbf{N}}$, and let $T = \{\omega\} \cup \mathbf{N}^{\mathbf{N}}$. We provide T with the topology that makes each point of $\mathbf{N}^{\mathbf{N}}$ open, and such that the neighborhoods of ω are the sets of the type $T \setminus B$, where B is the union of a finite set and finitely many elements of A. Then T is completely regular and $T \setminus \{\omega\}$ is open in any compactification of T.

Let us fix some notations, that we will use through this paper. Given a finite sequence s of integers, let $|s|$ be its length, and let A_s be the subset of $\mathbf{N}^{\mathbf{N}}$ of sequences such that their $|s|$ first terms coincide with those of s.

Denote by S the Stone-Cech compactification of T. We show that, (independently of the choice of $\mathfrak{A}$), T is a $K_{\sigma\delta}$ set in S, and more precisely that

$$T = \{\omega\} \cup \bigcap_n \bigcup_{|s|=n} \overline{A_s}$$

where the closure is in S. This implies in particular that T is $\mathcal{K}$-analytic.

First, the inclusion of T in the right hand side is obvious, so we prove the reverse inclusion. Let $s \neq s'$ with $|s| = |s'|$. We show first that $\overline{A_s} \cap \overline{A_{s'}} = \{\omega\}$. If $t \in \overline{A_s} \cap \overline{A_{s'}} \setminus \{\omega\}$, then $t \in \overline{A_s} \cap \overline{A_{s'}} \setminus T$, and there is $B \in \mathfrak{A}$ with $t \in \overline{B \cap A_s}$, $t \in \overline{B \cap A_{s'}}$. But since B is discrete for the topology of T, and since $A_s \cap A_{s'} \cap B = \emptyset$ this is impossible. It follows that if

$$t \in \bigcap \bigcup_{|s|=n} \overline{A_s} \setminus \{\omega\}$$

then there exists $\sigma \in \mathbf{N}^{\mathbf{N}}$ such that for each n we have $t \in \overline{A_{\sigma|n}}$, where $\sigma|n$ denotes the sequence of the first n terms of σ. Since $t \neq \omega$, there is $B \in A$ such that $t \in \overline{B \cap A_{\sigma|n}}$ for each n. Since B is closed discrete in $\mathbf{N}^{\mathbf{N}}$, there is a neighborhood of σ for the usual topology in $\mathbf{N}^{\mathbf{N}}$ which meets B in a finite set, that is, there is n such that $B \cap A_{\sigma|n}$ is finite, so $t \in T$.

Given the family $\mathfrak{A}$, we denote by $X(\mathfrak{A})$ the compactification of T such that the closen sets of $X(\mathfrak{A})$ can be identified to the algebra generated by $\mathfrak{A}$ and the finite sets of $\mathbf{N}^{\mathbf{N}}$. The closure of the sets of extreme points of K will be identified to $X(\mathfrak{A})$ for a suitably chosen family $\mathfrak{A}$. Among other properties, $\mathfrak{A}$ must be chosen so that T is not a $\mathcal{K}$-Borel set of $X(\mathfrak{A})$. Let first describe a family $\mathfrak{A}$ such that T is not a K_σ set (this is the family used in [7]). Let

$$\mathfrak{A}_0 = \{B \subset \mathbf{N}^{\mathbf{N}}, \exists n, \forall \sigma, \rho \in B, \sigma|n = \rho|n, \sigma|n+1 \neq \rho|n+1\}.$$

Then each element of $\mathfrak{A}_0$ is closed and discrete. Suppose now that $T = \bigcup_n K_n$. Then there is n such that (for the usual topology),

$\overset{\circ}{\overline{K}}_n \neq \phi$; it is easily seen that this implies that there is an infinite $B \in \mathcal{A}_0$ with $B \subset K_n$. If $x \in \overline{B} \setminus B$, then $x \in \overline{K}_n \setminus T$, so T is not a K_σ set.

Let us now try to construct $\mathcal{A}_1$ such that T is not a $K_{\sigma\delta}$ in the corresponding compactification $X(\mathcal{A}_1)$. A natural idea is to use the family closed and discrete

$$\mathcal{A}_1 = \{B \subset \mathbf{N}^{\mathbf{N}}, B = \bigcup_n B_n, \forall n, B_n \in \mathcal{A}_0, \forall \sigma \in B_n, \sigma(1) = n\}.$$

Suppose that we have $T \subset \bigcap_n \bigcup_q K_{qn}$ where K_{qn} is a compact subset of $X(\mathcal{A}_1)$. Let

$$A_n = \{\sigma \in \mathbf{N}^{\mathbf{N}}; \sigma(1) = n\}.$$

For each n, there is q_n such that the closure of $A_n \cap K_{q_n, n}$ has non-empty interior (for the usual topology). So there is $B_n \subset \overline{K}_{q_n, n}$ with $B_n \in \mathcal{A}_0$, $B_n \subset A_n$. It follows that $\bigcap_n \overline{B}_n \subset \bigcap_n \bigcup_q K_{qn}$. Unfortunately, the set $\bigcap \overline{B}_n$ is empty since for each p there is $C_p \in \mathcal{A}_1$ such that $C_p \cap (\bigcup_n B_n) = B_p$. We shall however be able to avoid this phenomenon by carefully restricting $\mathcal{A}_1$. Of course if we use for $\mathcal{A}$ a subfamily of $\underline{\mathcal{A}}_1$, T will be a $K_{\sigma\delta\sigma}$ of $X(\mathcal{A})$, so a construction of higher order is needed.

For two finite sequences $s = (s_1, \ldots, s_n)$, $t = (t_1, \ldots, t_m)$ let $s \frown t = (s_1, \ldots, s_n, t_1, \ldots, t_m)$. Suppose that for each $n \geqslant 1$ we are given a map ψ_n that associates a finite sequence $\psi_n(B_1, \ldots, B_n)$ to each n-uple $(B_1, \ldots, B_n)$ of countable sets of finite sequences. The specific choice of ψ_n will be described in section 3. By induction over the countable ordinal α, we construct families $\mathcal{B}_\alpha$ of countable sets of finite sequences, in the following manner. $\mathcal{B}_0$ consists of the sets containing one single finite sequence. If $\mathcal{B}_\beta$ has been constructed for $\beta < \alpha$, we define $\mathcal{B}_\alpha$ as the union of $\bigcup_{\beta<\alpha} \mathcal{B}_\beta$ and of the collection of the sets of type

$$B = \{u \frown (2n, 2n) \frown \psi_{n-1}(B_1, \ldots, B_{n-1}) \frown t; t \in B_n, n \geqslant 1\}$$

where u is a fixed finite sequence, and $(B_n)_{n\geqslant 1}$ is a sequence of $\bigcup_{\beta<\alpha} \mathcal{B}_\beta$. (For $n = 1$, $\psi_{n-1}(B_1, \ldots, B_{n-1})$ is defined as the empty sequence). We set $\mathcal{B} = \bigcup_\alpha \mathcal{B}_\alpha$.

Recall that a set is called of first category if it is contained in a countable union of closed sets of empty interior.

The motivation for this construction is the following :

LEMMA 1. – *Let* Z *be a* $\mathcal{K}$*-Borel set of* $X(\mathcal{C})$, *so, say,* $Z \in \mathcal{K}_\alpha(X(\mathcal{C}))$. *Let* t *be a finite sequence. Assume that for the usual topology of* $\mathbf{N}^{\mathbf{N}}$, $Z \cap A_t$ *is not of first category. Then there is* $B \in \mathcal{B}_\alpha$ *and a family* $(L_s)_{s \in B}$ *of compact sets of* $X(\mathcal{C})$, *with the following properties:*

1) $\bigcap_{s \in B} L_s \subset Z$.

2) *For each* $s \in B$, $L_s \cap A_{t \frown s}$ *is dense in* $A_{t \frown s}$ *for the usual topology.*

Proof. – It goes by induction over α. If $\alpha = 0$, Z is compact. The hypothesis implies that the closure of $Z \cap A_t$ has nonempty interior. So, there is a finite sequence s such that $Z \cap A_{t \frown s}$ is dense in $A_{t \frown s}$. We take $B = \{t \frown s\}$, $L_s = Z$.

Suppose now that the lemma has been proved for each $\beta < \alpha$. If α is limit, then $Z \in K_\beta(X(\mathcal{C}))$ for some $\beta < \alpha$ and there is nothing to prove. Suppose that $\alpha = \beta + 1$, where β is odd. Then $Z = \cup Z_n$, with $Z_n \in K_\beta(X(\mathcal{C}))$. Since there exists n such that $Z_n \cap A_t$ is not of first category for the usual topology, the conclusion follows by induction hypothesis. Suppose finally that $\alpha = \beta + 1$, where β is even, so $Z = \bigcap_{n \geq 1} Z_n$ where $Z_n \in \mathcal{K}_\beta(X(\mathcal{C}))$. Let u be a finite sequence such that (for the usual topology) Z is not of first category in any nonempty subset of $A_{t \frown u}$. By induction over n we construct sets $B_n \in \mathcal{B}_\beta$ and compact sets $(L_s^n)_{s \in B_n}$. Let $v_1 = t \frown u \frown (2,2)$. Then Z_1 is not of first category in A_{v_1}, so by induction hypothesis there exists $B_1 \in \mathcal{B}_\beta$ and a family $(M_s^1)_{s \in B_1}$ of compact subsets of $X(\mathcal{C})$ such that $\bigcap_{s \in B_1} M_s^1 \subset Z_1$ and for each $s \in B_1$, $M_s^1 \cap A_{v_1 \frown s}$ is dense in $A_{v_1 \frown s}$. Suppose now that $B_1, \ldots, B_{n-1}$ have been constructed. Let

$$v_n = t \frown u \frown (2n, 2n) \frown \psi_{n-1}(B_1, \ldots, B_{n-1}).$$

Then Z_n is not of first category in A_{v_n} so by induction hypothesis there exists $B_n \in \mathcal{B}_\beta$ and a family $(M_s^n)_{s \in B_n}$ of compact subsets of $X(\mathcal{C})$ such that $\bigcap_{s \in B_n} M_s^n \subset Z_n$ and that for each $s \in B_n, M_s^n \cap A_{v_n \frown s}$ is dense in $A_{v_n \frown s}$. This completes the construction of the B_n. By definition of $\mathcal{B}_\alpha$,

$$B = \bigcup_n \{u^\frown(2n, 2n)^\frown \psi_{n-1}(B_1, \ldots, B_{n-1})^\frown s ; s \in B_n\}$$

belongs to $\mathcal{B}_\alpha$. For $v \in B$, if v is of the type

$$u^\frown(2n, 2n)^\frown \psi_{n-1}(B_1, \ldots, B_{n-1})^\frown s, s \in B_n,$$

let $L_v = M_s^n$. Then, by construction, $A_{t \frown v} \cap L_v$ is dense in $A_{t \frown v}$. Moreover

$$\bigcap_{v \in B} L_v \subset \bigcap_n \bigcap_{s \in B_n} M_s^n \subset \bigcap Z_n \subset Z.$$

Remark. – We shall apply lemma 1 when t is the empty sequence.

Each element of $\mathcal{B}$ is countable. We fix an enumeration $(s_B^n)_n$ of each $B \in \mathcal{B}$. We also fix an enumeration $(\theta_1(n), \theta_2(n))$ of $\mathbf{N}^2$, where $\theta_1(n) \leqslant n$. Suppose that for each n, we are given a map ϕ_n that associates a finite sequence $\phi_n(\sigma_1, \ldots, \sigma_n)$ to each $\sigma_1, \ldots, \sigma_n \in \mathbf{N}^\mathbf{N}$. The explicit choice of ϕ_n will be described in section 3. For a finite sequence s and $\sigma \in \mathbf{N}^\mathbf{N}$, write $s < \sigma$ if $s = \sigma | n$ for $n = |s|$. We then describe $\mathcal{A}$ as the family of sets H for which there exists an enumeration (σ_n) of H and $B \in \mathcal{B}$ such that for each n we have

$$s_B^{\theta_1(n)\frown}(2n+1, 2n+1)^\frown \phi_{n-1}(\sigma_1, \ldots, \sigma_{n-1}) < \sigma_n.$$

We shall call this enumeration of H the *defining* enumeration of H, and B the *root* of H.

LEMMA 2. – *Each* $H \in \mathcal{A}$ *is closed discrete for the usual topology.*

Proof. – Suppose there is $H \in \mathcal{A}$ that is not closed discrete. Let (σ_n) be the defining enumeration of H and B the root of H. There exists a one to one sequence $n(k)$ and $\sigma \in \mathbf{N}^\mathbf{N}$ with $\sigma_{n(k)} \longrightarrow \sigma$. Let $m(k) = \theta_1(n(k))$. We have

$$s_B^{m(k)\frown}(2n(k)+1, 2n(k)+1) < \sigma_{n(k)}.$$

This shows that $m(k) \longrightarrow \infty$. So we have found B in $\mathcal{B}$, a sequence s_k in B, $\rho_k \in \mathbf{N}^\mathbf{N}$ with $s_k < \rho_k$ and $\rho_k \longrightarrow \sigma$. If

α is the smallest ordinal for which $B \in \mathcal{B}_\alpha$, it is routine to show by induction over α that this cannot happen.

LEMMA 3. – *Let* Z *be a* $\mathcal{B}$*-Borel set of* $X(\mathcal{A})$, *such that* $Z \cap \mathbf{N}^{\mathbf{N}}$ *is not of first category for the usual topology. Then there exists* $H \in \mathcal{A}$ *and a family* (L_s) *of compact sets of* $X(\mathcal{A})$ *such that* $\cap L_s \subset Z$ *and* $H \cap L_s$ *is infinite for each* s.

Proof. – We use lemma 1 to find $B \in \mathcal{B}$ and for $s \in B$ a compact set L_s of $X(\mathcal{A})$ such that $L_s \cap A_s$ is dense in A_s, and $\cap L_s \subset Z$. By induction over n, we construct $\sigma_n \in L_{u(n)}$, where $u(n) = s_B^{\theta_1(n)}$, such that (3) holds. This is possible since $L_{u(n)} \cap A_{u(n)}$ is dense in $A_{u(n)}$.

The cornerstone of the construction is the following lemma, that will be proved in section 3.

LEMMA 4. – *It is possible to choose the maps* ϕ_n *and* ψ_n *such that for* $H_1, H_2 \in \mathcal{A}$ *we have either* $H_1 = H_2$ or $H_1 \cap H_2$ *is finite.*

We assume that $\mathcal{A}$ has this property, and we finish the proof of theorem A.

For each H in $\mathcal{A}$, the trace on H of the algebra generated by H and by the finite sets is the algebra of sets that are either finite or cofinite. It follows that $\overline{H}\setminus T$ (where the closure is in $X(\mathcal{A})$) consists of a single point a_H, and that for each infinite subset G of H, we have $a_H \in \overline{G}$.

PROPOSITION 5. – T *is not* $\mathcal{K}$*-Borel in* $X(\mathcal{A})$. *Actually, if* $Z \subset T$ *is* $\mathcal{K}$*-Borel, then* $Z \cap \mathbf{N}^{\mathbf{N}}$ *is of first category for the usual topology.*

Proof. – Suppose Z is $\mathcal{K}$-Borel, but that $Z \cap \mathbf{N}^{\mathbf{N}}$ is not of first category for the usual topology. Let H and (L_s) be as in lemma 3. Since $H \cap L_s$ is infinite for each s, we have $a_H \in L_s$, so $a_H \in \cap L_s \subset Z$. Q.E.D.

We note also that the set $(a_H)_{H \in \mathcal{A}}$ is discrete, ω is its only cluster point. To prove theorem A, it remains only to construct a Choquet simplex K such that $\mathcal{E}$ can be identified with T and $\overline{\mathcal{E}}$ can be identified with $X(\mathcal{Q})$. Denote by R the subset of $\mathbf{N}^{\mathbf{N}}$ of sequences $\sigma = (\sigma(n))$ such that $\sigma(m) \neq \sigma(n)$ for $m \neq n$. We note that by construction $H \cap R$ is empty for $H \in \mathcal{Q}$. Since R and H both have the power of continuum, we can find for $H \in \mathcal{Q}$ points b_H, c_H in R such that these points are all distinct. Denote by Y the subspace of $C(X(\mathcal{Q}))$ consisting of those functions f such that

$$\forall H \in \mathcal{Q}, f(a_H) = \frac{1}{2}(f(b_H) + f(c_H)). \tag{4}$$

Note that $1 \in Y$. Let

$$K = \{x^* \in Y^* \,;\, \|x^*\| \leqslant 1, x^*(1) = 1\}.$$

Then, for the weak* topology, K is convex compact. Let M denote the set of probability measures on $X(\mathcal{Q})$ (provided with the weak* topology) and let θ be the natural map $\theta : M \longrightarrow K$. We identify $X(\mathcal{Q})$ to a subset of M. Let $u \in \mathbf{N}^{\mathbf{N}}$. If u is not equal to b_H or c_H for any $H \in \mathcal{Q}$, then $f = 1_{\{u\}} \in Y$. Since $f(u) > f(x)$ for x in $X(\mathcal{Q})$, $x \neq u$, $\theta(u)$ is actually an exposed point of K. If u is equal to b_H or c_H for some $H \in \mathcal{Q}$, then $f = 1_{\{u\}} + \frac{1}{2} 1_{H \cup \{a_H\}} \in Y$ so again $\theta(u)$ is an exposed point of K. This also shows that $\theta(\omega)$ is extreme. By the same type of arguments, one gets that θ is one to one, so is an isomorphism on its image. Moreover, $\theta(a_H)$ is not extreme since $\theta(a_H) = \frac{1}{2}(\theta(b_H) + \theta(c_H))$, and $\theta(b_H) \neq \theta(c_H)$. It follows that $\theta(T) = \mathcal{E}$, $\theta(X(\mathcal{Q})) = \overline{\mathcal{E}}$. It remains to show that K is a Choquet simplex. It is enough to show that for μ, ν two probability measures on T then

$$\forall f \in Y, \mu(f) = \nu(f) \Longrightarrow \mu = \nu$$

(it will then follow that each point of K is barycenter of a unique maximal measure). Note that μ and ν are atomic.

Let $\epsilon > 0$, and let F be a finite set with

$$\mu(\mathbf{N}^{\mathbf{N}} \setminus F) < \epsilon, \nu(\mathbf{N}^{\mathbf{N}} \setminus F) < \epsilon.$$

Let $u \in \mathbf{N}^{\mathbf{N}}$. Assume for example that u is of the type b_H. Then if $G = \{a_H\} \cup (H \setminus F)$

$$f = 1_{\{u\}} + \frac{1}{2} 1_G \in Y$$

so $|\mu(\{u\}) - v(\{u\})| \leqslant \epsilon$. Letting $\epsilon \longrightarrow 0$, we get $\mu(\{u\}) = \nu(\{u\})$ for $u \in \mathbf{N}^{\mathbf{N}}$, so $\mu = \nu$. Theorem A is proved.

3. Choice of ψ_n and ϕ_n.

The set $\mathfrak{F}$ of countable sets of finite sequences has the power of continuum, so there is a one to one map $B \longrightarrow \sigma(B)$ from $\mathfrak{F}$ to $\mathbf{N}^{\mathbf{N}}$. We define $\psi_n(B_1, \ldots, B_n)$ as the sequence of length n^2 obtained by taking the first n terms of $\sigma(B_1)$, then the first n terms of $\sigma(B_2)$, etc. The only two properties of ψ_n we shall use is that $|\psi_n(B_1, \ldots, B_n)|$ depends on n only, and that if $B_1, \ldots, B_n, \ldots, C_1, \ldots, C_n, \ldots$ are two sequences of $\mathcal{B}$ such that

$$\psi_n(B_1, \ldots, B_n) = \psi_n(C_1, \ldots, C_n)$$

for infinitely many integers n, then $C_i = B_i$ for each i.

We define $\phi_n(\sigma_1, \ldots, \sigma_n)$ as the sequence of length n^2 obtained by taking the first n terms of σ_1 then the first n terms of σ_2, etc. The only two properties of ϕ_n we shall use are again that $|\phi_n(\sigma_1, \ldots, \sigma_n)|$ depends on n only, and that if $\sigma_1, \ldots, \sigma_n, \ldots, \rho_1, \ldots, \rho_n, \ldots,$ are two sequences in $\mathbf{N}^{\mathbf{N}}$ such that

$$\phi_n(\sigma_1, \ldots, \sigma_n) = \phi_n(\rho_1, \ldots, \rho_n)$$

for infinitely many values of n, then $\sigma_i = \rho_i$ for each i.

LEMMA 6. – *Let* $B \in \mathcal{B}$. *and* (s_n) *be a sequence of elements of* B *with* $s_n \neq s_m$ *for* $n \neq m$. *Then there is a subsequence* (s'_k) *of* (s_n), *there is a finite sequence* t, *there is a strictly increasing sequence* $m(k)$ *of integers, a sequence* (B_p) *of* $\mathcal{B}$ *and a sequence* $x_k \in B_{m(k)}$ *such that for each* k

$$t^\frown(2m(k), 2m(k))^\frown \psi_{m(k)-1}(B_1, \ldots, B_{m(k)-1})^\frown x_k = s'_k .$$

Proof. – Suppose $B \in \mathcal{B}_\alpha$. The proof goes by induction over α. It is obvious for $\alpha = 0$. Suppose it has been proved for $\beta < \alpha$. By definition, there is a finite sequence u, and a sequence (C_i) of $\bigcup_{\beta<\alpha} \mathcal{B}_\beta$ such that B is the set of sequences of the type $t_n \frown v$, for $n \in \mathbf{N}$, $v \in C_n$, where

$$t_n = u \frown (2n, 2n) \frown \psi_{n-1}(C_1, \ldots, C_{n-1}).$$

If there exists a strictly increasing sequence $n(k)$ such that $t_{n(k)} < s_{n(k)}$, the conclusion holds. Otherwise, there is n_0 and a subsequence s_k'' of s_n with $t_{n_0} < s_k''$ for each k, so $s_k'' = t_{n_0} \frown v_k$ for $v_k \in C_{n_0}$. The induction hypothesis implies that there is a subsequence v_k' of v_k, a finite sequence u, a sequence (B_p) of $\mathcal{B}$, a strictly increasing sequence $m(k)$ of integers and a sequence $x_k \in B_k$ such that $v_k' = u \frown w_k \frown x_k$, where

$$w_k = (2m(k), 2m(k)) \frown \psi_{m(k)}(B_1, \ldots, B_{m(k)}).$$

If $s_k' = t_{n_0} \frown v_k'$, we have $t_{n_0} \frown u \frown w_k \frown z_k = s_k'$. The proof is complete.

LEMMA 7. – Let $B \in \mathcal{B}$. If $s, t \in B$, $s < t$, then $s = t$.

The obvious induction is left to the reader. As a consequence, if $\sigma \in H \in \mathcal{A}$ and B is the root of H, there is a unique $s \in B$ with $s < \sigma$.

We now start proving that if $G, H \in \mathcal{A}$ have an infinite intersection, then $G = H$. Let (σ_k) (resp. (ρ_k)) be the defining enumeration of G(resp. H) and B(resp. C) be the root of G(resp. H). So, we assume that we have two sequences $k(n)$, $\ell(n)$ such that $\sigma_{k(n)} = \rho_{\ell(n)}$ for each n, and we want to prove that $G = H$. Let s^n (resp. t^n) be the unique element of B(resp. C) such that $s^n < \sigma_{k(n)}$ (resp. $t^n < \rho_{\ell(n)}$). We have to distinguish four cases.

Case 1. – There exists an infinite $I \subset \mathbf{N}$, and s, t such that $s^n = s$, $t^n = t$ for $n \in I$.

In this case, we have for each $n \in I$

$$s^\frown(2k(n)+1\,,\,2k(n)+1)^\frown\phi_{k(n)-1}\,(\sigma_1\,,\,\ldots\,,\sigma_{k(n)-1}) < \sigma_{k(n)}$$

$$t^\frown(2\ell(n)+1\,,\,2\ell(n)+1)^\frown\phi_{\ell(n)-1}\,(\rho_1\,,\,\ldots\,,\rho_{t(n)-1}) < \rho_{\ell(n)}\,.$$

It follows that $s = t$, and $k(n) = \ell(n)$ for $n \in \mathrm{I}$. Since the length of $\phi_k(\cdot\,,\ldots,\cdot)$ depends only of k, this forces

$$\phi_{k(n)-1}\,(\sigma_1\,,\,\ldots\,,\sigma_{k(n)-1}) = \phi_{\ell(n)-1}\,(\rho_1\,,\,\ldots\,,\rho_{\ell(n)-1})$$

for each $n \in \mathrm{I}$. This implies that $\sigma_i = \rho_i$ for each i, i.e. $\mathrm{G} = \mathrm{H}$.

Case 2. There exists an infinite $\mathrm{I} \subset \mathbf{N}$ and t, such that $t^n = t$ for $n \in \mathrm{I}$, and $s^n \neq s^m$ for $n\,,\,m \in \mathrm{I}$, $n \neq m$.

From lemma 6, by restricting I one can assume that there is a finite sequence s, integers $m(n)$ such that for $n \in \mathrm{I}$,

$$s^\frown(2m(n)\,,\,2m(n)) < s^n < \sigma_{k(n)}\,.$$

On the other hand

$$t^\frown(2\ell(n)+1\,,\,2\ell(n)+1) < \rho_{\ell(n)} = \sigma_{k(n)}\,.$$

Since $2m(n)$ is even, while $2\ell(n)+1$ is odd, this is impossible.

Case 3. Same as Case 2, exchanging the role of G and H.

This case is impossible just as Case 2.

Case 4. There exists an infinite $\mathrm{I} \subset \mathbf{N}$ such that for $n\,,\,m \in \mathrm{I}$, $n \neq m$, we have $s^n \neq s^m$, $t^n \neq t^m$.

From lemma 6, by restricting I, one can assume that there exists finite sequences $s\,,\,t$, strictly increasing sequences $(m(n))$, $(p(n))$, sequences (D_p), (F_p) of $\mathcal{B}$, sequences $x_n \in \mathrm{D}_{m(n)}$, $y_n \in \mathrm{F}_{p(n)}$ such that for $n \in \mathrm{I}$ we have

$$s^n = s^\frown(2m(n)\,,\,2m(n))^\frown\psi_{m(n)-1}\,(\mathrm{D}_1\,,\,\ldots\,,\mathrm{D}_{m(n)-1})^\frown x_n$$

$$t^n = t^\frown(2p(n)\,,\,2p(n))^\frown\psi_{p(n)-1}\,(\mathrm{F}_1\,,\,\ldots\,,\mathrm{F}_{p(n)-1})^\frown y_n\,.$$

Since $s^n < \sigma_{k(n)}$, $t^n < \rho_{\ell(n)}$, and $\sigma_{k(n)} = \rho_{\ell(n)}$, it follows first that $s = t$, and $m(n) = p(n)$ for each n. It then follows that for $n \in \mathrm{I}$

$$\psi_{m(n)-1}\,(\mathrm{D}_1\,,\,\ldots\,,\mathrm{D}_{m(n)-1}) = \psi_{m(n)-1}\,(\mathrm{F}_1\,,\,\ldots\,,\mathrm{F}_{m(n)-1})$$

since these sequences have the same length. This in turns implies that $\mathrm{D}_i = \mathrm{F}_i$ for each i. We have $x_n\,,\,y_n \in \mathrm{D}_{m(n)}$. Since either

$x_n < y_n$ or $y_n < x_n$, lemma 7 shows $x_n = y_n$. We have proved that $s^n = t^n$ for each n. By definition of $\mathcal{A}$, we have for $n \in I$:

$$s^n \frown (2k(n) + 1, 2k(n) + 1) \frown \psi_{k(n)-1}(\sigma_1, \ldots, \sigma_{k(n)-1}) < \sigma_{k(n)}$$

$$t^n \frown (2\ell(n) + 1, 2\ell(n) + 1) \frown \psi_{\ell(n)-1}(\rho_1, \ldots, \rho_{\ell(n)-1}) < \rho_{\ell(n)}.$$

Since $s^n = t^n$, this shows $\ell(n) = k(n)$. This implies

$$\psi_{k(n)-1}(\sigma_1, \ldots, \sigma_{k(n)-1}) = \psi_{k(n)-1}(\rho_1, \ldots, \rho_{k(n)-1})$$

since these sequences have the same length. It follows that $\sigma_1 = \rho_i$ for each i, so $G = H$. The proof is complete.

BIBLIOGRAPHY

[1] G. CHOQUET, *Lectures on analysis,* New York, W.A. Benjamin, 1969 (Math. Lecture Notes series).

[2] Z. FROLICK, A survey of separable descriptive theory of sets and spaces, *Czechoslovak Math. J.,* 20 (1967), 406-467.

[3] R. HAYDON, An extreme point criterion for separability of a dual Banach space, and a new proof of a theorem of Corson, *Quarterly J. Math.,*27 (1976), 379-385.

[4] B. MACGIBBON, A criterion for the metrizability of a compact convex set in terms of the set of extreme points, *J. Funct. Anal.,* 11 (1972), 385-392.

[5] R. PHELPS, Lectures on Choquet's theorem, *Van Nostrand Math. studies,* 7 (1966).

[6] M. TALAGRAND, Géométrie du simplexe des moyennes, *J. Funct. Anal.,* 33 (1919), 304-333.

[7] M. TALAGRAND, Espaces de Banach faiblement $\mathcal{K}$-analytiques, *Ann. of Math.,*110 (1979), 407-438.

[8] M. TALAGRAND, Sur les convexes compacts dont l'ensemble des points extrémaux est $\mathcal{K}$-analytique, *Bull. Soc. Math. France,* 107 (1979), 49-53.

Manuscrit reçu le 11 octobre 1984.

Michel TALAGRAND,
Equipe d'Analyse – Tour 46
Université Paris VI
4 Place Jussieu
75230 Paris Cedex 05.