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AN L^p -VERSION OF A THEOREM OF D. A. RAIKOV

by Gero FENDLER(*)

1. Introduction.

Let G be a locally compact group, for $p \in (1, \infty)$, let $Pf_p(G)$ denote the closure of $L^1(G)$ in the convolution operator norm on $L^p(G)$. Denote by $W_p(G)$ the dual of $Pf_p(G)$ which is contained in the space of pointwise multipliers of the Figà-Talamanca Herz space $A_p(G)$. (See [5], [8], [9] for all this.)

It is shown in these notes that on the unit sphere of $W_p(G)$ the weak * (i.e. the $\sigma(W_p, Pf_p)$) topology and the A_p -multiplier topology coincide ($u_\beta \rightarrow u$ in the latter if $\|(u_\beta - u)v\| \rightarrow 0$ for each $v \in A_p(G)$).

If $p = 2$ and G is amenable then $W_2(G)$ is just the Fourier-Stieltjes algebra of G , denoted $B(G)$, and $A_2(G)$ is the Fourier algebra of G . From this point of view the above enunciation is an L^p -version of a theorem of D.A. Raikov, which asserts that on the positive face of the unit sphere of $B(G)$ the weak * topology coincides with the topology of uniform convergence on compact sets (since $A_p(G)$ always contains functions which take the value one on a given compact set the latter topology is clearly weaker than the $A_p(G)$ multiplier topology; and on norm bounded sets obviously stronger than the weak * topology).

The proof is based on a technique of G.C. Rota [10], first used in harmonic analysis by E.M. Stein; our application is close to the work of M. Cowling [3]. On the other hand this paper continues the line of studies taken up by E.E. Granirer and M. Leinert in [7].

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2. An estimate for the L^p -operator norm of the sum of two "spectrally disjoint" operators.

If R and S are two commuting normal (of course bounded) operators on an Hilbert space H then, via the Gelfand transform, R and S correspond to some continuous functions on a locally compact space X ; further R, S are spectrally disjoint, if the supports of those functions are disjoint. It then follows easily that $\|R + S\| = \max \{\|R\|, \|S\|\}$; we remark that there exists an orthogonal projection P with $PR = R = RP$ and $(1 - P)S = S = S(1 - P)$.

From this :

$$\begin{aligned} \|(R + S)\xi\| &= \|(R + S)(P + 1 - P)\xi\| \\ &= \|\text{PRP}\xi + (1 - P)S(1 - P)\xi\| \\ &= (\|\text{PRP}\xi\|^2 + \|(1 - P)S(1 - P)\xi\|^2)^{1/2} \\ &\leq (\|R\|^2 \|\text{P}\xi\|^2 + \|S\|^2 \|(1 - P)\xi\|^2)^{1/2} \\ &\leq \max \{\|R\|, \|S\|\} (\|\text{P}\xi\|^2 + \|(1 - P)\xi\|^2)^{1/2} \\ &\leq \max \{\|R\|, \|S\|\} \|\xi\| \quad \text{for all } \xi \in H. \end{aligned}$$

Now let (X, μ) be a σ -finite measure space ; an operator T acting on all L^p -spaces will be called special if :

- i) $Tf \geq 0$ if $f \geq 0$
- ii) $\|Tf\|_p \leq \|f\|_p \quad f \in L^p(X, \mu), 1 \leq p \leq \infty$
- iii) $T1_X = 1_X$
- iv) $\int_X Tf(x) \overline{g(x)} d\mu(x) = \int_X f(x) \overline{Tg(x)} d\mu(x) \quad f, g \in L^2(X, \mu).$

Those operators will serve as a substitute for orthogonal projections, since by a method due to G.C. Rota they may be seen as conditional expectations on a certain measure space.

We begin with the following observation :

PROPOSITION. — Let $(Y, \mathfrak{F}, \nu)$ be a σ -finite measure space, $\mathfrak{F}_1 \subset \mathfrak{F}$ a sub- σ -algebra of $\mathfrak{F}$ such that $(Y, \mathfrak{F}_1, \nu)$ is again σ -finite, (which ensures the existence of a conditional expectation operator E_1 with respect to $\mathfrak{F}_1$).

Then we have for $\xi, \eta \in L^p(Y, \mathfrak{F}, \nu)$:

$$\|E_1 \xi + (1 - E_1) \eta\|_p \leq (\|\xi\|_p^r + 2 \|\eta\|_p^r)^{1/r}$$

where $r = p$ if $1 \leq p \leq 2$ and $r = p'$, the index conjugate to p , if $2 \leq p \leq \infty$.

Proof. — Clearly

$$1) \|E_1 \xi + (1 - E_1) \eta\|_1 \leq \|\xi\|_1 + 2 \|\eta\|_1$$

$$2) \|E_1 \xi + (1 - E_1) \eta\|_2^2 \leq \|\xi\|_2^2 + \|\eta\|_2^2 \leq \|\xi\|_2^2 + 2 \|\eta\|_2^2$$

$$3) \|E_1 \xi + (1 - E_1) \eta\|_\infty \leq \|\xi\|_\infty + 2 \|\eta\|_\infty$$

and the assertion follows from interpolation between 1) and 2) (resp. 2) and 3)) on mixed $L^p(L^q)$ -spaces (see [1]).

Let (X, μ) and T be as above. Define $Y = X \times X$ and endow Y with the usual product σ -algebra denoted $\mathfrak{F}$. We define a measure ν on Y by requiring that

$$\nu(S_0 \times S_1) = \int_X \chi_{S_0}(x) T \chi_{S_1}(x) d\mu(x)$$

(whenever S_0, S_1 are measurable subsets of X).

Denote by $\mathfrak{F}_1$ and $\mathfrak{F}_0$ the σ -algebras of sets $X \times S$ ($S \subseteq X$ measurable), respectively of sets $S \times X$ ($S \subseteq X$ measurable), further denote by E_1, E_0 the corresponding conditional expectation operators. For a measurable function ξ on X we define for $x = (x_0, x_1) \in Y$

$$\xi^i(x_0, x_1) = \xi(x_i) \quad i = 0, 1.$$

Then $\xi \rightarrow \xi^0$ gives rise to an isometric isomorphism between $L^p(X, \mu)$ and the subspace of $\mathfrak{F}_0$ -measurable elements of $L^p(Y, \nu)$; whereas $\xi \rightarrow \xi^1$, from $L^p(X, \mu)$ to $L^p(Y, \nu)$, does not increase norms.

Further :

$$E_0(\xi^i) = \begin{cases} (T\xi)^0 & \text{if } i = 1 \\ \xi^0 & \text{if } i = 0 \end{cases}$$

$$E_1(\xi^0) = (T\xi)^1.$$

For a proof of these facts we refer the reader to the book of E.M. Stein [11].

PROPOSITION. — *Let (X, μ) be a σ -finite measure space, T a special operator and $1 < p < \infty$. Then for $\xi_1, \xi_2 \in L^p(X, \mu)$: $\|T^2 \xi_1 + (1 - T^2) \xi_2\| \leq (\|\xi_1\|_p^r + 2 \|\xi_2\|_p^r)^{1/r}$, with $r = \min\{p, p'\}$.*

Proof. — We apply the above procedure to T , then

$$\begin{aligned} \|T^2 \xi_1 + (1 - T^2) \xi_2\| &= \|E_0((T\xi_1)^1 + \xi_2^0 - (T\xi_2)^1)\| \\ &\leq \|(T\xi_1)^1 + \xi_2^0 - (T\xi_2)^1\| \\ &= \|E_1(\xi_1^0) + (1 - E_1)(\xi_2^0)\| \\ &\leq (\|\xi_1^0\|^r + 2 \|\xi_2^0\|^r)^{1/r}. \end{aligned}$$

COROLLARY. — *Let R, S be bounded operators on $L^p(X, \mu)$, then we have*

$$\|T^2 R + (1 - T^2) S\| \leq (\|R\|^r + 2 \|S\|^r)^{1/r}.$$

3. The weak* topology on the unit sphere of $W_p(G)$.

Let G be a locally compact group, with a fixed left Haar measure dg and modular function Δ . Let $L^p(G)$, $1 \leq p \leq \infty$, denote the usual Lebesgue spaces with respect to dg and for functions f, h on G let be defined $f * h(x) = \int_G f(g) h(g^{-1}x) dg$, $f^\sim(g) = f(g^{-1}) \Delta(g^{-1})$, $f^* = \bar{f}^\sim$, $f^v(g) = f(g^{-1})$.

For this section let now $p \in (1, \infty)$ be fixed and let $A_p(G)$ (as in [8]) be the algebra of functions u on G which can be represented as $u = \sum_{n=1}^{\infty} v_n * w_n^v$, where

$$\sum_n \|v_n\|_p \cdot \|w_n\|_{p' < \infty}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

The norm on A_p is defined as the $\inf \sum \|v_n\|_p, \|w_n\|_p$ taken over all such representations of u .

If f is an element of $L^1(G)$ then on one hand $w \mapsto f * w$ defines a convolution operator on $L^p(G)$ and on the other $u \mapsto \int_G f(g) u(g) dg$ a continuous linear functional on $A_p(G)$. From $\langle f, v * w^V \rangle = \langle f * w, v \rangle$ it follows that the corresponding norms of f coincide.

Let $Pf_p(G)$ denote the closure of $L^1(G)$ in the algebra of convolution operators on $L^p(G)$ and $W_p(G)$ the dual space of $Pf_p(G)$, which is contained in $L^\infty(G)$, and in which $A_p(G)$ is norm non-increasingly embedded.

If t is a nonnegative (almost everywhere) function with $\|t\|_1 = 1$ then $t * t^\sim$, as a convolution operator, is almost a special operator, except that (G, dg) might not be σ -finite.

Let U_α be an open relatively compact neighborhood base at the identity e of G . If $V_\alpha = V_\alpha^{-1}$ are open neighborhoods of e such that $V_\alpha^2 \subset U_\alpha$ then $\tau_\alpha = \lambda(V_\alpha)^{-1} \chi_{V_\alpha}$, where $\lambda(V)$ denotes the Haar measure of V and χ_V its characteristic function, $t_\alpha = \tau_\alpha * \tau_\alpha^\sim$ and $e_\alpha = t_\alpha * t_\alpha$ are approximate identities for $L^1(G)$, e_α being the square of a "special" operator. This last fact we seem really to need in the proof of the following

LEMMA. — Let $e_\alpha = t_\alpha * t_\alpha$ be as above, if u_β is a net in $W_p(G)$ such that $u_\beta \rightarrow u_0$ in the weak* topology of $W_p(G)$ and if $\|u_\beta\|_{W_p} \rightarrow \|u_0\|_{W_p}$, then for $\epsilon > 0$ there exist β_0, α_0 such that

$$i) \|e_{\alpha_0} * u_\beta - u_\beta\|_{W_p} \leq \epsilon \text{ for all } \beta \geq \beta_0$$

and

$$ii) \|e_{\alpha_0} * u_0 - u_0\|_{W_p} \leq \epsilon.$$

Proof. — Clearly ii) is a consequence of i), so it is enough to prove i) and we may assume that $\|u_0\| = 1$. We suppose now that there is a net u_β which converges to u_0 as described in the lemma and an $\epsilon > 0$ such that for all α_0, β_0 there exists $\beta > \beta_0$ with

$$\|e_{\alpha_0} * u_\beta - u_\beta\| > \epsilon.$$

We shall derive a contradiction.

Let $0 < \eta < \epsilon/2$, to be specified later, and choose $f \in L^1(G)$ with

$$\|f\|_{p_{f_p}} = 1, \langle f, u_0 \rangle \geq 1 - \eta,$$

then choose α_0 with

$$\|e_{\alpha_0} * f - f\|_{p_{f_p}} \leq \eta$$

and β_0 with

$$|\langle u_\beta, e_{\alpha_0} * f \rangle - \langle u_0, e_{\alpha_0} * f \rangle| \leq \eta,$$

$$\|u_\beta\| \leq 1 + \eta \text{ for all } \beta \geq \beta_0.$$

We may now fix $\beta > \beta_0$ with

$$\|e_{\alpha_0} * u_\beta - u_\beta\|_{w_p} > \epsilon$$

and find $g \in L^1(G)$, $\|g\|_{p_{f_p}} = 1$, with

$$\langle e_{\alpha_0} * u_\beta - u_\beta, g \rangle > \epsilon - \eta$$

i.e. $\langle u_\beta, (e_{\alpha_0} - 1) * g \rangle = \langle u_\beta, (1 - e_{\alpha_0}) * (-g) \rangle > \epsilon - \eta$.

Now, the supports of t_{α_0}, f, g are contained in a σ -finite open subgroup G_0 of G . Since for an $L^1(G)$ function h with support in G_0 : $\|h\|_{p_{f_p}(G_0)} = \|h\|_{p_{f_p}(G)}$, we may apply the estimation of the corollary of the last section to $e_{\alpha_0} * f - \lambda g + \lambda e_{\alpha_0} * g$, where $\lambda > 0$:

$$\|e_{\alpha_0} * f + (1 - e_{\alpha_0}) * (-\lambda g)\| \leq (\|f\|^r + 2\|-\lambda g\|^r)^{1/r} = (1 + 2\lambda^r)^{1/r}.$$

So on one hand

$$\begin{aligned} \langle u_\beta, e_{\alpha_0} * f + (1 - e_{\alpha_0}) * (-\lambda g) \rangle &\leq \|u_\beta\| (1 + 2\lambda^r)^{1/r} \\ &\leq (1 + \eta) (1 + 2\lambda^r)^{1/r}, \end{aligned}$$

and on the other

$$\begin{aligned} |\langle u_\beta, e_{\alpha_0} * f + (1 - e_{\alpha_0}) * (-\lambda g) \rangle| &= |\langle u_0, f \rangle + \langle u_0, e_{\alpha_0} * f - f \rangle \\ &\quad + \langle u_\beta - u_0, e_{\alpha_0} * f \rangle + \lambda \langle e_{\alpha_0} * u_\beta - u_\beta, g \rangle| \geq 1 - 3\eta + \lambda\epsilon/2. \end{aligned}$$

But $1 - 3\eta + \lambda\epsilon/2 \leq (1 + \eta)(1 + 2\lambda^r)^{1/r}$ cannot hold for all $\eta \in (\epsilon/2, 0)$, $\lambda > 0$.

We thank the referee for pointing out to us the following implication of the lemma (due to M. Cowling, theorem 3 of [3]; see [4] for a different proof).

COROLLARY. — *Translations act continuously on $W_p(G)$.*

Proof. — For $h \in G$ let ${}_h u(g) = u(h^{-1}g)$ and $u_h(g) = u(gh)$, $g \in G$.

We first consider left translations, if u is in $W_p(G)$, $\epsilon > 0$ then we find, by the lemma, an element e of $L^1(G)$ with

$$\|e * u - u\|_{W_p} \leq \epsilon.$$

Then

$$\begin{aligned} \|{}_h u - u\|_{W_p} &\leq \|{}_h u - {}_h(e * u)\|_{W_p} + \|{}_h(e * u) - e * u\|_{W_p} \\ &\quad + \|e * u - u\|_{W_p} \\ &\leq \|u - e * u\|_{W_p} + \|{}_h e - e\|_1 \|u\|_{W_p} + \|e * u - u\|_{W_p} \\ &\leq 3\epsilon \text{ if } h \text{ is in a neighborhood } V \text{ of the identity,} \\ &\text{choosen such that } \|{}_h e - e\|_1 \leq \epsilon \|u\|_{W_p}^{-1} \text{ for all } h \in V. \end{aligned}$$

From $\|f\|_{pf_p} = \|\tilde{f}\|_{pf_p'}$, for $f \in L^1(G)$, we infer that $\|u\|_{W_p} = \|u^v\|_{W_p'}$, for $u \in W_p(G)$, and hence the continuity of right translations, on W_p , follows from that of left translations on W_p' .

It has been proved by Herz [8], that for $v \in A_p(G)$ and $u \in W_p(G)$ the pointwise product $u \cdot v$ is in $A_p(G)$ and $\|u \cdot v\|_{A_p} \leq \|u\|_{W_p} \|v\|_{A_p}$.

We say that a net $u_\beta \in W_p(G)$ converges to $u \in W_p$ in the A_p -multiplier topology, if, for all $v \in A_p$, $u_\beta v \rightarrow uv$ in A_p norm.

THEOREM. — *On the unit sphere $S = \{u \in W_p / \|u\|_{W_p} = 1\}$ of $W_p(G)$ the weak* and the A_p -multiplier topology coincide.*

Proof. — Let $u_\beta, u \in S$ be such that $u_\beta \rightarrow u$ in the weak* topology. Let $e_\alpha = t_\alpha * t_\alpha$ be as in the lemma. Then for $v \in A_p(G)$

$$\begin{aligned} \|u_\beta v - uv\| &\leq \|(u_\beta - e_{\alpha_0} * u_\beta) v\| + \|[e_{\alpha_0} * (u_\beta - u)] v\| \\ &\quad + \|(e_{\alpha_0} * u - u) v\| \\ &\leq \epsilon \|v\| + \|[e_{\alpha_0} * (u_\beta - u)] v\| + \epsilon \|v\|, \end{aligned}$$

when $\beta \geq \beta_0$, where α_0, β_0 are choosen according to the lemma.

Since $t_{\alpha_0} \in L^1(G) \cap L^\infty(G)$ has compact support we may

apply lemma 6 of [7] and find $\beta_1 \geq \beta_0$ such that for $\beta \geq \beta_1$ $\| [e_{\alpha_0} * (u_\beta - u)] v \| \leq \epsilon$.

For the converse it is sufficient to note that $u_\beta \rightarrow u$ uniformly on compact sets, whenever $u_\beta \rightarrow u$ in the A_p -multiplier topology and $\|u\|_{W_p}$ is bounded. So, for a compact set K , let $v \in A_p(G)$ be a function which takes the value one on K (e.g. take $v = \lambda(U)^{-1} \chi_U * \chi_{K^{-1}U}$, where U is open, relatively compact) then

$$\sup_{g \in K} |(u_\beta - u)(g)| \leq \|(u_\beta - u)v\|_\infty \leq \|(u_\beta - u)v\|_{A_p} \rightarrow 0.$$

The following corollary is of interest with respect to the problems considered in [6]. To state it, let, for a compact set $K \subset G$, $A_K^p(G) = \{v \in A_p(G) / \text{supp } v \subset K\}$. This space we consider as a subspace of $W_p(G)$.

COROLLARY. — On the unit sphere of $(A_K^p(G), \|\cdot\|_{W_p})$ the weak * and the norm topology coincide.

Proof. — Let $u_\beta, u \in A_K^p(G)$ be such that $u_\beta \rightarrow u$ in the weak * topology and $\|u_\beta\|_{W_p} = 1 = \|u\|_{W_p}$. Then, for $v \in A_K^p(G)$ which is constant one on K ,

$$\|u_\beta - u\|_{W_p} = \|(u_\beta - u)v\|_{W_p} \leq \|(u_\beta - u)v\|_{A_p} \rightarrow 0$$

by our theorem. The converse is evident.

4. Addendum.

When the paper was already finished we realized that, by our method, we can improve a theorem of E.E. Granirer, theorem 3 of [6], which we think to be central in the cited paper.

Let $MA_p(G)$ be the algebra of (continuous, bounded) functions on G which pointwise multiply $A_p(G)$ into itself and let for $u \in MA_p(G)$ $\|u\|_{MA_p} = \sup \{ \|uv\|_{A_p} / \|v\|_{A_p} = 1 \}$.

THEOREM. — Let $u \in MA_p(G)$ be such that $u(g) = \|u\|_{MA_p}$ for an $g \in G$. If u_β is a net in $MA_p(G)$ such that

$$\|u_\beta\|_{MA_p} \rightarrow \|u\|_{MA_p}$$

and $u_\beta \rightarrow u$ in the $\sigma(MA_p(G), L^1(G))$ -topology then $u_\beta \rightarrow u$ in the A_p -multiplier topology.

To prove this theorem we need an auxiliary result for whose proof we use that we admit complex scalars for our linear spaces.

PROPOSITION. — *The linear span of $\{v \in A_p(G) / v(e) = \|v\|_{A_p}, v \text{ has compact support}\}$ is norm dense in $A_p(G)$.*

Proof. — The dual space of $A_p(G)$ is the ultra weak operator topology closure of $Pf_p(G)$ in the space of bounded operators on $L^p(G)$, the duality is given by

$$\langle T, u \rangle = \sum_{n=1}^{\infty} \int_G T w_n(g) v_n(g) dg$$

when $u = \sum_{n=1}^{\infty} v_n * w_n^v \in A_p(G)$, $T \in A_p(G)^*$ (see [9]).

By theorem 4.1 and theorem 9.4 of [2] we have

$$e^{-1} \|T\| \leq \sup \{ \langle Tf, f^\# \rangle / f \in L^p(G), \|f\|_p = 1 \},$$

where $f^\# = |f|^{p-1} \exp(-i \arg(f(\cdot)))$ is the unique element of $L^{p'}(G)$ with $\langle f, f^\# \rangle = 1$ and norm one.

If we approximate $f \in L^p(G)$ by $f \cdot \chi_K$, where $K \subset G$ is a suitable compact set, in the L^p -norm, then $(f \chi_K)^\# = f^\# \chi_K$ approximates $f^\#$ in $L^{p'}$ -norm. This is why we can restrict the supremum to be taken over the elements $f \in L^p(G)$ with compact support and norm one.

If $f \in L^p(G)$ has compact support then $v = f^\# * f^v$ will have compact support too, and if $\|f\|_p = 1$ then,

$$1 = \|f\|_p \|f^\#\|_{p'} \geq \|v\|_{A_p} \geq \|v\|_\infty = f^\# * f^v(e) = \|f\|_p^p = 1.$$

Hence for any $T \in A_p(G)^*$:

$$e^{-1} \|T\| \leq \sup \{ \langle T, v \rangle / v(e) = \|v\|_{A_p}, v \text{ has compact support} \},$$

and the proposition follows by an application of the Hahn-Banach theorem.

Proof of the theorem. — We may assume $\|u\|_{MA_p} = 1$ and, since translations are isometries of $MA_p(G)$, we may further assume $u(e) = \|u\|_{MA_p} = 1$.

Since there exists β_0 such that $\sup \{ \|u_\beta\|_{MA_p} / \beta \geq \beta_0 \} < \infty$ it suffices, by the above proposition, to show $u_\beta v \rightarrow uv$ when v has compact support, say K , and $v(e) = \|v\|_{A_p} = 1$. Now, the $u_\beta v$ and uv are elements of $A_K^p(G)$, and on this space the W_p -norm is equivalent to the A_p -norm (this follows from proposition 1 of [6] and proposition 3 of [8]). Thus we must only show $\|u_\beta v - uv\|_{W_p} \rightarrow 0$.

Clearly, $u_\beta v \rightarrow uv$ in the weak* topology of $A_K^p(G)$, and, if we can show that $\lim \|u_\beta v\|_{W_p} = \|uv\|_{W_p}$, then the corollary of the last section finishes the proof.

But,

$$1 = u(e)v(e) \leq \|uv\|_{W_p} \leq \liminf \|u_\beta v\|_{W_p}$$

and

$$\begin{aligned} 1 = u(e)v(e) &= \|u\|_{MA_p} \|v\|_{A_p} = \lim \|u_\beta\|_{MA_p} \|v\|_{A_p} \\ &\geq \limsup \|u_\beta v\|_{A_p} \geq \limsup \|u_\beta v\|_{W_p} \end{aligned}$$

from which $\lim \|u_\beta v\|_{W_p} = 1 = \|uv\|_{W_p}$ follows.

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