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POINCARÉ BUNDLES FOR PROJECTIVE SURFACES

by Nicole MESTRANO

Let X be a non singular projective variety defined over an algebrically closed field. For any ample divisor H on X, we note $M_H(c_1,c_2,\ldots,c_n;r)$ the moduli space of rank-r, H-stable (in the sense of Mumford-Takemoto, see [16] definition 1.3) vector bundles on X, with Chern classes $c_1, \ldots, c_n$ (in the numerical ring of X).

Does there exist a rank-r vector bundle P on $M_H(c_1,\ldots,c_n;r) \times X$ such that:

1) for each m in $M_H(c_1,\ldots,c_n;r)$, the bundle P(m) induced by the embedding $X \to M_H(c_1,\ldots,c_n;r) \times X$ defined by the point m is H-stable, with Chern classes $c_1, \ldots, c_n$,

2) for each m in $M_H(c_1,\ldots,c_n;r)$, the isomorphism class of P(m) is m?

Every bundle which satisfies these two conditions is called a Poincaré Bundle.

Ramanan (see [12] and [14]) has proved that, if X is a smooth projective curve, then, there exists a Poincaré Bundle on $M(d;r) \times X$ if and only if r and d are coprime. Newstead (see [13]) has shown that, in the degree 0 case, even topological obstructions exist for the existence of Poincaré Bundle. Then Maruyama has given a sufficient condition for the existence of Poincaré Bundle (see [8], theorem 6.11). Le Potier has proved (see [6]) that this condition is also necessary for rank-two vector bundles on $\mathbf{P}^2$ i.e. that, if $X = \mathbf{P}^2$, there exists a Poincaré Bundle on $M_H(c_1,c_2;2) \times X$ if and only if c_1 is odd or c_1 is even and $c_2 - \frac{1}{4}c_1^2$ is odd. More recently, Hirschowitz and Narasimhan have obtained the same result for $X = \mathbf{P}^3$ (see [5]).

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In this work X is a surface, H is a very ample divisor and $r = 2$; so, we note $M_H(c_1,c_2)$ instead of $M_H(c_1,c_2;r)$. We obtain the:

THEOREM (3.2.7). — *If c_1 is even and if $c_2 - \frac{1}{4}c_1^2$ is an even integer at least equal to $H^2 + HK + 4$, then, there is no Poincaré Bundle on $M_H(c_1,c_2) \times X$.*

We also give the following variation of Maruyama's condition:

COROLLARY (3.1.5). — *If $\frac{1}{2}c_1^2 - \frac{1}{2}c_1K - c_2$ is odd or if c_1 is numerically odd, then there exists a Poincaré Bundle on $M_H(c_1,c_2) \times X$.*

For $X = \mathbf{P}^2$, that gives another proof of Le Potier's theorem (see 3.3.1). When X is a surface fibered in irreducible curves, we obtain the theorem 3.3.3 which gives some complements to the theorem 3.2.7. In particular, using these two theorems, we prove that Maruyama's condition is necessary for rank-two H-stable vector bundles on quadric surfaces and on rational ruled surfaces when H is « well chosen » (see 3.3.7). For that we use one of Maruyama's results (see [7], theorem 4.15) which tells us that, if H is « well chosen » the condition $c_2 - \frac{1}{4}c_1^2 \geqslant H^2 + HK + 4$ is necessary for $M_H(c_1,c_2)$ to be not empty. Unfortunately this property cannot be generalized, even for not rational ruled surfaces (see 3.3.2 and 3.3.5). In 3.2.10, we give some examples of surfaces X and of Chern classes c_1, c_2 such that $M_H(c_1,c_2)$ is not empty but we don't know (independantly of the condition $c_2 - \frac{1}{4}c_1^2 \geqslant H^2 + HK + 4$) if there is a Poincaré Bundle on $M_H(c_1,c_2) \times X$.

Now let us explain Hirschowitz-Narasimhan's method. If X is a smooth projective variety, they study a special kind of vector bundles on X (the so called, special bundles) obtained from the trivial bundle by Hecke transformation (that's what Maruyama calls elementary transformation see [9] and Hartshorne, reduction step, see [4], proposition 9.1). For each special bundle E, we have the exact sequence $S(E)$:

$$0 \to \mathcal{O}_X \oplus \mathcal{O}_X \to E \to \mathscr{F}_E \to 0$$

where $\mathscr{F}_E$ is the extension by zero of a line bundle $\mathscr{F}_E$ on a smooth subvariety C_E of codimension one in X, chosen in a « special family »

$\mathscr{C}$. The sheaf $\mathscr{G}_E := \mathscr{E}xt^1(\mathscr{F}_E, \mathcal{O}_X)$ is the extension by zero of a line bundle $\mathscr{G}_E$ of degree d on C_E. The functor $\mathscr{H}om(-, \mathcal{O}_X)$ applied to $S(E)$ gives, when C_E is irreducible, two sections which generate $\mathscr{G}_E$ and then a pencil of divisors of degree d, without base point. This construction gives an identification between the moduli of special bundles and the variety M of pencils of divisors of degree d without base point, on the varieties of the special family $\mathscr{C}$. We construct the modular morphism from the functor of the special bundles to the functor represented by M and we are reduced to a problem on the special family $\mathscr{C}$ of hypersurfaces. For example, when X is the projective plane $\mathbf{P}^2$, Hirschowitz and Narasimhan study the special bundles E for which C_E is a smooth conic in $\mathbf{P}^2$. They show that a morphism $\varphi : T \to M$ is in the image of the modular morphism if and only if there exists, on the pullback on T of the universal conic of $\mathbf{P}^2$, a line bundle of degree d. But the universal conic of $\mathbf{P}^2$ is a non banal conic bundle so, M is a coarse moduli (via the modular morphism) which admits a Poincaré object if and only if d iseven. Using this idea, the natural way to study the existence of Poincaré Bundle when X is a surface, is to consider special bundles E for which C_E is a smooth irreducible curve of genus g. In this case, the existence of Poincaré Bundle on $M_H(c_1,c_2) \times X$ is reduced to the existence of Poincaré Bundle on $J^d \times_{\mathscr{C}} \mathscr{U}$, where $\mathscr{U} \to \mathscr{C}$ is the universal curve of the special family $\mathscr{C}$ of curves chosen in the surface X and $J^d \to \mathscr{C}$ is the relative Jacobian of degree d. But to solve this problem we need to know which are the degree of the divisors on $\mathscr{U}$ (see [11], theorem 2.5). So, we need another idea; let us consider a special family of *non connected curves*. Such a family is easy to find when the surface X is fibered in curves (see chapter 2, introduction). In the general case, we work in a surface S fibered in curves, obtained by blowing up (see chapter 3, introduction). On this surface S, we study the special bundles obtained by elementary transformation of center a bifiber i.e. the union of two distinct smooth fibers (see § 2.1). Then we define the notion of bipencil of divisors of degree d on a bifiber (see 2.2.2) and the bipencil variety of degree d, $\mathrm{Bi}(S/L)$ (see 2.2.4) which comes with a universal bipencil $U \to \mathrm{Bi}(S/L)$ which is a non banal conic bundle (see 2.2.6).

We identify the moduli of special bundles with $\mathrm{Bi}(S/L)$ and we show that a morphism $\varphi : T \to \mathrm{Bi}(S/L)$ is in the image of the modular morphism if and only if, the pullback of U on T is banal (see 2.3.2). So $\mathrm{Bi}(S/L)$ is a coarse moduli without Poincaré object (see 2.3.3).

To show that $U \to \mathrm{Bi}(S/L)$ is a non banal conic bundle, we give in chapter 1 a general construction of the universal *passerelle* associated to any conic bundle (see 1.2.3) and we show (see 1.2.8) that it is a non banal conic bundle.

This is a second version of the « thèse de 3^e^ cycle, soutenue a l'Université de Nice, 1983 » where one can find more details which are omit here.

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INDEX

CHAPTER 1. — **Isodiagonals and passerelles.**

1.1. Tautological isodiagonal.
1.2. Universal passerelle.

CHAPTER 2. — **Special bundles on surfaces fibered in curves.**

2.1. Special bundles.
2.2. Bipencil of divisors on reducible curves.
2.3. Modular morphism.
2.4. Very special bundles.

CHAPTER 3. — **Poincaré Bundles for projective surfaces.**

3.1. A sufficient condition of existence.
3.2. A necessary condition of existence.
3.3. Examples.

NOTATIONS

- For every scheme M, we note $\hat{M}$ the functor represented by M.
- All the functors considered have their values in the category of sets.

• If $\alpha : A \to B$ is a morphism, for each x in B, we note $A(x)$ the fiber $\alpha^{-1}(x)$. If $\beta : B \to Z$ is a morphism, to each Z-scheme $\mu : T \to Z$, we associate the cartesian diagram:

$$\begin{array}{ccc} A_T & \to & A \\ {\scriptstyle\alpha_T}\downarrow & & \downarrow{\scriptstyle\alpha} \\ B_T & \to & B \\ {\scriptstyle\beta_T}\downarrow & & \downarrow{\scriptstyle\beta} \\ T & \xrightarrow{\mu} & Z \end{array}$$

• For each variety X, we note $S^2(X)$ the second symmetric product of X and we set:

$$S^2_*(X) = \{(x,y) \in S^2(X)/x \neq y\}.$$

• For each vector bundle $\mathscr{E}$ on $T \times X$, we note $\mathscr{E}(t)$ the restriction of $\mathscr{E}$ to $\{t\} \times X$; we look at it as a vector bundle on X.

CHAPTER 1

ISODIAGONALS AND PASSERELLES

1.1. Tautological isodiagonal.

In this paragraph $A \xrightarrow{\alpha} B \xrightarrow{\beta} Z$ is given and fixed, α is a locally isotrivial morphism whose fibers are smooth irreducible projective curves and β is a non ramified finite morphism of degree n, $n \geqslant 1$, of irreducible quasi-projective non singular varieties.

We construct (see 1.1.1) a quasi-projective Z-scheme $m : M \to Z$ such that, for each z in Z, the fiber $M(z)$ is the product $\prod_{x \in \beta^{-1}(z)} A(x)$. Then we construct (see 1.1.4) a quasi projective Z-scheme $\nu : I \to Z$ such that, for each z in Z, the fiber $I(z)$ is the set of all subschemes Δ in the product $\prod_{x \in \beta^{-1}(z)} A(x)$ such that the projection of Δ on $A(x)$ is an isomorphism for each x in $\beta^{-1}(z)$. We process as follows:

It is easy to see that the functor, on the category of Z-schemes, which associates to T the set of all algebraic sections of the morphism $\alpha_T : A_T \to B_T$ is representable by a quasi projective Z-scheme $m : M \to Z$.

1.1.1. DEFINITION. — *We say that* M *is the isoproduct variety associated to* $A \xrightarrow{\alpha} B \xrightarrow{\beta} Z$.

1.1.2. *Remark.* — The variety M is not a product variety but there exists an etale covering $B^1 \to Z$ such that $M \times_Z B^1$ is a product variety.

Let z be in Z and $\{x_1, \ldots, x_n\}$ be the set $\beta^{-1}(z)$. Then the fiber of $B \times_Z M$ over z is $\{x_1, \ldots, x_n\} \times \prod_{i=1}^{n} A(x_i)$. Let $\gamma : B \times_Z M \to A$ be the morphism (such that the diagram

$$\begin{tikzcd} B \times_Z M \arrow[r, "\gamma"] \arrow[dr] & A \arrow[d, "\alpha"] \\ & B \end{tikzcd}$$

commutes) whose restriction to $\{x_{i'}\} \times \prod_{i=1}^{n} A(x_i)$ is the projection $\prod_{i=1}^{n} A(x_i) \to A(x_{i'})$ for each i' in $\{1, \ldots, n\}$.

1.1.3. *Notation.* — For each Z-scheme T, let $\gamma_T : B_T \times M_T \to A_T$ be the pullback of γ, and for each subscheme Δ of M_T let $\gamma_\Delta : B_T \times_T \Delta \to A_T$ be the restriction of γ_T to $B_T \times_T \Delta$.

Let $\mathscr{I}$ be the functor on the category of Z-schemes which associates to T the set of all T-proper and flat subschemes Δ of M_T such that the map $\gamma_\Delta : B_T \times_T \Delta \to A_T$ is an isomorphism. Then $\mathscr{I}$ is representable by a quasi projective Z-scheme $\nu : I \to Z$.

1.1.4. DEFINITION. — *We say that* I *is the isodiagonal variety associated to* $A \xrightarrow{\alpha} B \xrightarrow{\beta} Z$.

1.1.5. *Remark.* — Let $X = \prod_{i=1}^{n} X_i$ be the product of n same-dimensional varieties X_i for i in $\{1, \ldots, n\}$. Call diagonal of X, any subscheme of X such that the projection on X_i is an isomorphism for each i in $\{1, \ldots, n\}$. Then the pullback of I on B^1 (see 1.1.2) is the set of diagonals of $M \times_Z B^1$. By definition, the isodiagonal I comes with an universal scheme $\delta : U \to I$.

1.1.6. DEFINITION. — *We say that* $U \to I$ *is the tautological isodiagonal associated to* $A \xrightarrow{\alpha} B \xrightarrow{\beta} Z$.

1.2. Universal passerelle.

Until the end of this chapter, X and Y are non singular irreducible quasi projective varieties, $\dim X$ is at least one, and $f: Y \to X$ is a conic bundle (i.e. a locally isotrivial morphism of smooth algebraic varieties whose fibers are $\mathbf{P}^1$); when a conic bundle is locally trivial (in Zariski topology) we say that it is *banal.* Let us recall the following:

1.2.1. *Remark.* — A conic bundle $\Pi : C \to B$ is banal if and only if there exist on C a divisor whose relative degree is equal to one. So, if the conic bundle $\Pi/\Pi^{-1}(B^1) : \Pi^{-1}(B^1) \to B^1$ is banal for a (dense) open subset B^1 of B, then $\Pi : C \to B$ is banal.

1.2.2. *Notation.* — Set $\Delta := \{\{x,x\} \in X \times X, x \in X\}$, let $P : X \times X \backslash \Delta \to X$ be the first projection and $P^*Y \to X \times X\backslash\Delta$ be the pullback of $Y \to X$.

1.2.3. DEFINITION. — *The tautological isodiagonal* $U \to \mathrm{BiX}$ *associated to* $P^*Y \to X \times X\backslash\Delta \to S^2_*(X)$ *is called the universal passerelle associated to the conic bundle* $Y \to X$; *and* BiX *is the passerelle variety associated to* $Y \to X$.

1.2.4. *Remark.* — The fibers of $Y \to X$ are isomorphic but (in general) not in a canonical way. The passerelle variety BiX associated to $Y \to X$ is a $S^2_*(X)$-scheme. For each $\{x, y\}$ in $S^2_*(X)$, a point δ of BiX over $\{x,y\}$ is a diagonal of $Y(x) \times Y(y)$. So δ corresponds to an identification between the two distincts fibers $Y(x)$ and $Y(y)$. The fiber of $U \to \mathrm{BiX}$ over δ is the line which follows from this identification.

1.2.5. *Remark.* — Assume that $Y \to X$ is the trivial conic bundle, so $Y \simeq X \times \mathbf{P}^1$, and BiX is nothing but the quotient of

$(X \times X \backslash \Delta) \times \mathbf{PGL}(2)$ by the involution

$$\begin{array}{lcl} (X \times X \backslash \Delta) \times \mathbf{PGL}(2) & \to & (X \times X \backslash \Delta) \times \mathbf{PGL}(2) \\ (x,y,g) & \mapsto & (y,x,g^{-1}) \end{array}$$

and U is the quotient of $(X \times X \backslash \Delta) \times \mathbf{PGL}(2) \times \mathbf{P}^1$ by the involution :

$$\begin{array}{lcl} (X \times X \backslash \Delta) \times \mathbf{PGL}(2) \times \mathbf{P}^1 & \to & (X \times X \backslash \Delta) \times \mathbf{PGL}(2) \times \mathbf{P}^1 \\ (x,y,g,z) & \mapsto & (y,x,g^{-1}, g(z)) . \end{array}$$

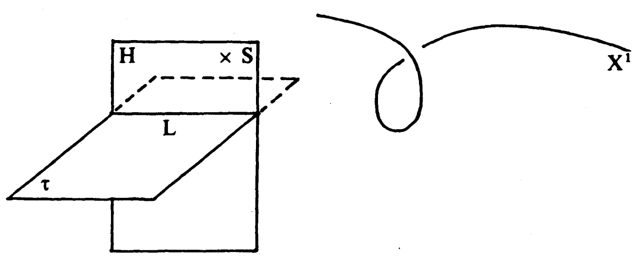
So, the universal passerelle associated to the trivial conic bundle is a conic bundle. As a conic bundle is locally isotrivial, by base change properties, we see that, the universal passerelle $U \to \mathrm{Bi}X$ associated to the conic bundle $Y \to X$ is a conic bundle.

The aim of the end of this chapter is to prove that the conic bundle $U \to \mathrm{Bi}X$ is not banal.

1.2.6. *Notations.*

i) We fix an embedding of X into a m-dimensional projective space $\mathbf{P}^m$, where $m \geqslant \dim X + 4$. So X is a locally closed subvariety of $\mathbf{P}^m$.

ii) Let H be an hyperplane in $\mathbf{P}^m$ which does not contain X. Then $X^1 := X \backslash (H \cap X)$ is a (dense) open subset of X.

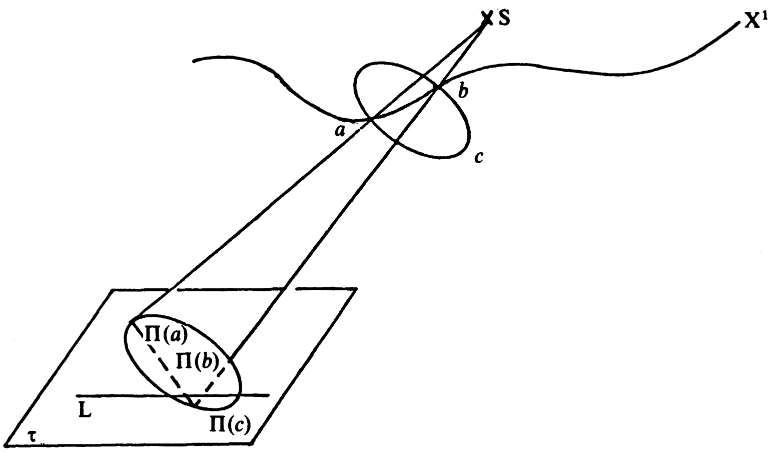


Let τ be a plane which meets H transversally in a line L.

Let S be a linear subvariety of codimension 3 in $\mathbf{P}^m$, contained in $H \backslash L$. Denote by Π the projection on τ, of center S.

iii) Let $v : U_\tau \to \mathscr{C}_\tau$ be the universal family of smooth conics in τ.

iv) Let $\mathscr{C}$ be the set of smooth conics in $\mathbf{P}^m \backslash S$ which meets the adherence $\bar{X}$ of X in exactly two distinct points lying in X^1, and such that the projection (by Π) on τ is a smooth conic of τ.



1.2.7. Lemma. — *Let* $U \to \mathrm{BiX}$ *be the universal passerelle associated to the trivial conic bundle* $X \times L \to X$. *The universal conic* $\mu : U \to \mathscr{C}$ *can be deduced from* $U \to \mathrm{BiX}$ *by base change.*

Proof. — Let C be in $\mathscr{C}$. Then C meets X^1 in two distinct points a and b. Put $C^1 = \Pi(C)$, then C^1 is a smooth conic of τ. Put $a^1 = \Pi(a)$ and $b^1 = \Pi(b)$. Let r_a (resp. r_b) be the stereographic projection on L of center a^1 (resp. b^1). As X^1 does not meet H, the points a^1 and b^1 are not on L, so r_a and r_b are isomorphisms. Put

$$\varphi(C) = \{(a,b,r_b \circ r_a^{-1}),\ (b,a,r_a \circ r_b^{-1})\}$$

and, for each x in C,

$$\Psi(x) = \{(a,b,r_b \circ r_a^{-1},r_a(x)),\ (b,a,r_a \circ r_b^{-1},r_b(x))\}.$$

Then the following diagram (see remark 1.2.5) is cartesian

$$\begin{array}{ccc} \mathscr{U} & \xrightarrow{\Psi} & U \\ \mu \downarrow & & \downarrow \\ \mathscr{C} & \xrightarrow{r} & \mathrm{BiX} \end{array}$$

1.2.8. Proposition. — *Let* $f : Y \to X$ *be a conic bundle of quasi projective varieties such that* $\dim X \geqslant 1$. *Then the universal passerelle* $U \to \mathrm{BiX}$ *associated to* $Y \to X$ *is a non banal conic bundle.*

Proof. — By base change properties of the universal passerelle, and as a conic bundle is locally isotrivial, we can assume that $Y \to X$ is the trivial conic bundle. Moreover, the universal family $\mu^1 : \mathscr{U}^1 \to \mathscr{C}^1$ of smooth conics in $\mathbf{P}^m$ meeting $\mathfrak{X}$ in exactly two distinct points is a non banal conic bundle (see [10], 4.3). As $\mathscr{C}$ is a dense open subset of $\mathscr{C}^1$, the universal conic $\mu : \mathscr{U} \to \mathscr{C}$ is not banal (see 1.2.1) so, by 1.2.7, $U \to \mathrm{B}i\mathrm{X}$ is not banal.

Chapter 2

SPECIAL BUNDLES ON SURFACES FIBERED IN CURVES

In this chapter S is a surface fibered in curves i.e. a smooth connected projective surface with a morphism $\Pi : S \to \mathfrak{L}$, onto a smooth irreducible curve $\mathfrak{L}$, such that the general fiber is an irreducible curve of genus g, $g \geqslant 0$. For each integer d, $d \geqslant g + 1$, we construct (see 2.1.2 and 2.1.8) special vector bundles E on S with Chern classes $c_1(E) = 2F$ and $c_2(E) = 2d$, where F is a smooth fiber of Π. Using the lemma 2.1.9, we construct for this special bundle a coarse moduli (see 2.3.3) namely, the bipencil variety Bi(S/L) of degree d (see 2.2.4). This bipencil variety comes with a non banal conic bundle (see 2.2.6), so, the coarse moduli Bi(S/L) has no Poincaré object (see 2.3.3). In paragraph 2.4 we assume that $\Pi : S \to \mathfrak{L}$ satisfies the following condition :

(*) $\mathfrak{L}$ is a projective line and there exists an integer m such that S contains m exceptional lines of the first kind $X_1, \ldots, X_m$ (mutually distinct)which cut transversally the fibers of Π at one point.

On such a surface, we introduce the « very special bundles » (see 2.4.1 and chapter 3, introduction). We construct for them a coarse moduli $\mathrm{Bi}^0(S/L)$, which is a dense open subset of Bi(S/L) (see 2.4.2). Then, we show (see 2.4.3) that $\mathrm{Bi}^0(S/L)$ has no Poincaré object.

2.1. Special bundles.

2.1.1. *Definition and Notations.*

Let $\Pi : S \to \mathbb{L}$ be the surface fibered in curves of genus g. Let L be the open subset of $\mathbb{L}$ which parametrizes the smooth fibers of Π. The set of any two distinct smooth fibers of Π is called a *bifiber*. Let $\mathscr{C} \to S^2_*(L)$ be the universal bifiber.

A line bundle of degree d on a bifiber C is a line bundle L on C such that the restriction of L to each irreducible component C_i of C is a line bundle of degree d on C_i for i in $\{1,2\}$.

2.1.2. Definition. — *A vector bundle* E *on* S *is* special *if there exists a bifiber* C_E *and a line bundle* $\mathscr{F}_E$ *of degree* $-d$ *on* C_E *such that* E *is extension of* $\tilde{\mathscr{F}}_E$ *by* $\mathcal{O}_S \oplus \mathcal{O}_S$, *where :*

1) $\tilde{\mathscr{F}}_E$ *is the extension by zero, to* S, *of the line bundle* $\mathscr{F}_E$.

2) *The dual line bundle* $\mathscr{F}_E^\vee$ *is not special and generated by its sections (this implies that* d *is at least equal to* $g+1$).

For each special bundle E, we have the exact sequence S(E) :

$$0 \to \mathcal{O}_S \oplus \mathcal{O}_S \to E \to \tilde{\mathscr{F}}_E \to 0$$

where $\tilde{\mathscr{F}}_E$ satisfies the hypothesis of the definition 2.1.2.

2.1.3. Proposition. — *Let* E *be a special bundle on* S. *Then the vector space* $H^0(S,E)$ *is two-dimensional,* C_E *and* $\mathscr{F}_E$ *are uniquely determined (upto isomorphism for* $\mathscr{F}_E$), *and* $\tilde{\mathscr{F}}_E$ *is the cokernel of the natural injection* $j_E : H^0(S,E) \otimes \mathcal{O}_S \to E$.

Proof. — The degree of the line bundle $\mathscr{F}_E$ is negative, so $\mathscr{F}_E$ has no global section. Writing the long exact cohomology sequence associated to the short exact sequence S(E), we see that $H^0(S,E)$ is a 2-dimensional vector space and that the given morphism $\mathcal{O}_S \oplus \mathcal{O}_S \to E$ can be identified with the morphism j_E. So the sheaf $\tilde{\mathscr{F}}_E$ is identified with the cokernel of j_E and C_E is the support of this cokernel.

2.1.4. Lemma. — *Let* C *be a smooth subvariety of* S *and* j : $\mathcal{O}_S \oplus \mathcal{O}_S \to \mathcal{O}_S \oplus \mathcal{O}_S$ *be an injection whose cokernel is* $\mathcal{O}_C$. *Then* $\det j$ *generates the ideal sheaf of* C.

Proof. — After base change, we can assume that $j(1,0) = (1,0)$ and $j(0,1) = (0,e)$, where e is the equation of C. So $\det j = e$.

2.1.5. PROPOSITION. — *Let* E *be a special bundle on* S. *Then:*

i) $c_1(\mathrm{E}) = 2\mathrm{F}$ *where* F *is a smooth fiber of* $\Pi : \mathrm{S} \to \mathrm{L}$ (*the equality holds in the Neron-Severi group of* S).

ii) $c_2(\mathrm{E}) = 2d$ *if* $deg\,(\mathscr{F}_{\mathrm{E}}) = -\,d$.

Proof. — The support of $\mathscr{F}_{\mathrm{E}}$ is a bififiber and $\mathscr{F}_{\mathrm{E}}$ is locally free on this bifiber, so (by lemma 2.1.4), $\det j_{\mathrm{E}}$ generates a bifiber, so, $c_1(\mathrm{E}) = 2\mathrm{F}$. Riemann Roch formula for rank two vector bundles on surfaces gives:

$$\chi(\mathrm{E}) = 2\quad \chi(\mathcal{O}_{\mathrm{S}}) + \frac{1}{2}\, c_1(\mathrm{E})^2 - \frac{1}{2}\, c_1(\mathrm{E}).\mathrm{K}_{\mathrm{S}} - c_2(\mathrm{E})$$

where K_{S} is the canonical divisor on S.

As $c_1(\mathrm{E}) = 2\mathrm{F}$, $\mathrm{F}^2 = 0$ and $\mathrm{F}^2 + \mathrm{K}_{\mathrm{S}}.\mathrm{F} = 2g - 2$ (by adjunction formula), we obtain, $\chi(\mathrm{E}) = 2\chi(\mathcal{O}_{\mathrm{S}}) - (2g-2) - c_2(\mathrm{E})$. On the other hand, by the exact sequence S(E), we have

$$\chi(\mathrm{E}) = 2\chi(\mathcal{O}_{\mathrm{S}}) + \chi(\mathscr{F}_{\mathrm{E}})$$

and Riemann Roch formula for line bundles on curves gives:

$$\chi(\mathscr{F}_{\mathrm{E}}) = 2(1-g+d)$$

so, $c_2(\mathrm{E}) = 2d$.

2.1.6. PROPOSITION. — *Let* T *be a variety,* $\tau : \mathrm{T} \times \mathrm{S} \to \mathrm{T}$ *the projection,* $\mathscr{E}$ *a vector bundle on* $\mathrm{T} \times \mathrm{S}$ *such that, for each* t *in* T, $\mathscr{E}(t)$ *is special with second Chern class* $c_2 = 2d$. *Note* $j_{\mathscr{E}}$ *the natural morphism from* $\tau^*\tau_*\mathscr{E}$ *to* $\mathscr{E}$. *Then the curves* $\mathrm{C}_{\mathscr{E}(t)}$ *form a flat family* $\mathrm{C}_{\mathscr{E}}$ *and the cokernel of* $j_{\mathscr{E}}$ *is the extension by zero of a line bundle* $\mathscr{F}_{\mathscr{E}}$ *on* $\mathrm{C}_{\mathscr{E}}$.

Proof. — Same than Hirschowitz-Narasimhan's proof of proposition 4.4 in [5].

2.1.7. LEMMA. — *Let* C *be a flat family of bififibers parametrized by a variety* T *and* $\mathscr{F}$ *be the extension by zero to* $\mathrm{T} \times \mathrm{S}$, *of a line bundle* $\mathscr{F}$

on C. *Then* $\mathscr{E}xt^1(\mathscr{F},\mathcal{O}_{T\times S})$ *is the extension by zero of the dual line bundle* $\mathscr{F}^\vee$.

Proof. – Let us remark first that, by dualyzing the exact sequence

$$0 \to \mathcal{O}_{T\times S}(-C) \to \mathcal{O}_{T\times S} \to \mathcal{O}_C \to 0$$

we get

$$0 \to \mathcal{O}_{T\times S} \to \mathcal{O}_{T\times S}(C) \to \mathscr{E}xt^1(\mathcal{O}_C,\mathcal{O}_{T\times S}) \to 0 .$$

Moreover, as C is a family of bifibers, $\mathcal{O}_C(C) = \mathcal{O}_C$.

So, $\mathscr{E}xt^1(\mathcal{O}_C,\mathcal{O}_{T\times S}) = \mathcal{O}_C$.

Now let $H := \mathscr{E}xt^1(-,\mathcal{O}_{T\times S})$ be the (contravariant) functor on the category of algebraic coherent sheaves on $T \times S$.

This functor induces the $\mathcal{O}_{T\times S}$-bilinear morphism :

$$f:\quad \mathscr{H}om(\mathscr{F},\mathcal{O}_C) \times \mathscr{E}xt^1(\mathcal{O}_C,\mathcal{O}_{T\times S}) \to \mathscr{E}xt^1(\mathscr{F},\mathcal{O}_{T\times S})$$

whose restriction to each g in $\mathscr{H}om(\mathscr{F},\mathcal{O}_C)$ is the morphism $H(g)$ in $\mathscr{H}om(\mathscr{E}xt^1(\mathcal{O}_C,\mathcal{O}_{T\times S}),\ \mathscr{E}xt^1(\mathscr{F},\mathcal{O}_{T\times S}))$.

But $\mathscr{E}xt^1(\mathcal{O}_C,\mathcal{O}_{T\times S})$ is an $\mathcal{O}_C$-Module and $\mathscr{F}$ is the extension by zero of a line bundle on C, so, $\mathscr{E}xt^1(\mathscr{F},\mathcal{O}_{T\times S})$ is also an $\mathcal{O}_C$-Module. Then, f is an $\mathcal{O}_C$-bilinear morphism and we get the morphism :

$$h:\quad \mathscr{H}om(\mathscr{F},\mathcal{O}_C) \otimes \mathcal{O}_C\mathscr{E}xt^1(\mathcal{O}_C,\mathcal{O}_{T\times S}) \to \mathscr{E}xt^1(\mathscr{F},\mathcal{O}_{T\times S}) .$$

But $\mathscr{H}om(\mathscr{F},\mathcal{O}_C) = \mathscr{F}^v$ and $\mathscr{E}xt^1(\mathcal{O}_C,\mathcal{O}_{T\times S}) = \mathcal{O}_C$, so, to prove the lemma, it is enough to show that h is an isomorphism. This is a local property, so, we can assume that $\mathscr{F} = \mathcal{O}_C$. Then, the elements of $\mathscr{H}om(\mathscr{F},\mathcal{O}_C)$ are the homotheties but $H(\mathrm{Id}_{\mathcal{O}_C}) = \mathrm{Id}_{\mathscr{E}xt^1(\mathcal{O}_C,\mathcal{O}_{T\times S})}$ so h is the identity. In particular, h is an isomorphism.

2.1.8. Lemma. – *here exists a special bundle on* S *with second Chern class equal to* $c_2 = 2d$ *if and only if* $d \geqslant g + 1$.

Proof. – Let C be a bifiber. The condition $d \geqslant g + 1$ is the sufficient and necessary condition for the existence of a non special line bundle $\mathscr{F}$ of degree d on C, generated by its sections ([3], ch. IV, Ex. 6.8).

Let $\mathscr{F}$ be such a line bundle, then $\mathscr{F}$ is generated by two linearly independant sections which define a surjection $\Omega : \mathcal{O}_S \oplus \mathcal{O}_S \to \mathscr{F}$.

Let $E := (\ker \Phi)^v$. Then E is locally free because $\mathscr{F}$ is of homological dimension one. By dualyzing the exact sequence :

$$0 \to E^v \to \mathcal{O}_S \oplus \mathcal{O}_S \to \mathscr{F} \to 0 .$$

We obtain the exact sequence (see lemma 2.1.7)

$$0 \to \mathcal{O}_S \oplus \mathcal{O}_S \to E \to \mathscr{F}^v \to 0$$

so, E is a special bundle on S with second Chern class $c_2 = 2d$.

Now we fix the integer d, $d \geqslant g + 1$ and we study the set of special bundles on S with second Chern class c_2, $c_2 = 2d$. This set is not empty by lemma 2.1.8, we are going to construct a non fine coarse moduli, namely the bipencil variety Bi(S/L), for special bundles. For that we will use the following lemma of Hirschowitz-Narasimhan (see [5], lemma 4.9) :

2.1.9. LEMMA. – *Let* F *be a functor on the category of varieties and* mod : F $\to \hat{\mathrm{M}}$ *a morphism from* F *to the functor* $\hat{\mathrm{M}}$ *represented by a variety* M. *Let us assume that :*

1) *For each variety* T, *if* α *and* β *are two elements of* F(T) *with* mod (α) = mod (β), *then* α *and* β *are locally equal in* T.

2) *There exist* T *and* u *in* F(T) *such that* mod (u) *is proper and smooth.*
Then M *is a coarse moduli for* mod.

2.2. Bipencil of divisors on reducible curves.

Until the end of this chapter the integer d, $d \geqslant g + 1$, is fixed. We construct the bipencil variety of degree d, Bi(S/L) (see 2.2.3). This variety comes with an universal bipencil U $\to$ Bi(S/L) which is a non banal conic bundle (see 2.2.6).

The idea for the definition of a bipencil of divisors of degree d comes from the following remark :

2.2.1. *Remark.* – For each special bundle E, there exist an unic bifiber $C_E = C_1 \cup C_2$, an unic (upto isomorphism) line bundle $\mathscr{G}_E$ on C_E

and an exact sequence :

$$0 \to E^{v} \xrightarrow{j_E^{v}} H^0(S,E)^{v} \otimes \mathcal{O}_S \xrightarrow{\Phi} \mathcal{G}_E \to 0$$

where $\mathcal{G}_E = \mathcal{F}_E^{v}$ (see 2.1.2, 2.1.3, 2.1.7).

Let us denote by $\mathcal{G}_i$ the restriction of $\mathcal{G}_E$ to C_i, for i in $\{1,2\}$. The morphism Φ induces the morphism :

$$\mathbf{P}(H^0(S,E)) \to \mathbf{P}H^0(C_1,\mathcal{G}_1) \otimes \mathbf{P}H^0(C_2,\mathcal{G}_2).$$

As Φ is surjective, the image P_i of $\mathbf{P}H^0(S,E)$ in $\mathbf{P}H^0(C_i,\mathcal{G}_i)$ is a pencil of divisors, without base point, for i in $\{1,2\}$. So $\mathbf{P}H^0(S,E)$ can be seen as a projective line in the product $P_1 \times P_2$, such that the projection of $\mathbf{P}H^0(S,E)$ on P_i is an isomorphism for each i in $\{1,2\}$ (i.e. as a diagonal of $P_1 \times P_2$).

2.2.2. Definition. — *Let* $C = C_1 \cup C_2$ *be a bifiber and, for each* i *in* $\{1,2\}$, *let* P_i *be a pencil of non special divisors of degree* d *on* C_i, *without base point. A* bipencil of divisors *of degree* d *(without base point) on the bifiber* C *is a subscheme* P *of the product* $P_1 \times P_2$, *such that the projection of* P *on each* P_i *is an isomorphism.*

In other words, a bipencil of divisors is given by :

i) Two distincts smooth fibers C_1 and C_2 of $\Pi : S \to L$.

ii) A pencil of non special divisors P_1 (resp. P_2) of degree d, without base point, on C_1 (resp. C_2).

iii) An identification between P_1 and P_2.

Let $\mathrm{Bi}(S/L)$ be the set of bipencil of divisors of degree d on the bifibers. One can describe $\mathrm{Bi}(S/L)$ in the following way :

2.2.3. *Bipencil variety, of degree* d, *of a reducible relative curve.*

Let $X \to T$ be a flat family of smooth reducible curves such that, for each t in T, the curve X_t has two connected components which are smooth curves of genus g (in our case, $X \to T$ will be the universal bifiber $\mathcal{C} \to S^2_*(L)$). Fix an integer d, $d \geqslant g+1$.

Let $\mathrm{Div}^{2d}(X/T)$ (resp. $\mathrm{Jac}^{2d}(X/T)$) be the T-scheme whose fiber over t in T is the set of non special effective divisors (resp. line bundle) of degree

d on each component of X_t, whose associated complete linear system has no base point (resp. which are generated by their global sections). The fibers of the natural map $f: \mathrm{Div}^{2d}(X/T) \to \mathrm{Jac}^{2d}(X/T)$ are $\mathbf{P}^{d-g} \times \mathbf{P}^{d-g}$. Let $\mathrm{Gr}(1,\mathrm{Div}^{2d}(X/T))$ be the relative Grassmannian (over $\mathrm{Jac}^{2d}(X/T)$) of lines in the fibers of f.

We define the *bipencil variety* of degree d, $\mathrm{Bi}(X/T)$, to be the open subset of $\mathrm{Gr}(1,\mathrm{Div}^{2d}(X/T))$ of subschemes D of the fibers of f, such that : (i) the projection on each factor is an isomorphism from D to its image, (ii) the image of D on each factor, is a pencil of divisors without base point.

Note that $\mathrm{Bi}(X/T)$ comes with a tautological bipencil of degree d, namely, the tautological sub-bundle $U_{X/T}$ of

$$\mathrm{Bi}(X/T) \times_{\mathrm{Jac}^{2d}(X/T)} \mathrm{Div}^{2d}(X/T).$$

2.2.4. Definition. — *The bipencil variety of degree d (associated to $\Pi : S \to L$) is the bipencil variety of degree d, $\mathrm{Bi}(\mathscr{C}/S^2_*(L))$, associated to the universal bifiber $\mathscr{C} \to S^2_*(L)$.*

For simplicity, we denote it by $\mathrm{Bi}(S/L)$ and by $U \to \mathrm{Bi}(S/L)$ the tautological bipencil of degree d.

We want to prove that $U \to \mathrm{Bi}(S/L)$ is a non banal conic bundle; for that, we interpret it as the universal passerelle associated to a conic bundle. We need to introduce some notations :

2.2.5. *Notations.* — Let $\mathrm{Div}^d(S/L)$ (resp. $\mathrm{Jac}^d(S/L)$) be the L-scheme whose fiber over l in L is the set of non special effective divisors (resp. line bundles) of degree d on the fiber $\Pi^{-1}(\ell)$, whose associated complete linear system has no base point (resp. which are generated by their global sections). The natural map $f: \mathrm{Div}^d(S/L) \to \mathrm{Jac}^d(S/L)$ is a $\mathbf{P}^{d-g}$-bundle. Let $\mathrm{Gr}(1,\mathrm{Div}^d(S/L))$ be the relative Grassmannian (over $\mathrm{Jac}^d(S/L)$) of the lines of the fibers of f. Let P be the open subset of $\mathrm{Gr}(1,\mathrm{Div}^d(S/L))$ of pencils without base point and $\mathscr{T} \to P$ be the tautological conic sub-bundle of $P \times_{\mathrm{Jac}^d(S/L)} \mathrm{Div}^d(S/L)$.

2.2.6. Proposition. — *The tautological bipencil of degree d, $U \to \mathrm{Bi}(S/L)$ is a non banal conic bundle.*

Proof. — The universal passerelle $U \to \mathrm{BiP}$ associated to the conic bundle $\mathscr{T} \to P$ (see 1.2.3) is a non banal conic bundle (see 1.2.8).

Moreover, BiP is a $S^2_*(P)$ – scheme, so, it is also a $S^2(L)$ – scheme and Bi(S/L) is the restriction of BiP to $S^2_*(L)$. As $S^2_*(L)$ is a dense open subset of $S^2(L)$, Bi(S/L) is an open subset of BiP so (remark 1.2.1), $U \to$ Bi(S/L) is also a non banal conic bundle.

2.3. Modular morphism.

Let SP be the functor on the category of varieties which associates to T the set of isomorphism classes of special bundles $\mathscr{E}$ on $T \times S$ (a vector bundle $\mathscr{E}$ on $T \times S$ is special, if, for each t in T, the vector bundle $\mathscr{E}(t)$ is special with second Chern class $c_2 = 2d$).

We construct the modular morphism mod from SP to the functor $\widehat{\mathrm{Bi(S/L)}}$ represented by the pencil variety Bi(S/L) (see 2.3.1). By using the lemma 2.1.9, we prove (see 2.3.3) that Bi(S/L) is a coarse moduli for SP which has no Poincaré object.

2.3.1. DEFINITION. – *In remark* 2.2.1, *we have associated, to each special bundle* E *on* S *a bipencil of divisors of degree* d. *Let us denote it by* $\langle E \rangle$. *For any special bundle* $\mathscr{E}$ *on* $T \times S$, *let* $\mathrm{mod}_{\mathscr{E}}$ *be the morphism :*

$$\begin{aligned} \mathrm{mod}_{\mathscr{E}} : T &\to \mathrm{Bi(S/L)} \\ t &\mapsto \langle \mathscr{E}(t) \rangle. \end{aligned}$$

This defines the morphism, mod, from SP to $\widehat{\mathrm{Bi(S/L)}}$.

2.3.2. LEMMA. – *Let* $U \to$ Bi(S/L) *be the tautological bipencil of degree* d. *A morphism* $\varphi : T \to$ Bi(S/L) *is in the image of* $\overline{\mathrm{mod}}$ (*i.e. there exists a special bundle* $\mathscr{E}$ *on* $T \times S$ *such that* $\varphi = \mathrm{mod}_{\mathscr{E}}$) *if and only if the conic bundle* φ^*U *is banal.*

Proof. – Necessary condition : let $\mathscr{E}$ be a special bundle on $T \times S$ and $\tau : T \times S \to T$ be the projection. By base change theorem and by proposition 2.1.3, $\tau_*(\mathscr{E})$ is a rank two vector bundle on T. As U is the universal scheme over Bi(S/L), $(\mathrm{mod}_{\mathscr{E}})^*U = \mathbf{P}(\tau_*\mathscr{E})$, so $(\mathrm{mod}_{\mathscr{E}})^*U$ is banal.

Sufficient condition : let $\varphi : T \to$ Bi(S/L) be a morphism such that φ^*U is a banal conic bundle. Let M be a line bundle on φ^*U of relative degree one and $\mu : \mathscr{C}_T \to \varphi^*U$ the $2d$-covering determined by φ where $\mathscr{C}_T : = \mathscr{C} \times_{s^2_*(L)} T$.

We have the following diagram :

$$\begin{tikzcd}
\tilde{\mathscr{F}} \arrow[d] & \mathscr{F} : = \mu^*M \arrow[d] & M \arrow[d] \\
T \times S \arrow[dr, "\tau"'] & \mathscr{C}_T \arrow[l] \arrow[r, "\mu"] \arrow[d, "P"] & \varphi^*U \arrow[dl, "\gamma"] \\
 & T &
\end{tikzcd}$$

Set $W : = \gamma_* M$ and $\mathscr{F} = \mu^*M$. The natural morphism

$$\gamma^*\gamma_*M \to M$$

induces by pullback (by μ) the morphism

$$P^*W \to \mathscr{F}$$

that is, the surjective morphism :

$$\Phi : \tau^*W \to \tilde{\mathscr{F}}.$$

Set $\mathscr{E} : = (\ker \Phi)^v$. Then $\mathscr{E}$ is locally free because $\tilde{\mathscr{F}}$ is of homological dimension one. By dualyzing the exact sequence :

$$0 \to \ker \Phi \to \tau^*W \to \tilde{\mathscr{F}} \to 0$$

we obtain

$$0 \to (\tau^*W)^v \to \mathscr{E} \to \tilde{\mathscr{F}}^v \to 0$$

(see lemma 2.1.7). So $\mathscr{E}$ is a special bundle and $\mathrm{mod}_{\mathscr{E}} = \varphi$.

2.3.3. Proposition. — *The bipencil variety* Bi(S/L) *is a coarse moduli for* SP *which has no Poincaré object.*

Proof. — To apply the lemma 2.1.9 let us verify its hypothesis.

1) Let $\mathscr{E}$ and $\mathscr{E}'$ be two special bundles on $T \times S$ such that

$\mathrm{mod}_{\mathscr{E}} = \mathrm{mod}_{\mathscr{E}'}$. We have the two exact sequences:

$$0 \to \mathscr{E}^v \to \tau^*(\tau_*\mathscr{E})^v \xrightarrow{\Phi} \mathscr{G}_{\mathscr{E}} \to 0$$

$$0 \to \mathscr{E}'^v \to \tau^*(\tau_*\mathscr{E}')^v \xrightarrow{\Phi'} \mathscr{G}_{\mathscr{E}'} \to 0$$

where $\tau : \mathrm{T} \times \mathrm{S} \to \mathrm{T}$ is the projection, $\mathscr{G}_{\mathscr{E}} = \mathscr{F}^v_{\mathscr{E}}$ and $\mathscr{G}_{\mathscr{E}'} = \mathscr{F}^v_{\mathscr{E}'}$.

For each t in T, the restrictions of $\mathscr{G}_{\mathscr{E}}$ and $\mathscr{G}_{\mathscr{E}'}$ to $\{t\} \times \mathrm{S}$ are isomorphic. By base change theorem the sheaf $\tau_* \mathscr{H}om(\mathscr{G}_{\mathscr{E}}, \mathscr{G}_{\mathscr{E}'})$ is then locally free (of rank two). So, we can assume that $\mathscr{G}_{\mathscr{E}}$ and $\mathscr{G}_{\mathscr{E}'}$ are isomorphic. We have seen (proof of lemma 2.3.2) that $(\mathrm{mod}_{\mathscr{E}})^*\mathrm{U} = \mathbf{P}(\tau_*\mathscr{E})$ and $(\mathrm{mod}_{\mathscr{E}'})^*\mathrm{U} = \mathbf{P}(\tau_*\mathscr{E}')$. So, we can assume that there exists an isomorphic $g : (\tau_*\mathscr{E})^v \to (\tau_*\mathscr{E}')^v$ which makes a commutative diagram:

$$\begin{tikzcd}
\mathbf{P}(\tau_*\mathscr{E}) \arrow[rr, "\mathbf{P}(g)"] \arrow[dr, "\simeq"'] & & \mathbf{P}(\tau_*\mathscr{E}^1) \arrow[dl, "\simeq"] \\
& (\mathrm{mod}_{\mathscr{E}})^*\mathrm{U} &
\end{tikzcd}$$

where $\mathbf{P}(g)$ is the morphism induced by g.

But $(\mathrm{mod}_{\mathscr{E}})^*\mathrm{U}$ can be identified with a subbundle of $\mathbf{P}(\mathscr{G}_{\mathscr{E}})$. So, there exist an automorphism h of $\mathscr{G}_{\mathscr{E}}$ such that the diagram:

$$\begin{tikzcd}
\tau^*(\tau_*\mathscr{E})^v \arrow[r, "\tau^*g"] \arrow[d, "\Phi"] & \tau^*(\tau_*\mathscr{E}')^v \arrow[d, "\Phi'"] \\
\mathscr{G}_{\mathscr{E}} \arrow[r, "\bar{h}"] & \overline{\mathscr{G}_{\mathscr{E}'}} = \mathscr{G}_{\mathscr{E}}
\end{tikzcd}$$

is commutative where $\bar{h}$ is the automorphism of $\mathscr{G}_{\mathscr{E}}$ induced by h and τ^*g is the pullback of g.

So $\mathscr{E}^v$ and $\mathscr{E}'^v$ (and then $\mathscr{E}$ and $\mathscr{E}'$) are isomorphic.

2) Let us remark that if $f : \mathrm{Y} \to \mathrm{X}$ is a conic bundle, then $f^*\mathrm{Y} \to \mathrm{Y}$ is a banal conic bundle. So (lemma 2.3.2), the morphism $\mathrm{U} \to \mathrm{Bi(S/L)}$ is in

the image of mod. So $\mathrm{Bi}(S/L)$ is a coarse moduli for SP, it has no Poincaré object because the identity of $\mathrm{Bi}(S/L)$ does not satisfy the condition of lemma 2.3.3 (see proposition 2.2.6).

2.4. Very special bundles.

In this paragraph, $\Pi : S \to \mathbf{L}$ is a surface fibered in curves of genus g, which satisfies the condition $(*)$ of chapter 2's introduction.

2.4.1. *Definitions and Notations.*

- A special bundle E on S is *very special* if, for every exceptional line X_i in S, the restriction E/X_i of E to X_i is isomorphic to the bundle $\mathcal{O}_{X_i}(1) \oplus \mathcal{O}_{X_i}(1)$.

- Let VSP be the subfunctor of SP which associates to T the set of isomorphism classes of very special bundles on $T \times S$ (a vector bundle $\mathscr{E}$ on $T \times S$ is very special, if for each t in T, $\mathscr{E}(t)$ is very special).

- Let $\mathrm{mod}\,(*) : \mathrm{SP}(*) \to \mathrm{Bi}(S/L)$ be the modular morphism (see 2.3.1) and set $\mathrm{Bi}^0(S/L) := \mathrm{mod}\,(*)(\mathrm{VSP}(*))$.

In order to prove that $\mathrm{Bi}^0(S/L)$ is a coarse moduli for VSP (via the restriction of the modular morphism mod, to VSP) which has no Poincaré object (see 2.4.3) we prove (see 2.4.2) that $\mathrm{Bi}^0(S/L)$ is a dense open subset of $\mathrm{Bi}(S/L)$.

2.4.2. LEMMA. – *The set* $\mathrm{Bi}^0(S/L)$ (*which corresponds to the very special bundles*) *is a dense open subset of* $\mathrm{Bi}(S/L)$.

Proof. – Let $U \xrightarrow{\delta} \mathrm{Bi}(S/L)$ be the tautological bipencil of degree d and $\mathscr{E}$ a vector bundle on $U \times S$ such that $\mathrm{mod}_{\mathscr{E}} = \delta$ (see proof of proposition 2.3.3,2)). For each t in U, and for each exceptional line X_i in S, the restriction $\mathscr{E}(t)/X_i$ is a rank two vector bundle of degree two on X_i. As there is a finite number of exceptional lines, the semi continuity theorem tells us that the set $U^0 := \{t \in U, \mathscr{E}(t) \text{ is very special}\}$ is open in U. Now $\mathrm{Bi}^0(S/L)$ is open in $\mathrm{Bi}(S/L)$ because δ is an open morphism and $Bi^0(S/L) = \delta(U^0)$. To finish we have to prove that $\mathrm{Bi}^0(S/L)$ is not empty. Let $E = (\ker \Phi)^v$ be the special bundle constructed in the proof of lemma 2.1.8 where Φ is a surjection form $\mathcal{O}_S \oplus \mathcal{O}_S$ into the sheaf $\mathscr{F}$. For each exceptional line X_i, let $\Phi_i : \mathcal{O}_{X_i} \oplus \mathcal{O}_{X_i} \to \mathcal{F}/X_i$ be the restriction of Φ to X_i. Then E is very special if and only if for each X_i the morphism $H^0(\Phi_i)$ induced on the global sections is surjective. This condition is dense and open, as there is a finite number of X_i, there exists a surjection $\Phi : \mathcal{O}_S \oplus \mathcal{O}_S \to \mathcal{F}$ such that $E = (\ker \Phi)^v$ is very special.

2.4.3. Proposition. — *The variety* $\mathrm{Bi}^0(S/L)$ *is a coarse moduli for* VSP *which has no Poincaré Bundle.*

Proof. — As $\mathrm{Bi}^0(S/L)$ is a dense open subset of $\mathrm{Bi}(S/L)$, the same proof as in 2.3.3, shows that $\mathrm{Bi}^0(S/L)$ is a coarse moduli for VSP. By using the remark 1.2.1 and the proposition 2.2.6, we also see that the identity of $\mathrm{Bi}^0(S/L)$ does not satisfy the condition of the lemma 2.3.3. So $\mathrm{Bi}^0(S/L)$ has no Poincaré object.

Chapter 3

POINCARÉ BUNDLES FOR PROJECTIVE SURFACES

In this chapter X is a smooth projective surface, H is a very ample divisor on X, K is the canonical divisor and $M_H(c_1,c_2)$ is the moduli space of rank two H-stable vector bundles on X with Chern classes (in the numerical ring of X) c_1 and c_2.

First, we give a sufficient condition for the existence of a Poincaré Bundle on $M_H(c_1,c_2) \times X$ (see 3.1.5) which is nothing but a variation of Maruyama's condition (see [8], theorem 6.11).

Then, in order to give a necessary condition, we associate to (X,H) a surface fibered in curves, which satisfies the condition (*) of chapter 2 introduction (see 3.2.1). In fact, S is the blowing up of X along a finite set of points. Let $\mathcal{L}\left(\sum_{i=1}^{m} X_i\right)$ be the line bundle on S, associated to the divisor $\sum_{i=1}^{m} X_i$, where $(X_i)_{1\leqslant i\leqslant m}$ are the exceptional lines on S. If E is very special on S (see 2.4.1), then, the bundle $E \otimes \mathcal{L}\left(\sum_{i=1}^{m} X_i\right)$ descends to X (see 3.2.3). As the coarse moduli $\mathrm{Bi}^0(\mathrm{S/L})$ of very special vector bundles on S has no Poincaré object (see 2.4.3), we show (theorem 3.2.7) that, for some suitable values of c_1 and c_2, there is no Poincaré Bundle on $\mathrm{M_H}(c_1,c_2) \times \mathrm{X}$.

3.1. A sufficient condition of existence.

3.1.1. *Notations.* Let $\mathrm{M_H}(c_1,c_2;r)$ be the moduli space of rank r, H-stable vector bundles on X with Chern classes c_1 and c_2. Let E in $\mathrm{M_H}(c_1,c_2;r)$ and set:

$$\chi := \chi(\mathrm{E})$$
$$\delta := gcd\{\chi(\mathrm{E}\otimes \mathrm{L}^m),\ \mathrm{L} \text{ very ample line bundle on } \mathrm{X},\ m\in \mathbf{Z}\}.$$

M. Maruyama has proved (see [8], theorem 6.11) that if δ equals 1, then there exists a Poincaré Bundle on $\mathrm{M_H}(c_1,c_2;r) \times \mathrm{X}$.

We want to prove that if δ and r are coprime, then there exists a Poincaré bundle on $\mathrm{M_H}(c_1,c_2;r) \times \mathrm{X}$. Note that one can find K-3 surfaces and Chern classes c_1, c_2 such that δ is coprime to $r = 2$ but δ is not equal to one.

For any line bundle L on X we will note $c_1.\mathrm{L}$ the intersection number $c_1.c_1(\mathrm{L})$ where $c_1(\mathrm{L})$ is the first Chern class (in the numerical ring of X) of L.

Let Num X be the group of divisors on X modulo numerical equivalence.

3.1.2. *Remark.*

$$\begin{aligned} gcd\{r,\chi(\mathrm{E}\otimes\mathrm{L}^m);\ \mathrm{L} \text{ very ample line bundle on } \mathrm{X},\ m\in\mathbf{Z}\} \\ = gcd\{r,\chi,c_1.\mathrm{L};\ \mathrm{L} \text{ very ample line bundle on } \mathrm{X}\} \\ = gcd\{r,\chi,c_1.\mathrm{L};\ \mathrm{L} \text{ line bundle on } \mathrm{X}\}. \end{aligned}$$

The first equality occurs because for every line bundle L on X,

$$\chi(\mathrm{E}\otimes\mathrm{L}) = \chi(\mathrm{E}) + r\left(\frac{\mathrm{L}^2-\mathrm{L.K}}{2}\right) + \mathrm{L}.c_1(\mathrm{E}),$$

where, by Riemann-Roch, $\mathrm{L}^2 - \mathrm{L.K}$ is even.

The second equality occurs because for every line bundle M on X, there exists a very ample line bundle L such that $L \otimes M$ is very ample (then note that $c_1.M = c_1.(L \otimes M) - c_1.L$).

3.1.3. THEOREM (*Variation of Maruyama's condition*). — *If* $gcd\{r,\chi,c_1.\ell;\ell \in \mathrm{Num}\, X) = 1$ *then there exists a Poincaré Bundle on* $M_H(c_1,c_2;r) \times X$.

Proof. — Because of 3.1.1 and 3.1.2, we have to prove that, if δ and r are coprime, then there exists a Poincaré Bundle on $M_H(c_1,c_2;r) \times X$. There exists a principal fibre bundle $q: R \to M_H(c_2,c_2;r)$ with group PGL(N) and a universal quotient sheaf F on $R \times X$. Maruyama has proved (proof of theorem 6.11 [8]) that, if there exists a line bundle L on R with a GL(N) – linearization such that the action on L of the center C of GL(N) is the multiplication by the inverse of constants, then there exists a Poincaré Bundle on $M_H(c_1,c_2;r) \times X$. Let us determine such a line bundle L. The line bundle $\det F$ on $R \times X$ is GL(N)-linearized and the action of C on $\det F$ is the multiplication by r^{th} power of constants. Let $P \to \mathrm{Pic}\, X \times X$ be a Poincaré line bundle. Set

$$L' := q_*(\det F \otimes p^* P^{\vee})$$

where $q: R \times X \to R$ is the projection and p is the morphism:

$$\begin{aligned} p: R \times X &\to \mathrm{Pic}\, X \times X \\ (t,x) &\to (\det(q(t)),x). \end{aligned}$$

By base change theorem, L′ is a line bundle, so it is GL(N)-linearized and the action of C on L′ is the multiplication by r^{th} power of constants.

Let L″ be the line bundle constructed by Maruyama (proof of theorem 6.11, [8]) such that the action of C on L″ is the multiplication by δ^{th} power of constants. By hypothesis, there exist two integers a and b such that $ar + b\delta = -1$. The line bundle $L := L'^{\otimes a} \otimes L''^{\otimes b}$ is the line bundle we were looking for.

We apply this result to rank-2 vector bundles on X. Let us recall that, Riemann Roch formula gives

$$\chi(E) = 2\chi(\mathcal{O}_X) + \frac{1}{2} c_1^2 - \frac{1}{2} c_1 K - c_2$$

for every E in $M_H(c_1,c_2)$.

3.1.4. DEFINITION. — *A divisor* Δ *is* even (*resp.* numerically even) *if there exists a divisor* Δ^1 *such that* Δ *is numerically equivalent to* $2\Delta^1$ (*resp. for each divisor* Δ^1, *the intersection number* $\Delta.\Delta^1$ *is an even number*). *Otherwise* Δ *is odd* (*resp. numerically odd*).

Let us remark that every even divisor is numerically even and every numerically odd divisor is odd.

One can rephrase the theorem 3.1.3 in the following way:

3.1.5. COROLLARY. — *If* $\frac{1}{2}c_1^2 - \frac{1}{2}c_1\mathrm{K} - c_2$ *is odd or if* c_1 *is numerically odd, then there exists a Poincaré Bundle on* $\mathrm{M_H}(c_1,c_2) \times \mathrm{X}$.

3.2. A necessary condition of existence.

The very ample divisor H on X defines an embedding from X to a projective space $\mathbf{P}^\mathrm{N}$. Let us identify X and its image in $\mathbf{P}^\mathrm{N}$. Call m the degree of the surface X in $\mathbf{P}^\mathrm{N}$.

3.2.1. *Fibered surface associated to* (X,H). — By Bertini's theorem, there exists a hyperplane V of $\mathbf{P}^\mathrm{N}$ (resp. W of V) such that the curve $\mathrm{C} := \mathrm{V} \cap \mathrm{X}$ is smooth and irreducible (resp. the 0-dimensional scheme $\mathrm{Y} := \mathrm{W} \cap \mathrm{C}$ is regular at every point).

Let $\mathfrak{L}$ be the projective line of hyperplanes of $\mathbf{P}^\mathrm{N}$ which contains W. We denote by L the (dense) open subset of $\mathfrak{L}$ such that if W^1 is in L, then the curve $\mathrm{W}^1 \cap \wedge \mathrm{X}$ is smooth and irreducible. Let S be the incidence variety in $\mathrm{X} \times \mathfrak{L}$, that is

$$\mathrm{S} = \{(x,\ell) \in \mathrm{X} \times \mathfrak{L} / x \in \ell\}.$$

Then the projection $\varepsilon : \mathrm{S} \to \mathrm{X}$ is the blowing up of X along Y, and the projection $\Pi : \mathrm{S} \to \mathfrak{L}$ is a surface fibered in curves of genus g, $g = \frac{1}{2}(\mathrm{H}^2 + \mathrm{HK}) + 1$, which satisfies the condition $(*)$ of chapter 2's introduction. We say that $\Pi : \mathrm{S} \to \mathfrak{L}$ is the fibered surface associated to (X,H).

3.2.2. *Notation.* — Fix an integer d, $d \geqslant g + 1$ and set $c_2' = 2d$.

Let G (resp. M) be the functor on the category of varieties which associates to T the set of isomorphism classes of algebraic coherent sheaves (resp. rank two, H-stable vector bundles with Chern classes $c_1 = 2\mathrm{H}$ Λ and $c_2 = c_2' + m$) on $\mathrm{T} \times \mathrm{X}$.

Let T be a variety and $\mathscr{E}$ be a very special bundle on $\mathrm{T} \times \mathrm{S}$. We note $\varepsilon_t : t \times \mathrm{S} \to \mathrm{T} \times \mathrm{X}$ the morphism $(\mathrm{id}_\mathrm{T} \times \varepsilon)$ where $\varepsilon : \mathrm{S} \to \mathrm{X}$ is the blowing up of X along Y (see 3.2.1),

$$\tau : \mathrm{T} \times \mathrm{S} \to \mathrm{S} \text{ the projection}$$

and

$$\mathscr{L}_\mathrm{T}\left(\sum_{i=1}^{m} \mathrm{X}_i\right) \text{ the pullback } \tau^* \mathscr{L}\left(\sum_{i=1}^{m} \mathrm{X}_i\right) \text{ on } \mathrm{T} \times \mathrm{S}$$

of the line bundle $\mathscr{L}\left(\sum_{i=1}^{m} \mathrm{X}_i\right)$ on S associated to the divisor $\sum_{i=1}^{m} \mathrm{X}_i$, where $(\mathrm{X}_i)_{1 \leqslant i \leqslant m}$ are theexceptional lines in S.

Let $\varepsilon_* : \mathrm{VSP} \to \mathrm{G}$ be the morphism which associates to each very special bundle $\mathscr{E}$ on $\mathrm{T} \times \mathrm{S}$ (such that, for each t in T, $c_2(\mathscr{E}(t)) = c_2'$) the isomorphism class of the sheaf $(\varepsilon_\mathrm{T})_*\left(\mathscr{E} \otimes \mathscr{L}_\mathrm{T}\left(\sum_{i=1}^{m} \mathrm{X}_i\right)\right)$.

We want to prove that the values of ε_* are in M. For that we will use the following four lemmas:

3.2.3. Lemma. — *Let* T *be a variety and* $\mathscr{E}$ *be a very special bundle on* $\mathrm{T} \times \mathrm{S}$. *Set* $\mathrm{V} := (\varepsilon_\mathrm{T})_*\left(\mathscr{E}_\mathrm{T} \otimes \mathscr{L}_\mathrm{T}\left(\sum_{i=1}^{m} \mathrm{X}_i\right)\right)$. *Then* V *is locally free and* $\mathscr{E} = \varepsilon_\mathrm{T}^* \mathrm{V} \otimes \mathscr{L}_\mathrm{T}\left(-\sum_{i=1}^{m} \mathrm{X}_i\right)$.

Proof. — Recall that $\varepsilon : \mathrm{S} \to \mathrm{X}$ is the blowing up along Y and that $\mathscr{E}$ is very special so, for each t in T and for each exceptional line X_i in S, the restriction to $\{t\} \times \mathrm{X}_i$ of the bundle $\mathscr{E} \otimes \mathscr{L}_\mathrm{T}\left(\sum_{i=1}^{m} \mathrm{X}_i\right)$ is isomorphic to the trivial rank two vector bundle on X_i.

If T is reduced to one point the lemma follows from Schwarzenberger's result (see [15], theorem 5). In the general case, the proof is similar to that of Ellinsgrud-Stromme (proposition 2.2, [2]).

3.2.4. LEMMA. — *Let* E *be a very special bundle on* S *such that, for each* t *in* T, $c_2(\mathscr{E}(t)) = c'_2$. *Set* $V := \varepsilon_*\left(E \otimes \mathscr{L}\left(\sum_{i=1}^m X_i\right)\right)$, *then* V *is a rank two vector bundle with Chern classes* $c_1(V) = 2H$ *and* $c_2(V) = c_2$.

Proof. — By lemma 3.2.3 we know that V is a rank two vector bundle on X and that $E = \varepsilon^* V \otimes \mathscr{L}\left(-\sum_{i=1}^m X_i\right)$. Let F be a smooth fiber of $\Pi : S \to \mathbb{L}$; as E is a special bundle on S, $c_1(E) = 2F$ so $c_1(V) = 2H$.

Moreover,

$$c_2(V) = c_2(E) + c_1(E).c_1\left(\mathscr{L}\left(\sum_{i=1}^m X_i\right)\right) + c_1\left(\mathscr{L}\left(\sum_{i=1}^m X_i\right)\right)^2$$

so, $$c_2(V) = c_2(E) + \sum_{i=1}^m c_1(E).X_i + \left(\sum_{i=1}^m X_i\right)^2$$

so, $$c_2(V) = c_2 + m = c_2.$$

Let us recall the following general lemma:

3.2.5. LEMMA. — *Let* S *be a smooth projective surface.* H *an ample divisor and* E *a rank two vector bundle on* S. *Then,* E *is* H-*stable if and only if, for every non trivial locally free subsheaf* V *of* E,

$$2c_1(V).H < c_1(E).H.$$

Proof. — The necessary condition is clear, let us prove the sufficient one. Let V be a non trivial torsion free subsheaf of E. As V is torsion free, the natural morphism $f : V \to V^{vv}$ from Λ to its bidual is injective so, $c_1(V).H \leqslant c_1(V^{vv}).H$.

Moreover E is locally free, so, $E = E^{vv}$ and the injection $i : V \to E$ factorizes through V^{vv}, i.e. there exists an injection $j : V^{vv} \to E$ such that the diagram

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & V^{vv} \arrow[dl, hook', "j"] \\ & E & \end{tikzcd}$$

is commutative. But V^{vv} is a rank one reflexive sheaf so ([1], lemma 1) it is locally free so :

$$2c_1(V^{vv}).H < c_1(E).H$$

so,
$$2c_1(V).H < c_1(E).H.$$

3.2.6. LEMMA. — *Let* E *be a very special bundle on* S. *The vector bundle* V, *where* $V := \varepsilon_*\left(E \otimes \mathscr{L}\left(\sum_{i=1}^{m} X_i\right)\right)$, *is* H-*stable.*

Proof. — Let V_1 be a locally free subsheaf of V. Set $E_1 = \varepsilon^* V_1 \otimes \mathscr{L}\left(-\sum_{i=1}^{m} X_i\right)$. As V_1 is a line bundle, the injection (of sheaves) from V_1 to V defines a non zero section σ of the vector bundle $V \otimes V_1^{v}$. But ε is surjective, so $\varepsilon^*(\sigma)$ is a non zero section of the bundle $\varepsilon^*(V \otimes V_1^{v}) = \varepsilon^* V \otimes \varepsilon^* V_1^{v}$. So $\varepsilon^* V_1$ is a subsheaf of $\varepsilon^* V$ and E_1 is a subsheaf of E, where $E = \varepsilon^* V \otimes \mathscr{L}\left(-\sum_{i=1}^{m} X_i\right)$.

As E is a special bundle on S, from the exact sequence S(E) :

$$0 \to \mathcal{O}_S \oplus \mathcal{O}_S \to E \to \mathscr{F}_E \to 0$$

we see that two cases can happen :

Ist case. E_1 is a subsheaf of $\mathcal{O}_S \oplus \mathcal{O}_S$.
That implies that V_1 is a subsheaf of $\mathcal{O}_X \oplus \mathcal{O}_X$ but $\mathcal{O}_X \oplus \mathcal{O}_X$ is semi-stable, so $c_1(V_1).H \leqslant 0$, in particular $c_1(V_1).H < m$.

2nd case. — There exists a non zero morphism $\varphi : E_{1/C} \to \mathscr{F}$ where $\mathscr{F}$ is the restriction of $\mathscr{F}_E$ to an irreducible component C of the bifiber c_E.

But $\deg(E_{1/C}) = c_1(E_1).F$ where F is a fiber of $\Pi : S \to \mathrm{L}$; as $\deg(\mathscr{F}) < 0$, we deduce that $c_1(E_1).F < 0$, but $c_1(E_1).F = c_1(V_1).H - m$, *so* $c_1(V_1).H < m$.

We have proved that, for every locally free subsheaf V_1 of V,

$$c_1(V_1).H < m.$$

But $c_1(V) = 2H$ (see 3.2.4) and $H^2 = m$ because m is, by definition, the degree of X. So

$$2c_1(V_1).H < c_1(V).H.$$

So V is H-stable.

3.2.7. Theorem. — *If c_1, is even and if $c_2 - \frac{1}{4}c_1^2$ is an even integer at least equal to $H^2 + HK + 4$, then, there is no Poincaré Bundle on $M_H(c_1,c_2) \times X$.*

Proof. — As c_1 is even, we can assume that $c_1 = 2H$. So $c_2 - \frac{1}{4}c_1^2 = c_2 - m = c_2' = 2d$ (see 3.2.2). As $g = \frac{1}{4}(H^2+HK)+1$, the condition $d \geqslant g + 1$ is equivalent to the condition

$$c_2 - \frac{c_1^2}{4} \geqslant H^2 + HK + 4.$$

Let M^1 be the image, in G, of the morphism ε_* (see notation 3.2.2). Then the two functors M^1 and VSP are isomorphic. So, by proposition 2.4.3, $Bi^0(S/L)$ is a coarse moduli for M^1, which has no Poincaré object. Moreover, by the lemmas 3.2.3, 3.2.4 and 3.2.6, M^1 is a subfunctor of M. So, there is no Poincaré Bundle on $M_H(c_1,c_2) \times X$.

3.2.8. *Remark.* — Every numerically even divisor is even if and only if the discriminant $\Delta(X)$ of the intersection pairing on Num X is odd. In fact, the intersection pairing gives a morphism from Num X to its dual $(\text{Num } X)^\vee$. By reduction modulo 2, we get the morphism :

$$(\text{Num } X) \otimes (\mathbf{Z}/2\mathbf{Z}) \to (\text{Num } X)^\vee \otimes (\mathbf{Z}/2\mathbf{Z})$$

which is an isomorphism if and only if the discriminant $\Delta(X)$ is inversible in $\mathbf{Z}/2\mathbf{Z}$ i.e. if and only if, $\Delta(X)$ is odd.

3.2.9. *Summary.* — Let χ be the Euler characteristic of rank-two vector bundles on X with Chern classes c_1 and c_2. Assume $c_2 - \frac{1}{4}c_1^2 \geqslant H^2 + HK + 4$. Then, we have the following diagram :

$$\begin{tikzcd}
\begin{array}{c}\text{EXISTENCE OF}\\ \text{POINCARE BUNDLE ON}\\ M_H(c_1,c_2) \times X\end{array} \arrow[r, Rightarrow, "\text{Theorem 3.2.7}"] & c_1 \text{ odd or } X \text{ odd} \arrow[d, "(1)"] \\
 & c_1 \text{ numerically odd or } \chi \text{ odd} \arrow[ul, Rightarrow, "\text{Corolarry 3.1.5}"]
\end{tikzcd}$$

where (1) is an equivalence if and only if $\Delta(X)$ is odd.

3.2.10. *Remark.* — If X is $\mathbf{P}^2$ or a ruled surface, then $\Delta(X)$ is odd. Unfortunately, for a lot of surfaces X, $\Delta(X)$ is even. In this case let c_1 be odd but numerically even, then if $\frac{1}{2}c_1^2 - \frac{1}{2}c_1.K - c_2$ is even, we do not know if there is a Poincaré Bundle on $M_H(c_1,c_2) \times X$. For example, if X is a general surface in $\mathbf{P}^3$ of even degree, at least equal to 4, or, if $X = L \times L$, where L is a general smooth curve, then $\Delta(X)$ is even.

3.3. Examples.

3.3.1. *Case of* $\mathbf{P}^2$. Let $X = \mathbf{P}^2$ and H be the hyperplane divisor. Then $H^2 + HK + 4 = 2$. Moreover (see [7], theorem 4.6), if c_1 is even $M_H(c_1,c_2)$ is not empty if and only if $c_2 - \frac{1}{4}c_1^2 \geqslant 2$. So 3.2.9 gives another proof of Le Potier's theorem (see [6] or the introduction). Note that Hirschowitz and Narasimhan have also proved this theorem (see [5]), their method, as well as Le Potier's one but unlike ours, makes distinction between the two cases $c_2 = 2$ and $c_2 > 2$.

3.3.2. *Remark.* – In the case where c_1 is even, we have proved that the condition $c_2 - \frac{1}{4}c_1^2 \geqslant H^2 + HK + 4$ is sufficient for $M_H(c_1,c_2)$ to be not empty (when $c_2 - \frac{1}{4}c_1^2$ is even). If $X = \mathbf{P}^2$, this condition is necessary.

Question 1. Is this condition necessary in general? Or can one find a surface X and c_1, c_2 such that c_1 is even, $c_2 - \frac{1}{4}c_1^2 < H^2 + HK + 4$ and $M_H(c_1,c_2) \neq \varnothing$?

Question 2. If $M_H(c_1,c_2)$ is not empty for some c_1 even and c_2 such that $c_2 - \frac{1}{4}c_1^2$ is even and smaller than $H^2 + HK + 4$, does there exist a Poincaré Bundle on $M_H(c_1,c_2) \times X$?

In order to give an answer (in the particular case where X is a ruled surface) to these questions (see 3.3.5) and also to give another example where Maruyama's condition (see 3.3.7) is necessary, we are going to give some complements to the theorem 3.2.7 (see 3.3.3) in the case where $\Pi : X \to L$ is a surface fibered in irreducible curves of genus g, F is a smooth fiber of Π and H is an ample divisor on X.

3.3.3. Theorem. – *Let* X *be as above.*

Assume that c_1 *is even and that* $c_2 - \frac{1}{4}c_1^2 = 2d$ *where* $d \geqslant g + 1$. *If* *the divisor* $d\mathrm{H} - (\mathrm{H.F})\mathrm{F}$ *is ample, then, does not exist a Poincaré Bundle on* $\mathrm{M_H}(c_1,c_2) \times \mathrm{X}$.

Proof. — As c_1 is even, we can assume that $c_1 = 2\mathrm{F}$ and $c_2 = 2d$ with $d \geqslant g+1$. By using the proposition 2.3.3, it is enough to prove that the special bundles on X are H-stable. Let E be a special bundle on X with second Chern class $c_2 = 2d$. As in remark 2.2.1, we have the exact sequence :

$$0 \to \mathrm{E}^v \to \mathcal{O}_{\mathrm{X}} \oplus \mathcal{O}_{\mathrm{X}} \xrightarrow{\Phi} \mathcal{G}_{\mathrm{E}} \to 0$$

where $\mathcal{G}_{\mathrm{E}}$ is a line bundle of degree d on the bifiber $\mathrm{C_E} = \mathrm{C_1} \cup \mathrm{C_2}$ and $\Phi = \operatorname{coker} j^v_{\mathrm{E}}$. Let us denote by $\mathcal{G}_i$ the restriction of $\mathcal{G}_{\mathrm{E}}$ to C_i for i in $\{1,2\}$. The morphism Φ is given by the matrix $\mathrm{M}(\Phi) = \begin{pmatrix} s_1 & \sigma_1 \\ s_2 & \sigma_2 \end{pmatrix}$, where σ_i and s_i are global sections of the line bundle $\mathcal{G}_i$.

Let V be a non trivial locally free subsheaf of E^v (see 3.2.5). We have the injection (of sheaves) $\psi : \mathrm{V} \to \mathcal{O}_{\mathrm{X}} \oplus \mathcal{O}_{\mathrm{X}}$ such that $\Phi \circ \psi \equiv 0$. By tensorising by the dual line bundle W of V we get the injection (of sheaves)

$$\begin{aligned} \varphi : \mathcal{O}_{\mathrm{X}} &\to \mathrm{W} \oplus \mathrm{W} \\ 1 &\to (\tau,\mu) = \varphi(1) \end{aligned}$$

where τ and μ are in $\mathrm{H^0(X,W)}$. As φ is injective, at least one of these two sections is non zero. For each i in $\{1,2\}$, let W_i (resp. τ_i, resp. μ_i) be the restriction of W (resp. τ, resp. μ) to the curve C_i. So τ_i and μ_i are in $\mathrm{H^0(C}_i,\mathrm{W}_i)$. The equality $\Phi \circ \psi \equiv 0$ becomes

$$s_i.\tau_i + \sigma_i.\mu_i = 0, \quad \text{for each } i \text{ in } \{1,2\}$$

where $s_i.\tau_i + \sigma_i.\mu_i$ is in $\mathrm{H^0}\ (\mathrm{C}_i,\mathcal{G}_i \oplus \mathrm{W}_i)$.

As $\deg(\mathcal{G}_i) = d$, the zero set of s_i (resp. σ_i) is a subset of C_i of order d. As Φ is surjective, these two sets are disjoint. So, τ (resp. μ) has d zeros on each C_i for i in $\{1,2\}$. There are two cases :

1*st case.* $c_1(\mathrm{W}) = \mathrm{F} + \Delta$ where Δ is an effective divisor.

As H is ample, $\Delta.\mathrm{H} > 0$ so, $c_1(\mathrm{W}).\mathrm{H} > \mathrm{F.H}$.

2*nd case.* $c_1(\mathrm{W}).\mathrm{F} \geqslant d$.

Then $d\,c_1(\mathrm{W}).\mathrm{H} = (d\mathrm{H}-(\mathrm{H.F}).\mathrm{F})c_1(\mathrm{W})+(c_1(\mathrm{W}).\mathrm{F})(\mathrm{H.F})$.

As $dH - (HF)F$ is ample $dc_1(W).H > (c_1(W).F)(H.F)$ so $c_1(W).H > H.F$.

As $c_1(E) = 2F$ we have proved that

$$2c_1(W).H > c_1(E).H$$

so,

$$2c_1(V).H < c_1(E^v).H.$$

So E^v (and then E) is H-stable.

3.3.4. Corollary. — *Let* $\Pi : X \to L$ *be a (geometrically) ruled surface. Denote by* F *a fiber of* Π, C_0 *the minimal section,* $H = mC_0 + rF$ *an ample divisor and* $e = -C_0^2$. *If one of the two conditions holds :*

i) $c_2 - \frac{1}{4} c_1^2 \geqslant H^2 + HK + 4$ and H is very ample.

ii) $c_2 - \frac{1}{4} c_1^2 \geqslant 2$, $m > 0$ and $\begin{cases} r > m\left(e + \dfrac{2}{c^2 - \frac{1}{4}c_1^2}\right) & \text{if } e \geqslant 0 \\ r > m\left(\dfrac{e}{2} + \dfrac{2}{c_2 - \frac{1}{4}c_1^2}\right) & \text{if } e < 0. \end{cases}$

Then, there exists a Poincaré Bundle on $M_H(c_1,c_2) \times X$ *if and only if* c_1 *is odd or* c_1 *is even and* $c_2 - \frac{1}{4}c_1^2$ *is odd.*

The part i) follows from 3.2.9 and the part ii) from 3.3.3.

3.3.5. *Remark.* — Let g_0 be the genus of L, then for any ample divisor H on X, $H^2 + HK + 4 \geqslant 2g_0 + 2$. Let H be a very ample divisor which satisfies the hypothesis, ii) of 3.3.4. Assume $g_0 \neq 0$. For every c_1 even and c_2 such that $c_2 - \frac{1}{4}c_1^2$ is even, if $2 \leqslant c_2 - \frac{1}{4}c_1^2 < H^2 + HK + 4$, then $M_H(c_1,c_2)$ is not empty (this answers to 3.3.2, question 1) and there is no Poincaré Bundle on $M_H(c_1,c_2) \times X$ (this answers, in this particular case, to 3.3.2, question 2).

3.3.6. Corollary. — *Let* X *be a rational ruled surface and* $H = aC_0 + bF$ *be an ample divisor on* X. *If one of the two following* *conditions holds :*

i) b *is a multiple of* a

ii) $b > ae + a$

then, there exists a Poincaré Bundle on $M_H(c_1,c_2) \times X$ *if and only if* c_1 *is odd or* c_1 *is even and* $c_2 - \frac{1}{4}c_1^2$ *is odd.*

Proof. — This follows from 3.3.4 because, on a rational ruled surface every ample divisor is very ample, moreover, the condition $c_2 - \frac{1}{4}c_1^2 \geqslant 2$ is necessary for $M_H(c_1,c_2)$ to be not empty (see [7], lemma 4.14 i)).

3.3.7. *Remark.* — This shows that for rational ruled surfaces Maruyama's condition is also necessary if H is « well chosen ». Moreover, if X is a quadric surface, there is no more condition on H, so Maruyama's condition is necessary for any choice of the ample divisor H.

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