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RESTRICTIONS OF FOURIER TRANSFORMS TO CURVES^(*)

by Stephen W. DRURY

Introduction.

Given a smooth curve in $\mathbf{R}^n$ and a smooth measure σ on the curve one may ask for which a and b does the restriction estimate $\left\{ \int |\hat{f}(x)|^b d\sigma(x) \right\}^{1/b} \leq C \|f\|_a$ ($f \in \mathcal{S}$) hold. Such an estimate implies that for f in $L^a(\widehat{\mathbf{R}^n})$ the restriction of $\hat{f}$ to the curve "makes sense". We refer the reader to [1] and [2] for general information about restriction theorems. The object of this article is to extend the restriction theorem of Prestini [3] to the full range of exponents.

Since $\mathcal{FL}^a$ is an affinely invariant space (that is invariant under the group of affine motions) we will consider only affine invariants of the curve. For a discussion of these invariants the reader may consult Guggenheimer [4] pp. 170-173. For the sake of simplicity in laying out the basic idea of this paper we will restrict attention to the special case of the non-compact curve $x(t) = \left(t, \frac{1}{2}t^2, \frac{1}{6}t^3 \right)$ in $\mathbf{R}^3$. This is essentially the unique curve for which the first and second affine curvatures vanish and the affine arc length measure is just dt .

THEOREM 1. — Let $1 \leq a < \frac{7}{6}$, let $a' = 6b$ (so that $\frac{7}{6} < b \leq \infty$) and let $x(t) = \left(t, \frac{1}{2}t^2, \frac{1}{6}t^3 \right)$. Then

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$$\left\{ \int |\hat{f}(x(t))|^b dt \right\}^{1/b} \leq C_a \|f\|_a \text{ for all } f \in \mathfrak{S}(\mathbf{R}^3).$$

The same techniques also yield the corresponding result in higher dimensions.

THEOREM 1'. — Let $n \geq 2$, $1 \leq a < (n^2 + n + 2)/(n^2 + n)$, $a' = \frac{1}{2}n(n+1)b$ and let $x(t) = \left(t, \frac{1}{2}t^2, \dots, \frac{1}{n!}t^n\right)$. Then

$$\left\{ \int |\hat{f}(x(t))|^b dt \right\}^{1/b} \leq C_a \|f\|_a$$

for all $f \in \mathfrak{S}(\mathbf{R}^n)$.

It is well known that the ranges of a and b in the above theorems are optimal at least for $n = 2$ and 3 . The theorem is well known in case $n = 2$ (Zygmund [5]).

Our methods can also be used to establish a local result. For this we demand that the curve possess an affine arc length parametrization, that is a parametrization $x(t)$ such that

$$\det(x^{(1)}(t), x^{(2)}(t), \dots, x^{(n)}(t)) = 1$$

for all t . Here $x^{(k)}$ denotes the k th derivative of x viewed as a column vector.

THEOREM 2. — Let $n \geq 2$ and let $x(t)$ be a $C^{(n)}$ curve in $\mathbf{R}^n$ defined for $\alpha < t < \beta$ and such that t is the affine arc length. Then for $1 \leq a < (n^2 + n + 2)/(n^2 + n)$ and $a' \leq \frac{1}{2}n(n+1)b$ we have $\left\{ \int_{\alpha'}^{\beta'} |\hat{f}(x(t))|^b dt \right\}^{1/b} \leq C_{\alpha', \beta', a} \|f\|_a$ ($f \in \mathfrak{S}$) for every compact subinterval $[\alpha', \beta']$ of (α, β) .

Proofs of the theorems. — We now seek to prove Theorem 1. We will adopt the dual formulation of the problem. Thus we will prove that

$$\|(\varphi \cdot \sigma)^\wedge\|_q \leq C \|\varphi\|_p \quad (1)$$

for $1 \leq p < 7$ and $p^{-1} + 6q^{-1} = 1$. Here σ denotes the affine arc length measure on the curve and φ is a function in $L^p(\sigma)$. We will prove this result by induction on the exponent p . Therefore we shall assume that equation (1) holds in the range $1 \leq p \leq p_0$ for some fixed $p_0 < 7$.

Because of the special geometry of the situation there is a 1-parameter group of affine motions of $\mathbf{R}^3$ given by

$$\alpha_s(x, y, z) = \left(x + s, y + sx + \frac{1}{2}s^2, z + sy + \frac{1}{2}s^2x + \frac{1}{6}s^3 \right)$$

which fix our curve and act on it by translation of the parameter t . The orbits of this action are curves affinely equivalent to the initial one. In fact let us parametrize the curves by y and z (taking $x = 0$) so that the corresponding curve is

$$t \longrightarrow \left(t, y + \frac{1}{2}t^2, z + ty + \frac{1}{6}t^3 \right).$$

By affine equivalence our induction hypothesis applies equally well to each of the orbits. For a function $f \in L^1_{\text{loc}}(\mathbf{R}^3)$ we introduce the auxiliary function F by defining

$$F(y, z; t) = f \left(t, y + \frac{1}{2}t^2, z + ty + \frac{1}{6}t^3 \right). \quad (2)$$

By disintegrating the function f on the family of orbits and applying the induction hypothesis on each orbit we have

LEMMA 1. — *Let $1 \leq p < p_0$ and $p^{-1} + 6q^{-1} = 1$. Then*

$$\|\hat{f}\|_q \leq C_p \|F\|_{L^1(L^p)}.$$

Here the mixed norm space is $L^1(\mathbf{R}^2_{y,z}, L^p(\mathbf{R}_t))$.

A simple change of variable and an application of the Plancherel Theorem also yield $\|\hat{f}\|_2 = \|f\|_2 = \|F\|_{L^2(L^2)}$. Thus by a routine interpolation argument (Benedeck and Panzone [6]) we have

LEMMA 2. — *For (a^{-1}, b^{-1}) in the triangle with vertices $(1, 1)$, $(1, p_0^{-1})$, $(\frac{1}{2}, \frac{1}{2})$ and c defined by $5a^{-1} + b^{-1} + 6c^{-1} = 6$*

we have $\|\hat{f}\|_c \leq C_{a,b} \|F\|_{L^a(L^b)}$.

This lemma may be viewed as a substitute for the Hausdorff-Young Theorem.

We now follow the method of Prestini. Let φ be a function on $\mathbf{R}$ satisfying $|\varphi| \leq 1_E$ and $\text{meas}(E) = m$. We consider

$(\varphi \cdot \sigma) * (\varphi \cdot \sigma) * (\varphi \cdot \sigma)$ and scale the resulting measure by a factor of $\frac{1}{3}$ so as to adapt it to the original curve. The scaled measure is given by a locally integrable density f such that

$$((\varphi \cdot \sigma)^\wedge (3u))^3 = \hat{f}(u) \quad (3)$$

and

$$f\left(\frac{1}{3}(x(t_1) + x(t_2) + x(t_3))\right) = cv^{-1}\varphi(t_1)\varphi(t_2)\varphi(t_3)$$

where v stands for the Vandermonde $|(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)|$ and c is an absolute constant. A calculation now leads to

$$\|F\|_{L^a(L^b)} = c \left\{ \int v^{-(a-1)} \left\{ \Phi(h_1, h_2) \right\}^{ab-1} dh_1 dh_2 \right\}^{a-1} \quad (4)$$

with v the Vandermonde $|h_1 h_2 (h_1 - h_2)|$ and

$$\Phi(h_1, h_2) = \int |\varphi(t)\varphi(t+h_1)\varphi(t+h_2)|^b dt.$$

Clearly $|\Phi(h_1, h_2)| \leq m$ and $\int |\Phi(h_1, h_2)| dh_1 dh_2 \leq m^3$. Combining these estimates gives

$$\|\Phi^{ab-1}\|_{L_{s,1}} \leq C_{s,a,b} m^{ab-1+2s-1} \quad (5)$$

where $L_{s,1}$ denotes the Lorentz space $L_{s,1}(dh_1 dh_2)$ (see Stein and Weiss [7] or Hunt [8]) and where $1 \leq ba^{-1} < s < \infty$. On the other hand routine calculations show that $v^{-(a-1)}$ lies in the dual Lorentz space $L_{s',\infty}(dh_1 dh_2)$ for $2 = 3(a-1)s'$ and $1 < a < \frac{5}{3}$.

Thus we obtain from (4) and (5) that for

$$1 < a < \frac{5}{3}, \quad 5a^{-1} - 2b^{-1} < 3, \quad a \leq b,$$

we have,

$$\|F\|_{L^a(L^b)} \leq C_{a,b} m^{5a^{-1}+b^{-1}-3}.$$

We are now in a position to apply lemma 2 for (a^{-1}, b^{-1}) in the quadrilateral defined by $a^{-1} \geq b^{-1}$, $a^{-1} > \frac{3}{5}$, $5a^{-1} - 2b^{-1} < 3$ and $(p_0 - 2)a^{-1} + p_0 b^{-1} \geq p_0 - 1$. Thus there exists a number a_0 depending only on p_0 with $a_0 < \frac{5}{3}$ so that lemma 2 can be applied

in case $a_0 < a < \frac{5}{3}$ and b is given by

$$(p_0 - 2)a^{-1} + p_0 b^{-1} = p_0 - 1.$$

We conclude that for suitable c_0 we have $\|\hat{f}\|_c \leq C_c m^{3-6c^{-1}}$ for all c in the range $30p_0(13p_0 - 1)^{-1} < c < c_0$. Thus by (3)

$$\|(\varphi \cdot \sigma)^\wedge\|_q \leq C_q m^{1-6q^{-1}}$$

for all q in the range $90p_0(13p_0 - 1)^{-1} < q < 3c_0$. Routine interpolation arguments now yield $\|\varphi \cdot \sigma^\wedge\|_q \leq C_p \|\varphi\|_p$ for $p^{-1} + 6q^{-1} = 1$ and $1 \leq p < 15p_0(2p_0 + 1)^{-1}$. This completes the induction step.

The induction starts trivially with $p_0 = 1$. One step of the induction yields the result for $1 \leq p < 5$ — that is the result of Prestini and with the same proof. With two steps we have the result for $1 \leq p < 75/11$ and it is clear that for any p with $1 \leq p < 7$ the result for that p will follow after only finitely many steps.

The proof of theorem 1' is entirely analogous.

We will leave the detailed proof of theorem 2 to the reader. Some comments however are in order. First of all in general there is no group action preserving the initial curve. Thus a typical curve of our family will be defined by

$$t \longrightarrow n^{-1} \sum_{k=1}^n x(t + h_k)$$

the family of curves being indexed by the $(n-1)$ -dimensional manifold of $(h_1, \dots, h_n)$ satisfying $\sum_{k=1}^n h_k = 0$. The inductive nature of the proof then leads in general to further curves of the form

$$y(t) = \sum_{k=1}^K \alpha_k x(t + \ell_k) \quad (6)$$

where $\alpha_k > 0$, $\sum_{k=1}^K \alpha_k = 1$ and the ℓ_k 's are sums of the h_k 's.

Let t_0 be a fixed point $\alpha < t_0 < \beta$. It will be necessary to establish uniform estimates for the curve (6) on an interval

$t_0 - \epsilon < t < t_0 + \epsilon$ and for $|\ell_k| < \epsilon$. Towards this we select convex neighbourhoods V_k of $x^{(k)}(t_0)$ such that

$$2 \geq \det(v_1, \dots, v_n) \geq \frac{1}{2}$$

for $v_k \in V_k$ ($1 \leq k \leq n$). It now follows from the fact that the initial curve is $C^{(n)}$ that there exists a number $\epsilon > 0$ such that

$$2 \geq \det(y^{(1)}(\tau_1), \dots, y^{(n)}(\tau_n)) \geq \frac{1}{2}$$

for $|\tau_k - t_0| < \epsilon$ and $|\ell_k| < \epsilon$. (In particular it follows that the measure dt is uniformly equivalent to the affine arc length measure of (6) for $|t - t_0| < \epsilon$, $|\ell_k| < \epsilon$). The vital estimate is a lower bound on the absolute value of the Jacobian J of the

barycentre map $(t_1, \dots, t_n) \longrightarrow \frac{1}{n} \sum_{k=1}^n y(t_k)$. Up to a constant

factor this is $|\det(y^{(1)}(t_1), \dots, y^{(1)}(t_n))|$ and by a generalization of the mean-value theorem (Polya, Szegő [9]. Vol. II, part V, Chap. 1, No. 95) this is equivalent to

$$\left(\prod_{1 \leq i < j \leq n} |t_i - t_j| \right) |\det(y^{(1)}(\tau_1), \dots, y^{(n)}(\tau_n))|$$

for suitable $\tau_1, \dots, \tau_n$. This now yields the uniform estimate

$$|J| \geq c_n \prod_{1 \leq i < j \leq n} |t_i - t_j| \quad (c_n > 0)$$

for $|t_0 - t_k| < \epsilon$ ($1 \leq k \leq n$), $|\ell_k| < \epsilon$. This completes our comments on Theorem 2.

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