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ORBIT GROWTH OF CONTACT STRUCTURES AFTER SURGERY

CROISSANCE DES ORBITES PÉRIODIQUES ET COMPLEXITÉ POUR DES STRUCTURES DE CONTACT CONSTRUITES PAR CHIRURGIE

Abstract. — This work is at the intersection of dynamical systems and contact geometry, and it focuses on the effects of a contact surgery adapted to the study of Reeb fields and on the effects of invariance of contact homology.

We show that this contact surgery produces an increased dynamical complexity for all Reeb flows compatible with the new contact structure. We study Reeb Anosov fields on closed 3-manifolds that are not topologically orbit-equivalent to any algebraic flow; this includes many examples on hyperbolic 3-manifolds. Our study also includes results of exponential growth in cases where neither the flow nor the manifold obtained by surgery is hyperbolic, as well as results when the original flow is periodic. This work fully demonstrates, in this context, the relevance of contact homology to the analysis of the dynamics of Reeb fields.

Keywords: Anosov flow, 3-manifold, contact structure, contact flow, Reeb flow, surgery, contact homology.
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Résumé. — Notre étude, à l'intersection des systèmes dynamiques et la géométrie de contact, porte sur les effets de la construction d'une chirurgie de contact adaptée à l'étude des champs de Reeb et sur les effets de l'invariance de l'homologie de contact.

Nous montrons que cette chirurgie de contact produit une complexité dynamique accrue pour tous les flots de Reeb compatibles avec la nouvelle structure de contact. Nous étudions des champs de Reeb Anosov sur des 3-variétés fermées qui ne sont topologiquement orbite-équivalents à aucun flot algébrique, ce qui inclut de nombreux exemples sur des 3-variétés hyperboliques. Notre étude comprend également des résultats de croissance exponentielle dans des cas où, ni le flot obtenu par chirurgie, ni la variété construite ne sont hyperboliques ainsi que des résultats quand le flot d'origine est périodique. Ce travail démontre pleinement, dans ce cadre, la pertinence de l'homologie contact pour analyser la dynamique des champs de Reeb.

1. Introduction

This paper is a sequel of [FH13] in which the authors described a surgery construction adapted to Reeb flows ("contact flows" to dynamicists; see Subsection 2.2). This construction was originally conceived as a source of uniformly hyperbolic Reeb flows. However it turns out that the surgered flows exhibit more noteworthy dynamical properties than originally observed and that interesting consequences of the surgery arise even when the initial or resulting flow are not hyperbolic. (This surgery was developed to modify geodesic flows, but here we also apply it to the (periodic!) fiber flow or vertical flow in a unit tangent bundle.) Thus the primary interest in this contact surgery may be as a rich source of Reeb flows exhibiting new phenomena from both dynamical and contact points of view.

The starting point of our surgery is the unit tangent bundle M of a surface of negative and (mainly, but not always necessarily) constant curvature equipped with its natural contact structures. This surgery is known to contact-symplectic topologists as a Weinstein surgery, and its description in [FH13] makes it easier to study dynamical properties (as opposed to, for instance, topological properties).

Our purpose is to expand the understanding of the dynamical effects achieved by the contact surgery from [FH13] in three main directions:

- we show that the complexity of the surgered flow exceeds that of the flow on which the surgery is performed (Theorems 3.1, 3.3, 3.9, and 3.14),
- we show that much of the complexity of the resulting flow is reflected in the cylindrical contact homology and is therefore realized in *any* Reeb flow associated to the contact structure resulting from the surgery (Theorems 3.9, 3.13, and 3.14), and
- we do this beyond the context of hyperbolic flows in more than one way: we obtain positivity of entropy even when the surgered flow is not hyperbolic (Theorem 3.13), and we produce nontrivial orbit growth by surgery on the strictly periodic fiber flow (Theorem 3.14).

Taken together, this reveals a much richer field of inquiry at the interface between contact geometry and dynamical systems than was apparent when the surgery construction was conceived.

Contact homology and its growth rate are relevant tools to describe dynamical properties of all Reeb flows associated to a given contact structure. Even if it is not always explicit in the statements, they play a crucial role in the proofs of Theorems 3.9, 3.13, and 3.14. A goal of this paper is to demonstrate to dynamicists the use of these powerful tools from contact geometry.

In addition to the dynamical point of view, our study is also motivated by contact geometry as we want to investigate connections between growth properties in Reeb dynamics (generally characterized by the growth rate of contact homology) and the geometry of the underlying manifold. The simplest model of such a connection is Colin and Honda's conjecture [CH13, Conjecture 2.10], and some surgeries under study give examples supporting it. Colin and Honda speculate that the number of Reeb-periodic orbits of universally tight(1) contact structures on hyperbolic manifolds grows at least exponentially with the period. More generally, one may look for sources of exponential or polynomial behavior of contact homology. Our starting point, the unit tangent bundle of an hyperbolic surface, is a transitional example as it carries two special contact structures, one with an exponential growth rate for contact homology and one with a polynomial growth rate. We prove (Theorems 3.13 and 3.14) that some surgeries lead to two coexisting contact forms on the surgered manifold with exponential and polynomial growth rates and therefore give new examples of transitional manifolds with respect to growth rate. Note that these examples do not include hyperbolic manifolds (and are therefore compatible with Colin and Honda's conjecture).

Colin, Dehornoy and Rechtman announced a major breakthrough after the completion of this paper: any nondegenerate Reeb flow on a closed irreducible oriented 3-manifold that is not a graph manifold has positive topological entropy [CDR20].

Structure of the paper

Section 2 gives some background, including Section 2.3, which presents and elaborates our earlier results [FH13]. Section 3 contains our main results. First, Section 3.1 introduces the resulting complexity increase of the surgered geodesic flow. Section 3.2 describes how cylindrical contact homology forces complexity of Reeb flows with the same contact structure, introduces our surgery on the fiber flow, and discusses the relation of our results to other works on contact surgery and Reeb dynamics.

The construction of contact surgery is recalled in Section 4, which also contains some preliminary results on the dynamics of the surgered flow and the proof of Theorem 3.1 on the complexity increase of the surgered geodesic flow.

In Section 5 we define contact homology and its growth rate. This enables us to show how cylindrical contact homology forces complexity of Reeb flows. We prove Theorem 3.9 in Section 6, Theorem 3.13 in Section 7 and Theorem 3.14 in Section 8.

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(1) see Definition 2.3.

Luminy, and we are deeply indebted to CIRM for providing excellent opportunities for this collaboration. We integrate these results with work by others that was done contemporaneously [Alv16a, Alv16b, Alv17]. We thank Marcelo Alves, Frédéric Bourgeois, Patrick Massot and Samuel Tapie for useful discussions and helpful advice. Boris Hasselblatt is also grateful for the support and hospitality of the Eidgenössische Technische Hochschule, which was important as we finalized this work. With respect to the latter, we also greatly owe the referees for insightful comments that led to significant improvements of this paper.

2. Background

This section provides some terminology, discusses the relationship between contact flows and Reeb flows, and recapitulates the surgery construction on which we build.

2.1. Definitions and notations

A manifold is said to be *closed* if it is compact and has no boundary.

Definition 2.1. — *A C^∞ 1-form α on a 3-manifold M is called a* contact form *if $\alpha\wedge d\alpha$ is a volume form. The associated plane field $\xi := \ker\alpha$ is a (cooriented)* contact structure*, and (M,ξ) is called a contact manifold (the geometric object under study in contact geometry is the contact structure as opposed to the contact form). A curve tangent to ξ is said to be* Legendrian*.*

For a given contact structure $\xi = \ker(\alpha)$ the contact forms with kernel ξ are exactly the forms $f\alpha$ where $f \in \mathcal{C}^\infty(M, \mathbb{R} \smallsetminus \{0\})$. Additionally, if $\alpha \wedge d\alpha$ is a volume form then $f\alpha \wedge d(f\alpha)$ is also a volume form for any $f \in \mathcal{C}^\infty(M, \mathbb{R} \smallsetminus \{0\})$.

Definition 2.2 (Reeb flow). — *The* Reeb vector field *associated to a contact form α is the vector field R_α such that $\iota_{R_\alpha}\alpha \equiv 1$ and $\iota_{R_\alpha}d\alpha \equiv 0$. Its flow is called the* Reeb flow *(and it preserves α because $\mathcal{L}_{R_\alpha}\alpha = \iota_{R_\alpha}d\alpha + d\iota_{R_\alpha}\alpha = 0$).* A *Reeb field on a contact manifold (M,ξ) is the Reeb field of any contact form α with $\xi = \ker\alpha$.*

A Reeb vector field (or the associated contact form) is said to be nondegenerate *if all periodic orbits are nondegenerate (*i.e.*, transverse: 1 is not an eigenvalue of the differential of the Poincaré map; one can always perturb a contact form into a nondegenerate contact form, this is a well-know fact, a proof can be found in* [CH13, Lemma 7.1]*).*

Note that the Reeb vector field is associated to a contact form α: if we consider another contact form $\alpha' = f\alpha$ where $f \in \mathcal{C}^\infty(M, \mathbb{R} \smallsetminus \{0\})$, then $d\alpha' = df \wedge \alpha + f d\alpha$, so the condition $\iota_{R_{\alpha'}}d\alpha' = 0$ implies that R_α and $R_{\alpha'}$ are not collinear unless f is constant.

Contact structures on 3-manifolds can be divided into two classes: tight contact structures and overtwisted contact structures. This fundamental distinction is due to Eliashberg [Eli89] following Bennequin [Ben83]; see [EG02, Gei08] for a discussion. Tight contact structures are the contact structures that reflect the geometry of the manifolds and this article focuses on them.

Definition 2.3 (Tight contact structure). — *A contact structure ξ is said to be* overtwisted *if there exists an embedded disk tangent to ξ on its boundary. Otherwise ξ is said to be* tight. Universally tight *contact structures are those with a tight lift to the universal cover. A Reeb vector field (or the associated contact form) is said to be* hypertight *if there is no contractible periodic Reeb orbit. A contact structure is said to be* hypertight *if it admits a hypertight contact form.*

Remark 2.4. — Universally tight and hypertight [Hof93] contact structures are always tight. All the contact structures considered in this paper are hypertight and therefore tight.

We recall from [FH13] a contact surgery on a Legendrian curve $\gamma \in S\Sigma$ derived from a closed geodesic $\mathfrak{c}$ on a hyperbolic surface Σ. This corresponds to a $(1,-q)$ Dehn-surgery and results in a new manifold M_S with a contact form α_A. The construction is presented in Section 4. The Reeb flow R_{α_A} is Anosov if q is positive—and only then (Proposition 4.3).

Definition 2.5 (Contact Anosov flow [KH95]). — *Let M be a closed manifold and $\varphi\colon \mathbb{R} \times M \to M$ a smooth flow with nowhere vanishing generating vector field X. Then φ (and also X) is said to be an Anosov flow if the tangent bundle TM (necessarily invariantly) splits as $TM = \mathbb{R}X \oplus E^+ \oplus E^-$ (the* flow, strong-unstable *and* strong-stable directions, *respectively), in such a way that there are constants $C > 0$ and $\eta > 1 > \lambda > 0$ for which*

$$\left\|D\varphi^{-t} \restriction E^+\right\| \leqslant C\eta^{-t} \quad \text{and} \quad \left\|D\varphi^{t} \restriction E^-\right\| \leqslant C\lambda^{t} \tag{2.1}$$

for $t > 0$. The weak-unstable *and* weak-stable *bundles are $\mathbb{R}X \oplus E^+$ and $\mathbb{R}X \oplus E^-$, respectively. ($E^\pm$ are then tangent to continuous foliations $W^\pm$ with smooth leaves.)*

An flow on a 3-manifold (3-flow for short) with the Anosov property is said to be of algebraic type *if it is finitely covered by the geodesic flow of a surface of constant negative curvature or the suspension of a diffeomorphism of the 2-torus.*

An Anosov flow is called a contact Anosov flow *if it is a Reeb flow, in which case $E^+ \oplus E^-$ is the contact structure, and the associated contact form α is said to be Anosov as well.*

Geodesic flows of Riemannian manifolds with negative sectional curvature are contact Anosov flows with the canonical contact form. For surfaces of constant negative curvature it is easy to verify the defining property directly, and we do so at the start of Section 3.3.

In this paper, we show that the complexity of the flow resulting from our surgery exceeds that of the flow on which the surgery is performed. We measure the complexity of the flow of X via its orbit growth, entropy and cohomological pressure.

Definition 2.6. — *For a contact form α, a free homotopy class ρ of loops and $T > 0$, we denote by $N_T^\rho(\alpha)$ the number of R_α-periodic orbits in ρ with period smaller than T, and by $N_T(\alpha)$ the total number of R_α-periodic orbits with period smaller than T. The* orbit growth *of R_α (or the associated flow) is the asymptotic behavior of $N_T(\alpha)$.*

For Anosov flows the exponential growth rate of $N_T(\alpha)$ is the topological entropy [FH19, Remark 8.3.13]. This is useful when we use entropy bounds to deduce periodic orbit growth for the Anosov flows we produce by surgery. Likewise, cohomological pressure drives orbit growth in a given homology class (Section 3.1.2).

We summarize the needed notions and facts in Section 3.1.1.

2.2. Contact flows versus Reeb flows

Since we hope that this work will be of interest to dynamicists as well as contact geometers, let us clarify the notion "contact flow" as used by dynamicists in comparison with "Reeb flow" (Definition 2.2), and the connection in our context. Dynamicists usually define a contact flow to be a flow that preserves a contact form. (Contact geometers may instead think of flows that preserve a contact *structure* ξ; this is called an infinitesimal automorphism of ξ or contact vector field, while a flow that preserves a contact *form* α is called an infinitesimal automorphism of α or a *strict* contact vector field [Gei08, Definition 1.5.7], [Gir91, Definition 2.1].) The flows we construct in this paper are Reeb flows. Hence our choice of terminology in Definition 2.5—so, except for "contact Anosov flow," we will mainly use the term "Reeb flow" henceforth. However, it seems opportune to establish in the following results (Propositions 2.7, 2.9, 2.10, and 2.14) that Anosov flows preserving a contact form or structure are Reeb flows. Most of the following results hold in arbitrary (odd) dimension; for pertinent definitions in that generality, see [Gei08].

The Anosov property implies that a flow commutes with no other flows [FH19, Corollary 9.1.4], so Proposition 2.8 below implies (in any dimension):

Proposition 2.7. — *An Anosov flow that preserves a contact* form *is its Reeb flow, up to constant rescaling.*(2)

Proposition 2.8. — *A flow with trivial flow-centralizer that preserves a contact form is up to constant scaling the Reeb flow of that contact form.*

Proof. — A flow that preserves a contact form α also preserves its Reeb vector field R_α, so its generating vector field commutes with R_α and is hence a constant multiple of R_α. □

We now establish that more generally, a vector field X which preserves a contact *structure* ξ is a Reeb field of ξ if X is transverse to ξ (Proposition 2.9) or if it is nowhere zero and generates a flow which is either topologically transitive (Proposition 2.10) or an expansive 3-flow (Proposition 2.14).

Libermann's Theorem [Lib59] on contact Hamiltonians (see for instance [Gei08, Theorem 2.3.1]) implies the following simple and well-known observation.

(2) An alternative argument for this is that an invariant 1-form for an Anosov flow must have kernel $E^+ \oplus E^-$ and be constant on the generating vector field. For volume-preserving Anosov 3-flows, the canonical 1-form defined by these constraints is C^1 only when the flow is a suspension or the form is a contact form and the flow its Reeb flow [FH03]. We note that for 3-flows the Anosov property itself can be characterized in contact terms [Hoz20, Mit95]; see also [Bar95].

Proposition 2.9. — *If the flow of X preserves the contact structure $\xi = \ker(\beta)$ and X is transverse to ξ, then X is the Reeb field of the contact form $\alpha := \frac{\beta}{\beta(X)}$.*

Proof. — By assumption, $\beta(X) \neq 0$ everywhere, so $\alpha := \frac{\beta}{\beta(X)}$ is a well-defined contact form (with $\iota_X \alpha = \alpha(X) \equiv 1$). By [Gei08, Lemma 1.5.8 (b)], invariance of $\xi = \ker \alpha$ gives a function f with

$$f \cdot \alpha = \mathcal{L}_X \alpha = \iota_X d\alpha + d\iota_X \alpha. \tag{2.2}$$

Since $d\iota_X \alpha \equiv 0$, contraction with X gives $f \equiv 0$, and $\iota_X d\alpha \equiv 0$, as required. □

Proposition 2.10. — *Consider a vector field X on M whose flow preserves a contact structure ξ. If there is an $x \in M$ with $0 \neq X(x) \in \xi(x)$, then the flow is not topologically transitive.*

Proof. — Write $\xi = \ker \alpha$. If $x \in \mathcal{N} := \{x \in M \mid X(x) \in \xi(x)\}$ is a critical point of $\iota_X \alpha$, then $X(x) = 0$ as follows: $\iota_X \alpha(x) = 0$, and $\iota_X d\alpha(x) = 0$ on $\xi(x)$ by (2.2). Now, $d\alpha$ being nondegenerate on ξ (this is the contact condition) implies that $X(x) = 0$.

By assumption, $\mathcal{N} = \{x \in M \mid \iota_X \alpha(x) = 0\}$ thus contains a regular point of $\iota_X \alpha$ and hence invariantly separates M into the *nonempty* open sets $A_\pm := \{x \mid \pm\iota_X \alpha(x) > 0\}$, so X is not topologically transitive. □

Remark 2.11. — The preceding leaves open what happens for flows with $X \equiv 0$ on $\mathcal{N} \neq \varnothing$, i.e., for which transversality fails, but only at fixed points.

Remark 2.12 (Characteristic hypersurface). — Let X be a vector field whose flow preserves a contact structure $\xi = \ker(\alpha)$ on M. The set $\mathcal{N}$ of points where X is tangent to ξ is a fundamental object in contact topology, called the *characteristic hypersurface*. It plays a crucial role in Giroux's seminal work [Gir91] on convexity in contact topology. In this paper, Giroux proves that it is a smooth surface if X has nondegenerate singularities [Gir91, Exemple 2.6]. The proof is elementary: when X has no singularities, then the contraposition of the first paragraph in the proof of Proposition 2.10 shows that $\mathcal{N}$ is a regular level set of the smooth function $\iota_X \alpha$ and hence a smooth hypersurface. Each connected component supports the nonvanishing vector field X and hence has Euler characteristic 0 [KH95, Poincaré–Hopf Index Theorem 8.6.6].

Remark 2.13. — By contraposition, a topologically transitive flow that preserves a contact structure fails to be transverse only at fixed points, and it is thus a Reeb flow for the contact structure if there are no fixed points. This applies in particular to *volume-preserving* Anosov flows, which have no fixed points and are transitive by ergodicity of volume.

Proposition 2.14. — *An Anosov 3-flow (in fact, an expansive 3-flow without fixed points) that preserves a contact* structure *is a Reeb flow for it (so it preserves volume and almost every point has a dense semiorbit).*

Proof. — If the generating vector field is transverse to the contact structure, we are done by Proposition 2.9. Otherwise, each connected component of $\mathcal{N} \neq \varnothing$ is a 2-torus (Remark 2.12), and the flow on it is a special flow over a circle diffeomorphism [KH95, Corollary 14.2.3], hence not expansive. So the flow is not expansive and hence also not an Anosov flow. □

Remark 2.15. — Proposition 2.14 implies that Anosov 3-flows that preserve a contact *structure* are topologically transitive, so, for instance, the Franks–Williams Anosov flow [FH19, Section 8.3] preserves no contact *structure* because it is not transitive.

2.3. New Reeb flows

We state a special case of the main result of the surgery construction from [FH13] in a way that points to the broader perspective of the present work and make a few initial observations that go further. We recall from [FH13] that this surgery per se (without "contact" or "Reeb") originated with Handel and Thurston [HT80] and has since proved flexible enough to produce infinitely many distinct (i.e., not mutually orbit-equivalent) Anosov flows on the same 3-manifold [BM19, CP20]. The new feature of the surgery from [FH13] is that it produces Reeb flows from Reeb flows.

Theorem 2.16 ([FH13, Theorems 1.6, 1.9]). — *On the unit tangent bundle M of a negatively curved surface, there is a family of smooth Dehn surgeries, including the Handel–Thurston surgery* [HT80], *that produce new Reeb flows from the geodesic flow. The geodesic flow has the following properties:*

1. *It acts on a manifold that is not a unit tangent bundle.*
2. *If it is Anosov, it is not orbit-equivalent to an algebraic Anosov flow.*
3. *If it is Anosov, then its topological and volume entropies differ, or, equivalently, the measure of maximal entropy is always singular* [Fou01].
4. *If it is Anosov and if the surgered manifold is hyperbolic, then each nonempty free homotopy class ρ of closed orbits is infinite, and it is an isotopy class;*[3] *moreover, there exist $a_1, c_1, a_2, c_2 > 0$ such that*
$$\frac{1}{a_2}\ln(T) - c_2 \leqslant N_T^\rho(\alpha_A) \leqslant a_1 \ln(T) + c_1$$
for all $T > 0$, where α_A is the contact form defined on the surgered manifold. [Fen94, Theorem A], [Bar12, Remark 5.1.16, Theorem 5.3.3], [BF14], [BF17, Theorem F].

That these surgeries produce Reeb flows on hyperbolic manifolds is a corollary of the two following theorems.

Theorem 2.17 (Thurston [Thu80, Theorem 5.8.2], [Thu82], Petronio and Porti [PP00]). — *For all but finitely many slopes, Dehn filling a hyperbolic 3-manifold gives rise to a hyperbolic manifold.*

(3) That is to say, each pair of freely homotopic closed orbits is actually related by isotopy, so in particular the pair is the boundary of an *immersed* cylinder. We note that, however, each closed orbit is related to at most finitely many others by the pair being the boundary of an *embedded* cylinder [BF14]. (This latter relation is neither transitive nor reflexive.)

Theorem 2.18 (Folklore [FH13, Theorem 1.12]). — *Suppose* Σ *is a hyperbolic surface,* $\pi\colon M \to \Sigma$ *its unit tangent bundle,* $\gamma\colon S^1 \to M$ *continuous such that* $\mathfrak{c}{:=}\pi\circ\gamma$ *is a closed geodesic that is not the same geodesic traversed more than once and such that* $\ell \cap \mathfrak{c} \neq \varnothing$ *whenever* ℓ *is a noncontractible closed curve. Then* $M \smallsetminus (\gamma(S^1))$ *is a hyperbolic manifold.*

Nonetheless, there exist infinitely many closed orientable hyperbolic manifolds of dimension 3 which do not support an Anosov flow [RSS03, Theorem A]. Additionally, Anosov Reeb flows arise from contact forms that are tight as they are hypertight ([PT72], [Bar06, p. 18]), and there are only finitely many homotopy classes of tight contact structures on a given 3-manifold [CGH09, Théorème 1].[4] On hyperbolic 3-manifolds the same goes for isotopy classes [CGH09, Théorème 2]. We do not know if the surgery from Theorem 2.16 can produce different contact structures on the same manifold, but this seems likely: this surgery can produce pairs of nonequivalent contact Anosov flows on the same hyperbolic manifold [BM19].

Remark 2.19. — The dynamical properties of the flow after surgery differ from the properties of Anosov algebraic flows. Indeed, for algebraic flows, free homotopy classes of closed orbits are finite. For geodesic flows no two (parametrized) orbits are homotopic, though rotating the tangent vector through π isotopes each to its flip, which has the same *image* as another orbit (the same geodesic run backwards), and only in suspensions are all free homotopy classes of *images* of orbits singletons [BF17, Corollary 4.3].

Our surgery corresponds to a $(1, -q)$-Dehn surgery and produces Anosov Reeb flows for $q > 0$. As part of our study focuses on the $q < 0$-case, it is important to note the following.

Proposition 2.20. — *Some surgeries from Theorem 2.16 produce flows that are not Anosov (Proposition 4.3).*

In the case $q = 1$, this surgery is the standard Weinstein surgery as defined by Weinstein [Wei91] in 1991 simplifying Eliashberg's work [Eli90] of 1990 (see [Gei08, Chapter 6] for more details). The surgery $(1, q)$ for any q can be deduced from this construction. A direct construction for any q using Giroux theory of convex surfaces can be found in [DG01].

In answer to a question of Serge Troubetzkoy we here note:

Proposition 2.21. — *There are analytic Anosov flows with the properties in Theorem 2.16.*

Proof. — The contact form is smooth and can hence be approximated by analytic ones. The contact property of the form and the Anosov property of its Reeb flow are open. □

Remark 2.22. — Another take on the connection with the Handel–Thurston construction is that our result implies in particular that the Handel–Thurston examples are topologically orbit-equivalent to Reeb flows.

[4] So there are only finitely many homotopy classes of Anosov vector fields on a 3-manifold [EG02].

Remark 2.23. — For context we recall here that contact Anosov flows have the *Bernoulli* property [CH96, KB94, OW98] and exponential decay of correlations [Liv04]. The Bernoulli property and the Ornstein Isomorphism Theorem [Orn74] imply that the flows we obtain from our surgery are measure-theoretically isomorphic to the original contact Anosov flow up to a constant rescaling of time, the constant being the ratio of the Liouville entropies. (This answers a question of Vershik.)

3. Main results

3.1. Production of closed orbits for contact Anosov flows

3.1.1. Impact on entropy

We continue with new results about the features of the contact Anosov flows from [FH13] to the effect that the surgery of Theorem 2.16 produces "exponentially many" closed orbits. We preface these statements by a brief summary of the needed notions and facts pertinent to entropy.

(1) The *topological entropy* of an Anosov flow (or of the vector field that generates it) equals the exponential growth rate of the number of periodic orbits; in our case this means that $h_{\text{top}}(R_\alpha) = \lim_{T\to\infty} \frac{1}{T} \log N_T(\alpha)$.

(2) The entropy $h_\mu(\varphi^t)$ of a flow φ^t with respect to an invariant Borel probability measure μ (also referred to as the entropy of μ with respect to φ^t) does not exceed the topological entropy of φ^t.(5)

(3) If a flow-invariant Borel probability measure μ is absolutely continuous with respect to a smooth volume, then we say that it is a Liouville measure and write $h_{\text{Liouville}} := h_\mu$.

(4) For the geodesic flow g^t of a surface we have $h_{\text{Liouville}}(g^t) = h_{\text{top}}(g^t)$ if (and, for genus larger than 1, only if [Fou01, Kat82, Kat88]) the curvature is constant.

(5) Time-scaling: if $s \in (0,\infty)$, then $h_{\text{Liouville}}(sX) = s h_{\text{Liouville}}(X)$ and $h_{\text{top}}(sX) = s h_{\text{top}}(X)$.

(6) More generally, there is Abramov's formula: the entropy of a time-change gX of a nonzero vector field X with respect to a gX-invariant probability measure μ_g canonically associated with an X-invariant Borel probability measure μ is

$$h_{\mu_g}(gX) = h_\mu(X) \int g d\mu. \tag{3.1}$$

This means that comparisons of the intrinsic dynamical complexity of these vector fields are meaningful only when $\int g = 1$.

(7) Pesin entropy formula [BP07, Theorem 10.4.1]: For a volume-preserving flow φ^t with 1-dimensional expanding direction, $h_{\text{Liouville}}(\varphi^t)$ equals the positive Lyapunov exponent of the flow [BP07], [KH95, Definition S.2.5], which is (a.e.) defined as the exponential growth rate of unstable vectors under the flow and as a function of time.

(5) Indeed, the topological entropy is the supremum of the entropies of invariant Borel probability measures (Variational Principle).

Theorem 3.1. — *If ψ^t is a contact Anosov flow obtained from the geodesic flow g^t of a compact oriented surface of constant negative curvature by the surgery in Theorem 2.16 (generated by the vector field in* (4.3)*), then its topological entropy is strictly larger. Indeed, $h_{top}(\psi^t) > h_{Liouville}(\psi^t) \geqslant h_{Liouville}(g^t) = h_{top}(g^t)$.*

Since h_{top} measures the exponential growth rate of periodic orbits of a hyperbolic dynamical system (item (1) *above), the number $N_T(\psi^t)$ of ψ^t-periodic orbits of period $t \leqslant T$ (of up to a given length) grows at a larger exponential rate than $N_T(g^t)$.*

Remark 3.2. — The strict inequality in Theorem 3.1 is obtained by contraposition of a rigidity result [Fou01], so we do not know by how much the topological entropy increases through our surgery. Bishop, Hughes, Vinhage and Yang suggested to provide lower bounds for this entropy-increase by using cutting sequences in the spirit of Series.

3.1.2. Growth in homology classes

In a self-contained digression, we can give rather more detailed information about orbit growth in homology classes.

Theorem 3.3. — *If ψ^t is a contact Anosov flow obtained from the geodesic flow g^t of a compact oriented surface of constant negative curvature by the surgery in Theorem 2.16 (generated by the vector field in* (4.3)*), then*

$$N_T^{\zeta}(\psi^t) \Big/ N_T^{\eta}(g^t) \xrightarrow[T\to\infty]{\text{exponentially}} \infty$$

for any homology classes ζ for ψ^t and η for g^t (where $N_T^{\zeta}(\psi^t)$ and $N_T^{\eta}(g^t)$ count the number of periodic orbits orbit with period $\leqslant T$ in the homology *classes ζ and η).*

The proof (equation (3.3) below) uses the notion of *cohomological pressure.*

Definition 3.4 ([Sha93, Theorem 1 (iii), p. 398]). — *The* cohomological pressure *of φ^t is*

$$P(\varphi^t) := \inf_{[b]\,\in\, H^1(M,\mathbb{R})} \left\{ \underbrace{\sup_{\mu\,\in\, Mc\,(\varphi^t)} \left\{ h_\mu(\varphi^t) + \int b(X)d\mu \right\}}_{\text{the usual pressure of the function } b(X)} \right\},$$

where $\mathcal{M}(\varphi^t)$ is the set of φ^t-invariant Borel probability measures.

Remark 3.5. — To see that this is well-defined, note that the pressure of $b(X)$ depends only on the cohomology class $[b]$ for the following reason. $[b] \in H^1(M,\mathbb{R})$, the first de Rham cohomology group, and the defining integral is the Schwartzman winding cycle, which is well-defined for a closed 1-form when μ is φ^t-invariant. Moreover, the supremum is unaffected by addition of an exact form to b.

Proof of Theorem 3.3. — Contact Anosov flows satisfy

$$h_{\text{top}}(\varphi^t) \geqslant P(\varphi^t) \geqslant h_{\text{Liouville}}(\varphi^t) \quad \text{[Fan09, Corollary 1].} \tag{3.2}$$

Theorem 3.6 ([Fan09, Theorem 5.3]). — *$P(\varphi^t) > h_{Liouville}(\varphi^t)$ in the context of Theorem 3.1.*

Thus, the conclusion of Theorem 3.1 is strengthened to

$$h_{\mathrm{top}}(\varphi^t) \geqslant P(\varphi^t) > h_{\mathrm{Liouville}}(\varphi^t) \geqslant h_{\mathrm{Liouville}}(g^t) = h_{\mathrm{top}}(g^t) \overset{(3.2)}{=} P(g^t).$$

Thus, $P(\varphi^t) > P(g^t)$, and Theorem 3.3 follows because contact Anosov flows are homologically full[(6)] [Fan09, Proposition 1], and, for homologically full flows, cohomological pressure drives orbit growth in a given homology class ζ [Sha93, Theorem 1]:

$$N_T^{\zeta}(\varphi_t) \sim C(\zeta)\frac{e^{TP(\varphi_t)}}{T^{1+\frac{b_1}{2}}} \text{ as } T \to \infty, \tag{3.3}$$

where b_1 is the first Betti number of the underlying manifold. □

3.2. Production of closed orbits for any Reeb flow

We now broaden the scope far beyond hyperbolic dynamics by beginning to involve contact geometry in a serious fashion. Specifically, the existence of well-understood Reeb flows, such as those in Theorem 2.16, allows us to control all the other Reeb flows associated to the same contact structure in terms of entropy or orbit growth. We transcend hyperbolicity because we describe here our results concerning dynamical properties of Reeb flows associated to all (or a subclass of) contact forms after a contact surgery. These flows need not be hyperbolic even if the contact structure arises from an Anosov flow.

3.2.1. Orbit growth from Anosov Reeb flows

This section presents an archetype of theorem deriving properties for all Reeb flows from stronger properties for one Reeb flow. Our results described in Section 3.2.2 can be seen as extension of this theorem. It can be applied to some of the Reeb flows described in Theorem 2.16.

The existence of Anosov Reeb flows is a source of exponential orbit growth for all Reeb flows as proved by Alves or Macarini and Paternain [MP12, Theorem 2.12.].

Theorem 3.7 (Alves [Alv16b, Corollary 3]). — *If one Reeb flow for a compact contact 3-manifold (M,ξ) is Anosov, then every Reeb flow on (M,ξ) has positive topological entropy. Indeed, if R_α is Anosov, then $h(R_{f\alpha}) \geqslant h(R_\alpha)/\max(f)$ for any $f > 0$.*

Remark 3.8. — These estimates can not be obtained by the Abramov formula, which determines the measure-theoretic entropy of a time-change because different Reeb fields for a contact structure need not be collinear.

(6) I.e., every homology class contains a closed orbit

The standard contact structure on the unit tangent bundle of a hyperbolic surface has an Anosov Reeb flow and therefore, by Theorem 3.7, all its other Reeb flows have positive entropy and their orbit growth is at least exponential. In particular, Theorem 3.7 applies to the contact structures obtained in Theorem 2.16 on hyperbolic manifolds: these are examples satisfying the Colin–Honda conjecture, and on non-hyperbolic manifolds, for instance, when the surgery is associated to a simple geodesic. We give a slightly different proof of this result in Section 5.

3.2.2. Orbit growth from contact homology

We now present our results and extend Theorem 3.7 in two different settings

(1) when the Reeb flow after surgery is Anosov, we study orbit growth in free homotopy classes;
(2) when the Legendrian knot associated to the surgery projects to a simple geodesic, we prove positivity of entropy for any contact form (and any surgery).

Let us describe our results in the first setting. The following result can be seen as a corollary of the invariance of contact homology and the Barthelmé–Fenley estimates from [BF17, Theorem F] in the nondegenerate case, and of Alves' proof of [Alv16b, Theorem 1] and the Barthelmé–Fenley estimates from [BF17, Theorem F] in the degenerate case.

Theorem 3.9. — *Let $(M_S, \xi_S = \ker(\alpha_A))$ be a contact manifold obtained after a nontrivial contact surgery such that α_A is Anosov. Let ρ be a primitive free homotopy class containing at least one R_{α_A}-periodic orbit. Then for all contact forms α on (M_S, ξ_S), ρ contains infinitely many R_α-periodic orbits. Additionally,*

(1) if α is nondegenerate, there exist $a > 0$ and $b \in \mathbb{R}$ such that $N_T^\rho(\lambda) \geqslant a\ln(T) + b$ for all $T > 0$,
(2) if α is degenerate and M_S is hyperbolic, there exist $a > 0$ and $b \in \mathbb{R}$ such that $N_T^\rho(\lambda) \geqslant a\ln(\ln(T)) + b$ for all $T > 0$.

Remark 3.10. — In fact, for nondegenerate α we will prove $N_T^\rho(\alpha) \geqslant N_{CT}^\rho(\alpha_A)$ for some $C > 0$ and for all $T > 0$ and use the Barthelmé–Fenley result. Therefore better control of $N_T^\rho(\alpha_A)$ in some free homotopy classes will lead to better estimates.

Remark 3.11. — There is no hope to obtain an upper bound on $N_T^\rho(\alpha_A)$ for all contact forms as the number of Reeb-periodic orbits can always be increased by creating many periodic orbits in a neighborhood of a preexisting periodic orbit.

Remark 3.12. — If the manifold is not hyperbolic in the second part of Theorem 3.9, the Barthelmé and Fenley estimates are weaker as the upper bound is linear. The proof of Theorem 3.9 can be adapted to this situation but leads to weak control of the growth of periodic orbits in a given homotopy class for degenerate contact forms.

We now turn to our second setting and assume that the Legendrian knot associated to the surgery projects to a simple geodesic. Note that we do not assume that the Reeb flow is Anosov and therefore consider any $(1, q)$-Dehn surgery. Additionally, note that M_S is never a hyperbolic manifold in this setting. Our main theorem is the following.

Theorem 3.13. — *If (M_S, α_A) is a contact manifold obtained from contact surgery along a Legendrian knot that projects to a simple geodesic, then any Reeb flow of $(M_S, \ker(\alpha_A))$ has positive topological entropy and the number of periodic orbits grows at least exponentially with respect to the period.*

The proof of this theorem is based on Alves' work [Alv16a]. In the same paper, Alves obtains the same result when the associated geodesic is separating [Alv16a, Section 4 and Theorem 2]. Our strategy of proof is similar to that of Alves.

Floer type homology and especially contact homology are the main tools to control Reeb-periodic orbits of all contact forms associated to a contact structure. The contact homology of a "nice" contact form α_0 is the homology of a complex generated by R_{α_0}-periodic orbits and therefore encodes dynamical properties of the Reeb vector field (contact homology is described in Section 5).

3.3. Coexistence of diverse Reeb flows

The growth rate of contact homology makes it possible define the polynomial behavior of a contact structure. We now focus on examples obtained by surgery exhibiting polynomial growth.

We first introduce the three Reeb flows that naturally appear on the unit tangent bundle of a constantly curved surface of higher genus. This is elementary but not commonly presented [FH19, Chapter 2]. On the unit tangent bundle of an oriented hyperbolic surface, there is a canonical framing consisting of X, the vector field on $S\Sigma$ that generates the geodesic flow, of V, the vertical vector field (pointing in the fiber direction and defined uniquely by a choice of orientation), and of $H := [V, X]$. It satisfies the classical *structure equations*

$$[V, X] = H, \quad [H, X] = V, \quad [H, V] = X. \tag{3.4}$$

One can check these by using that in the $\mathrm{PSL}(2, \mathbb{R})$-representation of $S\widetilde{\Sigma}$, these vector fields are given by

$$X \sim \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad H \sim \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad V \sim \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}.$$

The structure equations imply that $e^{\pm} := V \pm H$ satisfies $[X, V \pm H] = \mp e^{\pm}$, so if a vector field $f \cdot e^{\pm}$ along an orbit of X is invariant under the geodesic flow, then $0 = [X, fe^{\pm}] = (\dot{f} \mp f)e^{\pm}$, where $\dot{f}$ is the derivative along the orbit. This means that $\dot{f} = \pm f$, so $f(t) = \mathrm{const}\, e^{\pm t}$. Thus, the differential of the geodesic flow expands and contracts, respectively, the directions $e^{\pm}$; this is the Anosov property and $E^{\pm}$ is spanned by the vector $e^{\pm} = V \pm H$.

Of course, in the $\mathrm{PSL}(2,\mathbb{R})$-representation of $S\widetilde{\Sigma}$, these three flows are given by

$$X \rightsquigarrow \exp\left(\begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix}t\right) = \begin{pmatrix}e^{t/2} & 0\\ 0 & e^{-t/2}\end{pmatrix},$$
$$H \rightsquigarrow \exp\left(\begin{pmatrix}0 & 1/2\\ 1/2 & 0\end{pmatrix}t\right) = \begin{pmatrix}\cosh t/2 & \sinh t/2\\ \sinh t/2 & \cosh t/2\end{pmatrix},$$
$$V \rightsquigarrow \exp\left(\begin{pmatrix}0 & -1/2\\ 1/2 & 0\end{pmatrix}t\right) = \begin{pmatrix}\cos t/2 & -\sin t/2\\ \sin t/2 & \cos t/2\end{pmatrix}$$

To see in these terms that X is a Reeb field, define a 1-form α_0 by $\alpha_0(X) = 1$ and $\alpha_0(V) = 0 = \alpha_0(H)$. For $Z \in \{V, H\}$ we have

$$d\alpha_0(X,Z) = \underbrace{\mathcal{L}_X \overbrace{\alpha_0(Z)}^{\equiv 0}}_{=0} - \underbrace{\mathcal{L}_Z \overbrace{\alpha_0(X)}^{\equiv 1}}_{=0} + \underbrace{\alpha_0(\overbrace{[Z,X]}^{\in\{V,H\}})}_{=0} = 0,$$

so $\iota_X d\alpha_0 \equiv 0$. Additionally $\alpha_0 \wedge d\alpha_0(X,V,H) = \alpha_0(X)d\alpha_0(V,H) = 1$ because

$$d\alpha_0(V,H) = \underbrace{\mathcal{L}_V \overbrace{\alpha_0(H)}^{\equiv 0}}_{=0} - \underbrace{\mathcal{L}_H \overbrace{\alpha_0(V)}^{\equiv 0}}_{=0} + \underbrace{\alpha_0(\overbrace{[H,V]}^{=X})}_{=1} = 1.$$

Thus, $\alpha_0 \wedge d\alpha_0$ is a volume form; in fact a volume particularly well adapted to this canonical framing, and α_0 is a contact form with $X = R_{\alpha_0}$.

Likewise, one can check that the 1-forms $\beta = -d\alpha_0(H,\cdot)$ and $\gamma = d\alpha_0(V,\cdot)$ defined by $\beta(V) = 1$ and $\beta(X) = 0 = \beta(H)$, and $\gamma(H) = 1$ and $\gamma(X) = 0 = \gamma(V)$ are contact forms with Reeb vector fields $R_\beta = V$ and $R_\gamma = H$. Note that the orientation given by $\beta \wedge d\beta$ is the opposite of the orientation given by $\alpha_0 \wedge d\alpha_0$; therefore α_0 and β define different contact structures. By contrast, α_0 and γ define isotopic contact structures. Indeed, let ψ^t be the flow of V. Then,

$$(\psi_t)_*X = \cos t/2\; X + \sin t/2\; H \quad \text{and}$$
$$(\psi_t)_*H = \cos t/2\; H - \sin t/2\; X,$$

thus

$$(\psi_t)_*\alpha_0 = \cos t/2\; \alpha_0 + \sin t/2\; \gamma$$

as the two contact forms coincide on $(\psi_t)_*X$, $(\psi_t)_*H$ and $(\psi_t)_*V = V$. So it suffices to study the geodesic flow as the leading representative of this S^1-family of contact Anosov flows. Geometrically, this family of flows can be described as: rotate a vector by an angle, carry it along the geodesic it now defines, and rotate back by the same angle. In other words, it is parallel transport for a fixed angle.

Dynamically R_{α_0} and R_β are polar opposites: the geodesic flow is hyperbolic and the fiber flow is periodic. The surgery increases the complexity of both, and for the fiber flow this is the case whether or not the twist goes in the correct direction to produce hyperbolicity from the geodesic flow:

Theorem 3.14. — *Let $(M_S, \ker(\beta_S))$ be a contact manifold obtained from the contact form $\beta = -d\alpha_0(H,\cdot)$ for the fiber flow after a nontrivial contact surgery along a Legendrian knot that projects to a simple geodesic. Then the growth rate of* *contact homology for $(M_S, \ker(\beta_S))$ is quadratic. In particular, any nondegenerate Reeb flow of $(M_S, \ker(\beta_S))$ has at least quadratic orbit growth.*[(7)]

Remark 3.15. — The growth rate of contact homology appears quite difficult to handle when a surgery as in Theorem 3.14 is performed along a Legendrian knot that projects to a geodesic which is not simple. However, the dynamics of the resulting flow will be hyperbolic: Curtis Heberle (in preparation) establishes that this produces a flow with a horseshoe, i.e., a uniformly hyperbolic Cantor set, and Aritro Pathak (in preparation) shows that the restriction of the surgered flow to the set of orbits that meet the surgery annulus is nonuniformly hyperbolic and ergodic (plus the attendant higher mixing properties).

3.4. Relation to other works on contact surgery and Reeb dynamics

Weinstein surgery/handle attachment is an elementary building block and fundamental operation in contact/symplectic topology and has been largely studied from the topological point of view (for instance it can be used to construct specific or tight or fillable contact manifolds). We only mention here works focusing on the Reeb dynamics.

A description of contact surgery with control of the Reeb vector field can be found in [EG99], where Etnyre and Ghrist construct tight contact structures and prove tightness using dynamical properties of the Reeb vector field (their description is different from ours as they consider a surgery on a transverse knot and focus on the description of this surgery via tori).

In [BEE12], Bourgeois, Ekholm and Eliashberg describe the effect of a Weinstein surgery on Reeb dynamics and contact homology. More precisely, they prove the existence of an exact triangle in any dimension connecting contact homologies of the initial manifold and the surgered manifold and a third term associated to the attaching sphere and called Legendrian contact homology. However, explicit computations are delicate even for our explicit examples, for instance as the Legendrian contact homology is the homology of a huge complex. In contrast, our results give precise estimates in Reeb dynamics but for specific examples.

Our work is largely inspired by Alves' work on Reeb dynamics as explained above, note that he himself applied his methods to contact surgery. The study of Reeb flows with positive entropy comes from Macarini and Schlenk [MS11] who studied spherizations in the cotangent bundle. This has been developed by Macarini and Paternain [MP12], Alves [Alv16a, Alv16b, Alv17] and others. In [ACH19], Alves, Colin and Honda relate topological entropy of Reeb flows to the monodromy of an associated open book decomposition.

(7) This means that for any nondegenerate contact form β' such that $\ker(\beta') = \ker(\beta_S)$ the number $N_T(\beta')$ of $R_{\beta'}$-periodic orbits with period smaller than T satisfies $N_T(\beta') \geqslant aT^2$ for some positive real number a.

4. Surgery and production of closed orbits

The surgery in [FH13] on which this work is based came with some infelicitous conventions and an immaterial sign error, so we recapitulate some of the steps here with more explicit details. This is necessary also as a base for the proof of Theorem 3.1, and for a supplementary result (Proposition 4.4) that is needed later. Our surgery can be performed in a neighborhood of any Legendrian knot in a contact 3-manifold. We start with a description of the surgery in adapted coordinates near a Legendrian knot and then explain how to obtain such coordinates in the unit tangent bundle of a hyperbolic surface and how they are linked to the stable and unstable bundles.

4.1. The surgery from the contact viewpoint

Let $(M,\xi=\ker(\alpha)\,)$ be a contact manifold of dimension 3 and let γ be a Legendrian knot in M. Then there exist coordinates

$$(t,s,w)\in\Omega:=(-\eta,\eta)\times S^1\times(-\epsilon,+\epsilon),$$

with $0<\epsilon<\eta/2\pi$ on a neighborhood of γ in which $\alpha=dt+w\,ds$ and $\gamma=\{0\}\times S^1\times\{0\}$. The *surgery annulus* is $\{0\}\times S^1\times(-\epsilon,+\epsilon)$ (and we may occasionally conflate $S^1\simeq\mathbb{R}/2\pi\mathbb{Z}$ with its universal cover $\mathbb{R}$). Note that in these coordinates $\alpha\wedge d\alpha=dt\wedge dw\wedge ds$ and $R_\alpha=\frac{\partial}{\partial t}$, so Ω is a flow-box chart. The surgeries split this chart into two one-sided flow-box neighborhoods of the surgery annulus, and while the initial transition map between these on $\{0\}\times S^1\times(-\epsilon,+\epsilon)$ is the identity, the surgered manifold M_S is defined by imposing the desired twist (or shear) as the transition map on this annulus:

$$F\colon S^1\times(-\epsilon,\epsilon)\to S^1\times(-\epsilon,\epsilon),\quad (s,w)\mapsto(s+f(w),w) \tag{4.1}$$

with $f\colon[-\epsilon,\epsilon]\to S^1\simeq\mathbb{R}/2\pi\mathbb{Z}$, $w\mapsto qg(w/\epsilon)$, $q\in\mathbb{Z}$, $g\colon\mathbb{R}\to[0,2\pi]$ nondecreasing smooth, $0\leqslant g'\leqslant 4$ even, and $g((-\infty,-1])=\{0\}$, $g([1,\infty))=\{2\pi\}$. We specify that the transition map from $\{t<0\}$ to $\{t>0\}$ is used to identify points $(0^-,x)$ with $(0^+,F(x))$. With this choice one sees that $F^*\alpha=\alpha+wf'(w)\,dw$ and hence that

$$F^*d\alpha=d\alpha\quad\text{and}\quad F^*(\alpha\wedge d\alpha)=\alpha\wedge d\alpha,$$

so $\alpha\wedge d\alpha$ is a well-defined volume on M_S. The vector field R_α on M induces the *Handel–Thurston* vector field X_{HT} on M_S. Its flow preserves the Liouville volume defined by $\alpha\wedge d\alpha$ [FH13, Corollary 3.3], and the total volume of the manifold is not changed by the surgery.

However, we have not yet produced a *contact* flow: $F^*\alpha=\alpha+wf'(w)dw$, so α does not induce a contact form on M_S. A deformation yields a well-defined contact form $\alpha_h^{\mp}=\alpha\mp dh$ for $\pm t\geqslant 0$, where

$$h(t,w):=\begin{cases}\frac{1}{2}\underbrace{\lambda(t)}_{\lambda\colon\mathbb{R}\to[0,1]\text{ is a smooth bump function supported in }(-\eta,\eta)\text{ with }\lambda=1\text{ near }0}\int_{-\epsilon}^{w}xf'(x)\,dx & \text{on }(-\eta,\eta)\times(-\epsilon,\epsilon)\\ 0 & \text{outside.}\end{cases}$$

satisfies $dh = \frac{1}{2}wf'(w)dw$ on the surgery annulus and $h \equiv 0$ for t near $\pm\eta$. Hence $F^*(\alpha_h^+) = \alpha_h^-$, and $\alpha_h^\pm$ induces a contact form α_A on M_S. Its Reeb field is a time-change

$$R_{\alpha_A} := \frac{X_{HT}}{1 \pm dh(X_{HT})} \tag{4.2}$$

of X_{HT} [FH13, Theorem 4.2], which is well-defined because $|dh(X_{HT})| < 1$ if $0 < \epsilon < \eta/2\pi$ [FH13, Lemma 4.1].[8] For small enough ϵ, one can impose the condition $|dh(X_{HT})| < 1/2$, and we will do so in Section 8.

The time-change that defines R_{α_A} is a slow-down near the surgery annulus, which confounds comparisons of dynamical complexity because of the extra factor in Abramov's formula (3.1), so we study the vector field

$$X_h := cR_{\alpha_A} = R_{\alpha_A/c}, \tag{4.3}$$

where $c \in \mathbb{R}$ is such that $\int \frac{c}{1 \pm dh(X_{HT})} \alpha \wedge d\alpha = 1$, to compare entropies.

4.2. Surgery on the unit tangent bundle and Anosov flows

We now explain how to perform a contact surgery on the unit tangent bundle of a hyperbolic surface Σ. Select a closed geodesic $\mathfrak{c}\colon S^1 \to \Sigma$, $s \mapsto \mathfrak{c}(s)$ and consider the Legendrian knot γ obtained by rotating the unit vector field along $\mathfrak{c}$ by the angle $\theta = \pi/2$. This knot is Legendrian as H is tangent to γ (see Figure 4.1). Standard

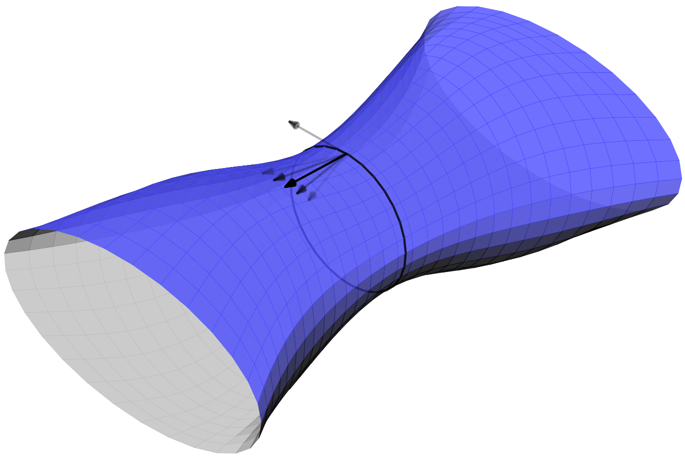

Figure 4.1. Surgery annulus in the base

coordinates for α_A near γ are obtained by flowing along the vertical field V and then along the geodesic vector field X [FH13, Lemma 5.1]: the surgery annulus is contained in the torus $\mathbb{T}$ above $\mathfrak{c}$ (see Figure 4.2); it consists of vectors that are almost orthogonal to a chosen geodesic in a surface. Along γ, E^+ is spanned by a vector $V + H$ in the first quadrant.

(8) We note that time-changes of contact flows are sometimes surprisingly flexible [Mat13].

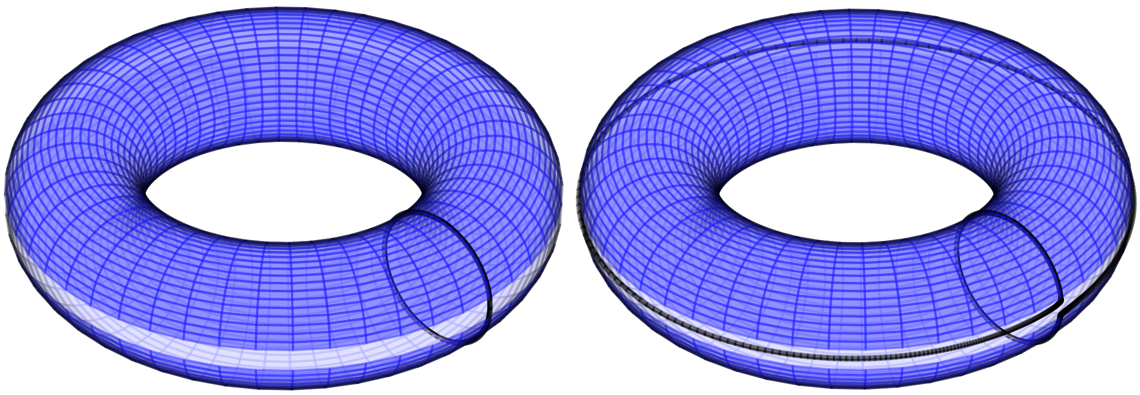

Figure 4.2. Surgery annulus before and after surgery ($q = 1$)

To prove that the surgered flow is Anosov, [FH13] uses Lyapunov–Lorentz metrics [FH13, Claim 4.5 and Appendix A].

Definition 4.1. — *The continuous Lorentz metrics Q^+ and Q^- on M are a pair of Lyapunov–Lorentz metrics for the flow φ^t generated by X if there exists constants $a, b, c, T > 0$ such that*

(1) $C^+ \cap C^- = \varnothing$ *where $C^\pm$ is the $Q^\pm$-positive cone;*
(2) $Q^\pm(X) = -c$;
(3) *for any $x \in M$, $v \in C^\pm(x)$ and $t > T$, $Q^\pm(D_x\varphi^{\pm t}(v)) \geqslant ae^{bt}Q^\pm(v)$*
(4) *for any $x \in M$ $D_x\varphi^{\pm T}(\overline{C^\pm(x)}) \smallsetminus \{0\} \subset C^\pm(\varphi^{\pm T}(x))$*

Proposition 4.2. — [FH13, Claim 4.5 and Appendix A] *A smooth flow φ^t is Anosov if and only if it admits a pair of Lyapunov–Lorentz metrics Q^- and Q^+. The unstable foliation of the flow is then contained in the positive cone Q^+ and the stable foliation in the positive cone of Q^-*

For the geodesic flow, one can choose $Q^\pm = \pm dwds - cdt^2$ in the coordinates (t, s, w). Understanding how the surgery affects the positive cones of $Q^\pm$ is crucial to understanding why the condition $q > 0$ is essential to obtaining an Anosov flow after surgery. We restrict attention to the trace of these cones in the sw-plane and consider the geometry of the action of F by differentiating (4.1) to see the twist (shear) in (s, w)-coordinates:

$$DF = \begin{pmatrix} 1 & f'(w) \\ 0 & 1 \end{pmatrix}.$$

Therefore, if $q > 0$, the image of the first and third quadrant (i.e., the trace of $\overline{C^+}$) is a subcone of the first and third quadrant that shares the horizontal axis (see Figure 4.3). Roughly speaking, this implies that the cone field C^+ is preserved by the surgery and one can define a new cone field on the surgered manifold by

$$\begin{aligned} Q_0^\pm &= \pm dwds - cdt^2, && \text{if } t < 0, \\ Q_1^\pm &= \pm\left(dwds - b(t)f'(w)dw^2\right) - cdt^2, && \text{if } t > 0, \end{aligned}$$

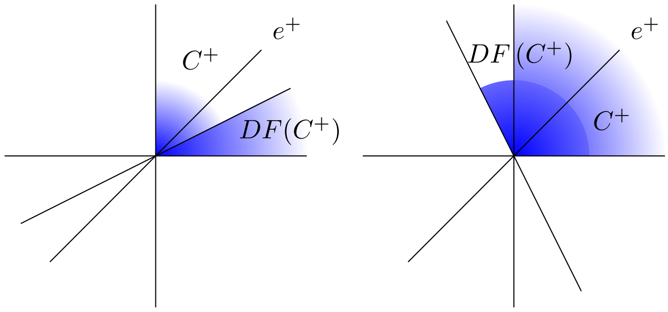

Figure 4.3. Action of a positive and negative twist (shear) on the first quadrant

where $b\colon \mathbb{R} \to \mathbb{R}^+$ smooth with $b((-\infty, 0]) = \{1\}$, $b([\eta, \infty)) = \{0\}$ and $b' < 0$ on $(-\eta, \eta)$. Then for $t = 0$, $F^*Q_1 = Q_0$ and $Q_0^\pm$ and $Q_1^\pm$ induces a pair of Lyapunov–Lorentz metrics on M_S. If $q < 0$, the cones are not preserved and the flow is not necessarily Anosov:

Proposition 4.3. — *The* $(1, q)$*-Dehn surgery defined by* F *in* (4.1) *does not produce an Anosov flow if* $-q/\epsilon$ *is large enough,* i.e., *if either* $q < 0$ *is fixed and* ϵ *is small enough or if* $\epsilon > 0$ *is fixed and* $q < 0$ *with* $|q|$ *big enough.*

Proof. — There is a lower bound on the return time to the surgery region, so there is a $K > 0$ such that the half-cone $a \leqslant -Kb \leqslant 0$ is mapped into the half-cone $0 \geqslant a \geqslant Kb$ by the differential of the return map (see Figure 4.4). Here, we use coordinates (a, b) in the (s, w)-plane. Now suppose that $q/\epsilon < -2K$ and that the function g in the definition of f (after (4.1)) is chosen with monotone derivative on $(0, \infty)$. Then $f'(0) < q/\epsilon$, so $f'(w) < q/\epsilon$ for small w. This has the effect that for such w, the half-cone around e^+ given by $0 \leqslant a \leqslant Kb$ is mapped by DF into the half-cone $a \leqslant -Kb \leqslant 0$, which is on the other side of e^-. The return map then sends it into the half-cone $0 \geqslant a \geqslant Kb$, which is the other half of the cone in which we started. This is incompatible with the existence of a *continuous* invariant cone field that extends to points that miss the surgery region, and hence with the Anosov property. □

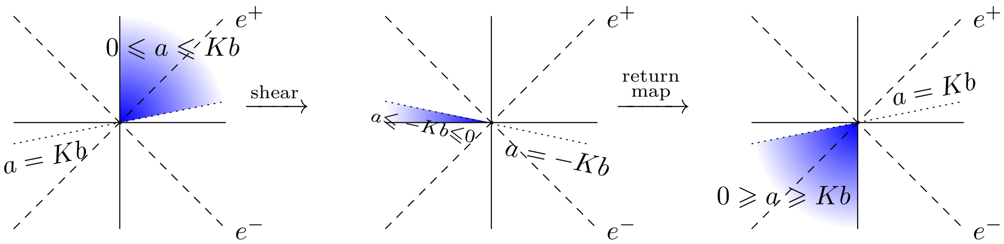

Figure 4.4. The cones in the proof of Proposition 4.3

Note that one can perform a positive surgery on an Anosov flow (and therefore obtain another Anosov flow) then undo it by performing a negative surgery and obtain again an Anosov flow. This is compatible with the statement of Proposition 4.3, as q and ϵ are fixed in the negative surgery (and thus Proposition 4.3 does not apply).

Returning to the case of positive q, we note from the preceding:

Proposition 4.4. — *The stable and unstable foliations of (M_S, α_A) as described in Theorem 2.16 are orientable.*

Proof. — The strong stable foliation is contained in the positive cone of Q^- and the strong unstable foliation in the positive cone of Q^+, so the stable foliation is orientable if and only if the positive cone of Q^- is orientable (an orientation of the positive cone is a choice of a connected component of this cone). The stable and unstable foliations of the unit tangent bundle over a hyperbolic surface are orientable. Additionally, $Q^-(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}) = 0$ and $F^*\frac{\partial}{\partial s} = \frac{\partial}{\partial s}$, so the surgery preserves the orientation of Q^-, and Q^- is orientable. This implies that Q^+ is orientable. □

4.3. Impact on entropy

The nature of the surgery map implies:

Proposition 4.5. — *If $q \geqslant 0$, then $h_{\mathit{Liouville}}(X_{HT}) \geqslant h_{\mathit{Liouville}}(X)$.*

Proof. — By the Pesin entropy formula it suffices to show that the positive Lyapunov exponent of X_{HT} is no less than that of X. Volume-preserving Anosov 3-flows are ergodic [KH95, Theorem 20.4.1], so the positive Lyapunov exponent, being a flow-invariant bounded measurable function, is a.e. constant. The earlier observation that for the geodesic flow on a hyperbolic surface the expanding vector is of the form $e^t e^+$ means that the Lyapunov exponent of (the normalized) Liouville measure is 1. Therefore, we will show that the positive Lyapunov exponent of X_{HT} is at least 1. To that end we verify that the differential of its time-1 map expands unstable vectors by at least a factor of e with respect to a suitable norm.

For the geodesic flow the Sasaki metric induces a natural norm, and this norm is what is called an *adapted* or *Lyapunov* norm: for unstable vectors, it grows by exactly e^t under the flow, and on each tangent space it is a product norm. Our argument involves only vectors in unstable cones, so we pass to a norm $\|\cdot\|_+$ that is (uniformly) equivalent when restricted to such vectors: the norm of the unstable component. Geometrically, this means that at each point we project tangent vectors to E^+ along $E^- \oplus \mathbb{R}X$ and take the length of this unstable projection as the norm of the vector. Thus, $\|Dg^t(v)\|_+ = e^t\|v\|_+$ for $t \geqslant 0$.

The proof of hyperbolicity of X_{HT} shows that the cone field defined by the Lyapunov–Lorentz functions is well-defined on the surgered manifold and invariant under X_{HT}. Thus, this adapted norm for the geodesic flow defines a (bounded, though discontinuous) norm $\|\cdot\|_+$ on unstable vectors for the flow φ^t defined byX_{HT}. We now show that $\|D\varphi^1(v)\|_+ \geqslant e\|v\|_+$ for any v in an unstable cone. This is clear (with equality) when the underlying orbit segment does not meet the surgery annulus because the action is that of the geodesic flow. If there is an encounter with the surgery annulus at time $t \in (0,1]$, then $v' := D\varphi^t(v)$ satisfies $\|v'\|_+ = e^t\|v\|_+$, and we will check that $v'' := DF(v')$ satisfies $\|v''\|_+ \geqslant \|v'\|$, which implies that $\|D\varphi^1(v)\|_+ = \|D\varphi^{1-t}(v'')\|_+ = e^{1-t}\|v''\|_+ \geqslant e^{1-t}\|v'\|_+ = e^{1-t}e^t\|v\|_+ = e\|v\|_+$, as required.

That $\|DF(v')\|_+ \geqslant \|v'\|$ follows from the same argument as hyperbolicity of X_{HT} as suggested by Figure 4.5, which superimposes the tangent spaces at some x and $F(x)$ in the surgery annulus (using the identification from the canonical isometries between these tangent spaces). DF is a positive shear, and in the H-V-frame in the figure the addition of a multiple of the projection of $\frac{\partial}{\partial s}$ (which is close to H) by a positive shear results in an increase in the projection to E^+, which is spanned by $e^+ = V + H$. □

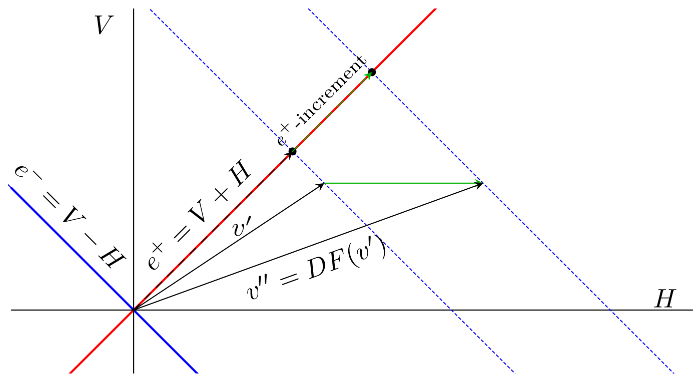

Figure 4.5. DF increases the unstable component

Remark 4.6. — Alternatively, let z be a point on the surgery annulus such that its orbit under the flow X_s crosses the surgery region infinitely often. In local coordinates, z is the identification of $p = (t, s, w)$ with $F(p) = (t, s + f(w), w)$. Consider a vector $v = ae^+(F(p)) + be^-(F(p))$ in the preserved cone at $F(p)$ given by $a > 0$, $0 \leqslant b \leqslant a$. Consider the first return, at time t. Let $q = \varphi_t(F(p)) = (0, s_t, w_t)$. The image of v by the flow differential at t before the identification between q and $F(q)$ is $v' = ae^te^+(q) + be^{-t}e^-(q)$. To compute its image $v'' = DF(v')$ after identification we need to consider the change of coordinate between the basis

$$(e_+(q), e_-(q), X(q)) \text{ and } \left(\frac{\partial}{\partial s}(q), \frac{\partial}{\partial w}(q), X(q)\right).$$

Let

$$\frac{\partial}{\partial s}(q) = a_0e^+(q) + b_0e^-(q) + c_0X(q) \text{ and } \frac{\partial}{\partial w}(q) = a_1e^+(q) + b_1e^-(q) + c_1X(q).$$

If the surgery is performed in a small annulus then $e_+(q)$ is arbitrarily close to $\frac{\partial}{\partial s}(q) + \frac{\partial}{\partial w}(q)$ and $e_-(q)$ is arbitrarily close to $-\frac{\partial}{\partial s}(q) + \frac{\partial}{\partial w}(q)$. We obtain

$$v'' = a'e^+(q) + b'e^-(q) + c'X(q)$$

with

$$a' = ae^t + \frac{df}{dw}(w_t)\, a_0 \frac{aa_0e^t - ba_1e^{-t}}{a_0b_1 - a_1b_0}.$$

As t is bounded below, we obtain $a' \geqslant ae^t$ for a surgery performed in a small annulus. This gives the desired inequality for the projected norm.

Remark 4.7. — We emphasize that the entropy-increase is manifested for X_{HT} and thus results from the surgery and not from the time-change that makes the flow contact.

We are now ready to pursue the growth of periodic orbits.

Proof of Theorem 3.1. — Abramov's formula (3.1) with $g := \frac{c}{1 \pm dh(X_{HT})}$ and μ_g the normalized volume defined by α_A gives

$$h_{\text{Liouville}}(X_h) = h_{\text{Liouville}}(X_{HT}) \int \frac{c}{1 \pm dh(X_{HT})} \alpha \wedge d\alpha = h_{\text{Liouville}}(X_{HT}).$$

Combined with our previous result, this gives

$$h_{\text{Liouville}}(\varphi^t) = h_{\text{Liouville}}(X_h) = \underbrace{h_{\text{Liouville}}(X_{HT}) \geqslant h_{\text{Liouville}}(X)}_{\text{Proposition 4.5}} = h_{\text{Liouville}}(g^t). \tag{4.4}$$

This in turn yields a comparison of topological entropies:

$$\underbrace{h_{\text{top}}(g^t) = h_{\text{Liouville}}(g^t)}_{\text{constant curvature}} \overset{(4.4)}{\leqslant} \underbrace{h_{\text{Liouville}}(\varphi^t) < h_{\text{top}}(\varphi^t)}_{\text{Theorem 2.16 (3)}}.$$

☐

Proof of Theorem 3.3. — By (3.3), increased cohomological pressure suffices:

$$\overbrace{h_{\text{Liouville}}(g^t) \leqslant P(g^t) \leqslant h_{\text{top}}(g^t)}^{(3.2)} \; \underbrace{= h_{\text{Liouville}}(g^t)}_{\text{constant curvature}} \overset{(4.4)}{\leqslant} \underbrace{h_{\text{Liouville}}(\varphi^t) < P(\varphi^t)}_{\text{Theorem 3.6}}.$$

Of course, applying (3.2) on the right-hand side reproves Theorem 3.1. ☐

5. Cylindrical contact homology and its growth rate

Contact homology is an invariant of the contact structure computed through a Reeb vector field and introduced in the vein of Morse and Floer homology by Eliashberg, Givental and Hofer in 2000 [EGH00]. The definition of contact homology is subtle and complicated. In this paper, we will consider it as a black box and only use the properties of contact homology described in Theorem 5.1.[9] We use the simplest version of contact homology: *cylindrical contact homology.*

Roughly speaking, contact homology is the homology of a complex generated by Reeb-periodic orbits of a (nice) contact form. Yet the homology does not depend on the choice of a contact form (but it depends on the underlying contact structure). Therefore a Reeb vector field provides us with information on contact homology and vice-versa. The differential of this complex "counts" rigid holomorphic cylinders in the symplectization $M \times \mathbb{R}$ of our contact manifold (this is the technical part of the definition). These cylinders are asymptotic to Reeb-periodic orbits when the $\mathbb{R}$-coordinate of the cylinder tends to $\pm\infty$. Roughly speaking, in the cases used in this paper, if a rigid cylinder is asymptotic to $\gamma_\pm$ at $\pm\infty$, then it contributes ± 1 to

(9) Plus, for Proposition 8.3 we also use (and elaborate in the proof) an elementary and standard application of the computation of contact homology in the Morse–Bott setting.

the coefficient of γ_- in the differential of γ_+. This can be seen as a generalization of the differential of Morse Homology where we “count” rigid gradient trajectories asymptotic to critical points of a Morse function. In particular, this implies that the differential of a periodic orbit only involves periodic orbits in the same free homotopy class and with smaller period. Moreover, the complex is *graded* and the differential decreases the degree by 1 (here we will only use the parity of this grading). Computing this differential is usually out of reach without strong control of homotopic periodic Reeb orbits.

Variants of contact homology can be defined by considering periodic orbits in specific free homotopy classes or periodic orbits with period bounded by a given positive real number T (this operation is called a filtration). In the latter situation, the limit $T \to \infty$ recovers the original homology. This process is fundamental to gathering information on the growth rate of Reeb-periodic orbits.

We recall that if γ is a nondegenerate T-periodic orbit of the Reeb flow φ^t of $(M, \xi = \ker(\alpha))$ and p is a point on γ, the orbit γ is said to be *even* if the symplectomorphism $d\varphi^T(p)\colon (\xi_p, d\alpha) \to (\xi_p, d\alpha)$ has two real positive eigenvalues, and *odd* otherwise.

Theorem 5.1 (Fundamental properties of cylindrical contact homology). — *Let (M, ξ) be a closed hypertight contact 3-manifold, α_0 a nondegenerate contact form on (M, ξ) and Λ a set of free homotopy classes of M,*

1. *Cylindrical contact homology $\mathrm{C}\mathbb{H}^{\Lambda}_{\mathrm{cyl}}(\alpha_0)$ is a $\mathbb{Q}$-vector space. It can be of finite or infinite dimension. It is the homology of a complex generated by R_{α_0}-periodic orbits in Λ.*
2. *The differential of an odd (resp. even) orbit contains only even (resp. odd) orbits.*
3. *If α is another nondegenerate contact form on (M, ξ), then $\mathrm{C}\mathbb{H}^{\Lambda}_{\mathrm{cyl}}(\alpha_0)$ and $\mathrm{C}\mathbb{H}^{\Lambda}_{\mathrm{cyl}}(\alpha)$ are isomorphic.*
4. *There exists a filtered version $\mathrm{C}\mathbb{H}^{\Lambda}_{\leqslant T}(\alpha_0)$ (for $T \geqslant 0$) of contact homology: the associated complex is generated only by periodic orbits in Λ with period $\leqslant T$. Therefore, $\mathrm{C}\mathbb{H}^{\Lambda}_{\leqslant T}(\alpha_0)$ is a $\mathbb{Q}$-vector space of finite dimension and*

$$\dim\left(\mathrm{C}\mathbb{H}^{\Lambda}_{\leqslant T}(\alpha_0)\right) \leqslant \sharp\left\{R_{\alpha_0}\text{-periodic orbits in } \Lambda \text{ with period} \leqslant T\right\} =: N^{\Lambda}_T(\alpha_0)$$

5. *$(\mathrm{C}\mathbb{H}^{\Lambda}_{\leqslant T}(\alpha))_T$ is a directed system and its direct limit is the cylindrical contact homology. Having a directed system means that for all $T \leqslant T'$, there exists a morphism $\varphi_{T,T'}\colon \mathrm{C}\mathbb{H}^{\Lambda}_{\leqslant T}(\alpha_0) \longrightarrow \mathrm{C}\mathbb{H}^{\Lambda}_{\leqslant T'}(\alpha_0)$ and*
   - *$\varphi_{T,T} = \mathrm{Id}$*
   - *if $T_0 \leqslant T_1 \leqslant T_2$, then $\varphi_{T_0,T_2} = \varphi_{T_1,T_2} \circ \varphi_{T_0,T_1}$.*

   As $\lim_{T\to\infty} \mathrm{C}\mathbb{H}^{\Lambda}_{\leqslant T}(\alpha_0) = \mathrm{C}\mathbb{H}^{\Lambda}(\alpha_0)$, there exist morphisms

$$\varphi_T\colon \mathrm{C}\mathbb{H}^{\Lambda}_{\leqslant T}(\alpha_0) \longrightarrow \mathrm{C}\mathbb{H}^{\Lambda}(\alpha_0)$$

   such that the following diagram commutes for $T \leqslant T'$:

$$\begin{array}{ccc} \mathrm{C}\mathbb{H}^{\Lambda}_{\leqslant T}(\alpha_0) & \xrightarrow{\varphi_{T,T'}} & \mathrm{C}\mathbb{H}^{\Lambda}_{\leqslant T'}(\alpha_0) \\ & \searrow^{\varphi_T} \quad \swarrow_{\varphi_{T'}} & \\ & \mathrm{C}\mathbb{H}^{\Lambda}(\alpha_0) & \end{array}$$

(6) *Let $\alpha = f\alpha_0$ be another nondegenerate contact form. Assume $f > 0$, and let B be such that $1/B \leqslant f(m) \leqslant B$ for all $m \in M$. There exist $C = C(B)$ and morphisms $\psi_T \colon \mathrm{C}\mathbb{H}^{\Lambda}_{\leqslant T}(\alpha_0) \longrightarrow \mathrm{C}\mathbb{H}^{\Lambda}_{\leqslant CT}(\alpha)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{C}\mathbb{H}^{\Lambda}_{\leqslant T}(\alpha_0) & \xrightarrow{\psi_T} & \mathrm{C}\mathbb{H}^{\Lambda}_{\leqslant CT}(\alpha) \\ {\scriptstyle \varphi_{T,T'}(\alpha_0)}\downarrow & & \downarrow{\scriptstyle \varphi_{CT,CT'}(\alpha)} \\ \mathrm{C}\mathbb{H}^{\Lambda}_{\leqslant T'}(\alpha_0) & \xrightarrow{\psi_{T'}} & \mathrm{C}\mathbb{H}^{\Lambda}_{\leqslant CT'}(\alpha) \end{array}$$

This defines a morphism of directed system.

Contact homology was introduced by Eliashberg, Givental and Hofer [EGH00]. The filtration properties come from [CH13]. The description in terms of directed systems takes its inspiration from [McL12] and is presented in [Vau15, Section 4]. Though commonly accepted, existence and invariance of contact homology remain unproven in general. This has been studied by many people using different techniques. This paper uses only proved results and follows the approaches of Dragnev and Pardon [Dra04, Par19]. If α is hypertight and Λ contains only primitive free homotopy classes, the properties of contact homology described in Theorem 5.1 derive from [Dra04] (see [Vau15, Section 2.3]). In the general case, Theorem 5.1 can be derived from [Par19]. Cylindrical contact homology for hypertight contact forms (and possibly nonprimitive homotopy classes) and the action filtration are described in [Par19, Section 1.8]. The case of a not hypertight contact form when there exists an hypertight contact form derives from the contact homology of contractible orbits [Par19, Section 1.8] and our invariant corresponds to $CH^{\Lambda}_{\bullet}$. Note that when computed through a hypertight contact form, $CH^{\mathrm{contr}}_{\bullet}$ is trivial and $CH^{\Lambda}_{\bullet}$ is the cylindrical contact homology. In the not hypertight case, our invariants can be interpreted geometrically using augmentations. This viewpoint is described in [Vau15, Section 2.4 and Section 4].

Combining the two commutative diagrams from Theorem 3.9 and the invariance of contact homology we obtain the following inequality.

Proposition 5.2. — *Let α_0 and $\alpha = f\alpha_0$ be two nondegenerate contact forms on (M, ξ), where M is a closed, 3-dimensional manifold and ξ is hypertight. Assume $f > 0$, and let B such that $1/B \leqslant f(m) \leqslant B$ for all $m \in M$. Then*

$$N^{\Lambda}_L(\alpha) \geqslant \operatorname{rank}(\varphi_L(\alpha)) \geqslant \operatorname{rank}(\varphi_{L/C(B)}(\alpha_0))$$

for all $L > 0$.

If $\mathrm{C}\mathbb{H}^{\Lambda}(\alpha_0)$ is well-understood, one can get an easier estimate.

Corollary 5.3. — *Let α_0 and $\alpha = f\alpha_0$ be two nondegenerate contact forms on (M,ξ) where M is a closed, 3-dimensional manifold and ξ is hypertight. Assume $f > 0$, and let B such that $1/B \leqslant f(m) \leqslant B$ for all $m \in M$. If*

$$\mathrm{C}\mathbb{H}^{\Lambda}(\alpha_0) = \bigoplus_{R_{\alpha_0}\text{-Reeb-periodic orbit } \gamma \text{ in } \Lambda} \mathbb{Q}\gamma$$

then, $N_L^{\Lambda}(\alpha) \geqslant N_{L/C(B)}^{\Lambda}(\alpha_0)$ for all $L > 0$.

In fact, one can derive another invariant of contact structures from these properties of contact homology. Two nondecreasing functions $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ and $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ have the same *growth rate type* if there exists $C > 0$ such that

$$f\left(\frac{x}{C}\right) \leqslant g(x) \leqslant f(Cx)$$

for all $x \in \mathbb{R}_+$ (for instance, a function grows exponentially is it is in the equivalence class of the exponential). The *growth rate type of contact homology* is the growth rate of $T \mapsto \operatorname{rank}(\varphi_T)$. Two nondegenerate contact forms associated to the same contact structure have the same growth rate type (by Proposition 5.2) and therefore, the growth rate type of contact homology is an invariant of the contact structure. The growth rate of contact homology was introduced in [BC05]. It "describes" the asymptotic behavior with respect to T of the number of Reeb-periodic orbits with period smaller than T that contribute to contact homology. For a more detailed presentation one can refer to [Vau15].

Colin and Honda's conjecture [CH13, Conjecture 2.10] (see Section 1) for the contact structures from Theorem 2.16, and Theorem 3.9 for nondegenerate contact forms follow from

Proposition 5.4. — *Let (M,ξ) be a compact contact 3-manifold and assume there exists a contact form α_0 on (M,ξ) whose Reeb flow is Anosov with orientable stable and unstable foliations. Then any R_{α_0}-periodic orbit is even and hyperbolic.*

Indeed, by Proposition 4.4, one can apply Proposition 5.4 to (M_S, α_A), which is hypertight as the Reeb flow is Anosov. Therefore, the differential in contact homology is trivial (Theorem 5.1 (2)) and for any set Λ of free homotopy classes,

$$\mathrm{C}\mathbb{H}^{\Lambda}_{\mathrm{cyl}}(\alpha_A) = \bigoplus_{R_{\alpha_A}\text{-Reeb-periodic orbit } \gamma \text{ in } \Lambda} \mathbb{Q}\gamma.$$

Let $\alpha = f\alpha_A$ be nondegenerate with $f > 0$ and let B be such that $1/B \leqslant f(m) \leqslant B$ for all $m \in M$. Applying Corollary 5.3 for $\Lambda = \{\rho\}$, we get $N_L^{\Lambda}(\alpha) \geqslant N_{L/C(B)}^{\Lambda}(\alpha_A)$ for any $L > 0$. Using the Barthelmé–Fenley estimates from [BF17, Theorem F] we obtain the desired logarithmic growth. This finishes the proof of Theorem 3.9 in the nondegenerate case. Additionally, the number of periodic orbits of an Anosov flow in primitive homotopy classes grows exponentially with the period. Applying Corollary 5.3 for Λ the set of all primitive free homotopy classes in M_S proves the Colin–Honda conjecture for contact structures from Theorem 2.16 and nondegenerate contact forms.

Proof of Proposition 5.4. — By definition of stable and unstable foliations, $D\varphi^T_{|\xi}(p)$ has real eigenvalues μ and $\frac{1}{\mu}$ and the associated eigenspaces are E^+ and E^-. As the strong stable foliation is orientable, the eigenvalues are positive. Thus γ is even and hyperbolic. □

6. Orbit growth in a free homotopy class for degenerate contact forms

We now prove Theorem 3.9 for degenerate contact form (the nondegenerate case is explained in the previous section). The proof derives from the proof of [Alv16b, Theorem 1]. Yet Alves' goal was to obtain one orbit with bounded period in some free homotopy class and not control the number of orbits in this class, and the following result is not explicit in [Alv16b].

Corollary 6.1. — *Let (M,ξ) be a closed manifold and α_0 an Anosov contact form on (M,ξ). Let ρ be a primitive free homotopy class of M such that*

$$\mathbb{CH}^{\rho}_{\mathrm{cyl}}(\alpha_0) = \bigoplus_{R_{\alpha_0}\text{-periodic orbit } \gamma \text{ in } \rho} \mathbb{Q}\gamma \neq \{0\}.$$

Then, for any contact form $\alpha = f_\alpha \alpha_0$ on (M,ξ) and for any R_{α_0}-periodic orbit of period T, there exists an R_α-periodic orbit in ρ of period T' with $e := \min|f_\alpha| \leqslant T'/T \leqslant E := \max|f_\alpha|$.

Proof of Corollary 6.1. — Fix $0 < \epsilon < e$. Without loss of generality, we may assume $f_\alpha > 0$. We follow Alves' proof of [Alv16b, Theorem 1] and consider $\alpha = f_\alpha\alpha_0$ on (M,ξ) nondegenerate. For any $R > 0$, Alves constructs (Step 1) a symplectic cobordism $\mathbb{R} \times M_S$ between $(E+\epsilon)\alpha_0$ and $(e-\epsilon)\alpha_0$ which corresponds to the symplectization of α on $[-R,R] \times M_S$, and a map

$$\Psi_R \colon \mathbb{CH}^{\rho}_{\mathrm{cyl}}((E+\epsilon)\alpha_0) \longrightarrow \mathbb{CH}^{\rho}_{\mathrm{cyl}}((e-\epsilon)\alpha_0)$$

by counting holomorphic cylinders in the symplectic cobordism. As $\mathbb{CH}^{\rho}_{\mathrm{cyl}}(C\alpha_0)$ is canonically isomorphic to $\mathbb{CH}^{\rho}_{\mathrm{cyl}}(\alpha_0)$ for any $C > 0$, Ψ_R induces an endomorphism of $\mathbb{CH}^{\rho}_{\mathrm{cyl}}(\alpha_0)$ and Alves proves that this endomorphism is, in fact, the identity.

Let γ be a R_{α_0}-periodic orbit of period T. For any $C > 0$, it induces a $R_{C\alpha_0}$-periodic orbit γ_C of period CT. As

$$\mathbb{CH}^{\rho}_{\mathrm{cyl}}(\alpha_0) = \bigoplus_{R_{\alpha_0}\text{-Reeb-periodic orbit } \gamma \text{ in } \rho} \mathbb{Q}\gamma,$$

$\Psi_R(\gamma_{E+\epsilon}) = \gamma_{e-\epsilon}$ and therefore, there exists a holomorphic cylinder between $\gamma_{E+\epsilon}$ and $\gamma_{e-\epsilon}$. Now as R tends to infinity (Step 2), SFT compactness (see [Alv16b]) shows that our family of cylinders breaks and an R_α-periodic orbit γ_ϵ of period T_ϵ appears in an intermediate level. By construction, $(e-\epsilon)T \leqslant T_\epsilon \leqslant (E+\epsilon)T$. Now, let ϵ tend to 0 and use the Arzelà–Ascoli Theorem to obtain an R_α-periodic orbit γ' with period T' such that $eT \leqslant T' \leqslant ET$.

If α is degenerate (Step 4), there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of nondegenerate contact forms converging to α and the Arzelà-Ascoli Theorem can again be applied to obtain the desired periodic orbit. □

Proof of Theorem 3.9 for degenerate contact forms. — As M_S is hyperbolic, there are positive real numbers $a_1, b_1, a_2, b_2 > 0$ such that

$$\frac{1}{a_2}\ln(T) - c_2 \leqslant N_T^\rho(\alpha_A) \leqslant a_1 \ln(T) + c_1$$

for all $T > 0$ [BF17, Theorem F]. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of R_{α_A}-periodic orbits in ρ of periods $(T_n)_{n\in\mathbb{N}}$ such that

- γ_0 is a R_{α_A}-periodic orbit in ρ with minimal period;
- for all $n \geqslant 0$, γ_{n+1} is a R_{α_A}-periodic orbit in ρ with period $T_{n+1} > \frac{E}{e}T_n$ and such that there exists no periodic orbit in ρ with period in $(\frac{E}{e}T_n, T_{n+1})$.

By Corollary 6.1, for any $n \geqslant 0$, there exists a R_α-periodic orbit γ'_n of period T'_n such that $eT_n \leqslant T'_n \leqslant ET_n$. Therefore, $T'_n \leqslant ET_n < eT_{n+1} \leqslant T'_{n+1}$ for all $n \geqslant 0$ and all the orbits γ'_n are distinct. Thus, $N^\rho_{T'_n}(\alpha) \geqslant n+1$ for all $n \geqslant 0$.

To control $N_T^\rho(\alpha)$, we now estimate the growth of $(T_n)_{n\in\mathbb{N}}$. By definition, for all $n \geqslant 0$,

$$T_{n+1} = \min\left\{T \middle| N_T^\rho(\alpha_A) \geqslant N^\rho_{E/eT_n}(\alpha_A) + 1\right\}.$$

Therefore, if T is such that

$$\frac{1}{a_2}\ln(T) - c_2 = a_1 \ln(E/eT_n) + c_1 + 1$$

then $T_{n+1} \leqslant T$ and

$$T_{n+1} \leqslant \left(\frac{E}{e}T_n\right)^{a_1 a_2} e^{(1+c_1+c_2)a_2}.$$

Therefore, there exist $a_3, c_3 > 1$ such that $T_{n+1} \leqslant (c_3 T_n)^{a_3}$ for all $n \geqslant 0$. Thus, there exists $c_4 > 0$ such that

$$\ln(T_{n+1}) \leqslant c_4 a_3^{n+1}$$

for all $n \geqslant 0$ and there exists $c_5 \in \mathbb{R}$ such that

$$\ln(\ln(T_{n+1})) \leqslant \ln(a_3)(n+1) + c_5$$

for all $n \geqslant 0$. Now, if $eT_{n-1} \leqslant T'_{n-1} \leqslant T \leqslant T'_n \leqslant ET_n$, then

$$N_T^\rho(\alpha) \geqslant n \geqslant \frac{1}{\ln(a_3)}\ln(\ln(T_n)) - c_5 \geqslant \frac{1}{\ln(a_3)}\ln(\ln(T)) - c_6$$

for some $c_6 \in \mathbb{R}$. This proves Theorem 3.9. □

Remark 6.2. — If $a_1 a_2 = 1$, one can get better estimates and obtain the same growth as in the nondegenerate case.

7. Exponential orbit growth after surgery on a simple closed geodesic

We now prove Theorem 3.13 using the following result by Alves about the *exponential homotopical growth* of cylindrical contact homology.

Definition 7.1 (Exponential homotopical growth [Alv16a]). — *Let (M, ξ) be a closed contact manifold and α_0 a hypertight contact form on (M, ξ). For $T > 0$, let $N_T^{\mathrm{cyl}}(\alpha_0)$ be the number of free homotopy classes ρ of M such that*

- *all the R_{α_0}-periodic orbits in ρ are simply-covered, nondegenerate and have period smaller than T;*
- $\mathrm{C}\mathbb{H}^{\rho}_{\mathrm{cyl}}(\alpha_0) \neq 0$.

We say that the cylindrical contact homology of (M, α_0) has exponential homotopical growth *if there exist $T_0 \geqslant 0$, $a > 0$ and $b \in \mathbb{R}$ such that, for all $T \geqslant T_0$,*

$$N_T^{\mathrm{cyl}}(\alpha_0) \geqslant e^{aT+b}.$$

Theorem 7.2 (Alves [Alv16a, Theorem 2]). — *Let α_0 be a hypertight contact form on a closed contact manifold (M, ξ) and assume that the cylindrical contact homology has exponential homotopical growth. Then every Reeb flow on (M, ξ) has positive topological entropy.*

If ρ is a free homotopy class containing only one R_{α_0}-periodic orbit and if this orbit is simply-covered and nondegenerate, it is a direct consequence of the definition of contact homology that $\mathrm{C}\mathbb{H}^{\rho}_{\mathrm{cyl}}(\alpha_0) = \mathbb{Q}$. Therefore, to prove Theorem 3.13, it suffices to prove the following propositions.

Proposition 7.3. — *The contact form α_A is hypertight in M_S.*

Proposition 7.4. — *Let (M_S, α_A) be a contact manifold obtained after a contact surgery along a Legendrian projecting to a simple closed geodesic and $N'_T(\alpha_A)$ the number of free homotopy classes ρ that contain only one R_{α_A}-periodic orbit and this orbit is simply-covered, nondegenerate and of period smaller than T. Then there exist $T_0 \geqslant 0$, $a > 0$ and $b \in \mathbb{R}$ such that, for all $T \geqslant T_0$, $N'_T(\alpha_A) \geqslant e^{aT+b}$.*

Indeed, the exponential growth of $N'_T(\alpha_A)$ with respect to T induces the exponential homotopical growth of (M_S, α_A) and we can apply Theorem 7.2.

We now turn to the proofs of Proposition 7.3 and Proposition 7.4. In $S\Sigma$, $\mathbb{T} = \pi^{-1}(c)$ is a torus, and our surgery preserves this torus. Let $\mathbb{T}_S$ denote the associated torus in M_S. Van Kampen's Theorem tells us that $\mathbb{T}_S$ is π_1-injective.

To prove Proposition 7.4, we want to find free homotopy classes with only one periodic Reeb orbit. We will consider free homotopy classes containing a periodic orbit disjoint from $\mathbb{T}_S$ and prove there are enough of such classes. First, we describe Reeb-periodic orbits and study the properties of free homotopies between them.

Claim 7.5. — *There are three types of R_{α_A}-periodic orbits:*

1. *periodic orbits contained in $\mathbb{T}_S$, the only periodic orbits of this kind are $\mathfrak{c}$, $-\mathfrak{c}$ ($\mathfrak{c}$ with the reverse orientation) and their covers,*
2. *periodic orbits disjoint from $\mathbb{T}_S$, these orbits correspond to closed geodesics in Σ disjoint from $\pi(\mathfrak{c})$ (this includes multiply-covered geodesics),*
3. *periodic orbits intersecting $\mathbb{T}_S$ transversely.*

Therefore, a free homotopy between two R_{α_A}-periodic orbits can always be perturbed to be transverse to $\mathbb{T}_S$.

Proposition 7.6. — *Let δ_0, δ_1 be two smooth loops in M_S and $H\colon [0,1] \times S^1 \to M_S$ be a free homotopy between δ_0 and δ_1 transverse to $\mathbb{T}_S$. $N := H^{-1}(\mathbb{T}_S)$ is a smooth manifold of dimension 1 properly embedded in $[0,1] \times S^1$. Therefore,*

(1) *one can modify H so that N does not contain contractible circles,*
(2) *if δ_0 is a R_{α_A}-periodic orbit transverse to $\mathbb{T}_S$, N does not contain a segment with both end-points on $\{0\} \times S^1$.*

Proof. — Consider an innermost contractible circle c_0 in $N \subset [0,1] \times S^1$, c_0 bounds a disk D_0 in $[0,1] \times S^1$. The image of c_0 is contractible in $\mathbb{T}_S$ as $\mathbb{T}_S$ is π_1-injective. Therefore, there exists a continuous $G\colon D_0 \to \mathbb{T}_S$ such that $H_{|c_0} = G_{|c_0}$ and one can replace $H_{|D_0}$ by G to obtain a new homotopy (still denoted by H) between δ_0 and δ_1. Now, consider a neighborhood $[-\nu, \nu] \times \mathbb{T}_S$ of $\mathbb{T}_S$ in M_S (with $\mathbb{T}_S \simeq \{0\} \times \mathbb{T}_S$) and a disk D_1 containing D_0 such that $H(D_1) \subset [0, \nu] \times \mathbb{T}_S$ and $H(D_1 \smallsetminus D_0) \subset (0, \nu] \times \mathbb{T}_S$. One can perturb H in $\operatorname{int}(D_1)$ so that $H(D_1) \subset (0, \nu] \times \mathbb{T}_S$. Performing this inductively on the contractible circles proves (1).

We now assume δ_0 is an R_{α_A}-periodic orbit transverse to $\mathbb{T}_S$. By contradiction, consider an innermost segment c_0 in N with end-points on $\{0\} \times S^1$. The end-points of c_0 correspond to consecutive intersection points of δ_0 with $\mathbb{T}_S$. Let c_1 be the segment in $\{0\} \times S^1$ joining these two end-points and homotopic (relative to end-points) to c_0. By construction, there exists a homotopy $(\eta_t)_{t \in [0,1]}\colon [0,1] \to M_S$ (relative to end-points) between $\eta_0 = H(c_0)$ et $\eta_1 = H(c_1)$ such that $\eta_t(s) \in \mathbb{T}_S$ if and only if $t = 1$ or $s = 0, 1$. Let M' be the manifold with boundary obtained by cutting M_S along $\mathbb{T}_S$. Note that M' can also be obtained by cutting $S\Sigma$ along $\mathbb{T}$. The projection $M' \to M_S$ is injective in the interior of M', therefore $\eta_t(s)$ is well-defined in M' if $t \neq 0$ and $s \neq 0, 1$. Thus, there exists a homotopy η'_t in M' lifting η_t. This homotopy induces a homotopy in $S\Sigma$ and, as a result, a homotopy in Σ between a geodesic arc contained in $\pi(\mathfrak{c})$ and a geodesic arc with end-points on $\pi(\mathfrak{c})$. As Σ is hyperbolic, this can only happen if our second geodesic arc is also contained in $\pi(\mathfrak{c})$, a contradiction. □

Proof of Proposition 7.3. — By contradiction, assume there exists a free homotopy H between δ, a R_{α_A}-periodic orbit, and a point $p \notin \mathbb{T}_S$. As $\mathbb{T}_S$ is π_1-injective, δ cannot be contained in $\mathbb{T}_S$. Without loss of generality we may assume that H is transverse to $\mathbb{T}_S$ and apply Proposition 7.6.

If δ is disjoint from $\mathbb{T}_S$, then $N \subset [0,1] \times S^1$ (see Proposition 7.6) can only contain circles parallel to the boundary. We will now prove that we can modify H so that N is empty. Let c_0 be the circle in N closest to $\{0\} \times S^1$ and let C be the closure of the connected component of $([0,1] \times S^1) \smallsetminus c_0$ containing $\{1\} \times S^1$. Then $H(c_0)$ is an immersed circle contractible in $\mathbb{T}_S$ and there exists a continuous map $G\colon C \to \mathbb{T}_S$ such that $G_{|c_0} = H$ and $G_{\{1\} \times S^1}$ is constant. We replace $H_{|C}$ with G to obtain a new homotopy H. Now, consider a neighborhood $[-\nu, \nu] \times \mathbb{T}_S$ of $\mathbb{T}_S$ in M_S and a neighborhood C_1 of C such that $H(C_1)$ is contained in $[0, \nu] \times \mathbb{T}_S$. We can perturb H so that $H(C_1)$ is contained in $(0, \nu] \times \mathbb{T}_S$. Therefore we may assume that N is empty and H is an homotopy in $M_S \smallsetminus \mathbb{T}_S$. It induces an homotopy in $S\Sigma$, a contradiction as the periodic orbits are not contractible in $S\Sigma$.

Finally, we consider the case of δ transverse to $\mathbb{T}_S$. In this case, N has boundary points on $\{0\} \times \mathbb{T}_S$ but not on $\{1\} \times \mathbb{T}_S$. This contradicts Proposition 7.6. □

Proposition 7.7. — *If δ is a R_{α_A}-periodic orbit disjoint from $\mathbb{T}_S$, then the free homotopy class of δ contains exactly one R_{α_A}-periodic orbit.*

Proof. — By contradiction, consider a free homotopy H from δ to δ_1, a distinct R_{α_A}-periodic orbit. Without loss of generality, we may assume that H is transverse to $\mathbb{T}_S$ (apply Proposition 7.6).

If δ_1 is disjoint from $\mathbb{T}_S$, then N can only contain circles parallel to the boundary. If N is empty, H induces a homotopy in $S\Sigma$ and therefore in Σ. Yet, two closed geodesics on a hyperbolic surface are not homotopic. This proves that $N \neq \varnothing$. Let c_0 be the circle in N closest to $\{0\}\times S^1$ and M' be the manifold with boundary obtained by cutting M_S along $\mathbb{T}_S$. The homotopy H induces a homotopy G between δ and $H(c_0) \subset \mathbb{T}_S$. The homotopy G lifts to M' and therefore induces a free homotopy in $S\Sigma$ and, as a result, a free homotopy in Σ between a closed geodesics and a loop contained in the geodesic $\pi(\mathfrak{c})$. This can happen only if our first geodesic is a cover of $\pi(\mathfrak{c})$. Yet this implies $\delta \subset \mathbb{T}_S$, a contradiction.

If δ_1 is transverse to $\mathbb{T}_S$, the manifold N is not empty and has end-points on $\{1\} \times S^1$ but cannot have end-points on $\{0\} \times S^1$. This contradicts Proposition 7.6.

Finally, the case δ_1 contained in $\mathbb{T}_S$ is similar to the case δ_1 disjoint from $\mathbb{T}_S$. In this case, N contains only circles parallel to the boundary and $\{1\} \times S^1$ is in N. ☐

Proof of Proposition 7.4. — If $\pi(\mathfrak{c})$ is nonseparating, by cutting Σ along $\pi(\mathfrak{c})$ we obtain a surface of genus at least 1 with two boundary components. Let ℓ_1 and ℓ_2 be two loops in $\Sigma \setminus \mathfrak{c}$, homotopically independent and with the same base-point. Then, any nontrivial word in ℓ_1 and ℓ_2 defines a nontrivial free homotopy class for Σ and there exists a closed geodesic on Σ representing this class. This R_{α_A}-periodic orbit is always nondegenerate. Additionally, we may assume that the orbits associated to ℓ_1 and ℓ_2 are simply-covered. If a word is not the repetition a smaller word, the associated orbit is therefore simply covered. As ℓ_1 and ℓ_2 are independent, all these geodesics are disjoint and their number grows exponentially with the period. Finally, these geodesics do not intersect $\mathfrak{c}$ as geodesics always minimize the intersection number.

If $\pi(\mathfrak{c})$ is separating, then by cutting Σ along $\pi(\mathfrak{c})$, we obtain two surfaces of genus at least 1 with one boundary component. The proof is similar. ☐

8. Surgery on a periodic Reeb flow

We now consider the coexistence of diverse Reeb flows and prove Theorem 3.14.

8.1. Dynamical properties of the periodic Reeb flow after surgery

We next apply the general construction of contact surgery along a Legendrian curve described in Section 4.1 to the contact structure with contact form β and periodic Reeb flow described in Section 3.3. Select a closed geodesic on a hyperbolic surface Σ and consider its lift $\mathfrak{c}\colon S^1 = \mathbb{R}/\mathbb{Z} \to M$ to the unit tangent bundle M of Σ. Consider the Legendrian knot γ in $(M, \ker(\beta))$ obtained by rotating the unit vector field along $\mathfrak{c}$ by the angle $\theta = \pi/2$. Note that the Legendrian knot γ is the same as in Section 4.2 (and is tangent to H). To obtain standard coordinates in a neighborhood of γ we first consider an annulus A in $S\Sigma$ transverse to the fibers with coordinates $(s, w) \in S^1 \times (-2\epsilon, 2\epsilon)$ such that $\beta_{|A} = w ds$ and then flow along the Reeb vector field R_β to obtain coordinates $(t, s, w) \in S^1 \times A = N$ such that $\beta = dt + w ds$ (to remain coherent with previous conventions our circles have different lengths, more precisely $t \in \mathbb{R}/2\pi\mathbb{Z}$ and $s \in \mathbb{R}/\mathbb{Z}$).[10] Note that N can be interpreted as the suspension of the annulus A by the identity map.

Our nontrivial surgery is defined by a twist (shear) F along A. We denote by M_S the manifold $S\Sigma$ after surgery and by $N_S \subset M_S$ the manifold (with boundary) N after surgery. Let β_S be the contact form on M_S as described in Section 4.1. Note that β and β_S coincide outside N and N_S respectively. The manifold N_S is the suspension of the annulus A by the shear map F. Moreover, the map $p_S \colon N_S \to (-2\epsilon, 2\epsilon)$ given by the w-coordinate is well-defined and is a trivial torus-bundle. For $w \in (-2\epsilon, 2\epsilon)$, the torus $p_S^{-1}(w)$ is foliated by closed Reeb orbits if and only if

$$f(w) = 2\pi \frac{p_w}{q_w} \in 2\pi\mathbb{Q}$$

where p_w and q_w are coprime. In this situation the orbits of $\frac{\partial}{\partial t}$ on $p_S^{-1}(w)$ are periodic of period q_w. The Reeb vector field is a renormalization of $\frac{\partial}{\partial t}$ (see (4.2)). Finally, let $\mathbb{T} = S^1 \times S^1 \times \{0\}$ in N and $\mathbb{T}_S$ be its image in M_S. By van Kampen's theorem, this torus is incompressible. Therefore the contact form β_S is hypertight. Note that if $f(w) \in 2\pi\mathbb{Q}$ and $f(w') \in 2\pi\mathbb{Q}$ but $f(w) \neq f(w')$, the associated periodic orbits are not freely homotopic.

8.2. Proof of Theorem 3.14

The contact form β_S is degenerate and the renormalization from the surgery makes the direct study a bit harder. So, to estimate the growth rate of its contact homology, we will standardize and perturb our contact form.

For any $w \in (-2\epsilon, 2\epsilon)$, the vector fields $\frac{\partial}{\partial t} + \frac{f(w)}{2\pi}\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial s}$ generate circles in the torus $p_S^{-1}(w)$. These circles induce a trivialisation of N_S. Let (τ, σ, w) be the coordinates on N_S associated to this trivialisation. Without loss of generality, we may assume that the map f defining the twist (shear) F is constant on $(-2\epsilon, -\epsilon) \cup (\epsilon, 2\epsilon)$,

(10) These coordinates along γ are different from the coordinates defined for the surgery associated to the contact form α as, for instance, the surgery annulus is different. It is possible to derive a contact form from β on the surgered manifold using the coordinates and surgery associated to α: write β in local coordinates, compute $F^*\beta$ and interpolate using bump and cut-off functions. Unfortunately, this construction yields a complicated Reeb vector field. Note that the contact structure obtained this way is isotopic to $\ker(\beta_S)$. This can be proved as follows. First the two surgeries result in the same manifold. Moreover, a surgery can be described as the gluing of a solid torus on an excavated manifold. Therefore we just need to prove that the contact structures on the glued tori are the same. This derives from the classification of contact structures on $\mathbb{D}^3$ by Eliashberg. See [ML98] for an application to the torus.

that $f'(w) \neq 0$ for any $w \in (-\epsilon, \epsilon)$ and that f is invariant under reflection with respect to the point $(0, q/2)$. Therefore, for w in $[-2\epsilon, -\epsilon]$,

$$\beta_S = d\tau + w d\sigma$$

and for w in $[\epsilon, 2\epsilon]$,

$$\beta_S = \left(1 + \frac{qw}{2\pi}\right) d\tau + w d\sigma.$$

Lemma 8.1. — *There exist smooth maps $h_0, k_0\colon (-2\epsilon, 2\epsilon) \to \mathbb{R}$ such that*

$$\beta_0 = h_0(w) d\tau + k_0(w) d\sigma$$

is a contact form such that $\beta_0 = \beta_S$ for w close to $\pm 2\epsilon$ and R_{β_0} and R_{β_S} are positively collinear on N_S. Therefore, β_0 and β_S are isotopic (through contact forms).

Proof. — Let h_0 and k_0 be the maps defined by $k_0(w) = w$ and

$$h_0(w) = 1 + \int_{-2\epsilon}^{w} f(u)/2\pi du$$

for $w \in (-2\epsilon, 2\epsilon)$. As $\int_{-\epsilon}^{\epsilon} f(u) du = q\epsilon$, $\beta_0 = \beta_S$ for $w \in (\epsilon, 2\epsilon)$. Moreover, the contact condition is

$$1 + \int_{-2\epsilon}^{w} f(u)/2\pi du - w f(w)/2\pi > 0$$

and this condition is always satisfied for ϵ small enough. Additionally, the Reeb vector field is positively collinear to $(k_0'(w), -h_0'(w)) = (1, -f(w)/2\pi)$. Finally, as R_{β_0} and R_{β_S} are positively collinear, $(u\beta_S + (1-u)\beta_0)_{u \in [0,1]}$ is a contact isotopy. ☐

The contact form β_0 is degenerate. To estimate the growth rate of its contact homology, we have to perturb it. Our perturbation draws its inspiration from Morse–Bott techniques. To describe our perturbation, we need to fix some notations. The manifold $S\Sigma \setminus p^{-1}((-\epsilon, \epsilon))$ is a trivial circle bundle. Let S' be a surface (with boundary) transverse to the fibers and intersecting each fiber once: S' provides us with a trivialisation $S' \times S^1$ of $S\Sigma \setminus p^{-1}((-\epsilon, \epsilon))$. The surface S' has two boundary components. Let $\varphi\colon S' \to \mathbb{R}$ be a Morse function such that $\varphi = 0$ on the boundary of S' and, if $q > 0$ (resp. $q < 0$), the connected component of $\partial S'$ corresponding to $w = -\epsilon$ is a maximum (resp. a minimum) and the connected component corresponding to $w = \epsilon$ a minimum (resp. a maximum). For any w such that $f(w) = 2\pi p(w)/q(w) \in 2\pi\mathbb{Q}$, we denote by $P(w)$ the period of the R_{β_0}-periodic orbits foliating $p_S^{-1}(w)$. Note that there exists $C_P > 0$ such that $q(w)/C_P \leqslant P(w) \leqslant C_P q(w)$, this implies that the number of torus with $w \in (-\epsilon, \epsilon)$ foliated by Reeb-periodic orbits with period smaller than L grows quadratically in L.

For a contact form α, let $\sigma(\alpha)$ denote the action spectrum: the set of periods of the periodic orbits of R_α.

Proposition 8.2. — *Let $T > 0$, $T \notin \sigma(\beta_0)$. There exists $\beta' = l\beta_0$ with $l\colon M_S \to \mathbb{R}_+$ arbitrarily close to 1 such that*

- *β' is hypertight and nondegenerate*
- *the periodic orbits of $R_{\beta'}$ with period $\leqslant T$ are exactly:*
  1. *the fibers associated to the critical points of φ and their multiple of multiplicity $\leqslant \lfloor \frac{T}{2\pi} \rfloor$*

(2) *for all* $w \in (-\epsilon, \epsilon)$ *such that* $P(w) < T$*, two orbits in* $p_S^{-1}(w)$ *and their multiple with multiplicity* $\leqslant \lfloor \frac{T}{P(w)} \rfloor$

- *if* δ *is a* $R_{\beta'}$*-periodic orbit of period* $\leqslant T$ *then all the* $R_{\beta'}$*-periodic orbit in the free homotopy class of* δ *are periodic orbits of period* $\leqslant T$*.*

Proposition 8.3. — *If* δ *is a simply-covered* $R_{\beta'}$*-periodic orbit of period* $\leqslant T$ *of the second type in Proposition 8.2, then*

$$\mathrm{C}\mathbb{H}_{cyl}^{[\delta]}(M, \ker(\beta_0)) = \mathbb{Q}^2.$$

Proof of Proposition 8.2. — There exists $\nu > 0$ such that for any $w \in (-\epsilon, -\epsilon + \nu] \cup [\epsilon - \nu, \epsilon)$, if $f(w) = 2\pi p(w)/q(w) \in 2\pi\mathbb{Q}$ then $q(w) > C_P T$. Let

$$N'_S = p_S^{-1}((-\epsilon, -\epsilon + \nu] \cup [\epsilon - \nu, \epsilon)).$$

Let S'' be a smooth surface in M_S with boundary obtained by adding to S' two annuli in N_S, transverse to R_{β_0} and projecting to $[-\epsilon, -\epsilon + \nu] \cup [\epsilon - \nu, \epsilon]$. We can therefore endow $S'' \smallsetminus S'$ with coordinates (s', w') such that w' lifts w. We now perturb φ and extend it to S'' so that $\varphi(s', w') = \varphi(w')$ on $S'' \smallsetminus S'$, $\varphi'(w') \neq 0$ for all $w' \in [-\epsilon, -\epsilon + \nu) \cup (\epsilon - \nu, \epsilon]$, φ is flat (all its derivative are equal to 0) for $w = \pm(\epsilon - \nu)$ and the critical points of φ are unaltered. Finally, we extend φ to M_S to obtain a smooth function, R_{β_0}-invariant and such that $\varphi \equiv 0$ in $N_S \smallsetminus N'_S$.

Let $\beta_\lambda = (1 + \lambda\varphi)\beta_0$. This is a standard Morse–Bott perturbation (see [Bou02, Lemma 2.3]) in $M_S \smallsetminus p_S^{-1}((-\epsilon, \epsilon))$, therefore, for $\lambda \ll 1$, the periodic orbits in this area correspond to the critical points of k.

In the coordinates (τ, σ, w), we have

$$\beta_\lambda = (1 + \lambda\varphi(w))(h_0(w)d\tau + k_0(w)d\sigma).$$

Therefore, in these coordinates, the Reeb vector field is positively collinear to

$$((1 + \lambda\varphi(w))k'_0(w) + \lambda\varphi'(w)k_0(w)) \frac{\partial}{\partial\tau} - ((1 + \lambda\varphi(w))h'_0(w) + \lambda\varphi'(w)h_0(w)) \frac{\partial}{\partial\sigma}.$$

The σ-coordinate is nonzero as φ and h have the same monotonicity. For $\lambda \ll 1$, the σ-coordinate is close to $-h'_0(w)$, the τ-coordinate to $k'_0(w)$ and R_{β_λ} is close to R_{β_0}. Therefore, for $\lambda \ll 1$, if there is a R_{β_λ}-periodic orbit in N'_S, this orbit has slope $2\pi p'(w)/q'(w) \in 2\pi\mathbb{Q}$ with $q'(w) > C_P T$. Thus there are no periodic orbit with period smaller than T in N'_S and the periodic orbits with period bigger than T are not in the free homotopy classes of orbits with period smaller than T as described in Proposition 8.2.

In $p_S^{-1}([-\epsilon + \nu, \epsilon - \nu])$, the periodic orbits with period $\leqslant T$ are contained in tori $p_S^{-1}(w)$ such that $P(w) \leqslant T$. These tori are foliated by periodic orbits. Morse–Bott techniques apply here and give the second type of periodic Reeb orbits: for any such w we perturb β in a neighborhood of $p_S^{-1}(w)$ with a function derived from a Morse function φ_w defined on $p_S^{-1}(w)/\text{Reeb flow} = S^1$ and the periodic orbits after perturbation correspond to the critical points of φ_w. For a given w we obtain two orbits (one associated to the maximum of φ_w and one associated to the minimum of φ_w), their covers and some orbits with period bigger than T and in the free homotopy class of arbitrarily large covers of our two simple orbits. This perturbation derives from [Bou02, Lemma 2.3] and is described for tori in [Vau15, Section 3.1].

Lastly, standard perturbation techniques prove there exists an arbitrarily small perturbation of β_λ with the following properties:

- it gives rise to a nondegenerate contact form,
- it does not change the periodic orbits with period smaller than T,
- it does not create periodic orbits of period bigger than T in the free homotopy classes of orbits of period smaller than T. □

Proof of Proposition 8.3. — Let $\delta \in p_S^{-1}(w)$ be a $R_{\beta'}$-periodic orbit of period $\leqslant T$ of the second type in Proposition 8.2. Then the R_{β_0}-periodic orbits in the class $[\delta]$ are exactly the orbits in $p_S^{-1}(w)$ (and all these orbits have the same period). As δ is simply-covered, Dragnev's [Dra04] results can be applied. Additionally, standard perturbations do not create contractible periodic Reeb orbits. Therefore, the differential for contact homology can be described using "cascades" from Bourgeois' work [Bou02]. The case of a unique torus of orbits is explained in [Bou02, Section 9.4]. The cascades used to describe the differential in this degenerate setting mix holomorphic cylinders between orbits and gradient lines for φ_w in $p_S^{-1}(w)/\text{Reeb flow} = S^1$ (for some generic metric). As all periodic orbits in this class have the same period, there is no homolorphic cylinder in the cascade and the differential coincides with the Morse–Witten differential for φ_w (ie the differential associated to Morse homology). Therefore, cylindrical contact homology in the free homotopy class ρ is 2-dimensional. The cascades of Morse–Bott homology are explicitly described in [BO09] (in a slightly different setting). □

Proof of Proposition 3.14. — Let $\beta' = f\beta$ be a nondegenerate hypertight contact form and B be such that $1/B < f < B$. Let $(T_i)_{i\in\mathbb{N}}$ be an increasing sequence such that $\lim_{i\to+\infty} T_i = +\infty$ and $T_i \notin \sigma(\beta_0)$. For all $i \in \mathbb{N}$, let $\beta_i = f_i\beta$ be the contact form given by Proposition 8.2 for $T = T_i$. We may assume,

$$\frac{1}{B} < \frac{f_i}{f} < B$$

as f_i is arbitrarily close to 1. By Proposition 8.2,

$$\dim\left(\mathbb{CH}_{T_i}(\alpha_i)\right) \leqslant \left\lfloor \frac{T_i}{2\pi} \right\rfloor C + 2 \sum_{w,\, P(w)\leqslant T_i} \left\lfloor \frac{T_i}{P(w)} \right\rfloor$$

where C is the number of critical points of k and

$$\sum_{w,\, P(w)\leqslant T} \left\lfloor \frac{T}{P(w)} \right\rfloor = O(T_i^2).$$

In addition, we have the following commutative diagram (see Theorem 5.1),

$$\begin{array}{ccc} \mathbb{CH}_{\leqslant T_i/C(B)}(\beta') & \longrightarrow & \mathbb{CH}_{\leqslant T_i}(\beta_i) \\ \Big\downarrow{\scriptstyle \varphi_{T_i/C(B)}(\beta')} & & \Big\downarrow{\scriptstyle \varphi_{T_i}(\beta_i)} \\ \mathbb{CH}(\beta') & \longrightarrow & \mathbb{CH}(\beta_i) \end{array}$$

thus

$$rk\left(\varphi_{T_i/C(B)}(\beta')\right) \leqslant rk\left(\varphi_{T_i}(\beta_i)\right) \leqslant \dim\left(\mathbb{CH}_{T_i}(\beta_i)\right) \leqslant a_1\left(T_i^2\right).$$

A symmetric commutative diagram implies

$$rk\left(\varphi_{T_i/C(B)}(\beta_i)\right) \leqslant rk\left(\varphi_{T_i}(\beta')\right)$$

Propositions 8.2 and 8.3 prove that $\varphi_{T_i/C(B)}$ is injective on the class of simply-covered periodic orbits of the second type (as defined in Proposition 8.2). Therefore, we have $rk(\varphi_{T_i/C(B)}(\beta_i)) \geqslant a_0 T_i^2$ and the growth rate of contact homology is quadratic. ☐

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