

ANNALES HENRI LEBESGUE

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## CORRIGENDUM: FINITENESS OF POLARIZED K3 SURFACES AND HYPERKÄHLER MANIFOLDS

In the proof of Proposition 2.8 in [Huy18] we consider isometric embeddings

$$
\varphi: T\left(S_{0}\right) \hookrightarrow T\left(S_{0}\right) \oplus \mathbb{Z} \cdot e
$$

with $\varphi(\sigma) \in \mathbb{C} \cdot \sigma \oplus \mathbb{C} \cdot \bar{\sigma} \oplus \mathbb{C} \cdot e$. At this point Lemma 2.5 is evoked, which, however, assumes the more restrictive and unrealistic condition (2.1) $\varphi_{\mathbb{C}}(\sigma)=\lambda \cdot \sigma+\mu \cdot e$, i.e. $\varphi_{\mathbb{C}}(\sigma) \in \mathbb{C} \cdot \sigma \oplus \mathbb{C} \cdot e$.
This is fixed by instead using the following lemma which is proved with essentially the same techniques as the original [Huy18, Lemma 2.5].

Lemma 0.1. - Assume that $\sigma \in T \otimes \mathbb{C}$ defines a general Hodge structure of $K 3$ type on $T$ such that $K_{\sigma}$ is a subfield of a $C M$ field $K$. Then there exist at most finitely many isometric embeddings $\varphi: T \hookrightarrow T \oplus \mathbb{Z} \cdot e$ satisfying

$$
\begin{equation*}
\varphi_{\mathbb{C}}(\sigma)=\lambda \cdot \sigma+\lambda^{\prime} \cdot \bar{\sigma}+\nu \cdot e . \tag{0.1}
\end{equation*}
$$

Proof. - Consider an isometric embedding $\varphi$ satisfying (0.1). We think of $\varphi$ in terms of the integral matrix $\left(a_{i j} \mid b_{j}\right)$, where $\varphi\left(\gamma_{i}\right)=\sum a_{i j} \cdot \gamma_{j}+b_{i} \cdot e$ with $\gamma_{i}$ a basis of $T(S)$ as above. Then writing $\sigma=\sum \mu_{i} \cdot \gamma_{i}$ the image $\varphi_{\mathbb{C}}(\sigma)$ corresponds
to $\left(a_{i j} \mid b_{j}\right) \cdot\left(\mu_{i}\right)$ and (0.1) becomes the system of equations $\sum a_{i j} \mu_{j}=\lambda \mu_{i}+\lambda^{\prime} \bar{\mu}_{i}$, $i=1, \ldots, n$, and $\sum b_{j} \mu_{j}=\nu$. This shows $\lambda, \lambda^{\prime}, \nu \in K$. Indeed, there is at most one solution $\lambda, \lambda^{\prime}$, which then is contained in $K$, unless $\operatorname{det}\binom{\mu_{i} \bar{\mu}_{i}}{\mu_{j} \bar{\mu}_{j}}=0$, i.e. $\mu_{i} \bar{\mu}_{j} \in \mathbb{R}$, for all $i \neq j$. But then $1=\sigma_{1}=\sum\left(\mu_{i} \gamma_{i} . \gamma_{1}\right)$ implies $\bar{\mu}_{j}=\sum \mu_{i} \bar{\mu}_{j}\left(\gamma_{i} . \gamma_{1}\right) \in \mathbb{R}$ for all $j$, which would yield the contradiction $\sigma=\bar{\sigma}$. Eventually use that $K_{\sigma}=\mathbb{Q}\left(\mu_{i}\right)$ and that the CM field $K$ is closed under complex conjugation.
Next observe that, as $a_{i j}, b_{j} \in \mathbb{Z}$, there exists an $N \in \mathbb{Z}$ independent of $\varphi$ such that $N \lambda, N \lambda^{\prime}, N \nu \in \mathcal{O}_{K}$. As $\varphi$ is an isometry, one also has $(\sigma \cdot \bar{\sigma})=(\varphi(\sigma) \cdot \varphi(\bar{\sigma}))$, which translates into

$$
\begin{equation*}
|\lambda|^{2}+\left|\lambda^{\prime}\right|^{2}+|\nu|^{2}(d /(\sigma . \bar{\sigma}))=1 \tag{0.2}
\end{equation*}
$$

As any embedding $g: K \hookrightarrow \mathbb{C}$ commutes with complex conjugation, $g$ applied to (0.2) also shows $|g(\lambda)|^{2}+\left|g\left(\lambda^{\prime}\right)\right|^{2}+|g(\nu)|^{2}(d / g(\sigma . \bar{\sigma}))=1$. Observe that $g(\sigma . \bar{\sigma})>0$ and that, therefore, the last summand is non-negative. Indeed, choose $z \in \mathbb{C}$ such that $(z \sigma, \bar{z} \bar{\sigma})=1$. Then $|g(z)|^{2} g(\sigma . \bar{\sigma})=(g(z \sigma), g(\bar{z} \bar{\sigma}))=1$, as $g$ commutes with complex conjugation. Hence, for $N \lambda, N \lambda^{\prime} \in \mathcal{O}_{K}$ one has $|g(N \lambda)| \leqslant N$ and $\left|g\left(N \lambda^{\prime}\right)\right| \leqslant N$ for all $g: K \hookrightarrow \mathbb{C}$. By Minkowski theory there are only finitely many such $N \lambda, N \lambda^{\prime} \in \mathcal{O}_{K}$.

As there are only finitely many possibilities for $\lambda$ and $\lambda^{\prime}$, it suffices to show that they determine $\varphi$ essentially uniquely. Indeed, if $\varphi$ and $\varphi^{\prime}$ are both isometric embeddings satisfying (0.1) with the same $\lambda$, $\lambda^{\prime}$, then $\sigma \in \operatorname{Ker}\left(\psi-\psi^{\prime}\right) \otimes \mathbb{C}$, where $\psi, \psi^{\prime}: T \longrightarrow T$ are the compositions of $\varphi, \varphi^{\prime}$ with the projection to $T$. Hence, using the assumption that the Hodge structure is general, one finds $\psi=\psi^{\prime}$. Using that $\varphi$ and $\varphi^{\prime}$ are both isometric embeddings allows one to conclude.

## BIBLIOGRAPHY

[Huy18] Daniel Huybrechts, Finiteness of polarized K3 surfaces and hyperkähler manifolds, Ann. Henri Lebesgue 1 (2018), 227-246. $\uparrow 273$

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