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LOCAL TRANSVERSELY PRODUCT SINGULARITIES

FEUILLETAGES À STRUCTURE PRODUIT LE LONG DU LIEU SINGULIER

Abstract. — In the main result of this paper we prove that a codimension one foliation of $\mathbb{P}^n$, which is locally a product near every point of some codimension two component of the singular set, has a Kupka component. In particular, we obtain a generalization of a known result of Calvo Andrade and Brunella about foliations with a Kupka component.

Résumé. — Nous démontrons qu'un feuilletage de codimension un de $\mathbb{P}^n$ qui est localement un produit autour de tous les points d'une composante de codimension 2 de l'ensemble singulier, a une composante de Kupka. En particulier, nous obtenons une généralisation d'un résultat déjà connu de Calvo Andrade et Brunella sur les feuilletages avec une composante de Kupka.

1. Basic definitions and results

It is known that a holomorphic codimension one foliation on $\mathbb{P}^n$, $n \geqslant 3$, with a Kupka component in the singular set has a rational first integral, which in homogeneous coordinates is of the form P^k/Q^ℓ, where P and Q are generic homogeneous polynomials with $k.\,deg(P) = \ell.\,deg(Q)$. The Kupka component for this specific example is the set $\Pi(P = Q = 0)$, where $\Pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the canonical projection (cf. [Bru09, CA16, CA99]). The aim of this paper is to generalize this

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result for codimension one foliations with a *local transversely product component* in the singular set. We will define this concept in a more general situation.

Let $\mathcal{F}$ be a holomorphic foliation of dimension $k \geqslant 2$ on a complex manifold M of dimension $n \geqslant k+1$, with singular set $Sing(\mathcal{F})$. We say that $\mathcal{F}$ is *a transversely product* at a point $p \in Sing(\mathcal{F})$ if the germ $\mathcal{F}_p$ of $\mathcal{F}$ at p is holomorphically equivalent to a product of a germ of singular foliation of dimension one with an isolated singularity by a regular foliation of dimension $k-1$. In other words, we can say that there exists a germ of submersion $\varphi\colon (M,p) \to (\mathbb{C}^{n-k+1},0)$ and a germ of a one dimensional foliation $\mathcal{G}$ at $0 \in \mathbb{C}^{n-k+1}$, with $Sing(\mathcal{G}) = \{0\}$, such that $\mathcal{F}_p = \varphi^*(\mathcal{G})$. In particular, the germ of the singular set of $\mathcal{F}$ at p is smooth of dimension $k-1$: $Sing(\mathcal{F}_p) = \varphi^{-1}(0)$.

Definition 1.1. — *We say that* Γ *is a* local transversely product component *(briefly l.t.p component) of* $Sing(\mathcal{F})$ *if* Γ *is an irreducible component of* $Sing(\mathcal{F})$ *and* $\mathcal{F}$ *is a transversely product at all points of* Γ.

Remark 1.2. — If Γ is a l.t.p. component of $Sing(\mathcal{F})$ then it follows from the definition that:

(a) Γ is smooth. Let $dim_{\mathbb{C}}(\Gamma) = m$.
(b) There exists a singular one dimensional foliation $\mathcal{G}$, on a polydisc V of $\mathbb{C}^{n-m}$, with an isolated zero at $0 \in V$, such that for any $p \in \Gamma$ there exists a local chart (U,z) around $p \in U$ satisfying the following conditions:
  (b1) $z = (x,y)\colon U \to \mathbb{C}^{n-m} \times \mathbb{C}^m$ with $x(U) = V$.
  (b2) $\mathcal{F}|_U = x^*(\mathcal{G})$.

In the chart $z = (x,y)$ the submersion of the definition is $\varphi = x\colon U \to V$ and the leaves of the non-singular foliation are the levels $x^{-1}(a)$, $a \in V$. Moreover, $\Gamma \cap U = x^{-1}(0)$.

The germ of $\mathcal{G}$ at $0 \in Q$ is called the *normal type* of $\mathcal{F}$ along Γ. Remark that, if T is a germ at $p \in \Gamma$ of $n-m$ manifold transverse to Γ then the restricted foliation $\mathcal{F}|_T$ is holomorphically equivalent to the normal type of $\mathcal{F}$ along Γ.

Moreover, since $\mathcal{G}$ is one dimensional we can assume that it is defined by a holomorphic vector field $X = \sum_{j=1}^{n-m} A_j(x)\frac{\partial}{\partial x_j}$, or by the $(n-m-1)$-form

$$(1.1)\quad \eta = i_X dx_1 \wedge \cdots \wedge dx_{n-m} = \sum_{j=1}^{n-m} (-1)^{j-1} A_j(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{n-m}\,,$$

where in (1.1) $\widehat{dx_j}$ means omission of dx_j in the product. The form η, considered as a form on U in the coordinates (x,y), defines $\mathcal{F}|_U$.

Let us see some examples:

Example 1.3. — Recall that a point $p \in M$ is a Kupka singularity of the foliation $\mathcal{F}$ if $p \in Sing(\mathcal{F})$ and $\mathcal{F}$ is represented in a neighborhood of p by an integrable $(n-k)$-form η such that $d\eta(p) \neq 0$. The form $d\eta$ defines a $k+1$ distribution $\mathcal{D} = ker(d\eta)$ in a neighborhood of p, where

$$\mathcal{D}(q) = ker(d\eta(q)) := \{v \in T_qM \mid i_v(d\eta(q)) = 0\}\ .$$

The distribution $\mathcal{D}$ is integrable and defines a regular foliation of dimension $k-1$ in a neighborhood of p. There exists a local chart $(U, z = (x,y))$, where

$x = (x_1, \ldots, x_{n-k+1})\colon U \to \mathbb{C}^{n-k+1}$, $y\colon U \to \mathbb{C}^{k-1}$ and $z(p) = (0,0)$, such that $d\eta = dx_1 \wedge \ldots \wedge dx_{n-k+1}$. In this case, the form η can be written as $\eta = i_X dx_1 \wedge \ldots \wedge dx_{n-k+1}$, where

$$X = \sum_{j=1}^{n-k+1} A_j(x) \frac{\partial}{\partial x_j}$$

defines the normal type of $\mathcal{F}_p$.

Note that $d\eta = \Delta(X)\, dx_1 \wedge \cdots \wedge dx_{n-k+1}$, where

$$\Delta(X) = div(X) = \sum_{j=1}^{n-k+1} \frac{\partial A_j}{\partial x_j},$$

so that $\Delta(X) \equiv 1$.

Definition 1.4. — *We say that K is a Kupka component of a foliation $\mathcal{F}$ (of dimension $k \geqslant 2$) if K is a l.t.p. component of $Sing(\mathcal{F})$ and the normal type of $\mathcal{F}$ along K is of Kupka type.*

Example 1.5. — Let P and Q be homogeneous polynomials on $\mathbb{C}^{n+1}$, $n \geqslant 3$, where $deg(P) = p$ and $deg(Q) = q$. The levels of the rational function $f = \frac{P^q}{Q^p}$ define a singular foliation of $\mathbb{P}^n$, that will be denoted by $\mathcal{F}(P,Q)$. We say that P and Q are transverse if the set

$$\left\{z \in \mathbb{C}^{n+1} \,\middle|\, P(z) = Q(z) = 0 \ \text{ and } \ dP(z) \wedge dQ(z) = 0\right\}$$

is either $\{0\}$, or empty (if $p = q = 1$). If P and Q are transverse then the subset Γ of $\mathbb{P}^n$ defined in homogeneous coordinates by $(P = Q = 0)$ is a Kupka component on $\mathcal{F}(P,Q)$. The normal type of $\mathcal{F}(P,Q)$ at the points of Γ is given by the linear vector field $X = p.\, x \frac{\partial}{\partial x} + q.\, y \frac{\partial}{\partial y}$.

In fact, the following result is known (cf. [Bru09, CA16, CA99, CLN94]):

Theorem 1.6. — *Let $\mathcal{F}$ be a holomorphic foliation of codimension one on $\mathbb{P}^n$, $n \geqslant 3$. If $\mathcal{F}$ has a Kupka component then $\mathcal{F} = \mathcal{F}(P,Q)$, where P and Q are transverse polynomials.*

Example 1.7. — Example 1.5 admits the following generalization: let $P_1, \ldots, P_m$ be homogeneous polynomials on $\mathbb{C}^{n+1}$ with $deg(P_j) = d_j$, $1 \leqslant j \leqslant m$. Assume that $n \geqslant m+1 \geqslant 4$ and that $P_1, \ldots, P_m$ are transverse; i.e. the set

$$\left\{z \in \mathbb{C}^{n+1} \,\middle|\, P_1(z) = \cdots = P_m(z) = 0 \ \text{ and } \ dP_1(z) \wedge \ldots \wedge dP_m(z) = 0\right\}$$

is either $\{0\}$, or empty (if $d_1 = \cdots = d_m = 1$). Let $(k_1, \ldots, k_m) \in \mathbb{N}^m$ be such that $gcd(k_1, \ldots, k_m) = 1$ and $k_1.\, d_1 = \cdots = k_m.\, d_m$. The levels of the rational map $\mathcal{P}\colon \mathbb{P}^n \to \mathbb{P}^{m-1}$, defined by

$$\mathcal{P} := \left[P_1^{k_1} : \ldots : P_m^{k_m}\right],$$

define a foliation of codimension $m-1$ on $\mathbb{P}^n$, denoted by $\mathcal{F}(P_1, \ldots, P_m)$. If $P_1, \ldots, P_m$ are transverse then the set $\Gamma \subset \mathbb{P}^n$, defined in homogeneous coordinates by $(P_1 = \cdots = P_m = 0)$, is a Kupka component of $\mathcal{F}(P_1, \ldots, P_m)$. The normal type of $\mathcal{F}(P_1, \ldots, P_m)$ at the points of Γ is given by the linear vector field $S = \sum_{j=1}^m d_j\, x_j \frac{\partial}{\partial x_j}$.

A natural problem is the following:

Problem 1.8. — *Let $\mathcal{F}$ be a holomorphic foliation of codimension $m-1$ on $\mathbb{P}^n$, where $n \geqslant m+1 \geqslant 4$. Assume that $\mathcal{F}$ has a Kupka component Γ. Are there transverse homogeneous polynomials $P_1, \ldots, P_m$ on $\mathbb{C}^{n+1}$ such that $\mathcal{F} = \mathcal{F}(P_1, \ldots, P_m)$ and Γ is defined by $(P_1 = \cdots = P_m = 0)$?*

Some partial results about this problem were proved (see for instance [CA09, CJCAFP14]).

In this paper we generalize Theorem 1.6:

Theorem 1.9. — *Let $\mathcal{F}$ be a holomorphic foliation of codimension one on $\mathbb{P}^n$, $n \geqslant 3$. Assume that $\mathcal{F}$ has a l.t.p. component Γ. Then Γ is a Kupka component of $\mathcal{F}$. In particular, $\mathcal{F}$ is like in Example 1.5.*

Let us state some consequences of Theorem 1.9.

Corollary 1.10. — *Let $\mathcal{F}$ be a codimension one holomorphic foliation on $\mathbb{P}^n$, $n \geqslant 4$. Assume that there is a linear embedding $i\colon \mathbb{P}^3 \to \mathbb{P}^n$ such that $i^*(\mathcal{F})$ has a l.t.p component. Then $\mathcal{F}$ has a rational first integral that can be written in homogeneous coordinates as P^q/Q^p, where P and Q are homogeneous polynomials on $\mathbb{C}^{n+1}$ with $deg(P) = p$ and $deg(Q) = q$.*

The proof of Corollary 1.10 is based in the fact that if there exists a linear embedding $i\colon \mathbb{P}^3 \to \mathbb{P}^n$ such that $i^*(\mathcal{F})$ has a first integral then $\mathcal{F}$ has also a first integral (see [CLN96]).

Corollary 1.11. — *Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n$, $n \geqslant 3$. Assume that all components of its singular set are l.t.p. Then $\mathcal{F}$ has degree zero: the first integral of Corollary 1.10 is of the form L_2/L_1, where L_1 and L_2 are linear.*

Corollary 1.12. — *Let η be an integrable 2-form on $\mathbb{C}^n$, $n \geqslant 4$, with homogeneous coefficients of the same degree $d \geqslant 1$. Then $dim_{\mathbb{C}}(sing(\eta)) \geqslant 1$.*

Remark 1.13. — Corollary 1.12 was proved in [CLN19] in the case $n = 4$. We would like to observe that the assertion is not true in the case of distributions of $\mathbb{C}^4$. The following example, due to Krishanu and Nagaraj [DN13]: define a 2-form θ on $\mathbb{C}^4$ by

$$\theta = x_3^2\, dx_2 \wedge dx_3 - x_1^2\, dx_3 \wedge dx_1 + (x_1\, x_2 + x_3\, x_4)\; dx_1 \wedge dx_2 + \left[x_4^2\, dx_1 + x_2^2\, dx_2 + (x_1\, x_2 - x_3\, x_4)\; dx_3\right] \wedge dx_4$$

has $Sing(\theta) = \{0\}$ and satisfies $\theta \wedge \theta = 0$. Hence, it generates a distribution of codimension two on $\mathbb{C}^4 \setminus \{0\}$. This distribution is not integrable.

Theorem 1.9 motivates the following problem:

Problem 1.14. — *Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{P}^n$ of codimension $\geqslant 2$ and dimension $\geqslant 2$. Assume that $\mathcal{F}$ has a l.t.p. component Γ. Is Γ a Kupka component of $\mathcal{F}$?*

A crucial point of our proof of Theorem 1.9 is the Camacho–Sad theorem on the existence of a separatrix for germs of holomorphic vector fields on $(\mathbb{C}^2, 0)$ [CS82]. The same type of argument cannot be used in the general case: there are examples of germs of vector fields on $(\mathbb{C}^m, 0)$, $m \geqslant 3$, without separatrices [GML92].

The proof of Theorem 1.9 will be done in Section 2. Since this proof is technical, in Section 2.1 we give an idea of the proof by stating the main objects and results that will be used. In Sections 2.2 and 2.4 we will prove the main auxiliary results used in the proof and stated in Section 2.1. Section 3 is dedicated to the proof of Corollaries 1.11 and 1.12.

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2. Proof of Theorem 1.9

2.1. Preliminaries and idea of the proof

Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n$, $n \geqslant 3$, with a l.t.p. component $\Gamma \subset Sing(\mathcal{F})$. The definition implies that $cod_{\mathbb{C}}(\Gamma) = 2$, so that the transversal type of $\mathcal{F}$ at the points of Γ is a germ of singular foliation at $(\mathbb{C}^2, 0)$ with an isolated singularity at $0 \in \mathbb{C}^2$ (see Remark 1.2 and Example 1.3). We can assume that this transversal type is given by germ at $0 \in \mathbb{C}^2$ of vector field $X = X_1(x,y)\frac{\partial}{\partial x} + X_2(x,y)\frac{\partial}{\partial y}$, where X_1, $X_2 \in \mathcal{O}_2$ and $X_1(0,0) = X_2(0,0) = 0$. Recall that Γ is a Kupka component if, and only if, we have $Tr(DX(0)) \neq 0$, where

$$Tr(DX(0)) := \frac{\partial X_1}{\partial x}(0) + \frac{\partial X_2}{\partial y}(0)$$

is the trace of the linear part $DX(0)$ of X at $0 \in \mathbb{C}^2$. In this case, as we have pointed out before, $\mathcal{F}$ is like in Example 1.5 (see Theorem 1.6).

Another useful ingredient is the normal Baum–Bott index of the component Γ, that we will denote as $BB(\mathcal{F}, \Gamma)$. Since Γ is a l.t.p. component of $Sing(\mathcal{F})$ then $BB(\mathcal{F}, \Gamma)$ coincides with the Baum–Bott index of X at the singularity 0 of X, denoted by $BB(X, 0)$ (see [CLN13]) (for the definition of $BB(X, 0)$ see [Bru00]). In [CLN13, Lemma 3.4, Section 3.2] it is proven that if $BB(\mathcal{F}, \Gamma) \neq 0$ and $DX(0) \not\equiv 0$ then Γ is a Kupka component and we are done.

One of the tools used in the proof of [CLN13, Lemma 3.4] is the existence of a smooth analytic separatrix along Γ. Below we define the concept of separatrix in a way that will be used in the proof of Theorem 1.9.

Definition 2.1. — *Let $\mathcal{F}$ be a holomorphic foliation of dimension k on a n dimensional compact complex manifold, $2 \leqslant k < n$, and Γ be l.t.p. component of $\mathcal{F}$ (recall that $dim(\Gamma) = k - 1$). A* separatrix Σ of dimension ℓ along Γ *of $\mathcal{F}$, where $k \leqslant \ell < n$, is a germ of ℓ analytic manifold along Γ which is $\mathcal{F}$-invariant in the sense that:*

(a) $\Sigma \sup \Gamma$.
(b) $\Sigma \setminus \Gamma$ *is contained in an union of leaves of* $\mathcal{F}$.

Remark 2.2. — Let Σ be a separatrix of $\mathcal{F}$ of dimension ℓ along Γ, as in Definition 2.1. Fix $p \in \Gamma$ and $(x, y)\colon U \to \mathbb{C}^{n-k+1} \times \mathbb{C}^{k-1}$, a local coordinate system around p as in Remark 1.2. It follows from the definition that $x^{-1}(x(\Sigma \cap U)) = \Sigma \cap U$.

Let T be a germ at p of a $n-k+1$ dimensional manifold transverse to Γ. As we have observed before, $\mathcal{F}|_T$ is equivalent to the normal type of $\mathcal{F}$ along Γ. In particular, the intersection $\Sigma \cap T$ is invariant by $\mathcal{F}|_T$.

In the case of Theorem 1.9, where $\mathcal{F}$ has codimension one, then $dim(T) = 2$ and the normal type is a germ $\mathcal{G}$ of one dimensional foliation on $(\mathbb{C}^2, 0)$. In this case $\Sigma \cap T$ is a finite number of analytic separatrices of $\mathcal{G}$ as considered in [CS82]. The next result will be used in proof of Theorem 1.9.

Lemma 2.3. — *Let $\mathcal{F}$ be a holomorphic codimension one foliation on a compact complex manifold M, where $dim(M) = n \geqslant 3$, and Γ be a l.t.p. component of $Sing(\mathcal{F})$. If the normal type of $\mathcal{F}$ along Γ is not equivalent to the radial foliation of $(\mathbb{C}^2, 0)$ then $\mathcal{F}$ admits an irreducible separatrix Σ along Γ with $dim(\Sigma) = n - 1$.*

Recall that the radial foliation of $\mathbb{C}^2$ is defined by the form $x\,dy - y\,dx$ and its leaves are the straight lines through 0. Lemma 2.3 will be proved in Section 2.2.

From now on, in this section, we will assume that $\mathcal{F}$ is a codimension one holomorphic foliation on the compact manifold M, $dim_{\mathbb{C}}(M) \geqslant 3$, with a l.t.p. component Γ and with a separatrix Σ along Γ, $dim(\Sigma) = n - 1$. Next we will introduce the *normal bundle of* Σ *along* Γ.

Since $dim(\Sigma) = n - 1$ we can find a Leray covering $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ of Γ by open sets and two collections $f = (f_\alpha)_{\alpha \in A}$ and $g = (g_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \emptyset}$ with the following properties:

(a) $f_\alpha \in \mathcal{O}(U_\alpha)$, $\forall\, \alpha \in A$, and $f_a = 0$ is a reduced equation of $\Sigma \cap U_\alpha$.
(b) $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ and $f_\alpha = g_{\alpha\beta}.\, f_\beta$ on $U_\alpha \cap U_\beta \neq \emptyset$.

Of course $g = (g_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \emptyset}$ is a multiplicative cocycle. We define the normal bundle of Σ along Γ as the line bundle on $Pic(\Gamma)$ induced on a tubular neighborhood $U \subset \bigcup_\alpha U_\alpha$ by the cocycle $g = (g_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \emptyset}$.

It will be denoted by N_Σ. Let $c_1(N_\Sigma)$ be the first Chern class of N_Σ, considered as an element of $H^2(U, \mathbb{R})$ via the homomorphism $H^2(U, \mathbb{Z}) \to H^2(U, \mathbb{R}) \simeq H^2_{DR}(U)$ induced by the inclusion $\mathbb{Z} \to \mathbb{R}$.

As we have seen in Remark 1.2, the normal type of $\mathcal{F}$ along Γ can be represented by a germ at $0 \in \mathbb{C}^2$ of holomorphic vector field $X = A_1(x, y)\,\frac{\partial}{\partial x} + A_2(x, y)\,\frac{\partial}{\partial y}$ with an isolated at 0. When we intersect Σ with a germ of transversal section $T \simeq (\mathbb{C}^2, 0)$ we obtain a separatrix of X, say $\gamma := \Sigma \cap T$ (in general γ is not irreducible). Let $f \in \mathcal{O}_2$ be a reduced analytic equation of γ. Since γ is X-invariant we can write

$$X(f) = h.\, f, \quad \text{where} \quad h \in \mathcal{O}_2\,. \tag{2.1}$$

Lemma 2.4. — *In the above situation, if $h(0) = 0$ then $c_1(N_\Sigma) = 0$.*

On the other hand, we have the following:

Lemma 2.5. — *If the ambient space is $M = \mathbb{P}^n$, $n \geqslant 3$, then $c_1(N_\Sigma) \neq 0$. In particular, if $X(f) = h.\, f$ then $h(0) \neq 0$.*

As a consequence of Lemma 2.5 we get the following:

Corollary 2.6. — *If $M = \mathbb{P}^n$, $n \geqslant 3$, then Σ is a Kupka component of $\mathcal{F}$.*

In particular, Theorem 1.6 will imply Theorem 1.9. Lemma 2.3 will be proved in the next section.

2.2. Proof of Lemma 2.3.

Let $\mathcal{F}$ be a holomorphic codimension one foliation on a compact complex manifold M with $dim(M) \geqslant 3$. Assume that $\mathcal{F}$ has a l.t.p. component Γ with normal type $\mathcal{G}$, where $\mathcal{G}$ is a germ of foliation on $(\mathbb{C}^2, 0)$ with an isolated singularity at $0 \in \mathbb{C}^2$. As before, we will assume that $\mathcal{G}$ is the foliation defined by a germ at $(\mathbb{C}^2, 0)$ of vector field $X = X_1 \frac{\partial}{\partial x} + X_2 \frac{\partial}{\partial y}$ with an isolated singularity at the origin of $\mathbb{C}^2$. The germ of foliation $\mathcal{G}$ can be defined also by the 1-form

$$\omega = i_X(dx \wedge dy) = X_1\, dy - X_2\, dx\,,$$

so that, $d\omega(0) = Tr(DX(0))\, dx \wedge dy$. We can assume that ω has a representative, denoted by $\widetilde{\omega}$, defined in the polydisc $Q = \mathbb{D}^2$ with an isolated singularity at $0 \in \mathbb{D}^2$.

By the definition of l.t.p. component, we can find a covering $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ of Λ by open sets biholomorphic to polydiscs, a collection of local charts $((z_\alpha,\, U_\alpha))_{\alpha \in A}$ and a multiplicative cocycle $(k_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \emptyset}$ with the following properties:

(1) $z_\alpha = (x_\alpha, y_\alpha)\colon U_\alpha \to \mathbb{C}^2 \times \mathbb{C}^{n-2}$, where $x_\alpha(U_\alpha) = Q$ and $\Gamma \cap U_\alpha = x_\alpha^{-1}(0)$, $\forall\, \alpha \in A$.

(2) $\mathcal{F}|_{U_\alpha}$ is defined by the integrable 1-form $\widetilde{\omega}_\alpha := x_\alpha^*(\widetilde{\omega})$. The germ of $\widetilde{\omega}_\alpha$ along $\Gamma \cap U_\alpha$ will be denoted by ω_α.

(3) $\widetilde{\omega}_a = k_{\alpha\beta}.\, \widetilde{\omega}_\beta$ on $U_\alpha \cap U_\beta \neq \emptyset$.

We will assume that $\mathcal{U}$ satisfies the following:

(4) If $U_\alpha \cap U_\beta \neq \emptyset$ then $\Gamma \cap U_\alpha \cap U_\beta \neq \emptyset$ and connected.

Remark 2.7. — Given $\alpha, \beta \in A$ such that $\Gamma \cap U_\alpha \cap U_\beta \neq \emptyset$ we can construct a germ $f_{\alpha\beta} \in \mathrm{Diff}(\mathbb{C}^2, 0)$ as follows: fix $p \in \Gamma \cap U_\alpha \cap U_\beta$ and a germ of plane $T = T_{\alpha,\beta} \simeq (\mathbb{C}^2, p)$ transverse to Γ at p. Note that $x_\alpha|_T, x_\beta|_T\colon (T, p) \to (\mathbb{C}^2, 0)$ are biholomorphisms. Therefore, we define

$$f_{\alpha\beta} = x_\alpha \circ (x_\beta|T)^{-1} = x_\alpha|T \circ (x_\beta|T)^{-1} \in \mathrm{Diff}\left(\mathbb{C}^2, 0\right).$$

Since $\omega_\alpha|_T = (x_\alpha|T)^*(\omega)$, $\omega_\beta|_T = (x_\beta|_T)^*(\omega)$ and $\omega_a = k_{\alpha\beta}.\, \omega_\beta$ we get $f^*_{\alpha\beta}(\omega) = h_{\alpha\beta}.\, \omega$, where

$$h_{\alpha\beta} = k_{\alpha\beta}|_T \circ (x_\beta|_T)^{-1} \in \mathcal{O}_2^*\,.$$

The biholomorphism $f_{\alpha\beta}$ can be interpreted as the *glueing map* of $\mathcal{F}|_{U_\beta}$ with $\mathcal{F}|_{U_\beta}$.

From now on, we fix a collection of germs $(f_{\alpha\beta})_{U_\alpha \cap U_\beta \neq \emptyset}$ as above.

Lemma 2.8. — *$\mathcal{F}$ admits a separatrix Σ along Γ if, and only if, X (or ω) has a separatrix γ (not necessarily irreducible) such that $f_{\alpha\beta}(\gamma) = \gamma$ for all $\Gamma \cap U_\alpha \cap U_\beta \neq \emptyset$.*

Terminology

We will say that the separatrix γ of $\mathcal{G}$ generates the separatrix Σ of $\mathcal{F}$.

Proof. — Assume that X has a separatrix γ such that $f_{\alpha\,\beta}(\gamma)=\gamma$ for all $\Gamma\cap U_\alpha\cap U_\beta\neq\emptyset$. Given $\alpha\in A$ define $\Sigma_\alpha := x_\alpha^{-1}(\gamma)$. We assert that if $U_\alpha\cap U_\beta\neq\emptyset$ then $\Sigma_\alpha\cap U_\beta=\Sigma_\beta\cap U_\alpha$. In fact, let $(x_\alpha,\, y_\alpha)$, $(x_\beta,\, y_\beta)$ and T be as before. Then

$$\Sigma_\alpha\cap T = x_\alpha^{-1}(\gamma)\cap T = (x_\alpha|_T)^{-1}(\gamma) = (x_\alpha|_T)^{-1}(f_{\alpha\beta}(\gamma)) = (x_\beta|_T)^{-1}(\gamma) = \Sigma_\beta\cap T\,.$$

This, of course, implies the assertion. In particular, the local separatrices Σ_a glue together forming a global separatrix Σ along Γ such that $\Sigma\cap U_\alpha=\Sigma_\alpha$, $\forall\,\alpha\in A$.

We leave the converse to the reader. □

Definition 2.9. — *Let $\mathcal{G}$ be a germ of foliation at $(\mathbb{C}^2,0)$ with an isolated singularity at 0. We say that a separatrix γ of $\mathcal{G}$ is* distinguished *if for any $f\in \mathrm{Diff}(\mathbb{C}^2,0)$ such that $f^*(\mathcal{G})=\mathcal{G}$ then $f(\gamma)=\gamma$.*

Lemma 2.10. — *Let $\mathcal{G}$ be a germ of foliation at $(\mathbb{C}^2,0)$ with an isolated singularity at 0 which is not equivalent to the radial foliation. Then $\mathcal{G}$ has a distinguished separatrix.*

Proof. — In the proof we use Seidenberg's resolution theorem [Sei68]. Let S be a smooth complex surface and $\mathcal{G}$ be a foliation by curves on S. Given $p\in Sing(\mathcal{G})\subset S$ we denote $\mathrm{Diff}(S,p)$ the set of germs at $p\in S$ of biholomorphisms $f\colon (S,p)\to S$ with a fixed point at p. Assume that the germ of $\mathcal{G}$ at p is defined by a germ of holomorphic vector field X with an isolated singularity at p. We use also the notations

$$\mathrm{Diff}_{\mathcal{G}}(S,p)=\{f\in \mathrm{Diff}(S,p)\,|\, f^*(\mathcal{G})=\mathcal{G}\}\ .$$

and

$$\mathrm{Diff}^0_{\mathcal{G}}(S,p)=\{f\in \mathrm{Diff}_{\mathcal{G}}(S,p)\,|\, f \text{ preserves the leaves of } \mathcal{G}\}\ . \qquad \square$$

Remark 2.11. — Note that:

(1) Given $f\in \mathrm{Diff}_{\mathcal{G}}(S,p)$, then $f^*(X)=h_X.\,X$, where $h_X\in\mathcal{O}_p^*$.
(2) $\mathrm{Diff}_{\mathcal{G}}(S,p)$ is a sub-group of $\mathrm{Diff}(S,p)$.
(3) Given $f\in \mathrm{Diff}_{\mathcal{G}}(S,p)$ and an irreducible separatrix γ of $\mathcal{G}$ through p then $f(\gamma)$ is also a separatrix of $\mathcal{G}$ through p.

Let $Sep(\mathcal{G})$ be the set of irreducible separatrices of $\mathcal{G}$ through p. By (3) of Remark 2.11, $\mathrm{Diff}_{\mathcal{G}}(S,p)$ acts in $Sep(\mathcal{G})$ as $(f,\delta)\in \mathrm{Diff}_{\mathcal{G}}(S,p)\times Sep(\mathcal{G})\to f(\delta)\in Sep(\mathcal{G})$. The idea of the proof is to find a finite subset $G_o:=\{\gamma_1,\,\dots,\,\gamma_k\}\subset Sep(\mathcal{G})$ such that $f(G_o)\subset G_o$ for all $f\in \mathrm{Diff}_{\mathcal{G}}(S,p)$. In this case, the set $\gamma:=\{f(\gamma_1)\,|\,f\in \mathrm{Diff}_{\mathcal{G}}(S,p)\}\subset G_o$ contains finitely many irreducible separatrices of $\mathcal{G}$ through p and can be considered as a germ of curve through p such that $f(\gamma)=\gamma$ for all $f\in \mathrm{Diff}_{\mathcal{G}}(S,p)$, and so γ is a distinguished separatrix of $\mathcal{G}$ through p. Let us prove the existence of the finite set G_o.

First of all, we observe that there are two possibilities for the foliation $\mathcal{G}$:

(I) $\mathcal{G}$ has finitely many irreducible separatrices through p. This case is trivial and the details are left to the reader.

(II) $\mathcal{G}$ has infinitely many irreducible separatrices through p. Let us prove Lemma 2.10 in this case.

We will consider a blowing-up process used to resolve the foliation $\mathcal{G}$ (see [CS82]). The first case, is when $\mathcal{G}$ has a simple singularity at p and no blowing-ups are needed in the process. Let λ_1 and λ_2 be the eigenvalues of $DX(p)$. The singularity is simple if:

(a) $\lambda_1 . \lambda_2 \neq 0$ and $\frac{\lambda_2}{\lambda_1} \notin \mathbb{Q}_+$.
(b) $\lambda_1 \neq 0$ and $\lambda_2 = 0$ (or vice-versa). In this case, p is a saddle-node.

In both cases $\mathcal{G}$ has one or two separatrices through p and so Lemma 2.10 is true.

When the singularity is not simple, Seidenberg's theorem says that after a finite process of blowing-ups $\Pi\colon (\widetilde{S}, E) \to (S, p)$ then all the singularities of the strict transform $\Pi^*(\mathcal{G})$ in the exceptional divisor E are simple. The blowing-up process Π can be considered as a composition blowing-ups of points

$$(2.2)\quad \left(\widetilde{S}, E\right) := \left(\widetilde{S}_k, E_k\right) \xrightarrow{\Pi_k} (\widetilde{S}_{k-1}, E_{k-1}) \xrightarrow{\Pi_{k-1}} \dots \xrightarrow{\Pi_2} \left(\widetilde{S}_1, E_1\right) \xrightarrow{\Pi_1} \left(\widetilde{S}_0, E_0\right) = (S, p)$$

where in the j^{th} step $\Pi_j\colon (\widetilde{S}_j, E_j) \to (\widetilde{S}_{j-1}, E_{j-1})$, $j \geqslant 2$, we blow-up in a point $p_{j-1} \in E_{j-1}$. The exceptional divisor obtained in this step will be denoted as $\mathbb{P}^1 \simeq \widetilde{E}_j \subset E_j$, so that $\Pi_j(\widetilde{E}_j) = p_{j-1}$. We use also the notation $\widetilde{\Pi}_j := \Pi_1 \circ \ldots \circ \Pi_j$. We will denote also $\widetilde{\mathcal{G}}_j := \widetilde{\Pi}^*(\mathcal{G})$. The point $p_{j-1} \in E_{j-1}$ is chosen between the non simple singularities of $\widetilde{\mathcal{G}}_{j-1}$ on E_{j-1}. Seidenberg's theorem can be stated as follows

Theorem 2.12. — *It is possible to choose a blowing-up process as above in such a way that all singularities of the strict transform $\widetilde{\mathcal{G}}_k = \widetilde{\Pi}_k^*(\mathcal{G})$ are simple.*

Remark 2.13. — There are two possibilities in each step

$$\Pi_j\colon \left(\widetilde{S}_j, \widetilde{E}_j\right) \to \left(\widetilde{S}_{j-1}, p_{j-1}\right).$$

We assume that p_{j-1} is a non simple singularity of $\widetilde{\mathcal{G}}_{j-1}$. Let X_{j-1} be a germ at p_{j-1} of holomorphic vector field that represents the germ of $\widetilde{\mathcal{G}}_{j-1}$ at p_{j-1}. Let $X_\nu = P_\nu(x,y)\frac{\partial}{\partial x} + Q_\nu(x,y)\frac{\partial}{\partial y}$ be the first non-zero jet of X_{j-1} at p_{j-1}, where P_ν and Q_ν are homogeneous polynomials of degree $\nu \geqslant 1$. Set $F_{\nu+1}(x,y) = x . Q_\nu(x,y) - y\, P_\nu(x,y)$.

(i) If $F_{\nu+1} \not\equiv 0$ then $F_{\nu+1}$ is homogeneous of degree $\nu+1$ and the blowing-up is called *non-dicritical.* The divisor $\widetilde{E}_j$ is invariant for the foliation $\widetilde{\mathcal{G}}_j$ and the singularities of $\widetilde{\mathcal{G}}_j$ on $\widetilde{E}_j$ are the directions correspondent to the directions defined by $F_{\nu+1}(x,y) = 0$.
(ii) If $F_{\nu+1} \equiv 0$ then $X_\nu = F_{\nu-1}(x,y)\, R$, where $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is the radial vector field in $\mathbb{C}^2$ and $F_{\nu-1}$ is homogeneous of degree $\nu-1$. In this case, the blowing-up is called *dicritical.* The divisor $\widetilde{E}_j$ is non-invariant for $\widetilde{\mathcal{G}}_j$ and it is transverse to $\widetilde{E}_j$ outside the set $V_j \subset \widetilde{E}_j$ corresponding to the directions defined by the equation $F_{\nu-1}(x,y) = 0$.

If $\nu = 1$ then $\widetilde{\mathcal{G}}_{j-1}$ is equivalent to the radial foliation at p_{j-1}. We will say that p_{j-1} is a *radial* singularity of $\widetilde{\mathcal{G}}_{j-1}$. If p_{j-1} is not radial for $\widetilde{\mathcal{G}}_{j-1}$ then $V_j \neq \emptyset$ and we can divide it into two disjoint subsets $V_j = \tau_j \cup \sigma_j$, where

- $\sigma_j = Sing(\widetilde{\mathcal{G}}_j) \cap \widetilde{E}_j$.
- $\tau_j = V_j \setminus Sing(\widetilde{\mathcal{G}}_j)$. We call τ_j the set of tangencies of $\widetilde{\mathcal{G}}_j$ with $\widetilde{E}_j$.

Remark also that $Sep(\mathcal{G})$ is finite if, and only if, all blowing-ups in the process are non-dicritical.

Since in the blowing-up process, in each step, $1 \leqslant j \leqslant k$, we blow-up in some non-simple singularity of $\widetilde{\mathcal{G}}_{j-1}$, if $\widetilde{E}_j$ is dicritical, at the end the tangencies τ_j "survive", in the sense that there exists a set $\tau \subset E_k$ such that for any $1 \leqslant j < k$ such that $\tau_j \neq \emptyset$ then

$$\tau_j \subset \Pi_k \circ \cdots \circ \Pi_{j+1}(\tau)$$

For each $1 \leqslant j \leqslant k$ denote by $\mathrm{Diff}(\widetilde{S}_j, E_j)$ the set of germs of biholomorphisms $f\colon (\widetilde{S}_j, E_j) \to (\widetilde{S}_j, E_j)$.

Definition 2.14. — *We say that $f \in \mathrm{Diff}(S,p)$ can be lifted to $\mathrm{Diff}(\widetilde{S}_j, E_j)$ if there exists a germ of biholomorphism $\widetilde{f}_j \in \mathrm{Diff}(\widetilde{S}_j, E_j)$ such that the diagram below commutes:*

$$\begin{tikzcd}
\left(\widetilde{S}_j, E_j\right) \arrow[r, "\tilde{f}_j"] \arrow[d, "\widetilde{\Pi}_j"'] & \left(\widetilde{S}_j, E_j\right) \arrow[d, "\widetilde{\Pi}_j"] \\
(S,p) \arrow[r, "f"] & (S,p)
\end{tikzcd}$$

Remark 2.15. — Observe that, if the lift $\widetilde{f}_j$ of f exists then it is unique. When $j = 1$ (just one blowing-up) the lifting exists for any $f \in \mathrm{Diff}(S,p)$, but if $j \geqslant 2$ then there are germs $f \in \mathrm{Diff}(S,p)$ that cannot be lifted to $\mathrm{Diff}(\widetilde{S}_j, E_j)$. However, we have the following:

Claim 2.16. — *The blowing-up process can be done in such a way that any $f \in \mathrm{Diff}_{\mathcal{G}}(S,p)$ can be lifted to the last step in an unique $\widetilde{f} = \widetilde{f}_k \in \mathrm{Diff}(\widetilde{S}_k, E_k)$. Moreover, $\widetilde{f}$ preserves $\widetilde{\mathcal{G}}_k$ in the sense that $\widetilde{f}^*(\widetilde{\mathcal{G}}_k) = \widetilde{\mathcal{G}}_k$.*

Proof. — We say that the j^{th} step of the blowing-up process is admissible if any $f \in \mathrm{Diff}_{\mathcal{G}}(S,p)$ has a lifting $\widetilde{f}_j \in \mathrm{Diff}(\widetilde{S}_j, E_j)$. We will obtain by induction a blowing-up process, as in (2.2), for which there are steps $1 = \ell_1 < \ell_2 < \cdots < \ell_r = k$ such that the ℓ_j^{th} step is admissible, for any $1 \leqslant j \leqslant r$, and $\widetilde{\Pi}_k\colon (\widetilde{S}_k, E_k) \to (S,p)$ is a resolution of the foliation $\mathcal{G}$.

First of all, the first step is admissible, because any $f \in \mathrm{Diff}(S,p)$ admits a lifting $\widetilde{f}_1 \in \mathrm{Diff}(\widetilde{S}_1, E_1)$.

Assume that we have found some process for which the $\ell := \ell_s$ step is admissible, $\ell \geqslant 1$, so that any $f \in \mathrm{Diff}_{\mathcal{G}}(S,p)$ admits a lifting $\widetilde{f} = \widetilde{f}_\ell \in Diff(\widetilde{S}_\ell, E_\ell)$ satisfying $\widetilde{f}^*(\widetilde{\mathcal{G}}_\ell) = \widetilde{\mathcal{G}}_\ell$. Given $f \in \mathrm{Diff}_{\mathcal{G}}(S,p)$, with lifting $\widetilde{f}$, and $q \in E_\ell$ then $\widetilde{f}$ is an equivalence between the two germs of $\widetilde{\mathcal{G}}_\ell$ at q and at $\widetilde{f}(q)$. In particular, $\widetilde{f}$ preserves the set of non simple singularities of $\widetilde{\mathcal{G}}_\ell$.

If $\widetilde{\mathcal{G}}_\ell$ is not a resolution of $\mathcal{G}$ then it has at least one non simple singularity q_1. Let $Sat(q_1) = \{\widetilde{f}(q_1) \mid f \in \mathrm{Diff}_{\mathcal{G}}(S,p)\} = \{q_1, \ldots, q_m\}$. We then blow-up once at all points $q_j \in Sat(q_1)$, passing from the $\ell = \ell_s$ step to the $\ell_{s+1} := \ell_s + m$ step directly. Let $\widehat{E}_j$ be the divisor obtained by the blowing-up at q_j.

Given $f \in \mathrm{Diff}_{\mathcal{G}}(S,p)$ and its lifting $\widetilde{f}_s$, let $\widetilde{f}_s(q_j) = q_{i(j)}$, $1 \leqslant j \leqslant m$. Then we can obtain a lifting $\widetilde{f}_{s+1}$ of $\widetilde{f}_s$ such that $\widetilde{f}_{s+1}(\widehat{E}_j) = \widehat{E}_{i(j)}$, $1 \leqslant j \leqslant m$. By Seidenberg's theorem this process must end at some step, when the final foliation $\widetilde{\mathcal{G}}_k = \widetilde{\Pi}_k^*(\mathcal{G})$ has all singularities simple. □

Proof of Lemma 2.10. — Let us finish the proof of the lemma. We will consider two cases:

(1) There is $q_1 \in Sing(\widetilde{\mathcal{G}}_k) \cap E_k$ that has some separatrix $\widetilde{\gamma}$ not contained in E_k.
(2) All the separatrices of the singularities of $\widetilde{\mathcal{G}}_k$ are contained in E_k.

In the first case, let $Sat(q_1) = \{\widetilde{f}(q_1) \mid f \in \mathrm{Diff}_{\mathcal{G}}(S,p)\} = \{q_1, \ldots, q_m\}$. Given $f \in \mathrm{Diff}_{\mathcal{G}}(S,p)$ and $q_j = \widetilde{f}(q_1)$ then $\widetilde{f}(\widetilde{\gamma}) := \widetilde{\gamma}_f$ is a separatrix of $\widetilde{\mathcal{G}}_k$ not contained in E_k. Since $\widetilde{\gamma}_f$ is not contained in E_k, its image $\gamma_f := \widetilde{\Pi}_k(\widetilde{\gamma}_f)$ is a separatrix of $\mathcal{G}$ through p. Moreover, since $q_1, \ldots, q_m$ are simple singularities of $\widetilde{\mathcal{G}}_k$ the set $\{\widetilde{\gamma}_f \mid f \in \mathrm{Diff}_{\mathcal{G}}(S,p)\}$ is finite. Therefore, if we set

$$\gamma = \bigcup_{f \in \mathrm{Diff}_{\mathcal{G}}(S,p)} \gamma_f \tag{2.3}$$

then $f(\gamma) = \gamma$, $\forall\, f \in \mathrm{Diff}_{\mathcal{G}}(S,p)$, and γ is a distinguished separatrix of $\mathcal{G}$.

In the second case necessarily there are dicritical irreducible divisors of $\widetilde{\mathcal{G}}_k$, say $\widetilde{E}_1, \ldots, \widetilde{E}_m$, contained in E_k (by Camacho–Sad theorem). This case will be divided into two sub-cases:

- (2.1) The set of tangencies τ is not empty.
- (2.2) $\tau = \emptyset$.

In case (2.1) let $q_o \in \tau$ and $\widetilde{\gamma}_{id}$ be the leaf of $\widetilde{\mathcal{G}}_k$ through q_o. Then, for any $f \in \mathrm{Diff}_{\mathcal{G}}(S,p)$ we have $q_f := \widetilde{f}(q_o) \in \tau$ and $\widetilde{\gamma}_f := \widetilde{f}(\widetilde{\gamma}_{id})$ is the leaf of $\widetilde{\mathcal{G}}_k$ through $\widetilde{f}(q_o)$. Since $q_f \in E_k$, but $\widetilde{\gamma}_f$ is not contained in E_k, $\forall\, f \in \mathrm{Diff}_{\mathcal{G}}(S,p)$, the image $\widetilde{\Pi}_k(\widetilde{\gamma}_f) := \gamma_f$ is an irreducible separatrix of $\mathcal{G}$ through p. Therefore, if we define γ as in (2.3) then γ is a distinguished separatrix of $\mathcal{G}$.

We will divide case (2.2) into two subcases:

$$E_k \setminus \bigcup_j \widetilde{E}_j \neq \emptyset. \tag{2.2.1}$$

$$E_k = \bigcup_j \widetilde{E}_j. \tag{2.2.2}$$

In case (2.2.1), let $\widehat{E}$ be a connected component of $E_k \setminus \bigcup_j \widetilde{E}_j$. Let $\bigcup_{i=1}^r D_i$ be decomposition of $\widehat{E}$ into irreducible components, $D_i \simeq \mathbb{P}^1$. Note that:

(i) The graph formed by the divisors D_i is a tree.
(ii) The intersection matrix $(D_i \,.\, D_j)_{1 \leqslant i,j \leqslant r}$ is negative.

(iii) If D is an irreducible divisor of E_k such that $D \not\subset \widehat{E}$ but $D \cap \widehat{E} \neq \emptyset$ then $D = \widetilde{E}_j$ for some j. In particular, D is dicritical.

In this case, $Sing(\widetilde{\mathcal{G}}_k) \cap \widehat{E} \neq \emptyset$ and contains a singularity q with a separatrix $\tilde{\gamma}$ not contained in $\widehat{E}$. This is a consequence of Sebastiani's version of Camacho–Sad theorem (see [Seb97]). In fact, $\tilde{\gamma}$ is not contained in E_k, for otherwise it would be contained in some irreducible divisor D of E_k not contained in $\widehat{E}$, and D is non dicritical, which contradicts (iii). Therefore, we reduce the problem to case (1).

In case (2.2.2) all irreducible divisors $\widetilde{E}_j$ of E_k are dicritical. We can assume that $Sing(\widetilde{\mathcal{G}}_k) = \emptyset$. In fact, if $q_o \in Sing(\widetilde{\mathcal{G}}_k)$ then q_o is simple and any of their separatrices cannot be contained in E_k, for otherwise some of the divisors $\widetilde{E}_j$ would be non-dicritical. Therefore, we are again in case (1). In particular, we can assume that all divisors $\widetilde{E}_j$ are radial, in the sense that for any $q \in \widetilde{E}_j$ the leaf of $\widetilde{\mathcal{G}}_k$ through q is transverse to $\widetilde{E}_j$. Moreover, $m \geqslant 2$ because otherwise p would be a radial singularity of $\mathcal{G}$. In particular, we can assume that $\widetilde{E}_1 \cap \widetilde{E}_2 = \{q_o\} \neq \emptyset$. Let $\tilde{\gamma}_{id}$ be the leaf of $\widetilde{\mathcal{G}}_k$ through q_o. Note that, for any $f \in \mathrm{Diff}_{\mathcal{G}}(S,p)$ then

$$q_f := \tilde{f}(q_o) = \tilde{f}(\widetilde{E}_1) \cap \tilde{f}(\widetilde{E}_2) \in E_k\,,$$

so that $\tilde{\gamma}_f := \tilde{f}(\tilde{\gamma}_{id})$ is the leaf of $\widetilde{\mathcal{G}}_k$ through q_f. For each $f \in \mathrm{Diff}_{\mathcal{G}}(S,p)$ the projection $\gamma_f := \Pi_k(\tilde{\gamma}_f)$ is an irreducible separatrix of $\mathcal{G}$. Since

$$A := \{q_f \mid f \in \mathrm{Diff}_{\mathcal{G}}(S,p)\} \subset \bigcup_{i \neq j} \widetilde{E}_i \cap \widetilde{E}_j$$

then A is finite. Therefore, we can construct a distinguished separatrix γ of $\mathcal{G}$ as in (2.3). □

Finally, note that Lemma 2.3 is a consequence of Lemmas 2.8 and 2.10.

2.3. Proof of Lemma 2.4.

Since U is a tubular neighborhood of Γ the map $\Theta \in H^2_{Dr}(U) \mapsto \Theta|_\Gamma \in H^2_{Dr}(\Gamma)$ is an isomorphism. Therefore, it is sufficient to prove that $c_1(N_\Sigma)|_\Gamma = 0$.

Recall that the germ of $\mathcal{F}$ at any $q \in \Gamma$ is equivalent to a product of a singular foliation by curves on $(\mathbb{C}^2, 0)$ by a regular foliation of dimension $n-2$. This implies that there exist a local coordinate system around q, $z = (x,y)\colon U \to \mathbb{C}^2 \times \mathbb{C}^{n-2}$, $x = (x_1, x_2)$, $y = (y_1, \ldots, y_{n-2})$, and a holomorphic vector field $X = P(x)\frac{\partial}{\partial x_1} + Q(x)\frac{\partial}{\partial x_2}$, with an isolated singularity at $0 \in \mathbb{C}^2$, such that

- $\mathcal{F}|_U$ is generated by the $n-1$ commuting vector fields X, $Y_1 := \frac{\partial}{\partial y_1}, \ldots, Y_{n-2} := \frac{\partial}{\partial y_{n-2}}$.

Moreover, the separatrix Σ of $\mathcal{F}$ along Γ is induced by a separatrix $\gamma = (f(x_1,x_2) = 0)$ of X, such that $X(f) = h.\,f$, where we have assumed $h(0) = 0$. It follows that we can find a Leray covering $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ of Γ by open sets with the following properties:

(a) For each $\alpha \in A$, there exists a coordinate system $z_\alpha = (x_\alpha, y_\alpha)\colon U_\alpha \to \mathbb{C}^2 \times \mathbb{C}^{n-2}$, where $\Sigma \cap U_\alpha = (x_\alpha = 0)$, $x_\alpha = (x_{\alpha 1}, x_{\alpha 2})$ and $y_\alpha = (y_{\alpha 1}, \ldots, y_{\alpha\, n-2})$.

(b) For each $\alpha \in A$, $\mathcal{F}|_{U_\alpha}$ is generated by the $n-1$ holomorphic vector fields

$$X_\alpha = P(x_\alpha)\frac{\partial}{\partial x_{\alpha 1}} + Q(x_\alpha)\frac{\partial}{\partial x_{\alpha 2}}\,,\; Y_{\alpha j} = \frac{\partial}{\partial y_{\alpha j}},\; j = 1, \ldots, n-2\,.$$

(c) $\Sigma \cap U_\alpha$ has the reduced equation $f_\alpha = 0$, where $f_\alpha = f(x_\alpha)$. In particular, if we set $h_\alpha = h(x_\alpha)$ then

$$X_\alpha(f_\alpha) = h_\alpha \cdot\, f_a\,,\, Y_{\alpha j}(f_\alpha) = 0, \quad \forall\; 1 \leqslant j \leqslant n-2\,.$$

Consider the multiplicative cocycle $g = (g_{\alpha\,\beta})_{U_\alpha \cap U_\beta \neq \emptyset}$ such that $f_\alpha = g_{\alpha\,\beta}.\, f_\beta$ on $U_\alpha \cap U_\beta \neq \emptyset$.

Claim 2.17. — *If $U_\alpha \cap U_\beta \neq \emptyset$ then $g_{\alpha\,\beta}$ is locally constant on $U_\alpha \cap U_\beta \cap \Gamma$: $dg_{\alpha\,\beta}|_{U_\alpha \cap U_\beta \cap \Sigma} \equiv 0$. In particular $c_1(N_\Sigma) = 0$.*

Proof. — Let $U_\alpha \cap U_\beta \cap \Sigma \neq \emptyset$. We assert that there exists a $(n-1) \times (n-1)$ matrix $A_{\alpha\beta}$, with entries in

$$\mathcal{O}(U_\alpha \cap U_\beta), A_{\alpha\beta} = \left(a^{ij}_{\alpha\beta}\right)_{0 \leqslant i, j \leqslant n-2}\,,$$

such that

$$\begin{cases} X_\alpha & = a^{00}_{\alpha\,\beta}.\, X_\beta + \displaystyle\sum_{j=1}^{n-2} a^{0j}_{\alpha\,\beta}.\, Y_{\beta\, j} \\ Y_{\alpha i} & = a^{i0}_{\alpha\,\beta}.\, X_\beta + \displaystyle\sum_{j=1}^{n-2} a^{ij}_{\alpha\,\beta}.\, Y_{\beta\, j},\; 1 \leqslant i \leqslant n-2 \end{cases} \tag{2.5}$$

In fact, since $\mathcal{F}|_{U_\alpha \cap U_\beta}$ is generated by both systems $\langle X_\alpha, Y_{\alpha\, i} \,|\, 1 \leqslant i \leqslant n-2\rangle$ and $\langle X_\beta, Y_{\beta\, i} \,|\, 1 \leqslant i \leqslant n-2\rangle$, we can find a matrix $A_{\alpha\,\beta}$ with entries in $\mathcal{O}(U_\alpha \cap U_\beta \setminus \Sigma)$ as in (2.5). But since $cod(\Sigma) = 2$ the entries of $A_{\alpha\,\beta}$ can be extended to $U_\alpha \cap U_\beta$ by Hartog's theorem.

Now, from (c) we get

$$0 = Y_{\alpha i}(f_a) = Y_{\alpha i}(g_{\alpha\beta}.\, f_\beta) = Y_{\alpha i}(g_{\alpha\beta}).\, f_\beta + g_{\alpha\beta}.\, Y_{\alpha i}(f_\beta)$$

and from (c) and (2.5)

$$\begin{aligned} Y_{\alpha\, i}(f_\beta) = a^{i0}_{\alpha\,\beta}.\, X_\beta(f_\beta) + \sum_{j=1}^{n-2} a^{ij}_{\alpha\beta}.\, Y_{\beta\, j}(f_\beta) = a^{i0}_{\alpha\,\beta}.\, h_\beta.\, f_\beta \\ \Longrightarrow \left(Y_{\alpha\, i}(g_{\alpha\,\beta}) + a^{i0}_{\alpha\,\beta}.\, h_\beta\right)\, f_\beta = 0 \;\Longrightarrow\; Y_{\alpha\, i}(g_{\alpha\,\beta}) = -a^{i0}_{\alpha\,\beta}.\, h_\beta\,. \end{aligned}$$

Now, $h_\beta|_{U_\alpha \cap U_\beta \cap \Sigma} = h(0) = 0$ and so

$$Y_{\alpha\, i}(g_{\alpha\,\beta})|_{U_\alpha \cap U_\beta \cap \Sigma} = \frac{\partial g_{\alpha\,\beta}}{\partial y_{\alpha i}}(0, y_\alpha) = 0\,,\; 1 \leqslant i \leqslant n-2 \;\Longrightarrow\; dg_{\alpha\,\beta}|_{U_\alpha \cap U_\beta \cap \Sigma} = 0\,.$$

This finishes the proof of Lemma 2.4. □

2.4. Proof of Lemma 2.5.

The case in which Σ is smooth was proved in [CLN13]. Here we give a more general proof (suggested by J. V. Pereira). Let us consider first the case $n = 3$: $M = \mathbb{P}^3$. In this case, Γ is a compact algebraic curve so that $H^2_{DR}(\Gamma) \simeq \mathbb{R}$ and the map

$$\Theta \in H^2_{DR}(\Gamma) \mapsto \int_\Gamma \Theta \in \mathbb{R}$$

is an isomorphism. In fact, we will prove that

$$\int_\Gamma c_1(N_\Sigma) \in \mathbb{N} \implies c_1(N_\Sigma) \neq 0\ .$$

We will see that $\int_\Gamma c_1(N_\Sigma)$ represents the intersection number of a small deformation Γ_t of Γ with Σ.

Let $\mathcal{X}(\mathbb{P}^3)$ be the vector space of holomorphic vector fields on $\mathbb{P}^3$: $dim(\mathcal{X}(\mathbb{P}^3)) = 15$. Given $Z \in \mathcal{X}(\mathbb{P}^3)$ we will denote by $(t,q) \in \mathbb{C} \times \mathbb{P}^3 \mapsto Z_t(q) \in \mathbb{P}^3$ its flow and $\Gamma_t := Z_t(\Gamma)$. Let U be a tubular neighborhood U of Γ with $U \subset \bigcup_\alpha U_\alpha$.

Remark 2.18. — There exist $Z \in \mathcal{X}(\mathbb{P}^3)$ and $\epsilon > 0$ with the following properties:

(a) If $t \in D_\epsilon \subset \mathbb{C}$ then $\Gamma_t \subset U$, where $D_\epsilon = \{t \,|\, |t| < \epsilon\}$.

(b) If $t \in D^*_\epsilon := D_\epsilon \setminus \{0\}$ then $\Gamma_t \cap \Gamma = \emptyset$.

(c) The set $B := \{t \in D_\epsilon \,|\, \Gamma_t$ is not transverse to $\Sigma\}$ is discrete in D^*.

(d) There exists $t_o \in D^* \setminus B$ such that $\Gamma_{t_o} \cap \Sigma \neq \emptyset$.

We leave the proof of Remark 2.18 for the reader. Let us finish the proof of Lemma 2.5 in the case of $\mathbb{P}^3$.

Proof of Lemma 2.5. — The idea is to prove that, if $t \in D^*_\epsilon \setminus B$ then $\int_\Gamma c_1(N_\Sigma) = \#(\Gamma_t \cap \Sigma)$, the intersection number of Γ_t with Σ. By (d) of Remark 2.18 $\#(\Gamma_t \cap \Sigma) > 0$ and so $c_1(N_\Sigma) \neq 0$.

First of all, note that

$$\Gamma_t \cap \Sigma = Z_t\left(\Gamma \cap Z_{-t}(\Sigma)\right) \implies \#\left[\Gamma_t \cap \Sigma\right] = \#\left[\Gamma \cap Z_{-t}(\Sigma)\right]\ .$$

On the other hand, $Z_{-t}(\Sigma)$ can be defined in the covering $\mathcal{U}_t := (Z_{-t}(U_\alpha))_{\alpha \in A}$ by the divisor $(f_\alpha \circ Z_t)_{\alpha \in A}$, with associated cocycle $g_t := (g_{\alpha\beta} \circ Z_t)_{U_\alpha \cap U_\beta \neq \emptyset}$. Since $t \in D^*_\epsilon \setminus B$, Γ is transverse to $Z_{-t}(\Sigma)$ and so $\Gamma \cap Z_{-t}(\Sigma)$ is finite and is defined by the divisor

$$\left(f_\alpha \circ Z_t|_{\Gamma \cap Z_{-t}(U_\alpha)}\right)_{\alpha \in A}$$

$$\text{with associated cocycle} \quad g_t|_\Gamma = \left(g_{\alpha\beta} \circ Z_t|_{\Gamma \cap Z_{-t}(U_\alpha \cap U_\beta)}\right)_{U_\alpha \cap U_\beta \neq \emptyset}\ .$$

This divisor can be interpreted as a holomorphic section of the line bundle induced by $g_t|_\Gamma$ on $Pic(\Gamma)$. In particular, if $c_1(g_t|_\Gamma)$ is its first Chern class then its degree is given by

$$\int_\Gamma c_1(g_t|_\Sigma) = \#\left[\Gamma \cap Z_{-t}(\Sigma)\right] = \#\left[\Gamma_t \cap \Sigma\right]\ .$$

Since the map

$$t \in D_\epsilon \mapsto \int_\Gamma c_1(g_t|_\Gamma)$$

is continuous and constant in $D_\epsilon^* \setminus B$, we get $\int_\Gamma c_1(g_0|_\Sigma) > 0 \implies c_1(g_0|_\Sigma) = c_1(N_\Sigma) \neq 0$. This finishes the proof of Lemma 2.5 in the case of $\mathbb{P}^3$.

The case of $\mathbb{P}^n$, $n \geqslant 4$, can be reduced to the previous by taking sections by generic 3-planes linearly embedded in $\mathbb{P}^n$. We leave the details to the reader. □

2.5. Proof of Corollary 2.6.

Proof. — Recall that $X(f) = h.\,f$, where X represents the normal type $\mathcal{G}$ of $\mathcal{F}$ along Γ and $f \in \mathcal{O}_2$ is reduced. By lemma 2.5 we have $h(0) \neq 0$. Let f_μ and X_ν be the first non-zero jets of f and X at $0 \in \mathbb{C}^2$, respectively. Then

$$X(f) = h.\,f \quad \implies \quad X_\nu(f_\mu) = h(0).\,f_\mu \quad \implies \quad \nu = 1$$

and $X_\nu = X_1$ is not nilpotent; has at least one non-zero eigenvalue. On the other hand, we have seen that Γ is a Kupka component of $\mathcal{F}$ if, and only if, $tr(X_1) \neq 0$. If $tr(X_1) = 0$ and X_1 has a non-zero eigenvalue, then we can assume that $X_1 = \lambda\,(x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2}), \lambda \neq 0$. In this case, X has exactly two separatrices through $0 \in \mathbb{C}^2$ which are smooth and tangent to $x_1 = 0$ and $x_2 = 0$. We can assume that these separatrices have equations $f_1(x_1, x_2) = x_1 + h.o.t$ and $f_2(x_1, x_2) = x_2 + h.o.t$. Consider the separatrix $\gamma = (f_1.\,f_2 = 0)$ of X. Note that $f(\gamma) = \gamma$, $\forall\, f \in \mathrm{Diff}_{\mathcal{G}}(\mathbb{C}^2, 0)$. By Lemma 2.8 γ generates a separatrix Σ of $\mathcal{F}$ along Γ. However $X(f_1.\,f_2) = h.\,f_1.\,f_2$ where $h(0) = 0$, because $X_1(x_1.\,x_2) = 0$. Therefore, we must have $tr(X_1) \neq 0$ and Γ is a Kupka component of $\mathcal{F}$. □

3. Corollaries 1.11 and 1.12

3.1. Proof of Corollary 1.11.

A codimension one foliation $\mathcal{G}$ on $\mathbb{P}^n$ of degree zero has a rational first integral of degree one. It is defined in some coordinate system $(x_1, \ldots, x_{n+1}) \in \mathbb{C}^{n+1}$ by a the form $\omega = x_1\,dx_2 - x_2\,dx_1$. In particular, $\Pi^{-1}(Sing(\mathcal{G})) = (x_1 = x_2 = 0)$, which is a l.t.p component.

Conversely, let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n$, $n \geqslant 3$. It is known that $Sing(\mathcal{F})$ has at least one irreducible component of codimension two [LN99]. Assume that all components of $Sing(\mathcal{F})$ are l.t.p. Let Ω be a 1-form on $\mathbb{C}^{n+1}$ that represents $\mathcal{F}$ in homogeneous coordinates: $\mathcal{F}_\Omega = \Pi^*(\mathcal{F})$. Then

(a) $i_R\Omega = 0$, where R is the radial vector field on $\mathbb{C}^{n+1}$.
(b) The coefficients of Ω are homogeneous of degree $d + 1$, where $d = deg(\mathcal{F})$.
(c) $i_R d\Omega = (d + 2)\,\Omega$ (see [CLN94]). In particular, $Sing(d\Omega) \subset Sing(\Omega)$.

Claim 3.1. — *Let $q \in Sing(\mathcal{F})$ and $p \in \Pi^{-1}(q) \subset \mathbb{C}^{n+1} \setminus \{0\}$. Then $d\Omega_p \neq 0$. In particular, $Sing(d\Omega) = \emptyset$ and $deg(\mathcal{F}) = 0$.*

Proof. — Let ω be a holomorphic 1-form that represents $\mathcal{F}$ in a neighborhood of q. The hypothesis and Theorem 1.9 imply that $d\omega(q) \neq 0$.

On the other hand, $\Pi^*(\omega)$ represents $\mathcal{F}_\Omega$ in a neighborhood, say U, of p. It follows that $\Pi^*(\omega) = \varphi.\,\Omega$ on U, where $\varphi \in \mathcal{O}^*(U)$. Therefore,

$$\begin{aligned}\Pi^*(d\omega) &= d\,\Pi^*(\omega) = d\varphi \wedge \Omega + \varphi.\,d\Omega \\ &\Longrightarrow \forall\ u, v \in T_p\mathbb{C}^{n+1} \quad \text{we get } \varphi(p).\,d\Omega_p(u,v) = \Pi^*(d\omega)_p\,(u,v) \\ &\qquad\qquad\qquad\qquad\qquad\qquad = d\omega_q\ (d\Pi(p).u, d\Pi(p).v)\ .\end{aligned}$$

Since Π is a submersion, it follows that $d\Omega_p \neq 0$. Therefore, the coefficients of Ω must be of degree one and $\mathcal{F}$ has degree zero, as asserted in Corollary 1.11. $\square$

3.2. Proof of corollary 1.12.

The idea is to use Corollary 1.11. Assume that there exists an integrable 2-form η on $\mathbb{C}^n$, $n \geqslant 4$, with homogeneous coefficients of degree $d \geqslant 1$ and such that $Sing(\eta) = \{0\}$. Denote by $\mathcal{F}_\eta$ the holomorphic codimension two foliation of $\mathbb{C}^n$ generated by η. By assumption $Sing(\mathcal{F}_\eta) = \{0\}$. Note also that the codimension two distribution of $\mathbb{C}^n \setminus \{0\}$ tangent to $\mathcal{F}_\eta$ is given by

$$ker(\eta)(p) = \{v \in T_p\mathbb{C}^n \mid i_v\,\eta(p) = 0\}\ ,\ \forall\ p \neq 0\,,$$

where i_v denotes the interior product. The fact that $ker(\eta)$ has codimension two is equivalent to

$$\eta \wedge \eta = 0\,. \tag{3.1}$$

Let $\omega = i_R\eta$, where $R = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$ is the radial vector field on $\mathbb{C}^n$. We have two possibilities: either $\omega \equiv 0$, or $\omega \not\equiv 0$.

In the first case, η generates a codimension two foliation on $\mathbb{P}^{n-1}$: there exists a codimension two foliation $\mathcal{F}$ on $\mathbb{P}^{n-1}$ such that $\Pi^*(\mathcal{F}) = \mathcal{F}_\eta$, where $\Pi\colon \mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ denotes the canonical projection. However, any codimension two foliation on $\mathbb{P}^{n-1}, n \geqslant 4$, has at least one singularity: if $q \in Sing(\mathcal{F})$ then the line $\overline{\Pi^{-1}(q)} \subset \mathbb{C}^n$ is contained in the singular set of η.

In the second case ω is a 1-form on $\mathbb{C}^n$ with homogeneous coefficients of degree $d+1$.

Lemma 3.2. — *The form ω is integrable: $\omega \wedge d\omega = 0$.*

Proof. — The following is equivalent to the integrability of the distribution $ker(\eta)$:

(I) for any $p \in \mathbb{C}^n \setminus \{0\}$ there exists a germ coordinate system $(x, y)\colon (\mathbb{C}^n, p) \to (\mathbb{C}^2, 0) \times (\mathbb{C}^{n-2}, 0)$, with $x = (x_1, x_2)$, such that $\eta_p = \varphi(x, y)\,dx_1 \wedge dx_2$, where η_p is the germ of η at p and $\varphi \in \mathcal{O}_p^*$.

Since the coefficients of η are homogeneous of degree d we have $L_R\eta = (d+2)\,\eta$, where L_R denotes the Lie derivative in the direction of R. From this we get

$$\begin{aligned}(d+2)\,\eta = L_R\eta = i_Rd\eta + d\,i_R\eta = i_Rd\eta + d\omega \\ \Longrightarrow\ \omega \wedge d\omega = i_R\eta \wedge d\omega = (d+2)\,i_R\eta \wedge \eta - i_R\eta \wedge i_Rd\eta\,.\end{aligned}$$

Now, from (3.1) we get

$$0 = i_R(\eta \wedge \eta) = 2\,i_R\eta \wedge \eta = 2\,\omega \wedge \eta\ \Longrightarrow\ \omega \wedge d\omega = -i_R\eta \wedge i_Rd\eta\,.$$

If we consider a coordinate system as in (I) we have $\eta_p = \varphi\, dx_1 \wedge dx_2$ and $d\eta_p = d\varphi \wedge dx_1 \wedge dx_2$ and this implies that $i_R\eta_p \wedge i_R d\eta_p = 0$, as the reader can check. □

Write $\omega = \phi.\, \omega_1$, where ϕ is homogeneous and $cod(Sing(\omega_1)) \geqslant 2$.

Remark 3.3. — Note that:

(a) $\omega_1 \wedge \eta = 0$. This is a consequence of $\omega \wedge \eta = 0$.

(b) $\omega_1 \wedge d\omega_1 = 0$ and $i_R\omega_1 = 0$. This is a consequence of $\omega \wedge d\omega = 0$ and $i_R\omega = 0$.

Denote by $\mathcal{F}_{\omega_1}$ the foliation generated by ω_1. It follows from (b) of Remark 3.3 that there exists a codimension one foliation $\mathcal{F}$ on $\mathbb{P}^{n-1}$ such that $\Pi^*(\mathcal{F}) = \mathcal{F}_{\omega_1}$.

Lemma 3.4. — *All irreducible components of $Sing(\mathcal{F})$ are l.t.p.*

Proof. — Fix $q \in Sing(\mathcal{F})$ and $p \in \mathbb{C}^n \setminus \{0\}$ with $\Pi(p) = q$. Note that $p \in Sing(\mathcal{F}_{\omega_1})$, the foliation generated by ω_1. Let $(x, y)\colon (\mathbb{C}^n, p) \to (\mathbb{C}^2, 0) \times (\mathbb{C}^{n-2}, 0)$ be as in (I), so that $\eta = \varphi.\, dx_1 \wedge dx_2$, $\varphi \in \mathcal{O}_p^*$. It follows from $\omega_1 \wedge \eta = 0$ that in these coordinates we have $\omega_1 = A(x, y)\, dx_1 + B(x, y)\, dx_2$ and from $\omega_1 \wedge d\omega_1 = 0$ that

$$(A\, dB - B\, dA) \wedge dx_1 \wedge dx_2 = 0 \implies \omega_1 = h(x, y).\, (C(x_1, x_2)\, dx_1 + D(x_1, x_2)\, dx_2)\ .$$

Since $cod(Sing(\omega_1)) \geqslant 2$ we get $h \in \mathcal{O}_p^*$ and the germ of $Sing(\omega_1)$ at p is defined by $(x_1 = x_2 = 0)$. Moreover, the germ of $\mathcal{F}_{\omega_1}$ at p is defined by the form $C(x_1, x_2)\, dx_1 + D(x_1, x_2)\, dx_2$ and so $\mathcal{F}_{\omega_1}$ is a transversely product at p. Since $p \in \Pi^{-1}(q)$ and Π is a submersion at p, $\mathcal{F}$ is a transversely product at q. □

Corollary 1.11 implies that ω_1 has a linear rational first integral that we can assume to be x_2/x_1, so that $\omega_1 = x_1\, dx_2 - x_2\, dx_1 = i_R(dx_1 \wedge dx_2)$. Let $\eta = \sum_{i<j} \eta_{ij}\, dx_i \wedge dx_j$, where η_{ij} is homogeneous of degree d, $\forall\ i < j$. From $\omega_1 \wedge \eta = 0$ we get $\eta_{ij} = 0$, $\forall\ j > i \geqslant 3$. Therefore, we can write $\eta = dx_1 \wedge \alpha + dx_2 \wedge \beta + \gamma\, dx_1 \wedge dx_2$, where $\alpha = \sum_{j\geqslant 3} \eta_{1j}\, dx_j$, $\beta = \sum_{j\geqslant 3} \eta_{2j}\, dx_j$ and $\gamma = \eta_{12}$. Hence,

$$0 = \omega_1 \wedge \eta = (x_1\, dx_2 - x_2\, dx_1) \wedge (\alpha \wedge dx_1 + \beta \wedge dx_2 + \gamma\, dx_1 \wedge dx_2) \implies$$
$$(x_1\, \alpha + x_2\, \beta) \wedge dx_1 \wedge dx_2 = 0 \implies x_1\, \alpha = -x_2\, \beta \implies$$

there exists 1-form μ with homogeneous coefficients of degree $d - 1$ such that $\alpha\alpha = -x_2\, \mu$ and $\beta = x_1\, \mu$. In particular, we get

$$\eta = \omega_1 \wedge \mu + \gamma\, dx_1 \wedge dx_2 = (x_1\, dx_2 - x_2\, dx_1) \wedge \mu + \gamma\, dx_1 \wedge dx_2 \implies$$
$$Sing(\eta)\sup(x_1 = x_2 = \gamma = 0) \implies$$

$d = 0$ and γ is a constant, for otherwise $cod(Sing(\eta)) \leqslant 3$ and $Sing(\eta) \supsetneq \{0\}$. This finishes the proof of Corollary 1.12. □

BIBLIOGRAPHY

[Bru00] Marco Brunella, *Birational geometry of foliations*, Monografías de Matemática, IMPA – Instituto de Matemática Pura e Aplicada, 2000. ↑489

[Bru09] ______, *Sur les feuilletages de l'espace projectif ayant une composante de Kupka*, Enseign. Math. **55** (2009), no. 3-4, 227–234. ↑485, 487

[CA99] Omegar Calvo-Andrade, *Foliations with a Kupka component on algebraic manifolds*, Bull. Braz. Math. Soc. (N.S.) **30** (1999), no. 2, 183–197. ↑485, 487

[CA09] ———, *Foliations of codimension greater than one with a Kupka component*, Qual. Theory Dyn. Syst. **8** (2009), no. 2, 241–253. ↑488

[CA16] ———, *Foliations with a radial Kupka set on projective spaces*, Bull. Braz. Math. Soc. (N.S.) **47** (2016), no. 4, 1071–1083. ↑485, 487

[CJCAFP14] Mauricio Corrêa Jr., Omegar Calvo-Andrade, and Arturo Fernández-Pérez, *Highter codimension foliations and Kupka singularities*, `https://arxiv.org/abs/1408.7020`, 2014. ↑488

[CLN94] Dominique Cerveau and Alcides Lins Neto, *Codimension-one foliations in $\mathbb{C}p^n$, $n \geqslant 3$ with Kupka components*, Complex analytic methods in dynamical systems. Proceedings of the congress held at Instituto de Matemática Pura e Aplicada, IMPA, Rio de Janeiro, Brazil, January 1992, Astérisque, vol. 222, Société Mathématique de France, 1994, pp. 93–132. ↑487, 499

[CLN96] ———, *Irreducible components of the space of holomorphic foliations of degree two in $\mathbb{CP}(n)$, $n \geqslant 3$*, Ann. Math. **143** (1996), no. 3, 577–612. ↑488

[CLN13] ———, *A structural theorem for codimension-one foliations on $\mathbb{P}^n$, $n \geqslant 3$, with an application to degree three foliations*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. **12** (2013), no. 1, 1–41. ↑489, 498

[CLN19] ———, *Codimension two holomorphic foliations*, J. Differ. Geom. **113** (2019), no. 3, 385–416. ↑488

[CS82] Cesar Camacho and Paulo Sad, *Invariant varieties through singularities of holomorphic vector fields*, Ann. Math. **115** (1982), 579–595. ↑489, 490, 493

[DN13] Krishanu Dan and Donihakkalu Shankar Nagaraj, *Null correlation bundle on projective three space*, J. Ramanujan Math. Soc. **28A** (2013), 75–80. ↑488

[GML92] Xavier Gómez-Mont and I. Luengo, *Germs of holomorphic vector fields in $\mathbb{C}^3$ without a separatrix*, Invent. Math. **109** (1992), no. 2, 211–219. ↑489

[LN99] Alcides Lins Neto, *A note on projective Levi flats and minimal sets of algebraic foliations*, Ann. Inst. Fourier **49** (1999), no. 4, 1369–1385. ↑499

[Seb97] Marcos Sebastiani, *Sur l'existence de séparatrices locales des feuilletages des surfaces*, Anais Acad. Brasil. Ci. **69** (1997), no. 2, 159–162. ↑496

[Sei68] Abraham Seidenberg, *Reduction of singularities of the differential equation $Ady = Bdx$*, Am. J. Math. **90** (1968), 248–269. ↑492

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