ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 35, nº 2 (1985), p. 117-143
<http://www.numdam.org/item?id=AIF_1985__35_2_117_0>

Ann. Inst. Fourier, Grenoble
35, 2 (1985), 117-143.

EXISTENCE OF STAR-PRODUCTS ON EXACT SYMPLECTIC MANIFOLDS

by M. DE WILDE and P. B. A. LECOMTE

Introduction.

The study of formal deformations of the Lie algebra structures associated to symplectic manifolds has been initiated by Flato, Lichnerowicz and Sternheimer [7], where the case of 1-differentiable deformations was discussed. The case of differentiable deformations has been studied by Vey who has obtained [13] such deformations when the third de Rham cohomology space of the symplectic manifold vanishes. With the same assumption, Neroslavsky and Vlassov [11] have shown that the manifold admits star-products, which are special deformations of the associative algebra of smooth functions of the manifold. As the skew-symmetric part of a star-product induces a formal deformation of the Poisson algebra, the latter result implies the former. In fact, these results are equivalent. Indeed, as can be deduced from [9] which is a general reference for the subject, a symplectic manifold admits a star-product if and only if its Poisson bracket admits a formal deformation.

Due to the possibility to build up a phase-space formulation of quantum mechanics in which quantization will manifest itself in a deformation of the algebra of classical observables [2 for instance], an important work has been done to obtain and to describe formal deformations of the Poisson algebra and star-products on symplectic manifolds whose third de Rham cohomology space is not necessarily vanishing. In particular, it has been shown that every cotangent bundle of a parallelizable manifold has star-products [3], next that it is so for any cotangent bundle [6]. In the present paper, we extend this result to every symplectic manifold whose symplectic form is exact by constructing a deformation of the Poisson algebra of these manifolds.

Nijenhuis and Richardson [12] have endowed the space of alternating multilinear maps from a vector space into itself with a graded Lie algebra

Key-words : Symplectic manifold - Formal deformations - Lie algebra, star-products - Derivations.

structure which they have shown to be particularly useful in the study of deformations of Lie algebras. In section 1, we construct similarly a graded Lie algebra structure on the space of all multilinear maps from a vector space into itself which reduces by antisymmetrization to the Nijenhuis-Richardson structure. It allows a unified presentation of the theories of deformations of associative and Lie algebras and it turns out be a very convenient tool for the study of star-products.

In section 2, we describe the Hochschild and Chevalley cohomology spaces associated to a symplectic manifold in the precise way which we need for our purpose. In particular, for exact symplectic manifolds, we point out properties of the second and third Chevalley cohomology spaces which are basic for the proof of our main results.

Section 3 is devoted to the proof of the existence of star-products on each exact symplectic manifold. The star-products constructed here have the special property that they admit non formal derivations. This was the occasion to classify the non formal derivations of an arbitrary star-product or formal deformation of the Poisson algebra, thus completing the results of Lichnerowicz [10]. It turns out (section 4) that the existence of such derivations is a rather restrictive property : if a star-product admits a non formal derivation, then the manifold is exact and this star-product is unique up to equivalence. The situation is similar (but more delicate) for the formal deformations of the Poisson Lie algebra.

1. Graded Lie algebras associated to a vector space.

DEFINITION. – *Let* V *be a vector space. Denote by* $M^p(V)$ $(p \geqslant -1)$ *the space of* $(p+1)$*-linear maps from* V^{p+1} *into* V *and set*

$$M(V) = \bigoplus_{p \geqslant -1} M^p(V).$$

For $R \in M^r(V)$ *and* $S \in M^s(V)$, *we define* $R \,\underline{\vee}\, S \in M^{r+s}(V)$ *by* $R \,\underline{\vee}\, S = 0$ if $r = -1$ *and*

$$(R \,\underline{\vee}\, S)(x_0, \ldots, x_{r+s}) = \sum_{i=0}^{r} (-1)^{is} R(x_0, \ldots, x_{i-1}, S(x_i, \ldots, x_{i+s}), \; x_{i+s+1}, \ldots, x_{r+s})$$

if $r \geqslant 0$ *and we set*

$$R \Delta S = R \,\underline{\vee}\, S + (-1)^{rs+1} S \,\underline{\vee}\, R.$$

The main properties of the bilinear maps $\underline{\vee}$ and Δ from $M(V)^2$ into $M(V)$ are given in the following proposition.

PROPOSITION 1.1. – *If* $R \in M^r(V)$, $S \in M^s(V)$ *and* $T \in M^t(V)$, *then*

(i) $R \underline{\vee} (S \underline{\vee} T) - (R \underline{\vee} S) \underline{\vee} T = (-1)^{st}[R \underline{\vee} (T \underline{\vee} S) - (R \underline{\vee} T) \underline{\vee} S]$,

(ii) $S \Delta R = (-1)^{rs+1} R \Delta S$,

(iii) $(-1)^{rt} R \Delta (S \Delta T) + (-1)^{sr} S \Delta (T \Delta R) + (-1)^{ts} T \Delta (R \Delta S) = 0$.

Proof. – (i) follows from straightforward computations, (ii) is obvious and (iii) is easily deduced from (i).

Remark 1.2. – Properties (ii) and (iii) of Δ mean that Δ *equippes* $M(V)$ *with a graded Lie algebra structure* : (iii) is called *the graded Jacobi identity*; it may be written in the following equivalent form

(iii)′ $(-1)^{rt}(R \Delta S) \Delta T + (-1)^{sr}(S \Delta T) \Delta R + (-1)^{ts}(T \Delta R) \Delta S = 0$.

Denote by $\alpha : M(V) \to M(V)$ the antisymmetrization, that is the projector defined on $M^r(V)$ by

$$\alpha(R)(x_0,\ldots,x_r) = (1/(r+1)!)\Sigma_\nu \varepsilon_\nu R(x_{\nu_0},\ldots,x_{\nu_r})$$

where ν runs over the permutations of $(0,\ldots,r)$ and ε_ν denotes the signature of ν. Set $A^p(V) = \alpha(M^p(V))$ and $A(V) = \bigoplus_{p \geq -1} A^p(V)$. The products $\bar{\wedge}$ and $[\,,]$ defined on $A(V)$ by Nijenhuis and Richardson in [11] are now given by

$$R \bar{\wedge} S = ((r+s+1)!/(r+1)!(s+1)!)\,\alpha(R \underline{\vee} S)$$

and

$$[R,S] = ((r+s+1)!/(r+1)!(s+1)!)\,\alpha(R \Delta S)$$

where $R \in A^r(V)$ and $S \in A^s(V)$. In particular

PROPOSITION 1.3 [12]. – *The graded space* $A(V)$ *is equipped by* $[\,,]$ *with a structure of graded Lie algebra.*

Graded Lie algebra structure of the Hochschild and Chevalley cohomology spaces

For $\mathcal{M} \in M^1(V)$, one has

$$(\mathcal{M} \Delta \mathcal{M})(x,y,z) = 2(\mathcal{M}(\mathcal{M}(x,y),z) - \mathcal{M}(x,\mathcal{M}(y,z)))$$

so that $\mathcal{M}$ is an associative product on V if and only if $\mathcal{M} \Delta \mathcal{M} = 0$.

Moreover, in this case the Hochschild coboundary operator δ of the associative algebra $(V,\mathcal{M})$ is given by

$$\delta C = (-1)^p \mathcal{M} \,\Delta\, C$$

for $C \in M^p(V)$. We will denote by $\tilde{H}(A)$ the Hochschild cohomology space of the associative algebra $A = (V,\mathcal{M})$.

Similarly, as shown in [12], $\mathcal{L} \in A^1(V)$ is a Lie bracket on V if and only if $[\mathcal{L},\mathcal{L}] = 0$ in which case the Chevalley coboundary operator ∂ of the adjoint representation of $(V,\mathcal{L})$ is given by

$$\partial C = (-1)^p [\mathcal{L},C]$$

for $C \in A^p(V)$. We will denote by $H(L)$ the Chevalley cohomology space of the adjoint representation of the Lie algebra $L = (V,\mathcal{L})$.

We now denote by P a couple $(V,\mathcal{P})$ where $\mathcal{P}$ is either an associative product or a Lie bracket on the space V.

In the first case, we set $P(V) = M(V)$, $\square = \Delta$, $D = \delta$ and $H(P) = \tilde{H}((V,\mathcal{P}))$ while in the second we set $P(V) = A(V)$, $\square = [\,,]$, $D = \partial$ and $H(P) = H((V,\mathcal{P}))$. With these notations, we have

Proposition 1.4. – *f* $R \in P^r(V)$ *and* $S \in P^s(V)$, *then*

$$D(R \,\square\, S) = R \,\square\, DS + (-1)^s (DR) \,\square\, S.$$

In particular, the graded Lie algebra structure $\square$ of $P(V)$ *induces a graded Lie algebra structure, also denoted by* $\square$, *on* $H(P)$ *in which the grading is the natural grading of the cohomology spaces reduced by* 1.

Proof. – Straightforward, using the graded Jacobi identity. The case of a Lie algebra structure has been given in [12].

Formal deformations of associative and Lie algebras.

The notations are the same as above. We now define the formal deformations of the algebra P. Denote by $E(P,\nu)$ the space of formal series

$$x_\nu = \sum_{k=0}^{\infty} \nu^k x_k, \quad x_k \in V.$$

An element $\mathscr{A}_\nu$ of $M^p(E(P,\nu))$ is said to be *formal* if it is of the form

$$\mathscr{A}_\nu : (x_\nu^{(0)}, \ldots, x_\nu^{(p)}) \to \sum_{k=0}^{\infty} \nu^k \sum_{r+s_0+\ldots+s_p=k} A_r(x_{s_0}^{(0)}, \ldots, x_{s_p}^{(p)})$$

where $A_r \in M^p(V)$ for each $r \geqslant 0$; A_r is called the *r-th component* of $\mathscr{A}_\nu$ and we write

$$\mathscr{A}_\nu = \sum_{k=0}^{\infty} \nu^k A_k .$$

If $\mathscr{A}_\nu$, $\mathscr{A}_\nu^{(0)}, \ldots, \mathscr{A}_\nu^{(p)}$ are formal maps, then the composed map $\mathscr{A}_\nu(\mathscr{A}_\nu^{(0)}, \ldots, \mathscr{A}_\nu^{(p)})$ is obviously a formal map and its *k-th* component reads

$$\mathscr{A}_\nu(\mathscr{A}_\nu^{(0)}, \ldots, \mathscr{A}_\nu^{(p)})_k = \sum_{r+s_0+\ldots+s_p=k} A_r(A_{s_0}^{(0)}, \ldots, A_{s_p}^{(p)}) .$$

In particular, for formal $\mathscr{R}_\nu$ and $\mathscr{S}_\nu$

$$(\mathscr{R}_\nu \square \mathscr{S}_\nu)_k = \sum_{r+s=k} R_r \square S_s .$$

We now define the formal deformations of the algebra P. A formal map $\mathscr{C}_\nu \in P^1(V)$ is a *formal deformation* of P if $C_0 = \mathscr{P}$ and if $\mathscr{C}_\nu \square \mathscr{C}_\nu = 0$ (this last condition means that $\mathscr{C}_\nu$ is an associative or a Lie algebra structure on $E(P,\nu)$ according as $\mathscr{P}$ is an associative or a Lie algebra structure on V). In a similar way $\mathscr{C}_\nu$ is a *formal deformation of order* k *of* P if $C_0 = \mathscr{P}$ and $(\mathscr{C}_\nu \square \mathscr{C}_\nu)_i = 0$ for $i = 0, \ldots, k$.

One may try to construct a formal deformation of P step by step: starting with $\mathscr{C}_\nu^{(0)} = \mathscr{P}$ and having constructed a formal deformation

$$\mathscr{C}_\nu^{(k)} = \sum_{i=0}^{k} \nu^i C_i$$

of order k, one tries to find C_{k+1} such that $\mathscr{C}_\nu^{(k+1)} = \mathscr{C}_\nu^{(k)} + \nu^{k+1} C_{k+1}$ is a formal deformation of order $k+1$. In this procedure, the following well-known proposition is useful [9, § 3 and 5 for instance]. Its proof is just a matter of computations. We indicate how the above formalism makes it straightforward.

PROPOSITION 1.5. — *A bilinear formal map*

$$\mathscr{C}_\nu^{(k)} = \sum_{i=0}^{k} \nu^i C_i$$

where $C_i \in P^1(V)$ *and* $C_0 = \mathscr{P}$ *is a formal deformation of order* k *of* P *if and only if*

$$2\,DC_i = J_i, \qquad \forall i \leqslant k,$$

where

$$J_i = \sum_{\substack{r+s=1\\ r,s>0}} C_r \square C_s .$$

Under these conditions, $DJ_{k+1} = 0$.

Proof. – The map $\mathscr{C}_v^{(k)}$ is a formal deformation of order k if and only if

$$(\mathscr{C}_v^{(k)} \square \mathscr{C}_v^{(k)})_i = \sum_{r+s=i} C_r \square C_s = \mathscr{P} \square C_i + C_i \square \mathscr{P} + \sum_{\substack{r+s=i\\ r,s>0}} C_r \square C_s$$

$$= -\,2DC_i + J_i$$

vanishes for $i = 0, \ldots, k$. Hence the first part of the proposition.

By the graded Jacobi identity for the graded Lie algebra $M(E(P,v))$, we have

$$\mathscr{C}_v^{(k)} \square (\mathscr{C}_v^{(k)} \square \mathscr{C}_v^{(k)}) = 0 .$$

Taking the $(k+1)-th$ component, we get

$$C_0 \square J_{k+1} + \sum_{\substack{r+s=k+1\\ r,s>0}} C_r \square (\mathscr{C}_v^{(k)} \square \mathscr{C}_v^{(k)})_s = 0 .$$

Hence the result.

2. Cohomology.

The Hochschild cohomology of the algebra of functions.

Let M be a smooth connected, Hausdorff and second countable manifold. Denote by $\mathscr{X}(M)$ the Lie algebra of smooth vector fields on M and by N the algebra of smooth real valued functions on M the product of which will be denoted by $\mathscr{M}$. All the objects on M considered in the sequel are assumed to be smooth.

A map $C \in M^p(N)$ is *local* if

$$\operatorname{supp} C(u_0,\ldots,u_p) \subset \bigcap_{i \leqslant p} \operatorname{supp} u_i$$

where supp denotes the support. It is said to be *vanishing on the constants* (in short *nc*) if $C(u_0,\ldots,u_p) = 0$ when u_i is constant for some $i \leqslant p$. As easily seen, the Hochschild coboundary operator δ of N stabilizes the space $M_{loc}(N)$ [resp. $M_{loc,nc}(N)$] of all $C \in M(N)$ which are local [resp. local and vanishing on the constants]. The corresponding cohomology space will be denoted by $\tilde{H}_{loc}(N)$ [resp. $\tilde{H}_{loc,nc}(N)$].

PROPOSITION 2.1. [13]. – *The spaces* $\tilde{H}^p_{loc}(N)$ *and* $\tilde{H}^p_{loc,nc}(N)$ *are isomorphic to the space of contravariant skew-symmetric p-tensor fields of* M.

Given a p-tensor T, we set

$$C_T(u_0,\ldots,u_{p-1}) = T(du_0,\ldots,du_{p-1}).$$

Given a cocycle $C \in M^{p-1}_{loc}(N)$, $\alpha(C)$ is *nc* and of order 1 in each argument, hence it is of the form C_T for some p-tensor T. Moreover $C - \alpha(C)$ is a coboundary. The isomorphism of the above proposition maps the class of C onto T.

In view of proposition 1.4, the space of contravariant skew-symmetric tensor fields of M has a graded Lie algebra structure defined by

$$C_{T \Delta T'} = \alpha(C_T \Delta C_{T'}).$$

If T is a p-tensor and T′ a q-tensor, it can be shown that

$$T \Delta T' = ((p+q-1)!/p!q!)(-1)^{p(q+1)}[T,T']$$

where $[T,T']$ is the Nijenhuis-Schouten bracket of T and T′.

The Chevalley cohomology of the Poisson algebra.

We now suppose that M is a symplectic manifold of dimension $m > 2$. We denote by L [resp. L^*] its Lie algebra of symplectic [resp. Hamiltonian] vector fields, by F its symplectic form and by μ the canonical isomorphism between the spaces of contravariant and covariant tensor fields of M induced by $X \to -i(X)F$ on $\mathscr{H}(M)$.

Given two vector spaces E and E′, $C^p(E,E')$ is the space of p-linear alternating maps from E^p into E′. In particular, $C^{p+1}(N,N) = A^p(N)$. For $C \in C^p(\mathscr{H}(M),N)$, we define $\mu^*C \in C^p(N,N)$ by

$$(\mu^*C)(u_0,\ldots,u_{p-1}) = C(X_{u_0},\ldots,X_{u_{p-1}})$$

where $X_u = \mu^{-1}(du)$ is the Hamiltonian field corresponding to u. As usual, we put $\Lambda = \mu^{-1}F$. The Poisson bracket of $u, v \in N$ is thus given by

$$P(u,v) = \Lambda(du,dv).$$

It is well-known that P equippes N with a Lie algebra structure, called the *Poisson algebra* of the symplectic manifold (M,F).

The Chevalley coboundary operator ∂ of the adjoint representation of the Poisson algebra N stabilizes the space $A_{loc,nc}(N)$ of local cochains on N vanishing on the constants. We will denote by $H_{loc,nc}(N)$ its cohomology space. Let ∂' be the Chevalley coboundary operator associated to the representation of $\mathscr{H}(M)$ by Lie derivatives on the space $\Lambda(M)$ of differential forms on M. It is easily checked that, for cochains with values in $N = \Lambda^0(M)$,

$$\mu^* \circ \partial' = \partial \circ \mu^*. \tag{1}$$

Moreover $\Omega \in \Lambda^p(M)$ can be viewed as an element of $C^p(\mathscr{H}(M),N)$. With that identification, $\partial'\Omega = d\Omega$ (d: the exterior differential).

Let us now give some basic examples of cocycles for ∂.

(i) In view of (1), if C is a p-cocycle for ∂', then μ^*C is a cocycle for ∂. In particular, $\mu^*\Omega$ is a cocycle for ∂ provided that $d\Omega = 0$. Observe that for $X \in L$, the cocycle $u \to L_X u$ is of that type, the corresponding closed 1-form Ω being $i_X F$.

Given a connection 1-form θ on the principal bundle of linear frames $\pi : L(M) \to M$ of M, one may construct $T'_\theta \in C^3_{loc}(\mathscr{H}(M),N)$ such that $\partial' T'_\theta = \tau_\theta$ is the representative associated to θ of the image by the Chern-Weil homomorphism of the symmetric function on $gl(m,\mathbf{R})$

$$(A_0,A_1,A_2,A_3) \to \sigma \operatorname{tr} A_0A_1A_2A_3$$

(where σ denotes the symmetrization). The cohomology class $[\tau_\theta]$ of τ_θ is thus the first trace-class of M. If $[\tau_\theta] = 0$, adding to T'_θ a suitable 3-form, we obtain a 3-cocycle T_θ for ∂' which is never exact [5].

(ii) If Φ is a p-cocycle for ∂' valued in $\Lambda^2(M)$, then

$$\mu'\Phi = \mu^*\langle\Lambda,\Phi\rangle$$

where $\langle\, ,\rangle$ indicates the dual pairing is clearly a cocycle for ∂. It has been shown [5] that there is a $\Lambda^2(M)$-valued 2-cocycle Φ_θ such that

$$\pi^*(\Phi_\theta)_{X,Y} = \operatorname{tr} \mathscr{L}_X^*\theta \wedge \mathscr{L}_Y^*\theta$$

where $\mathscr{L}^*\theta : \mathscr{H}(M) \to \Lambda^1(L(M), gl(m,\mathbf{R}))$ maps X onto $L_{X^*}\theta$, X^* being the canonical lift of X on $L(M)$. When the connection Γ associated to θ is symplectic, the corresponding cocycle $\mu'\Phi_\theta$ is the well-known cocycle S_Γ^3 of Vey [13,9 for a more classical construction of S_Γ^3].

We are now in position to describe the spaces $H^p_{loc,nc}(N)$, $p = 2,3$ in the precise way needed in the sequel.

PROPOSITION 2.2 [8]. – *Each cocycle* $C \in C^2_{loc,nc}(N,N)$ *has a decomposition*

$$C = r\mu'\Phi_\theta + \mu^*\Omega + \partial E$$

where $r \in \mathbf{R}$ *and* $\Omega \in \Lambda^2(M)$ *is closed. Moreover the mapping* $[C] \to (r,[\Omega])$ *([ ] denoting cohomology classes) is an isomorphism between* $H^2_{loc,nc}(N)$ *and* $\mathbf{R} \times H^2_{de\,Rham}(M)$. *In particular, if* $\mu^*\Omega = \partial C$ *for* $\Omega \in \Lambda^3(M)$ *and* $C \in C^2_{loc,nc}(N,N)$, *there exists* $\Omega' \in \Lambda^2(M)$ *such that* $\mu^*\Omega = \partial\mu^*\Omega'$.

PROPOSITION 2.3 [4,8]. – (i *If* $[\tau_\theta] = 0$, *each cocycle* $C \in C^3_{loc\,nc}(N,N)$ *has a decomposition*

$$C = \mu'(\Phi_\theta \wedge i_X F) + \mu^*(rT_\theta + \Omega) + \partial E$$

where $X \in L$, $r \in \mathbf{R}$ *and* $\Omega \in \Lambda^3(M)$ *is closed. Moreover, the mapping* $[C] \to ([X],r,[\Omega])$ *is an isomorphism between* $H^3_{loc,nc}(N)$ *and* $(L/L^*) \times \mathbf{R} \times H^3_{de\,Rham}(M)$.

(ii) *If* $[\tau_\theta] \neq 0$, *the same conclusion holds with* $r = 0$, $H^3_{loc,nc}(N)$ *being isomorphic to* $(L/L^*) \times H^3_{de\,Rham}(M)$.

COROLLARY 2.4. – *The spaces* $H^p_{loc,nc}(N)$, $p = 2,3$, *are spanned by cohomology classes of cocycles of the form* μ^*C *and* $\mu'\Phi$.

The case of exact symplectic manifolds.

We now suppose that M is an exact symplectic manifold. This means that F is an exact form. It is of course equivalent to the existence of $\xi \in \mathscr{H}(M)$ such that $L_\xi F = F$.

The main properties of such a ξ used in the sequel are summarized in the following proposition.

PROPOSITION 2.5. – *If* $\xi \in \mathscr{H}(M)$ *is such that* $L_\xi F = F$, *then*

(i) $L_\xi \circ \partial = \partial \circ L_\xi - \partial$,

(ii) $L_\xi X_u = X_{L_\xi u} - X_u, \ \forall u \in N$,

(iii) $L_\xi \mu^* C = \mu^* L_\xi C - p\mu^* C, \ \forall C \in C^p(\mathscr{H}(M), N)$,

(iv) $L_\xi \mu' \Phi = \mu' L_\xi \Phi - (p+1)\mu'\Phi, \ \forall \Phi \in C^p(\mathscr{H}(M), \Lambda^2(M))$.

Proof. – If $C \in A^p(M)$, $\partial C = (-1)^p [P,C]$. Thus, since $L_\xi P = -P$,

$$L_\xi \partial C = (-1)^p [L_\xi, [P,C]]$$
$$= (-1)^p ([[L_\xi, P], C] + [P, [L_\xi, C]]) = -\partial C + \partial L_\xi C$$

by the graded Jacobi identity. Similar computations achieve the proof.

It follows from the previous proposition that L_ξ maps cocycles and coboundaries onto cocycles and coboundaries respectively. This means that for each k, $L_\xi + k$ induces a linear map from $H_{loc,nc}(N)$ into itself. We will denote this map also by $L_\xi + k$.

Observe moreover that since $L_\xi \circ \partial' = \partial' \circ L_\xi$ and $L_\xi = i_\xi \circ \partial' + \partial' \circ i_\xi$, proposition 2.5 shows that for cocycles $D \in C^p(\mathscr{H}(M), N)$ and $\Phi \in C^p(\mathscr{H}(M), \Lambda^2(M))$, $(L_\xi + p)\mu^* D$ and $(L_\xi + (p+1))\mu'\Phi$ are the coboundaries of $\mu^* i_\xi D$ and $\mu' i_\xi \Phi$ respectively.

PROPOSITION 2.6. – *f* $k \neq 2,3$ [*resp.* $k \neq 3,4$], *then* $L_\xi + k$ *is a bijection from* $H^2_{loc,nc}(N)$ [*resp.* $H^3_{loc,nc}(N)$] *into itself.*

Proof. – Assume that $k \neq 2,3$ and let

$$C = r\mu'\Phi_\theta + \mu^*\Omega + \partial E$$

and

$$C' = r'\mu'\Phi_\theta + \mu^*\Omega' + \partial E'$$

be cocycles. One has $(L_\xi + k)C = (k-3)r\mu'\Phi_\theta + (k-2)\mu^*\Omega + \partial E''$. Thus, in view of proposition 2.2, $(L_\xi + k)C$ and C' are cohomologous if and only if $r' = (k-3)r$ and $\Omega' = (k-2)\Omega$. This shows that for $k \neq 2,3$, $L_\xi + k : H^2_{loc,nc}(N) \to H^2_{loc,nc}(N)$ is a bijection. Using proposition 2.3, similar computations show that for $k \neq 3,4$, $L_\xi + k$ is a bijection from $H^3_{loc,nc}(N)$ into itself.

Remark 2.7. – Observe that the kernel of $L_\xi + 3 :$ $H^2_{\mathrm{loc},nc}(N,N) \to H^2_{\mathrm{loc},nc}(N,N)$ is $\mathbf{R}[\mu'\Phi_\theta]$.

COROLLARY 2.8. – *Let* $A \in C^3_{\mathrm{loc},nc}(N,N)$ *be a cocycle and* $B \in C^2_{\mathrm{loc},nc}(N,N)$. *Assume that* $(L_\xi + k)A = \partial B$. *If* $k \neq 3,4$, *there exists* $C \in C^2_{\mathrm{loc},nc}(N,N)$ *such that* $A = \partial C$ *and* $(L_\xi + k - 1)C - B$ *is a coboundary. Such a* C *is unique up to a coboundary.*

Proof. – Since $L_\xi + k$ is injective on $H^3_{\mathrm{loc},nc}(N)$, A is the coboundary of some C_1. One has then

$$\partial((L_\xi + k - 1)C_1 - B) = 0.$$

Thus, the surjectivity of $L_\xi + k - 1$ on $H^2_{\mathrm{loc},nc}$ means that for some cocycle C_2, $(L_\xi + k - 1)(C_1 + C_2) - B$ is a coboundary. The cochain $C_1 + C_2$ has the properties required for C.

The uniqueness of C up to a coboundary results immediatly from proposition 2.6.

3. Existence of formal deformations of the associative or Lie algebra N.

Formal deformations of the Poisson algebra.

We will only deal with formal deformations

$$\mathscr{L}_\nu = \sum_{k=0}^{\infty} \nu^k C_k$$

of the Poisson algebra (N,P) where the *cochains* C_k *are local and vanishing on the constants.*

Let $T_\nu = \sum_k \nu^k T_k \in M^0(E(N,\nu))$ be formal. If $T_0 = \mathbf{1}$, the identity map on N, then T_ν is non singular and T_ν^{-1} is formal. Moreover, if, for each $k > 0$, T_k is local and vanishing on the constants, then so does $(T_\nu^{-1})_k$. In this case, if $\mathscr{L}_\nu$ is a formal deformation of (N,P), then this is also the case for

$$T_\nu^* \mathscr{L}_\nu : \quad (u_\nu, v_\nu) \to T_\nu(\mathscr{L}_\nu((T_\nu^{-1} u_\nu, T_\nu^{-1} v_\nu)).$$

Two formal deformations $\mathscr{L}_\nu$ and $\mathscr{L}'_\nu$ are said to be *equivalent* if $\mathscr{L}'_\nu = T_\nu^* \mathscr{L}_\nu$ for such a T_ν.

Denote by Θ the element of $M^0(E(N,\nu))$ defined by

$$\Theta(u_\nu) = \sum_{k=0}^{\infty} \nu^k (2k-1) u_k .$$

As easily checked, for a formal $\mathscr{A}_\nu = \Sigma \nu^k A_k \in M^p(E(N,\nu))$, $[\Theta,\mathscr{A}_\nu]$ is formal and its *k-th* component is $(2k+p)A_k$.

In the sequel, we will deal with non formal linear maps from $E(N,\nu)$ into itself of the form $\mathscr{D} = \mathscr{D}_\nu + \Theta$, where $\mathscr{D}_\nu$ is a formal map whose components D_k are local and vanishing on the constants. Such a map $\mathscr{D}$ will be called *of type* Θ.

LEMMA 3.1. — *If a formal deformation* $\mathscr{L}_\nu$ *of* (N,P) *has a derivation* $\mathscr{D}$ *of type* Θ, *then* F *is exact,* $D_0 = L_\xi$ *where* $L_\xi F = F$ *and* C_1 *is of the form* $r\mu'\Phi_\theta + \partial E$ *for some* $E \in M_{loc,nc}(N)$.

Proof. — Observe that $\mathscr{D}$ is a derivation of $\mathscr{L}_\nu$ if and only if $[\mathscr{D},\mathscr{L}_\nu] = 0$.

The first and second components of this equality read $[D_0,P] + P = 0$ and $[D_0,C_1] + 3C_1 = \partial D_1$. From the first and from the structure of the local derivations of P, it follows that $D_0 = L_\xi$ with $L_\xi F = F$. From the second and from remark 2.7, we see that C_1 is of the announced form.

Remark 3.2. — Recall that the cohomology class of $\mu'\Phi_\theta$ does not depend on the connection 1-form θ. Therefore, the cohomology class of C_1 is also independent of θ. We call it the *class* of $\mathscr{L}_\nu$.

THEOREM 3.3. — *Let* (M,F) *be an exact symplectic manifold. For each* $r \in \mathbf{R}$ *and* $E \in C^1_{loc,nc}(N,N)$, *there exists a formal deformation*

$$\mathscr{L}_\nu = P + \nu(r\mu'\Phi_\theta + \partial E) + \sum_{k>1} \nu^k C_k$$

of (N,P) *which admits at least one derivation of type* Θ.

Proof. — We prove by induction on k the existence of $C_i \in C^2_{loc,nc}(N,N)$ and $D_i \in C^1_{loc,nc}(N,N)$ such that

$$\mathscr{L}_\nu^{(k)} = \sum_{i=0}^{k} \nu^i C_i$$

is a formal deformation of order k of (N,P) and such that, if

$$\mathscr{D}^{(k)} = \sum_{i=0}^{k} \nu^i D_i + \Theta,$$

then

$$[\mathscr{D}^{(k)}, \mathscr{L}_\nu^{(k)}]_i = 0, \qquad \forall i \leqslant k. \tag{2}$$

Choose ξ such that $L_\xi F = F$ and set $C_0 = P$, $C_1 = r\mu'\Phi_\theta + \partial E$ and $D_0 = L_\xi$. As

$$(L_\xi + 3)C_1 = \partial(r\mu' i_\xi \Phi_\theta + L_\xi E + 2E),$$

setting $D_1 = r\mu' i_\xi \Phi_\theta + L_\xi E + 2E$, (2) is true for $k = 1$. Assume it is also verified for some $k > 1$. By the graded Jacobi identity, we have

$$[\mathscr{D}^{(k)}, [\mathscr{L}_\nu^{(k)}, \mathscr{L}_\nu^{(k)}]] = 2[[\mathscr{D}^{(k)}, \mathscr{L}_\nu^{(k)}], \mathscr{L}_\nu^{(k)}].$$

The $(k+1)$-*th* component of this equality reads

$$(L_\xi + 2k + 4)J_{k+1} = 2\partial \sum_{\substack{r+s=k+1 \\ r,s>0}} [C_r, D_s]$$

where

$$J_{k+1} = \sum_{\substack{r+s=k+1 \\ r,s>0}} [C_r, C_s].$$

As $\mathscr{L}_\nu^{(k)}$ is a formal deformation of order k, J_{k+1} is a cocycle (prop. 1.5). Thus, by corollary 2.8, there exist C_{k+1} and D_{k+1} such that $J_{k+1} = 2\partial C_{k+1}$ and

$$(L_\xi + 2k + 3)C_{k+1} = \sum_{\substack{r+s=k+1 \\ r,s>0}} [C_r, D_s] + \partial D_{k+1}.$$

Thus $\mathscr{L}_\nu^{(k+1)} + \nu^{k+1}C_{k+1}$ is a formal deformation of order $k + 1$ of (N,P) and the last equality implies that

$$[\mathscr{D}^{(k)} + \nu^{k+1} D_{k+1}, \mathscr{L}_\nu^{(k+1)}]_i = 0, \qquad \forall i \leqslant k + 1.$$

Hence the result.

Let $\mathscr{L}_\nu$ be a formal deformation of (N,P) and let $X \in L$ be given. If

$U \subset M$ is open and contractile, then $X|_U = X_f$ for some $f \in N$, defined up to a constant. Following [10], we define $D_X^{\mathscr{L}_\nu} \in M^1(E(N,\nu))$ by

$$\text{(3)} \qquad D_X^{\mathscr{L}_\nu}(u_\nu)|_U = \mathscr{L}_\nu(f,u_\nu).$$

Since the components of $\mathscr{L}_\nu$ are vanishing on the constants, $D_X^{\mathscr{L}_\nu}$ is well-defined. Observe that it is a formal map with $D_0 = L_X$ and that *it is a derivation of the Lie algebra* $(E(N,\nu), \mathscr{L}_\nu)$. More details on those derivations will be given in the next section.

THEOREM 3.4. — *Two formal deformations of* (N,P) *admitting derivations of type* Θ *are equivalent if and only if they have the same class.*

Proof. — (i) Assume that $\mathscr{L}_\nu$ and $\mathscr{L}'_\nu$ are equivalent, that is $\mathscr{L}'_\nu = T_\nu^* \mathscr{L}_\nu$ for some $T_\nu = 1 + \sum_{k>0} \nu^k T_k$ where the T_k's are local and vanishing on the constants. As

$$(T_\nu^* \mathscr{L}_\nu)_1 = C_1 - \partial T_1,$$

we see that $\mathscr{L}_\nu$ and $\mathscr{L}'_\nu$ have the same class.

(ii) Assume now that $\mathscr{L}_\nu$ and $\mathscr{L}'_\nu$ have the same class and admit derivations of type Θ $\mathscr{D}$ and $\mathscr{D}'$ respectively. From lemma 3.1, it follows that $D_0 = L_\xi$ and $D'_0 = L_{\xi'}$ where $\xi, \xi' \in \mathscr{H}(M)$ are such that $L_\xi F = L_{\xi'} F = F$. Since $X_0 = \xi' - \xi \in L$, we may assume that $\xi' = \xi$ by replacing $\mathscr{D}'$ by $\mathscr{D}' - \mathscr{D}_{X_0}^{\mathscr{L}'_\nu}$.

Suppose then that $C'_i = C_i$ and $D'_i = D_i$, $\forall i < k$.

The *k-th* component of the equality $[\mathscr{D}', \mathscr{L}'_\nu] - [\mathscr{D}, \mathscr{L}_\nu] = 0$ reads, setting $\alpha = C'_k - C_k$ and $\beta = D'_k - D_k$,

$$(L_\xi + 2k + 1)\alpha = \partial\beta.$$

On the other hand, the *k-th* component of $[\mathscr{L}'_\nu, \mathscr{L}'_\nu] - [\mathscr{L}_\nu, \mathscr{L}_\nu] = 0$ shows that $\partial\alpha = 0$. If $k = 1$, α is a coboundary because $\mathscr{L}_\nu$ and $\mathscr{L}'_\nu$ have the same class. If $k > 1$, it follows from proposition 2.6, that α is also a coboundary. Thus by corollary 2.8, $\alpha = \partial E$ for some $E \in C^1_{\mathrm{loc},nc}(N,N)$ and

$$\partial((L_\xi + 2k)E - \beta) = 0.$$

Therefore

$$D'_k = D_k + (L_\xi + 2k)E + L_X$$

for some $X \in L$. We may assume that $X = 0$, by replacing $\mathscr{D}'$ by $\mathscr{D}' - \mathscr{D}_X^{\mathscr{L}_\nu}$.

Set then $T_\nu = 1 + \nu^k E$. As easily seen,

$$(T_\nu^* \mathscr{L}'_\nu)_i = C_i, \qquad \forall i \leqslant k$$

and $T_\nu \circ \mathscr{D}' \circ T_\nu^{-1} = \mathscr{D}''_\nu + \Theta$ where $\mathscr{D}''_\nu$ is formal, local, *nc* and such that

$$(\mathscr{D}''_\nu)_i = D_i, \qquad \forall i \leqslant k.$$

It is then easily seen that the initial $\mathscr{L}_\nu$ and $\mathscr{L}'_\nu$ are equivalent by a formal map which is the product of maps of type $1 + \nu^k E_k$.

Star-products.

A formal deformation

$$\mathscr{M}_\lambda = \mathscr{M} + \lambda P + \sum_{k=2}^{\infty} \lambda^k C_k$$

of the associative algebra $(N,\mathscr{M})$ is a *weak star-product* if for each $k > 1$, $C_k(v,u) = (-1)^k C_k(u,v)$, C_k is local and C_{2k+1} is vanishing on the constants. It is a *star-product* if C_{2k} also vanishes on the constants for $k > 1$. We set $C_0 = \mathscr{M}$ and $C_1 = P$.

Equivalence of star-products is defined in the same way as equivalence of formal deformations of $(N,P) : \mathscr{M}'_\lambda = T_\lambda^* \mathscr{M}_\lambda$ for some formal map $T_\lambda : E(N,\lambda) \to E(N,\lambda)$ of the form $T_\lambda = 1 + \sum_{k>1} \lambda^k T_k$ where the T_k's are local and vanishing on the constants. For weak star-products, the T_k's are no more required to vanish on the constants.

Clearly, if $\mathscr{M}_\lambda$ is a weak star-product, then

$$\mathscr{L}_\nu(u,v) = [(\lambda/2)(\mathscr{M}_\lambda(u,v) - \mathscr{M}_\lambda(v,u))]_{\lambda^2=\nu}$$

is a formal deformation of (N,P), We will say that $\mathscr{L}_\nu$ *derives* from $\mathscr{M}_\lambda$. Observe that if $\mathscr{M}_\lambda = \sum_k \lambda^k C_k$, then $\mathscr{L}_\nu = \sum_k \nu^k C_{2k+1}$.

The question to know whether a formal deformation of (N,P) derives from some weak star-product has been solved by Lichnerowicz [9, § 16].

His result is obtained by a pretty long proof, checking the result for $M = \mathbf{R}^m$ and globalizing it by a careful analysis of equivalences. It seems to be worth to give a direct proof inspired by the Neroslavsky-Vlassov existence theorem [11]. Recall the cocycle $S^3_\Gamma = \mu'\Phi_\theta$ associated to a symplectic connection Γ of M of connection 1-form θ.

THEOREM 3.5 [9]. – *A formal deformation*

$$\mathscr{L}_\nu = \sum_{k=0}^{\infty} \nu^k C_{2k+1}$$

of (N,P) *derives from a weak star product* $\mathscr{M}_\lambda$ *if and only if*

$$C_3 = S^3_\Gamma/3! + \mu^*\Omega + \partial E$$

where $d\Omega = 0$. *The weak star-product* $\mathscr{M}_\lambda$ *is unique.*

Proof. – (i) It is known that, given a symplectic connection Γ, the most general weak star-product of order 3 reads

$$\mathscr{M} + \lambda P + \lambda^2(P^2_\Gamma/2+\delta E) + \lambda^3(S^3_\Gamma/3! + \mu^*\Omega + \partial E)$$

where $E \in C^1_{loc}(N,N)$ and $\Omega \in \Lambda^2(M)$. If it extends to a star-product of order 4, Ω must be closed. Hence the form of C_3 in $\mathscr{L}_\nu$.

(ii) Assume now that C_3 is of this type and let us prove that $\mathscr{L}_\nu$ derives from a weak star-product. To do that, we show that, if

$$\mathscr{M}_\lambda = \sum_{i=0}^{2k-1} \lambda^i C_i$$

is a weak star-product of order $2k-1$, by modifying $C_{2(k-1)}$ if necessary, there exists $C_{2k} \in C^2_{loc}(N,N)$ such that $\mathscr{M}_\lambda + \lambda^{2k}C_{2k} + \lambda^{2k+1}C_{2k+1}$ is a weak star product of order $2k+1$. The induction starts with $k=2$, the case $k=1$ being obvious by (i).

The two key steps in the Neroslavsky-Vlassov existence theorem are the following:

(*a*) $(\mathscr{M}_\lambda \Delta \mathscr{M}_\lambda)_{2k} = 2\delta A$ for some symmetric $A \in C^2_{loc,nc}(N,N)$ if and only if $[\mathscr{L}_\nu, \mathscr{L}_\nu]_{k-1} = 0$;

(*b*) if $\mathscr{M}'_\lambda$ is a weak star-product of order $2k$, there always exists some antisymmetric $B \in C^2_{loc,nc}(N,N)$ such that $(\mathscr{M}'_\lambda \Delta \mathscr{M}'_\lambda)_{2k+1} = 2\delta B$.

Moreover $[\mathcal{L}'_v, \mathcal{L}'_v]_k - 2\partial B = \mu^*\eta$ for some $\eta \in \Lambda^2(M)$ if $\mathcal{L}'_v$ derives from $\mathcal{M}'_\lambda$.

We thus obtain A and $B \in C^2_{loc,nc}(N,N)$ such that $\mathcal{M}_\lambda + \lambda^{2k}A + \lambda^{2k+1}B$ is a weak star-product of order $2k+1$ and that

$$\sum_{\substack{i+j=k\\ i,j>0}} [C_{2i+1}, C_{2j+1}] = 2\partial B + \mu^*\eta .$$

The left-hand side also equals $2\partial C_{2k+1}$. Thus $2\partial(C_{2k+1} - B) = \mu^*\eta$. By proposition 2.2, taking account of the form of C_3,

$$C_{2k+1} = B + 2aC_3 + \mu^*\Omega' + \partial E'$$

for some $a \in \mathbf{R}$, some $\Omega' \in \Lambda^2(M)$ and some $E' \in C^1_{loc,nc}(N,N)$. It is now a trivial matter to show that

$$\mathcal{M}_\lambda - \lambda^{2k-2}a\mathcal{M} + \lambda^{2k}(aC_2 + A + \delta E') + \lambda^{2k+1}C_{2k+1}$$

is a weak star-product of order $2k+1$.

(iii) Uniqueness follows from corollary 3.7 below.

Part (i) of next lemma is a slight improvement of a result of [10], while part (ii) is implicitly contained in the proof of the previous theorem in [9].

LEMMA 3.6. — *Let* $\mathcal{M} + \lambda P + \lambda^2 C_2 + \lambda^3 C_3$ *be a weak star-product of order* 3 *and let* A, B *belong to* $C^2_{loc}(N,N)$.

(i) *if* $P \Delta A = 0$, *then* $A = a\mathcal{M}$ *for some* $a \in \mathbf{R}$.

(ii) *if, in addition,*

$$\mathcal{M} \Delta B + C_2 \Delta A = 0$$

and

$$P \Delta B + C_3 \Delta A = 0 \qquad \textit{or} \qquad P \Delta B + C_3 \Delta A + \frac{1}{2} A \Delta A = 0,$$

then $A = 0$ *and* $B = b\mathcal{M}$ *for some* $b \in \mathbf{R}$.

Proof. — (i) Denote by σ the symbol of A (in the lexicographical order) and by (r_0, r_1) its bi-degree. We have to show that $r_0 = r_1 = 0$. Suppose that $\wedge r_0 > 0$. If $r_1 > 1$, the symbol of $P \Delta A$ as a function of ξ, η, ζ is $\Lambda(\xi,\zeta)\sigma(\xi,\eta)$, if $r_1 = 1$, it is $\Lambda(\xi,\zeta)\sigma(\xi,\eta) + \Lambda(\xi,\eta)\sigma(\xi,\zeta)$ and if $r_1 = 0$, it is $\Lambda(\xi,\eta)\sigma(\xi)$.

In all cases, $P\Delta A = 0$ obviously implies that $\sigma = 0$. Thus $r_0 = 0$. A similar proof shows that $r_1 = 0$. Now $A = f\mathcal{M}$ for some $f \in N$ and $P\Delta A = 0$ implies that f is constant.

(ii) $A = a\mathcal{M}$ by (i); thus $A\Delta A = 0$ and the second equation of (ii) reads

$$0 = P\Delta B + C_3\Delta A = P\Delta(B - aC_2)$$

because $C_3\Delta\mathcal{M} + C_2\Delta P = 0$. By (i), $B = aC_2 + b\mathcal{M}$ for some $b \in \mathbf{R}$. The first equation of (ii) becomes

$$0 = \mathcal{M}\Delta B + C_2\Delta A = 2a\mathcal{M}\Delta C_2$$

which implies $a = 0$ since $\mathcal{M}\Delta C_2 = -P\Delta P \neq 0$. Hence the result.

COROLLARY 3.7. – *Let* $\mathcal{M}_\lambda$ *be a weak star-product and let* $\mathcal{A}_\lambda = \sum_{k=0}^{\infty} \lambda^{2k} A_k \in M^1(E(N,\lambda))$ *where the* A_k*'s are local. If* $\mathcal{M}_\lambda \Delta \mathcal{A}_\lambda = 0$ *or* $2\mathcal{M}_\lambda \Delta \mathcal{A}_\lambda + \mathcal{A}_\lambda \Delta \mathcal{A}_\lambda = 0$, *then* $\mathcal{A}_\lambda = 0$. *In particular, if* $\mathcal{L}_v$ *derives from* $\mathcal{M}_\lambda$ *and* $\mathcal{M}'_\lambda$, *then* $\mathcal{M}_\lambda = \mathcal{M}'_\lambda$.

Proof. – If A_k is the first non vanishing term of $\mathcal{A}_\lambda$, taking the terms of order $2k+1$, $2k+2$ and $2k+3$ of $\mathcal{M}_\lambda \Delta \mathcal{A}_\lambda = 0$ or $2\mathcal{M}_\lambda \Delta \mathcal{A}_\lambda + \mathcal{A}_\lambda \Delta \mathcal{A}_\lambda = 0$, we see by lemma 3.6, that $A_k = 0$. Hence a contradiction.

If $\mathcal{L}_v$ derives from $\mathcal{M}_\lambda$ and $\mathcal{M}'_\lambda$, setting $\mathcal{M}'_\lambda = \mathcal{M}_\lambda + \mathcal{A}_\lambda$, we have

$$0 = \mathcal{M}'_\lambda \Delta \mathcal{M}'_\lambda - \mathcal{M}_\lambda \Delta \mathcal{M}_\lambda = 2\mathcal{M}_\lambda \Delta \mathcal{A}_\lambda + \mathcal{A}_\lambda \Delta \mathcal{A}_\lambda,$$

hence the result since $\mathcal{A}_\lambda$ is of the above-mentioned type.

A further useful result of [9] is

LEMMA 3.8. – *If* $\mathcal{M}_\lambda$ *is a weak star-product, its terms* C_{2k} *have the form* $\bar{C}_{2k} + a_{2k}\mathcal{M}$, *with* $\bar{C}_{2k} \in C^2_{\mathrm{loc},nc}(N,N)$ *and* $a_{2k} \in \mathbf{R}$.

Denote by π the element of $M^0(E(N,\lambda))$ defined by

$$\pi(u_\lambda) = \sum_{k=0}^{\infty} k\lambda^k u_k.$$

As easily checked, for a formal $\mathcal{A}_\lambda = \sum_k \lambda^k \mathcal{A}_k \in M^p(E(N,\lambda))$, $\pi\Delta\mathcal{A}_\lambda$ is formal and its *k-th* components is kA_k. A linear map from $E(N,\lambda)$ into itself of the form $\mathscr{D}_\lambda + \pi$, where $\mathscr{D}_\lambda$ is formal, local and *nc* will be called *of type* π. Note also that if $\mathscr{A}_\lambda = \sum_k \lambda^k A_k$ is a formal map, then $\mathscr{A}_{\lambda^2}$ denotes the formal map $\sum_k \lambda^{2k} A_k$.

THEOREM 3.9. — *Let* $\mathscr{L}_\nu$ *be a formal deformation of* (N,P) *admitting a derivation* $\mathscr{D} = \mathscr{D}_\nu + \Theta$ *of type* Θ. *If the class of* $\mathscr{L}_\nu$ *is* $[S^3_\Gamma]/3!$, *then* $\mathscr{L}_\nu$ *derives from a unique star-product* $\mathscr{M}_\lambda$. *Moreover, the map of type* π $\mathscr{D}' = \mathscr{D}_{\lambda^2} + \pi$ *is a derivation of* $\mathscr{M}_\lambda$.

Proof. — It follows from theorem 3.5 that $\mathscr{L}_\nu$ derives from a unique weak star-product $\mathscr{M}_\lambda$. We have to show that $\mathscr{M}_\lambda$ is a star-product and that $\mathscr{D}' \Delta \mathscr{M}_\lambda = 0$. Set $\mathscr{M}_\lambda = \sum_{k=0}^{\infty} \lambda^k C_k$ and $\mathscr{D}_\nu = \sum_{k=0}^{\infty} \nu^k D_k$. One has

$$(\mathscr{D}' \Delta \mathscr{M}_\lambda)_{2k} = \sum_{r+s=k} D_r \Delta C_{2s} + 2k C_{2k},$$

$$(\mathscr{D}' \Delta \mathscr{M}_\lambda)_{2k+1} = \sum_{r+s=k} D_r \Delta C_{2s+1} + (2k+1) C_{2k+1}$$

and

$$[\mathscr{D}, \mathscr{L}_\nu]_k = \sum_{r+s=k} [D_r, C_{2s+1}] + (2k+1) C_{2k+1} = (\mathscr{D}' \Delta \mathscr{M}_\lambda)_{2k+1}$$

because, for $D \in M^0(N)$ and $C \in M^1(N)$, $D \Delta C = [D,C]$. As $\mathscr{D}$ is a derivation of $\mathscr{L}_\nu$, $[\mathscr{D}, \mathscr{L}_\nu]_k = 0$ hence the odd components of $\mathscr{D}' \Delta \mathscr{M}_\lambda$ are vanishing. It follows then from corollary 3.7 and from the identity

$$\mathscr{M}_\lambda \Delta (\mathscr{D}' \Delta \mathscr{M}_\lambda) = 0$$

that $\mathscr{D}' \Delta \mathscr{M}_\lambda = 0$.

Recall that $D_0 = L_\xi$, where ξ is such that $L_\xi F = F$. Assume that C_{2i} is *nc* for $i < k$. In view of lemma 3.8, $C_{2k} = \bar{C}_{2k} + a_{2k} \mathscr{M}$ with $\bar{C}_{2k} \in C^2_{\mathrm{loc},\, nc}(N,N)$ and $a_{2k} \in \mathbf{R}$. Thus

$$0 = (\mathscr{D}' \Delta \mathscr{M}_\lambda)_{2k} = \sum_{\substack{r+s=k \\ r>0}} D_r \Delta C_{2s} + (L_\xi + 2k) \bar{C}_{2k} + 2k a_{2k} \mathscr{M}$$

yields $a_{2k} = 0$. Since C_2 is *nc*, it follows by induction on k that the C_{2k}'s are *nc*. Hence the result.

4. Derivations.

The case of star-products.

The formal derivations of star-products and of formal deformations of (N,P) have been determined in [10]. It has been seen in § 3 that, when F is exact, there exist star-products and formal deformations which admit some special type of non formal derivations. We describe here the space of all derivations for arbitrary weak star-products and, if M is non compact, for arbitrary formal deformations of (N,P).

Denote by $\mathscr{M}_\lambda$ a weak star-product, by $\operatorname{der}\mathscr{M}_\lambda$ its algebra of derivations and by $\operatorname{der}_\lambda \mathscr{M}_\lambda$ its subalgebra of formal derivations. For $X \in L$, with the notation of (3),

$$\mathscr{D}_X^{\mathscr{M}_\lambda} = \mathscr{D}_X^{\mathscr{L}_v}|_{v=\lambda^2}$$

where $\mathscr{L}_v$ is the formal deformation of (N,P) deriving from $\mathscr{M}_\lambda$ belongs to $\operatorname{der}_\lambda \mathscr{M}_\lambda$ and its 0-*th* component is L_X.

Suppose now that $\mathscr{M}_\lambda$ is a star-product and let $\mathscr{D}$ be a derivation of $\mathscr{M}_\lambda$. It stabilizes the center of $E(N,\lambda)$ which is $E(\mathbf{R},\lambda)$, $\mathbf{R}$ being identified with the center of N. In particular, $a_\lambda = \mathscr{D}(\lambda 1) \in E(\mathbf{R},\lambda)$. Since $\lambda u_\lambda = \mathscr{M}_\lambda(\lambda 1, u_\lambda)$, one has

$$\mathscr{D}(\lambda u_\lambda) - \lambda \mathscr{D}(u_\lambda) = a_\lambda \circ u_\lambda$$

where

$$a_\lambda \circ u_\lambda = \sum_{k=0}^{\infty} v^k \sum_{i+j=k} a_i u_j .$$

It is easily seen that $\mathscr{E} \in M^0(E(N,\lambda))$ is formal if and only if $\mathscr{E}(\lambda u_\lambda) = \lambda \mathscr{E}(u_\lambda)$ for each $u_\lambda \in E(N,\lambda)$. On the other hand, defining

$$\pi' : E(N,\lambda) \to E(N,\lambda) : u_\lambda \to \sum_{k=1}^{\infty} \lambda^{k-1} k u_k ,$$

one has $\pi'(\lambda u_\lambda) - \lambda \pi'(u_\lambda) = u_\lambda$. Thus $\mathscr{D} - a_\lambda \circ \pi'$ is formal:

$$\mathscr{D} = \mathscr{D}_\lambda + a_\lambda \circ \pi' ,$$

with $\mathscr{D}_\lambda = \sum_{k=0}^{\infty} \lambda^k D_k$.

Taking $u_\lambda = u \in \mathrm{N}$ and $v_\lambda = v \in \mathrm{N}$, the 0-*th* component of $\mathcal{D} \,\Delta\, \mathcal{M}_\lambda(u,v) = 0$ shows that $\mathrm{D}_0 \,\Delta\, \mathcal{M} + a_0\mathrm{P} = 0$. Since $\mathrm{D}_0 \,\Delta\, \mathcal{M}$ is symmetric, it follows that $a_0 = 0$. Setting $a_\lambda = \lambda a'_\lambda$, we get $\mathcal{D} = \mathcal{D}_\lambda + a'_\lambda \circ \pi$.

The second component of $\mathcal{D} \,\Delta\, \mathcal{M}_\lambda = 0$ gives us $\mathrm{D}_0 \,\Delta\, \mathrm{P} + a'_0\mathrm{P} + \mathrm{D}_1 \,\Delta\, \mathcal{M} = 0$. We have already seen that $\mathrm{D}_0 \,\Delta\, \mathcal{M} = 0$, thus $\mathrm{D}_0 = \mathrm{L}_\mathrm{X}$ for some $\mathrm{X} \in \mathscr{H}(\mathrm{M})$. If $a'_0 = 0$, the antisymmetric part of the preceeding equality reads $[\mathrm{D}_0,\mathrm{P}] = 0$, hence $\mathrm{X} \in \mathrm{L}$. Then $\mathcal{D} - \mathcal{D}_\mathrm{X}^{\mathcal{M}_\lambda} = \lambda\mathcal{D}'$ is a new derivation of $\mathcal{M}_\lambda$.

If $\mathcal{D}$ is not formal, $\mathcal{D}'$ is not formal. By induction, we find thus a derivation $\mathcal{D}'' = \mathcal{D}''_\lambda + a''_\lambda \circ \pi$ with $a''_0 \neq 0$; a''_λ admits an inverse b_λ in $\mathrm{E}(\mathbf{R},\lambda)$. Then $b_\lambda \circ \mathcal{D}''$ is another derivation

$$\mathcal{E} = \mathcal{E}_\lambda + \pi, \tag{4}$$

where $\mathcal{E}_\lambda = \sum_{k=0}^{\infty} \lambda^k \mathrm{E}_k$ is formal.

Now, for our initial $\mathcal{D}$, $\mathcal{D} - a'_\lambda \circ \mathcal{E}$ is formal and still a derivation. We are thus left to describe the formal derivations. This was done in [10] by the argument we have used here in the case when $a'_0 = 0$. It shows that, if $\mathcal{D}_\lambda$ is a formal derivation,

$$\mathcal{D}_\lambda = \sum_{k=0}^{\infty} \lambda^k \mathcal{D}_{\mathrm{X}_k}^{\mathcal{M}_\lambda} \quad (\mathrm{X}_k \in \mathrm{L},\ \forall k).$$

Let us now try to find out which star-products admit non formal derivations. Let us take the non formal $\mathcal{E}$ obtained in (4).

We may assume that the odd components of $\mathcal{E}_\lambda$ are vanishing. Indeed, assume that $\mathrm{E}_{2\ell-1} = 0$ for $\ell < k$. One has

$$0 = (\mathcal{E} \,\Delta\, \mathcal{M}_\lambda)_{2k-1} = \mathrm{E}_{2k-1} \,\Delta\, \mathcal{M} + \sum_{\substack{r+s=k \\ s>0}} \mathrm{E}_{2r} \,\Delta\, \mathrm{C}_{2s-1} + (2k-1)\mathrm{C}_{2k-1}.$$

The symmetric part of this equality reads $\mathrm{E}_{2k-1} \,\Delta\, \mathcal{M} = 0$. Similarly, the antisymmetric part of $(\mathcal{E} \,\Delta\, \mathcal{M}_\lambda)_{2k} = 0$ reads $\mathrm{E}_{2k-1} \,\Delta\, \mathrm{P} = 0$. This shows that $\mathrm{E}_{2k-1} = \mathrm{L}_\mathrm{X}$ for some $\mathrm{X} \in \mathrm{L}$. Thus $(\mathcal{E}_\lambda - \mathcal{D}_\mathrm{X}^{\mathcal{M}_\lambda})_{2\ell-1} = 0$ for $\ell \leqslant k$. The result follows by induction.

We need the following lemma, partly in [10].

LEMMA 4.1. — (i) *If* $E \in M^0(N)$ *and if* δE *is local, then* E *is local;* δE *is* nc *if and only if* E *is* nc.

(ii) *Let* M *be non compact and* E *belong to* $M^0(N)$; *if* ∂E *is local,* E *is local; if* ∂E *is local and* nc, $E = \bar{E} + a\mathbf{1}$, *with* $\bar{E}$ nc *and* $a \in \mathbf{R}$; *if moreover* $[F] = 0$, *then there exists* $E' \in M^0_{\mathrm{loc},\,nc}(N)$ *such that* $\partial E = \partial E'$.

Proof. — (i) is contained in [10], as well as the first part of (ii). If ∂E is local and nc, $E = \bar{E} + f.\mathbf{1}$ for some $\bar{E} \in M^0_{\mathrm{loc},\,nc}(N)$ and some $f \in N$. Then $\mu^*\, df \wedge \mathbf{1} = \partial E - \partial \bar{E} + fP$ is nc, thus $df = 0$ and f is constant. If $[F] = 0$, $F\wedge = d\omega$ for some ω, hence $P = \mu^* F = \partial \mu^* \omega$ and $\partial E = \partial(\bar{E} - f\mu^*\omega)$.

Let us come back to the non formal derivation $\mathscr{E} = \mathscr{E}_\lambda + \pi$ for which we may assume that $E_{2\ell-1} = 0$ for $\ell \geqslant 1$. Each $E_{2\ell}$ is local and nc. Indeed, we have already seen that it is true for E_0. If it is true for $E_{2\ell}(\ell < k)$, it is true for E_{2k} using $(\mathscr{E}\,\Delta\,\mathscr{M}_\lambda)_{2k} = 0$ and lemma 4.1 (i).

If $\mathscr{L}_\nu$ is the formal deformation which derives from $\mathscr{M}_\lambda$, $\mathscr{E}' = \sum_{k=0}^{\infty} \nu^k E_{2k} + \Theta$ is a derivation of type Θ of $\mathscr{L}_\nu$. It follows then from lemma 3.1 that F is exact.

Conversely, by theorem 3.3 and 3.9, if F is exact, there exists a star-product with a non formal derivation.

If two star-products $\mathscr{M}_\lambda$ and $\mathscr{M}'_\lambda$ admit non formal derivations, the corresponding formal deformations $\mathscr{L}_\nu$ and $\mathscr{L}'_\nu$ admit a derivation of type Θ. Having the same class (thm. 3.5), they are equivalent (thm. 3.4): $\mathscr{L}_\nu = T^*_\nu \mathscr{L}'_\nu$. Then $\mathscr{L}_\nu$ derives from both $\mathscr{M}_\lambda$ and $T^*_{\lambda^2}\mathscr{M}'_\lambda$, hence (thm. 3.9) $\mathscr{M}_\lambda = T^*_{\lambda^2}\mathscr{M}'_\lambda$.

The preceeding results are summarized in the following:

THEOREM 4.2. — *Let* (M,F) *be a symplectic manifold.*

(i) *If* $[F] \neq 0$, *for each star-product* $\mathscr{M}_\lambda$ *of* (M,F)

$$\operatorname{der} \mathscr{M}_\lambda = \operatorname{der}_\lambda \mathscr{M}_\lambda = \left\{ \sum_{k=0}^{\infty} \lambda^k \mathscr{D}^{\mathscr{M}_\lambda}_{X_k} : X_k \in L,\ \forall k \right\}.$$

(ii) *If* $[F] = 0$, *there exists a unique equivalence class* $\mathbf{M}$ *of star-products admitting non formal derivations. If* $\mathscr{M}_\lambda \in \mathbf{M}$, *it admits a derivation*

$\mathscr{D}_\pi$ *of type* $\mathscr{D}_{\lambda^2} + \pi$ *where* $\mathscr{D}_{\lambda^2}$ *is formal, local and nc. Moreover*

$$\operatorname{der} \mathscr{M}_\lambda = \operatorname{der}_\lambda \mathscr{M}_\lambda \oplus \mathrm{E}(\mathbf{R}\mathscr{D}_\pi,\lambda).$$

The case of weak star-products.

If $\mathscr{M}_\lambda$ is a weak star-product, it follows easily from lemma 3.8 that there exists a star-product $\overline{\mathscr{M}_\lambda}$ and $a_\lambda \in \mathrm{E}(\mathbf{R},\lambda)$ such that $a_0 = 1$ and $\mathscr{M}_\lambda = \mathrm{T}^*_{a_\lambda}\overline{\mathscr{M}_\lambda}$, where $\mathrm{T}_{a_\lambda}u_\lambda = a_\lambda \circ u_\lambda$. Then $\mathscr{D}$ is a derivation of $\overline{\mathscr{M}_\lambda}$ if and only if $\mathrm{T}^*_{a_\lambda}\mathscr{D}$ is a derivation of $\mathscr{M}_\lambda$. Since $a_\lambda \in \mathrm{E}(\mathbf{R},\lambda)$, $\mathrm{T}_{a_\lambda}\pi = \pi + \mathrm{T}_{b_\lambda}$, where

$$b_\lambda = \sum_{k=0}^{\infty} \lambda^k \sum_{i+j=k} j a_i (a_\lambda^{-1})_j .$$

It is now a trivial matter to formulate the analogue of theorem 4.2, for weak star-products, the only difference being that in the derivation $\mathscr{D}_\pi = \mathscr{D}_{\lambda^2} + \pi$, $\mathscr{D}_{\lambda^2}$ is still formal and local, but no longer *nc*.

The case of formal deformations.

We shall assume that M is not compact. If it is compact, on one side there exist some non local derivations which, even when they are formal, are uneasy to handle (see [10]). On the other side, we are specially interested in exact manifolds, which are never compact.

LEMMA 4.3. — *If* M *is not compact and if* $\mathscr{L}_\nu$ *is a formal deformation of* (N,P), *then the algebra* E(N,ν) *is equal to its derived ideal.*

Proof. — The so-called Calabi's lemma states that, given U open in M, if $\operatorname{supp} u \subset \mathrm{M}\setminus\mathrm{U}$, there exist $u_i, v_i \in \mathrm{N}$ with support in $\mathrm{M}\setminus\mathrm{U}$ such that $u = \sum_{i=1}^{n} \mathrm{P}(u_i,v_i)$ [see 1]. It is important to note that the v_i's and the number n can be fixed independently of u.

For $u_\nu \in \mathrm{E}(\mathrm{N},\nu)$, one has, for suitable u_i^0, $u_0 = \sum_{i=1}^{n} \mathrm{P}(u_i^0,v_i)$ hence $u_\nu - \sum_{i=1}^{n} \mathscr{L}_\nu(u_i^0,v_i) = \nu u'_\nu$. Repeating the argument with u'_ν and continuing by induction, we obtain elements $u_{i,\nu} \in \mathrm{E}(\mathrm{N},\nu)$ such that $u_\nu = \sum_{i=1}^{n} \mathscr{L}_\nu(u_{i,\nu}, v_i)$.

We denote by $\operatorname{der}\mathscr{L}_\nu$ the algebra of derivations of $\mathscr{L}_\nu$ and by $\operatorname{der}_\nu \mathscr{L}_\nu$ its subalgebra of formal derivations.

THEOREM 4.4. — *Let* M *be a non compact manifold and* $\mathscr{L}_\nu$ *be a formal deformation of* (N,P).

a) If $\mathscr{L}_\nu$ *is equivalent to* P,

$$\operatorname{der}_\nu \mathscr{L}_\nu = \left\{ \sum_{k=0}^{\infty} \nu^k \mathrm{D}_k : \mathrm{D}_k \in \operatorname{der}(\mathrm{N},\mathrm{P}) \right\}$$

and

$$\operatorname{der}\mathscr{L}_\nu = \operatorname{der}_\nu \mathscr{L}_\nu \oplus \mathrm{E}(\mathbf{R}\pi',\nu).$$

b) If $\mathscr{L}_\nu$ *is not equivalent to* P,

$$\operatorname{der}_\nu \mathscr{L}_\nu = \left\{ \sum_{k=0}^{\infty} \nu^k \mathscr{D}_{\mathrm{X}_k}^{\mathscr{L}_\nu} : \mathrm{X}_k \in \mathrm{L} \right\}. \tag{5}$$

If $[\mathrm{F}] \neq 0$,

$$\operatorname{der}\mathscr{L}_\nu = \operatorname{der}_\nu \mathscr{L}_\nu .$$

If $[\mathrm{F}] = 0$, *there may exist a non formal derivation*

$$\mathscr{E} = \mathscr{E}_\nu + \pi ,$$

where $\mathscr{E}_\nu$ *is formal and its components* E_k *belong to* $\mathrm{M}^0_{\mathrm{loc},nc}(\mathrm{N}) \oplus \mathbf{R}1$; *moreover*

$$\operatorname{der}\mathscr{L}_\nu = \operatorname{der}_\nu \mathscr{L}_\nu \oplus \mathrm{E}(\mathbf{R}\mathscr{E},\nu). \tag{6}$$

Proof. — Let $\mathscr{D}$ be a derivation of $\mathscr{L}_\nu$. Denote by $\operatorname{Ent}\mathscr{L}_\nu$ the space of intertwinning maps T of the adjoint action with itself: $\mathrm{T}\mathscr{L}_\nu(u_\nu,v_\nu) = \mathscr{L}_\nu(\mathrm{T}u_\nu,v_\nu) = \mathscr{L}_\nu(u_\nu,\mathrm{T}v_\nu)$ for all $u_\nu, v_\nu \in \mathrm{E}(\mathrm{N},\nu)$. It is easily checked that $[\mathscr{D},\mathrm{T}] \in \operatorname{Ent}\mathscr{L}_\nu$ whenever $\mathrm{T} \in \operatorname{Ent}\mathscr{L}_\nu$ and that $[\mathrm{T},\mathrm{T}'] = 0$ on the derived ideal of $\mathrm{E}(\mathrm{N},\nu)$ hence (lemma 4.3) on $\mathrm{E}(\mathrm{N},\nu)$ itself for all $\mathrm{T},\mathrm{T}' \in \operatorname{Ent}\mathscr{L}_\nu$.

Since $\nu\mathbf{1} \in \operatorname{Ent}\mathscr{L}_\nu$, it follows from the latter that, for each $\mathrm{T} \in \mathrm{E}(\mathrm{N},\nu)$, $\mathrm{T}(\nu u_\nu) = \nu\mathrm{T}(u_\nu)$. Thus T is formal. It is easily seen that $\operatorname{Ent}(\mathrm{N},\mathrm{P}) = \mathbf{R}\mathbf{1}$. If $\mathrm{T} = \sum_k \nu^k \mathrm{T}_k$, it follows by induction on k that $\mathrm{T}_k = a_k\mathbf{1}$ for some $a_k \in \mathbf{R}$. Thus $\mathrm{T}u_\nu = a_\nu \circ u_\nu$ for some $a_\nu \in \mathrm{E}(\mathbf{R},\nu)$.

We have thus $\mathscr{D}(\nu u_\nu) - \nu\mathscr{D}(u_\nu) = a_\nu \circ u_\nu$ for some $a_\nu \in \mathrm{E}(\mathbf{R},\nu)$. As in the case of star-products, $\mathscr{D} - a_\nu \circ \pi'$ is formal, thus

$$\mathscr{D} = \mathscr{D}_\nu + a_\nu \circ \pi'$$

with $\mathscr{D}_\nu = \sum_{k=0}^{\infty} \nu^k D_k$.

For $Q_\nu = \mathbf{1} + \nu^k E$, $Q_\nu^* \mathscr{L}_\nu$ has the same components as $\mathscr{L}_\nu$ up to $k-1$ and its *k-th* components is $C_k - \partial E$. Moreover, $P = \partial \mathbf{1}$. It follows that $\mathscr{L}_\nu$ is always equivalent either to P, either to a formal deformation of type $P + \sum_{k=k_0}^{\infty} \nu^k C_k$, with $[C_{k_0}] \notin \mathbf{R}[P]$, by a map R_ν which is formal and whose components are local and *nc* up to constant multiples of $\mathbf{1}$.

Such a R_ν transforms the sets (5) and (6) into sets of the same type. We may thus replace $\mathscr{L}_\nu$ by $R_\nu^* \mathscr{L}_\nu$.

Assume first that $\mathscr{L}_\nu = P$. It is trivial that π' is a derivation of $(E(N,\nu),P)$. Thus $\mathscr{D}_\nu$ is a derivation. Since $[\mathscr{D}_\nu, \mathscr{L}_\nu] = \sum_k \nu^k [D_k, P] = 0$, the D_k's are derivations of (N,P). Hence *a*).

Assume that

$$\mathscr{L}_\nu = P + \sum_{k=k_0}^{\infty} \nu^k C_k,$$

with $[C_{k_0}] \notin \mathbf{R}[P]$. The component $k_0 - 1$ of $[\mathscr{D}, \mathscr{L}_\nu] = 0$ gives

$$[D_{k_0-1}, P] + a_0 k_0 C_{k_0} = 0.$$

Since $[C_{k_0}] \notin \mathbf{R}[P]$, by lemma 4.1, it implies $a_0 = 0$. Thus $\mathscr{D}$ takes the form

$$\mathscr{D} = \mathscr{D}_\nu + a'_\nu \circ \pi$$

for some $a'_\nu \in E(\mathbf{R},\nu)$.

The 0-*th* component of $[\mathscr{D}, \mathscr{L}_\nu] = 0$ reads $[D_0, P] = 0$.

If $[F] \neq 0$, it follows that $D_0 = L_X$ for some $X \in L$ and thus $\mathscr{D} - \mathscr{D}_X^{\mathscr{L}_\nu} = \nu \mathscr{D}'$. Repeating the same argument for the new derivation $\mathscr{D}'$, it follows by induction that

$$\mathscr{D} = \sum_{k=0}^{\infty} \nu^k \mathscr{D}_{X_k}^{\mathscr{L}_\nu} \quad (X_k \in L, \ \forall k). \tag{7}$$

We have thus proved the first part of *b*).

If $[F] = 0$, either D_0 is always L_X for some $X \in L$ and all the derivations take the form (7), or correcting $\mathscr{D}$ by some $\mathscr{D}_X^{\mathscr{L}_\nu}$ $(X \in L)$, we may assume that $D_0 = L_\xi - \mathbf{1}$. If $a'_0 = 0$,

$$0 = [\mathscr{D},\mathscr{L}_\nu]_{k_0} = [D_0,C_{k_0}] + [D_{k_0},P]$$

contradicts $[C_{k_0}] \neq 0$. Thus a'_ν is invertible and $\mathscr{E} = a'^{-1}_\nu \circ \mathscr{D}$ is a derivation. We have shown that either der $\mathscr{L}_\nu$ = der$_\nu$ $\mathscr{L}_\nu$ or there exists a derivation

$$\mathscr{E} = \mathscr{E}_\nu + \pi,$$

where $\mathscr{E}_\nu = \sum_{k=0}^{\infty} \nu^k E_k$ and $E_0 = a'^{-1}_0(L_\xi - \mathbf{1})$.

Substracting $a'_0\mathscr{E}$ to $\mathscr{D}$, we obtain a derivation $\nu\mathscr{D}'$. Thus, by induction, $\mathscr{D}$ takes the form

$$\mathscr{D} = \sum_{k=0}^{\infty} \nu^k \mathscr{D}_{X_k}^{\mathscr{L}_\nu} + b_\nu \circ \mathscr{E},$$

with $X_k \in L$ and $b_\nu \in E(\mathbf{R},\nu)$.

Let us give some more details about $\mathscr{E}$: the E_k's take the form $\bar{E}_k + a_k\mathbf{1}$, with $\bar{E}_k$ local and *nc*. It is true for $k = 0$. Assume that it is true for $k < \ell$. Then $[\mathscr{E},\mathscr{L}_\nu]_\ell = 0$ shows that $[E_\ell,P]$ is local and *nc*. The conclusion follows from lemma 4.1.

The question of determining which equivalence classes of formal deformations admit non formal derivations is more intricated than for star-products. Of course the formal deformations constructed in § 3 provide interesting examples. It should be observed that $\Theta = 2\pi + \mathbf{1}$. So in part *b*) of theorem 4.4, π can be replaced by Θ. The term E_0 of the derivation $\mathscr{E}_\nu$ has the form $\theta L_\xi + (1-\theta)\mathbf{1}$. The main difference with the derivation $\mathscr{D}$ of type Θ considered in § 3, for instance in lemma 3.1, is that the formal part of $\mathscr{D}$ is assumed to be *nc*. This requires that $D_0 = L_\xi$. It fixes $[C_1]$ and finally $\mathscr{L}_\nu$ up to equivalence, while the existence of $\mathscr{E}$ provides only a much weaker information about $\mathscr{L}_\nu$.

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Manuscrit reçu le 14 juin 1984.

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