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CORRIGENDUM: FINITENESS OF POLARIZED K3 SURFACES AND HYPERKÄHLER MANIFOLDS

CORRIGENDUM : FINITUDE DES SURFACES K3 ET DES VARIÉTÉS HYPERKÄHLERIENNES POLARISÉES

In the proof of Proposition 2.8 in [Huy18] we consider isometric embeddings

$$\varphi \colon T(S_0) \hookrightarrow T(S_0) \oplus \mathbb{Z} \cdot e$$

with $\varphi(\sigma) \in \mathbb{C} \cdot \sigma \oplus \mathbb{C} \cdot \bar{\sigma} \oplus \mathbb{C} \cdot e$. At this point Lemma 2.5 is evoked, which, however, assumes the more restrictive and unrealistic condition (2.1) $\varphi_{\mathbb{C}}(\sigma) = \lambda \cdot \sigma + \mu \cdot e$, i.e. $\varphi_{\mathbb{C}}(\sigma) \in \mathbb{C} \cdot \sigma \oplus \mathbb{C} \cdot e$.

This is fixed by instead using the following lemma which is proved with essentially the same techniques as the original [Huy18, Lemma 2.5].

LEMMA 0.1. — Assume that $\sigma \in T \otimes \mathbb{C}$ defines a general Hodge structure of K3 type on T such that K_{σ} is a subfield of a CM field K. Then there exist at most finitely many isometric embeddings $\varphi \colon T \hookrightarrow T \oplus \mathbb{Z} \cdot e$ satisfying

(0.1)
$$\varphi_{\mathbb{C}}(\sigma) = \lambda \cdot \sigma + \lambda' \cdot \bar{\sigma} + \nu \cdot e.$$

Proof. — Consider an isometric embedding φ satisfying (0.1). We think of φ in terms of the integral matrix $(a_{ij}|b_j)$, where $\varphi(\gamma_i) = \sum a_{ij} \cdot \gamma_j + b_i \cdot e$ with γ_i a basis of T(S) as above. Then writing $\sigma = \sum \mu_i \cdot \gamma_i$ the image $\varphi_{\mathbb{C}}(\sigma)$ corresponds

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to $(a_{ij}|b_j) \cdot (\mu_i)$ and (0.1) becomes the system of equations $\sum a_{ij}\mu_j = \lambda \mu_i + \lambda' \bar{\mu}_i$, $i = 1, \ldots, n$, and $\sum b_j \mu_j = \nu$. This shows $\lambda, \lambda', \nu \in K$. Indeed, there is at most one solution λ, λ' , which then is contained in K, unless det $\begin{pmatrix} \mu_i & \bar{\mu}_i \\ \mu_j & \bar{\mu}_j \end{pmatrix} = 0$, i.e. $\mu_i \bar{\mu}_j \in \mathbb{R}$, for all $i \neq j$. But then $1 = \sigma_1 = \sum (\mu_i \gamma_i.\gamma_1)$ implies $\bar{\mu}_j = \sum \mu_i \bar{\mu}_j (\gamma_i.\gamma_1) \in \mathbb{R}$ for all j, which would yield the contradiction $\sigma = \bar{\sigma}$. Eventually use that $K_{\sigma} = \mathbb{Q}(\mu_i)$ and that the CM field K is closed under complex conjugation.

Next observe that, as $a_{ij}, b_j \in \mathbb{Z}$, there exists an $N \in \mathbb{Z}$ independent of φ such that $N\lambda, N\lambda', N\nu \in \mathcal{O}_K$. As φ is an isometry, one also has $(\sigma.\bar{\sigma}) = (\varphi(\sigma).\varphi(\bar{\sigma}))$, which translates into

(0.2)
$$|\lambda|^2 + |\lambda'|^2 + |\nu|^2 (d/(\sigma.\bar{\sigma})) = 1.$$

As any embedding $g: K \hookrightarrow \mathbb{C}$ commutes with complex conjugation, g applied to (0.2) also shows $|g(\lambda)|^2 + |g(\lambda')|^2 + |g(\nu)|^2(d/g(\sigma,\bar{\sigma})) = 1$. Observe that $g(\sigma,\bar{\sigma}) > 0$ and that, therefore, the last summand is non-negative. Indeed, choose $z \in \mathbb{C}$ such that $(z\sigma, \bar{z}\bar{\sigma}) = 1$. Then $|g(z)|^2 g(\sigma,\bar{\sigma}) = (g(z\sigma), g(\bar{z}\bar{\sigma})) = 1$, as g commutes with complex conjugation. Hence, for $N\lambda, N\lambda' \in \mathcal{O}_K$ one has $|g(N\lambda)| \leq N$ and $|g(N\lambda')| \leq N$ for all $g: K \hookrightarrow \mathbb{C}$. By Minkowski theory there are only finitely many such $N\lambda, N\lambda' \in \mathcal{O}_K$.

As there are only finitely many possibilities for λ and λ' , it suffices to show that they determine φ essentially uniquely. Indeed, if φ and φ' are both isometric embeddings satisfying (0.1) with the same λ, λ' , then $\sigma \in \operatorname{Ker}(\psi - \psi') \otimes \mathbb{C}$, where $\psi, \psi' \colon T \longrightarrow T$ are the compositions of φ, φ' with the projection to T. Hence, using the assumption that the Hodge structure is general, one finds $\psi = \psi'$. Using that φ and φ' are both isometric embeddings allows one to conclude.

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