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MATEUSZ KWAŚNICKI

RANDOM WALKS ARE DETERMINED BY THEIR TRACE ON THE POSITIVE HALF-LINE

LES MARCHES ALÉATOIRES SONT DÉTERMINÉES PAR LEUR TRACE SUR LA DEMI-DROITE POSITIVE

Abstract. — We prove that the law of a random walk X_n is determined by the one-dimensional distributions of $\max(X_n, 0)$ for $n = 1, 2, \ldots$, as conjectured recently by Loïc Chaumont and Ron Doney. Equivalently, the law of X_n is determined by its upward space-time Wiener–Hopf factor. Our methods are complex-analytic.

Résumé. — Nous démontrons que la loi d'une marche aléatoire X_n est déterminée par les distributions de $\max(X_n, 0)$ pour $n = 1, 2, \ldots$, comme l'avaient conjecturé récemment Loïc Chaumont et Ron Doney. De manière équivalente, la loi de X_n est déterminée par son facteur de Wiener–Hopf espace-temps ascendant. Nos méthodes relèvent de l'analyse complexe.

1. Introduction and main result

In this note we give an affirmative answer to a question posed by Loïc Chaumont and Ron Doney in [CD20], inspired by Vincent Vigon's conjecture in [Vig01]. The

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main result was previously stated without proof in a more general form in [OU90], and an erroneous proof was given in [Ula92].

A random walk X_n is said to be *non-degenerate* if $\mathbb{P}(X_n > 0) \neq 0$. Similarly, a finite signed Borel measure μ on $\mathbb{R}$ is said to be *non-degenerate* if the restriction of μ to $(0, \infty)$ is a non-zero measure.

Theorem 1.1. — *If X_n and Y_n are non-degenerate random walks such that $\max(X_n, 0)$ and $\max(Y_n, 0)$ are equal in distribution for all $n = 1, 2, \ldots$, then X_n and Y_n are equal in distribution for $n = 1, 2, \ldots$*

More generally, if μ and ν are non-degenerate finite signed Borel measures and their n-fold convolutions μ^{*n} and ν^{*n} agree on $(0, \infty)$ for $n = 1, 2, \ldots$, then $\mu = \nu$.

Following [CD20], we remark that various reformulations of the above result are possible. A non-degenerate random walk X_n is determined by any of the following objects:

- The law of the ascending ladder process (T_k, S_k); here $S_k = X_{T_k}$ is the k^{th} running maximum of the random walk.
- The upward space-time Wiener–Hopf factor $\Phi_+(q, \xi)$, that is, the characteristic function of (T_1, S_1).
- The distributions of the running maxima $\max(0, X_1, X_2, \ldots, X_n)$ for all $n = 1, 2, \ldots$

Theorem 1.1 clearly implies that a non-degenerate Lévy process X_t is determined by any of the following objects:

- The distributions of $\max(X_t, 0)$ for all $t > 0$ (or even for $t = 1, 2, \ldots$).
- The law of the ascending ladder process (T_t, S_t).
- The upward space-time Wiener–Hopf factor $\kappa_+(q, \xi)$, that is, the characteristic exponent of (T_t, S_t).
- The distributions of the running suprema $\sup\{X_s : s \in [0, t]\}$ for all $t > 0$.

For further discussion, we again refer to [CD20], where Theorem 1.1 was proved under various relatively mild additional conditions. For related research, see [CD20, LMS76, Ost85, OU90, Ula90, Ula92] and the references therein.

Theorem 1.1 was given without proof in [OU90] in a more general form: Theorem 4 therein claims that $\mu = \nu$ if μ and ν are non-degenerate finite Borel measures on $\mathbb{R}$ and the restrictions of μ^{*n_k} and ν^{*n_k} to $(0, \infty)$ are equal for $k = 1, 2, \ldots$, where $n_1 = 1$ and $n_2 - 1, n_3 - 1, \ldots$ are distinct and have no common divisor other than 1. Noteworthy, this result is stated for measures on the Euclidean space of arbitrary dimension, and their restrictions to the half-space. A proof is given in [Ula92] under the additional condition $n_2 = 2$, and only in dimension one. However, the argument in [Ula92] contains a gap, that we describe at the end of this article.

2. Proof

All measures considered below are finite, signed Borel measures. For a measure μ on $\mathbb{R}$, we denote the restrictions of μ to $(0, \infty)$ and $(-\infty, 0]$ by $\mu_+ = \mathbb{1}_{(0,\infty)}\, \mu$ and $\mu_- = \mathbb{1}_{(-\infty,0]}\, \mu$. This should not be confused with the Hahn decomposition of μ into the positive and negative part. By μ^{*n} we denote the n-fold convolution of μ, and we define μ^{*0} to be the Dirac measure δ_0. For brevity, we write $\mu_\pm^{*n} = (\mu_\pm)^{*n}$, as opposed to $(\mu^{*n})_\pm$. We record the following elementary identities: $(\delta_0 * \sigma_-)_+ = (\sigma_-)_+ = 0$, $(\pi_- * \sigma_-)_+ = 0$, and $(\pi * \sigma_-)_+ = (\pi_+ * \sigma_-)_+$.

We denote the characteristic function of a measure μ by $\widehat{\mu}$:

$$\widehat{\mu}(z) = \int_{\mathbb{R}} e^{izx}\mu(dx)$$

for $z \in \mathbb{R}$, and also for those $z \in \mathbb{C}$ for which the integral converges. We recall that $\widehat{\mu}_+$ is a bounded holomorphic function in the upper complex half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, continuous on the boundary. Similarly, $\widehat{\mu}_-$ is a bounded holomorphic function on the lower complex half-plane $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$.

Lemma 2.1. — *Suppose that μ, ν are measures on $\mathbb{R}$ satisfying*

$$(\mu^{*n})_+ = (\nu^{*n})_+ \quad \text{for } n = 1, 2, \ldots, N.$$

Then $\mu_+ = \nu_+$ and

$$\left(\mu_+^{*n} * \mu_-^{*k}\right)_+ = \left(\nu_+^{*n} * \nu_-^{*k}\right)_+ \quad \text{for } n = 0, 1, \ldots, N-1 \text{ and } k = 1, 2, \ldots \tag{2.1}$$

Proof. — We proceed by induction with respect to N. For $N = 1$ the result is trivial: we have $\mu_+ = (\mu^{*1})_+ = (\nu^{*1})_+ = \nu_+$ and $(\mu_+^{*0} * \mu_-^{*k})_+ = (\delta_0 * \mu_-^{*k})_+ = 0 = (\delta_0 * \nu_-^{*k})_+ = (\nu_+^{*0} * \nu_-^{*k})_+$ for $k = 1, 2, \ldots$ Suppose that the assertion of Lemma 2.1 holds for some N, and suppose that $(\mu^{*n})_+ = (\nu^{*n})_+$ for $n = 1, 2, \ldots, N$, $N + 1$. By the induction hypothesis, formula (2.1) holds for $n = 0, 1, \ldots, N - 1$ and $k = 1, 2, \ldots$, and we have $\mu_+ = \nu_+$. Therefore, we only need to prove (2.1) for $n = N$ and $k = 1, 2, \ldots$

By the binomial theorem,

$$\begin{aligned} 0 &= \left(\mu^{*N+1} - \nu^{*N+1}\right)_+ = \left((\mu_+ + \mu_-)^{*N+1} - (\nu_+ + \nu_-)^{*N+1}\right)_+ \\ &= \sum_{j=0}^{N+1} \binom{N+1}{j} \left(\mu_+^{*j} * \mu_-^{*N+1-j} - \nu_+^{*j} * \nu_-^{*N+1-j}\right)_+ . \end{aligned}$$

We already know that $\mu_+^{*N+1} = \nu_+^{*N+1}$ and $(\mu_+^{*j} * \mu_-^{*N+1-j})_+ = (\nu_+^{*j} * \nu_-^{*N+1-j})_+$ for $j = 1, 2, \ldots, N-1$. Furthermore, $(\mu_-^{*N+1})_+ = 0 = (\nu_-^{*N+1})_+$. It follows that all terms corresponding to $j \neq N$ in the above sum are zero. Thus,

$$0 = \binom{N+1}{N} \left(\mu_+^{*N} * \mu_- - \nu_+^{*N} * \nu_-\right)_+ ,$$

which proves (2.1) for $n = N$ and $k = 1$. The proof for $n = N$ and $k > 1$ proceeds again by induction. Suppose that (2.1) holds for $n = N$ and $k = 1, 2, \ldots, K$. By the identity $(\pi * \sigma_-)_+ = (\pi_+ * \sigma_-)_+$,

$$\left(\mu_+^{*N} * \mu_-^{*K+1}\right)_+ = \left(\mu_+^{*N} * \mu_-^{*K} * \mu_-\right)_+ = \left(\left(\mu_+^{*N} * \mu_-^{*K}\right)_+ * \mu_-\right)_+ .$$

Applying (2.1), with $n = N$ and $k = K$, and then again the identity $(\pi * \sigma_-)_+ = (\pi_+ * \sigma_-)_+$, we find that

$$\left(\left(\mu_+^{*N} * \mu_-^{*K}\right)_+ * \mu_-\right)_+ = \left(\left(\nu_+^{*N} * \nu_-^{*K}\right)_+ * \mu_-\right)_+ = \left(\nu_+^{*N} * \nu_-^{*K} * \mu_-\right)_+ .$$

Recall that $\mu_+ = \nu_+$, so that

$$\left(\nu_+^{*N} * \nu_-^{*K} * \mu_-\right)_+ = \left(\mu_+^{*N} * \mu_- * \nu_-^{*K}\right)_+ .$$

We use the identity $(\pi * \sigma_-)_+ = (\pi_+ * \sigma_-)_+$ for the third time, and then we again apply (2.1), with $n = N$ and $k = 1$:

$$\left(\mu_+^{*N} * \mu_- * \nu_-^{*K}\right)_+ = \left(\left(\mu_+^{*N} * \mu_-\right)_+ * \nu_-^{*K}\right)_+ = \left(\left(\nu_+^{*N} * \nu_-\right)_+ * \nu_-^{*K}\right)_+ .$$

Finally, once again we apply the identity $(\pi * \sigma_-)_+ = (\pi_+ * \sigma_-)_+$:

$$\left(\left(\nu_+^{*N} * \nu_-\right)_+ * \nu_-^{*K}\right)_+ = \left(\nu_+^{*N} * \nu_- * \nu_-^{*K}\right)_+ = \left(\nu_+^{*N} * \nu_-^{*K+1}\right)_+ .$$

The above chain of equalities implies that $(\mu_+^{*N} * \mu_-^{*K+1})_+ = (\nu_+^{*N} * \nu_-^{*K+1})_+$, which is just (2.1) with $n = N$ and $k = K + 1$. We conclude that (2.1) holds for $n = N$ and every $k = 1, 2, \ldots$, and the proof of Lemma 2.1 is complete. □

A holomorphic function f on $\mathbb{C}_-$ is said to be of *bounded type* (or belong to the *Nevanlinna class*) if $\log|f(x)|$ has a harmonic majorant on $\mathbb{C}_-$. Equivalently, f is of bounded type if it is a ratio of two bounded holomorphic functions on $\mathbb{C}_-$. We recall the following fundamental factorisation theorem for holomorphic functions on $\mathbb{C}_-$ which are bounded or of bounded type, and we refer to [Gar07, Mas09] for further details.

Theorem 2.2. — [Gar07, Theorem II.5.5 and Corollary II.5.7]; [Mas09, Theorem 13.15]

Let f be a holomorphic function of bounded type on the lower complex half-plane, and suppose that f is not identically zero. Let α_0 be the multiplicity of the zero of f at $z = -i$ (possibly $\alpha_0 = 0$), and let $z_1, z_2, \ldots$ be the (finite or infinite) sequence of all zeros of f in the lower complex half-plane, with corresponding multiplicities $\alpha_1, \alpha_2, \ldots$ Then f admits a factorisation

$$f(z) = f_{\mathrm{b}}(z) f_{\mathrm{o}}(z) f_{\mathrm{s}}(z) \tag{2.2}$$

(unique, up to multiplication of f_{o} and f_{s} by a constant of modulus 1), with the following factors. The function f_{b} is a Blaschke product, *determined uniquely by the zeros of f:*

$$f_{\mathrm{b}}(z) = \left(\frac{z+i}{z-i}\right)^{\alpha_0} \prod_j \left(\frac{|1+z_j^2|}{1+z_j^2} \, \frac{z-z_j}{z-\bar{z}_j}\right)^{\alpha_j} . \tag{2.3}$$

The function f_{o} is an outer function, *a holomorphic function determined uniquely up to multiplication by a constant of modulus 1 by the formula:*

$$|f_{\mathrm{o}}(z)| = \exp\left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{-\operatorname{Im} z}{|z-x|^2} \log|f(x)| \, dx\right). \tag{2.4}$$

Finally, the function f_{s} is a singular inner function, *a holomorphic function determined uniquely up to multiplication by a constant of modulus 1 by the expression:*

$$|f_{\mathrm{s}}(z)| = \exp\left(a \operatorname{Im} z - \frac{1}{\pi} \int_{\mathbb{R}} \frac{-\operatorname{Im} z}{|z-x|^2} \, \sigma(dx)\right), \tag{2.5}$$

where $a \in \mathbb{R}$ is a constant and σ is a signed measure, singular with respect to the Lebesgue measure.

Furthermore, for almost all $x \in \mathbb{R}$ with respect to both the Lebesgue measure and the measure σ, the limit $f(x)$ of $f(x+iy)$ as $y \to 0^-$ exists. This boundary limit $f(x)$ is non-zero almost everywhere with respect to the Lebesgue measure and zero almost everywhere with respect to σ. The symbol $f(x)$ used in the definition of the outer function f_o refers precisely to this boundary limit. Additionally, we have

$$\sum_j \alpha_j |\operatorname{Im} z_j| \left(1+|z_j|^2\right)^{-1} < \infty, \int_{-\infty}^{\infty} \left(1+x^2\right)^{-1} |\log|f(x)||dx < \infty$$

and

$$\int_{\mathbb{R}} \left(1+x^2\right)^{-1} |\sigma|(dx) < \infty\,,$$

and any parameters α_j, z_j, a, σ and boundary values $|f(x)|$, $x \in \mathbb{R}$, which satisfy these conditions, correspond to some function f of bounded type.

Finally, f is a bounded holomorphic function in the lower complex half-plane if and only if $a \geqslant 0$, σ is a non-negative measure and the boundary values $|f(x)|$ are bounded for $x \in \mathbb{R}$.

Lemma 2.3. — *Suppose that μ is a measure on $\mathbb{R}$ such that μ_- is a non-zero measure and $(\mu_+ * \mu_-)_+ = 0$. Then $\widehat{\mu}_+$ has a holomorphic extension φ to the connected open set*

$$D = \mathbb{C} \setminus \{z \in \mathbb{C}_- \cup \mathbb{R} : \widehat{\mu}_-(z) = 0\}\,,$$

and φ is a meromorphic function on $\mathbb{C} \setminus \{z \in \mathbb{R} : \widehat{\mu}_-(z) = 0\}$. Furthermore, $\varphi\widehat{\mu}_-$ extends to a function which is holomorphic on $\mathbb{C}_-$ and continuous on $\mathbb{C}_- \cup \mathbb{R}$, namely, the characteristic function of $\mu_+ * \mu_-$.

Proof. — Denote $\nu = \mu_+ * \mu_-$; by the assumption, $\nu = \nu_-$. Let $f = \widehat{\mu}_+$, $g = \widehat{\mu}_-$ and $h = \widehat{\nu} = \widehat{\nu}_-$. Clearly, $h(z) = f(z)g(z)$ for $z \in \mathbb{R}$. Let

$$A = \{z \in \mathbb{R} : g(z) = 0\}, \quad B = \{z \in \mathbb{C}_- : g(z) = 0\},$$

so that $D = \mathbb{C} \setminus (A \cup B)$.

We note basic properties of A and B. By continuity of g, A and $A \cup B$ are closed sets, and D is an open set. Since g is holomorphic on $\mathbb{C}_-$ (and not identically zero), B is a countable (possibly finite) set with no accumulation points on $\mathbb{C}_-$. By Theorem 2.2, A has zero Lebesgue measure (as a subset of $\mathbb{R}$). In particular, D is connected. Indeed: the sets $D \cap \mathbb{C}_+ = \mathbb{C}_+$ and $D \cap \mathbb{C}_- = \mathbb{C}_- \setminus B$ are clearly path-connected, the set $D \cap \mathbb{R} = \mathbb{R} \setminus A$ is non-empty, and since D is open, each point of $D \cap \mathbb{R}$ is path-connected with points from both $D \cap \mathbb{C}_+$ and $D \cap \mathbb{C}_-$.

We define a function φ on D by the formula

$$\varphi(z) = \begin{cases} f(z) & \text{if } z \in \mathbb{C}_+ \cup (\mathbb{R} \setminus A), \\ \dfrac{h(z)}{g(z)} & \text{if } z \in \mathbb{C}_- \setminus B\,. \end{cases}$$

By definition, φ is holomorphic both on $\mathbb{C}_+$ and on $\mathbb{C}_- \setminus B$, as well as meromorphic on $\mathbb{C}_-$. Furthermore, φ is continuous at each point $z \in \mathbb{R} \setminus A$, because both f (defined on $\mathbb{C}_+ \cup \mathbb{R}$) and h/g (defined on $(\mathbb{C}_- \setminus B) \cup (\mathbb{R} \setminus A)$) are continuous at

z and $f(z) = h(z)/g(z)$. By a standard application of Morera's theorem (see [Con73, Theorem IV.5.10 and Exercise IV.5.9], or [Gar07, Exercise II.12]), φ is holomorphic in D. It remains to note that $\varphi(z)g(z) = h(z)$ for $z \in \mathbb{C}_- \setminus B$. $\square$

LEMMA 2.4. — *If μ is a measure on $\mathbb{R}$ such that $(\mu_+^{*n} * \mu_-)_+ = 0$ for all $n = 1, 2, \ldots$, then either μ_+ or μ_- is a zero measure.*

Proof. — Let μ be such a measure, and suppose that both μ_+ and μ_- are non-zero measures. Let $\varphi, f, g, h, A, B, D$ be as in the proof of Lemma 2.3. Clearly, φ^n is the holomorphic extension of f^n, the characteristic function of μ_+^{*n}. An application of Lemma 2.3 to the measure $\mu_+^{*n} + \mu_-$ implies that for all $n = 1, 2, \ldots$, the function $\varphi^n g$ extends from $\mathbb{C}_- \setminus B$ to a function h_n which is bounded and holomorphic on $\mathbb{C}_-$ and continuous on $\mathbb{C}_- \cup \mathbb{R}$, namely, h_n is the characteristic function of $\mu_+^{*n} * \mu_-$.

Consider the factorisations $g = g_{\mathrm{b}}\, g_{\mathrm{o}}\, g_{\mathrm{s}}$ and $h_n = h_{n,\mathrm{b}}\, h_{n,\mathrm{o}}\, h_{n,\mathrm{s}}$ given in Theorem 2.2, and let σ_g, a_g and $\sigma_{h,n}$, $a_{h,n}$ denote the corresponding non-negative measures σ and constants a for g and h_n, respectively. Note that Theorem 2.2 applies both to g and to $h_n = \varphi^n g$, as these functions are not identically zero: f and g are characteristic functions of non-zero measures μ_+ and μ_-, while h_n is the product of g and the holomorphic extension of f^n.

Recall that $\varphi^n = h_n/g$ on $\mathbb{C}_- \setminus B$. It follows that if $\varphi_{n,\mathrm{b}} = h_{n,\mathrm{b}}/g_{\mathrm{b}}$, $\varphi_{n,\mathrm{o}} = h_{n,\mathrm{o}}/g_{\mathrm{o}}$ and $\varphi_{n,\mathrm{s}} = h_{n,\mathrm{s}}/g$, then

$$\varphi^n = \varphi_{n,\mathrm{b}}\, \varphi_{n,\mathrm{o}}\, \varphi_{n,\mathrm{s}}$$

on $\mathbb{C}_- \setminus B$. Let us examine the above factors in more detail.

By definition, $\varphi_{n,\mathrm{o}}$ and $\varphi_{n,\mathrm{s}}$ have no zeros in $\mathbb{C}_-$. This means that if $z_0 \in \mathbb{C}_-$ is a pole of φ of order α_0, then z_0 is a pole of $\varphi_{n,\mathrm{b}} = h_{n,\mathrm{b}}/g_{\mathrm{b}}$ of order $n\alpha_0$, and therefore g_{b} has a zero at z_0 of multiplicity at least $n\alpha_0$ for all $n = 1, 2, \ldots$ Since all zeroes of g_{b} have finite multiplicity, φ has no poles in $\mathbb{C}_-$. In particular, φ extends to a holomorphic function on $\mathbb{C} \setminus A$, which will be denoted again by φ, and $\varphi_{n,\mathrm{b}} = h_{n,\mathrm{b}}/g_{\mathrm{b}}$ has no poles in $\mathbb{C}_-$. Therefore, the zeros of $h_{n,\mathrm{b}}$ must cancel the zeros of g_{b}, and $\varphi_{n,\mathrm{b}}$ is a Blaschke product.

Since $h_n(x)/g(x) = (f(x))^n$ for $x \in \mathbb{R} \setminus A$ and A has Lebesgue measure zero, we have

$$\begin{aligned} |\varphi_{n,\mathrm{o}}(z)| &= \exp\left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{-\operatorname{Im} z}{|z-x|^2} \left(\log|h_n(x)| - \log|g(x)|\right) dx\right) \\ &= \exp\left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{-\operatorname{Im} z}{|z-x|^2} \log|f(x)|^n \, dx\right). \end{aligned}$$

In particular, $\varphi_{n,\mathrm{o}}$ is a bounded outer function, namely, the outer function in the factorisation of the bounded holomorphic function $(\overline{f(\bar{z})})^n$ on the lower complex half-plane.

Finally $\varphi_{n,\mathrm{s}}$ is the ratio of two singular inner functions, and hence a singular inner function. If we denote $a_{\varphi,n} = a_{h,n} - a_g$ and $\sigma_{\varphi,n} = \sigma_{h,n} - \sigma_g$, then

$$|\varphi_{n,\mathrm{s}}(z)| = \exp\left(-a_{\varphi,n} \operatorname{Im} z - \frac{1}{\pi} \int_{\mathbb{R}} \frac{-\operatorname{Im} z}{|z-x|^2}\, \sigma_{\varphi,n}(dx)\right).$$

The above properties imply that φ^n is of bounded type, and therefore the factors $\varphi_{n,\mathrm{b}}$, $\varphi_{n,\mathrm{o}}$, $\varphi_{n,\mathrm{s}}$, the signed measure $\sigma_{\varphi,n}$ and the constant $a_{\varphi,n} \in \mathbb{R}$ are uniquely determined (up to multiplication by a constant of modulus 1 in case of $\varphi_{n,\mathrm{o}}$ and $\varphi_{n,\mathrm{s}}$).

By comparing the factorisations of φ and φ^n, we find that $\varphi_{n,\mathrm{s}} = c_n(\varphi_{1,\mathrm{s}})^n$ for some constant c_n with modulus 1. It follows that $a_{\varphi,n} = na_{\varphi,1}$ and $\sigma_{\varphi,n} = n\sigma_{\varphi,1}$. This, however, implies that $a_{\varphi,1} = \frac{1}{n}a_{\varphi,n} \geqslant -\frac{1}{n}a_g$ for all $n = 1, 2, \ldots$, and so $a_{\varphi,1} \geqslant 0$. Similarly, the negative part of $\sigma_{\varphi,1} = \frac{1}{n}\sigma_{\varphi,n}$ is dominated by $\frac{1}{n}\sigma_g$ for any $n = 1, 2, \ldots$ This is not possible if the negative part of $\sigma_{\varphi,1}$ is non-zero, and therefore $\sigma_{\varphi,1}$ is a non-negative measure. We conclude that $\varphi = \varphi_{1,\mathrm{b}}\,\varphi_{1,\mathrm{o}}\,\varphi_{1,\mathrm{s}}$ is a bounded holomorphic function on $\mathbb{C}_-$.

Since $\varphi = f$ on $\mathbb{C}_+$ and f is a bounded holomorphic function on $\mathbb{C}_+$, we have proved that φ is a bounded holomorphic function on $\mathbb{C} \setminus A$. However, A has zero Lebesgue measure (as a subset of $\mathbb{R}$). By Painlevé's theorem (see [You15, Theorem 2.7]), φ extends to a bounded holomorphic function on $\mathbb{C}$. This, in turn, implies that φ is constant, and so $\widehat{\mu}_+$ is constant, contradicting the assumption that μ_+ is a non-zero measure on $(0, \infty)$. ☐

Proof of Theorem 1.1. — Suppose that $(\mu^{*n})_+ = (\nu^{*n})_+$ for $n = 1, 2, \ldots$ for some measures μ and ν such that μ_+ and ν_+ are non-zero measures. By Lemma 2.1, $\mu_+ = \nu_+$ and $(\mu_+^{*n} * \mu_-)_+ = (\nu_+^{*n} * \nu_-)_+$ for $n = 1, 2, \ldots$ Let $\eta = \mu_+ + \mu_- - \nu_-$, so that $\eta_+ = \mu_+ = \mu_-$ and $\eta_- = \mu_- - \nu_-$. Then $(\eta_+^{*n} * \eta_-)_+ = 0$ for $n = 1, 2, \ldots$, and therefore, by Lemma 2.4, either η_+ or η_- is a zero measure. Since $\eta_+ = \mu_+$ is a non-zero measure, we must have $\eta_- = 0$, that is, $\mu_- = \nu_-$. ☐

3. An error in [Ula92]

In [Ula92] an analogue of Theorem 1.1 is given, with equality of μ^{*n} and ν^{*n} on $(-\infty, 0)$ rather than on $(0, \infty)$. In [Ula92, Page 3001, line 16], it is claimed that the measures μ and ν satisfy [Ula92, condition (B) of Theorem A], as a consequence of the results of [LO77, Section 11.2]. This reasoning would have been correct if the holomorphic extensions of $\widehat{\mu}$ and $\widehat{\nu}$ to the upper complex half-plane had been known to be continuous on the boundary. However, this is not verified in [Ula92].

More precisely, it is observed in [Ula92] that $\widehat{\mu} = (\widehat{\chi}_2 - (\widehat{\chi}_1)^2)/(2\widehat{\chi}_1)$ almost everywhere on $\mathbb{R}$, where $\chi_1 = \mu - \nu$ and $\chi_2 = \mu^{*2} - \nu^{*2}$ are measures concentrated on $(0, \infty)$. Since $\widehat{\chi}_1$ and $\widehat{\chi}_2$ extend to holomorphic functions on $\mathbb{C}_+$, $\widehat{\mu}$ extends to a meromorphic function on $\mathbb{C}_+$. Equality of μ^{*n} and ν^{*n} on $(-\infty, 0)$ for $n \geqslant 3$ is used only to show that the extension of $\widehat{\mu}$ has no poles in $\mathbb{C}_+$. However, the extension of $\widehat{\mu}$ can have singularities near $\mathbb{R}$ and thus fail to satisfy [Ula92, condition (B) of Theorem A].

To be specific, observe that $\widehat{\mu}(z) = z^2(z+i)^{-4}\exp(i/z)$ is the characteristic function of a measure μ on $\mathbb{R}$. Namely, μ is the convolution of $\frac{1}{6}x^3e^{-x}\,\mathbb{1}_{(0,\infty)}(x)dx$ and $\frac{1}{6}\,{}_0F_1(4;x)\,\mathbb{1}_{(-\infty,0)}(x)dx - \frac{1}{2}\delta_0(dx) - \delta_0'(dx) - \delta_0''(dx)$ (in the sense of distributions; ${}_0F_1$ is the hypergeometric function; we omit the details). Clearly, $\widehat{\mu}$ extends holomorphically to the upper complex half-plane, but this extension is not continuous on the boundary, and thus μ does not satisfy [Ula92, condition (B) of Theorem A]. Furthermore, $\widehat{\mu}(z)$ is the ratio of two characteristic functions of finite measures supported in $[0,\infty)$: $z^4/(z+i)^8$ and $z^2(z+i)^{-4}\exp(-i/z)$.

The author of the present article was not able to correct the error in [Ula92]. The proof given above uses a related, but essentially different idea.

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Mateusz KWAŚNICKI
Department of Pure Mathematics,
Wrocław University of Science and Technology,
Wybrzeże Wyspiańskiego 27,
50-370 Wrocław (Poland)
mateusz.kwasnicki@pwr.edu.pl