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MARTIN BOUNDARY OF RANDOM WALKS IN CONVEX CONES

FRONTIÈRE DE MARTIN DE MARCHES ALÉATOIRES DANS DES CÔNES CONVEXES

This article is dedicated to the memory of Émile Le Page (1946–2021).

Abstract. — We determine the asymptotic behavior of the Green function for zero-drift random walks confined to multidimensional convex cones. As a consequence, we prove that there is a unique positive discrete harmonic function for these processes (up to a multiplicative constant); in other words, the Martin boundary reduces to a singleton.

Résumé. — Nous déterminons le comportement asymptotique de la fonction de Green pour des marches aléatoires à dérive nulle confinées dans des cônes convexes multidimensionnels. Comme conséquence, nous prouvons qu'il existe une unique fonction harmonique discrète et positive pour ces processus aléatoires, c'est-à-dire que leur frontière de Martin se réduit à un singleton.

Keywords: Random walk; cone; exit time; Green function; harmonic function; Martin boundary; Brownian motion; coupling.
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1. Introduction and main results

The primary motivation of the present paper is to solve the following uniqueness problem for discrete harmonic functions: take a lattice Λ (a linear transform of $\mathbf{Z}^d$), a convex cone K in $\mathbf{R}^d$ and a discrete Laplacian operator

$$L(f)(x) := \sum_{x \sim y} p_{y-x} \left(f(y) - f(x)\right),$$

where the weights $\{p_z\}_{z \in \Lambda}$ sum to 1, have zero drift (meaning that $\sum_{z \in \Lambda} z p_z = 0$) and satisfy some minimal moment assumptions (we will be more specific later). We prove that up to multiplicative constants, there is a unique function $f : \Lambda \to \mathbf{R}$ which is positive, harmonic in $\Lambda \cap K$, i.e., $L(f) = 0$, and equal to zero outside K.

In terms of potential theory for random walks, we show that the Martin boundary of killed, zero-mean random walks in cones is reduced to one point. Our solution to this uniqueness problem is fully based on Martin boundary theory and requires the thorough asymptotic computation of the Green function for killed random walks in multidimensional cones. These asymptotics represent actually the main contribution of the paper.

Green functions and Martin boundary of random walks in cones

Random walks conditioned to stay in multidimensional cones are a very popular topic in probability. Indeed, they appear naturally in various situations: nonintersecting paths [DW10, EK08, Ste90], which can be seen as random walks in Weyl chambers, random walks in the quarter plane [FIM17, Ras11], queueing theory [CB83], branching processes and random walks in random environment [AGKV05], finance [CdL13], modelling of some populations in biology [BT12], etc. As these random walk models are in bijection with many other discrete models (maps, permutations, trees, Young tableaux, partitions), they are also intensively studied in combinatorics [BBMKM16, BMM10, DHRS18].

Let us now briefly review the literature regarding asymptotics of Green functions and Martin boundary for killed random walks in cones (see [Saw97] for a general introduction to Martin boundary theory). In the one-dimensional case, Doney [Don98] describes the harmonic functions and the Martin boundary of a random walk $\{S(n)\}$ on $\mathbf{Z}$ killed on the negative half-line (obviously there is essentially a unique cone in dimension 1, namely $\mathbf{N} = \{0, 1, 2, \ldots\}$). Alili and Doney [AD01] extend this result to the corresponding space-time random walk $\{(S(n), n)\}$.

In the higher dimensional case, let us start by quoting the famous Ney and Spitzer result [NS66] on the Green function asymptotics of drifted, unconstrained random walks in $\mathbf{Z}^d$. As a consequence, the Martin boundary is shown to be homeomorphic to the unit sphere $\mathbf{S}^{d-1}$. By large deviation techniques and Harnack inequalities, Ignatiouk-Robert [IR08, IR09], then Ignatiouk-Robert and Loree [IRL10], find the Martin boundary of random walks in half-spaces $\mathbf{N} \times \mathbf{Z}^{d-1}$ and orthants $\mathbf{N}^d$, with non-zero drift and killing at the boundary; they also derive the asymptotics of ratios of Green functions. For small step walks in the quarter plane, Lecouvey and

Raschel [LR16] show that generating functions of harmonic functions are strongly related to certain conformal mappings.

The results on Green functions and Martin boundaries are rarer for driftless random walks, and typically require a strong underlying structure: the random walks are Cartesian products in [PW92]; they are associated with Lie algebras in [Bia91, Bia92]; certain reflection groups are supposed to be finite in [BBO05]. Varopoulos [Var99, Var09] derives upper and lower bounds for the tail of the survival probability in cones under the assumption that the increments of the random walk are bounded. He also proves various statements on the growth of harmonic functions. Raschel [Ras11, Ras14] obtains the asymptotics of the Green function and the Martin boundary in the case of small step quadrant random walks related to finite reflection groups. Bouaziz, Mustapha and Sifi [BMS15] prove the existence and uniqueness of the positive harmonic function for random walks satisfying finite range, centering and ellipticity conditions, killed at the boundary of the orthant $\mathbf{N}^d$. Mustapha and Sifi [MS19] extend these results to Lipschitz domains, under similar hypotheses. Ignatiouk-Robert [IR21] shows the uniqueness of the harmonic function in a convex cone, under the assumption that the first exit time has infinite expectation. Finally, in the paper [RT20], Raschel and Tarrago derive a local limit theorem for zero-drift random walks confined to multidimensional convex cones, when the endpoint is close to the boundary.

As we will see below, our theorems unify and extend all these results in the context of convex cones, under optimal moment assumptions.

Exit time, Green functions, harmonic functions and reverse random walk

Consider a random walk $\{S(n)\}_{n\geqslant 1}$ on $\mathbf{R}^d$, $d\geqslant 1$, where

$$S(n) = X(1) + \cdots + X(n)$$

and $\{X(n)\}_{n\geqslant 1}$ is a family of independent and identically distributed (i.i.d) copies of a random variable $X=(X_1,\ldots,X_d)$. The support of the increments is supposed to generate a lattice, which we denote by Λ.

Given a cone K, let τ_x be the first exit time from the cone K of the random walk with starting point $x\in K$, i.e.,

$$\tau_x := \inf\left\{n\geqslant 1 : x+S(n)\notin K\right\}. \tag{1.1}$$

By definition, the Green function of $S(n)$ killed at τ_x is

$$G_K(x,y) := \sum_{n=0}^{\infty} \mathbf{P}(x+S(n)=y, \tau_x>n). \tag{1.2}$$

A function $h : K \to \mathbf{R}$ is said to be (discrete) harmonic with respect to K and $\{S(n)\}$ if for every $x\in K$ and $n\geqslant 1$,

$$h(x) = \mathbf{E}\Big[h(x+S(n)), \tau_x>n\Big].$$

Observe that the above identity for $n=1$ implies all the other relations for $n\geqslant 2$. In the sequel, a harmonic function with respect to K and $\{S'(n)\}$ will simply be called a harmonic function.

For cones which are either convex or with $\mathcal{C}^2$ boundary, Denisov and Wachtel proved [DW15, DW19] the existence of a positive harmonic function $V : K \to \mathbf{R}_+$ defined by

$$V(x) = \lim_{n\to\infty} \mathbf{E}\Big[u\,(x + S(n)), \tau_x > n\Big], \tag{1.3}$$

where $u(x)$ is the unique (up to a constant factor) positive solution to the Dirichlet problem

$$\begin{cases} \Delta u(x) = 0, & x \in K, \\ u(x) = 0, & x \in \partial K. \end{cases} \tag{1.4}$$

This function u is unique up to scalar multiplication, see [GSC11, Corollary 6.10 and Remark 6.11], and is called the réduite of the cone K. It is homogeneous (or radial) in the sense that $u(tx) = t^p u(x)$ for all $t > 0$ and $x \in K$. The homogeneity exponent p is always non-negative and is called the exponent of the cone K. When K is convex, which is assumed in our paper (see (H4) below), we also have $p \geqslant 1$, see [BS97].

The harmonic function V in (1.3) is of central importance in the present paper, since it will ultimately be identified with the Martin boundary of the random walk in K.

We denote by $\{S'(n)\}_{n\geqslant 1}$ the reverse random walk, which is the sum of the increments $\{X'(n)\}_{n\geqslant 1}$, i.i.d, independent from $\{X(n)\}_{n\geqslant 1}$ and such that $X'(n)$ is distributed as $-X$. In the sequel, every quantity involving S' will be denoted similarly as the same quantity involving S, with a prime added at the right.

Notations and assumptions on cones and random walks

Our hypotheses are of three types: some of them only concern the random walk (see (H1), (H2) and (H3)), the assumption (H4) is a convexity restriction on the cone, while the last ones, namely, (H5), (M1) and (M2) (moment assumptions) concern the behavior of the random walk in the cone.

(H1) $\mathbf{E}[X_i] = 0$ (zero drift),

(H2) $\operatorname{cov}(X_i, X_j) = \delta_{i,j}$ (identity covariance matrix),

(H3) the random walk is strongly aperiodic, i.e., defining

$$A = \{x \in \Lambda : \mathbf{P}(X = x) > 0\},$$

the smallest subgroup containing the set $z + A$ is Λ, for all $z \in \Lambda$.

Notice that (H2) is not a restriction: we may always perform a linear transform so as to decorrelate the random walk (obviously this linear transform impacts the cone in which the walk is defined).

Denote by $\mathbf{S}^{d-1}$ the unit sphere of $\mathbf{R}^d$ and by Σ an open, connected subset of $\mathbf{S}^{d-1}$. Let K be the cone generated by the rays emanating from the origin and passing through Σ, i.e., $\Sigma = K \cap \mathbf{S}^{d-1}$; see Figure 1.1 for two examples. In this paper, we shall suppose that

(H4) the cone K is convex.

We further require a form of irreducibility of the random walk, which is an adaptation to unbounded random walks of the concept of reachability condition from infinity introduced in [BBMM21]. For $z \in \mathbf{R}^d$ and $R > 0$, denote by $B(z,R) := \{y \in \mathbf{R}^d : |y - z| < R\}$.

(H5) The random walk S is asymptotically strongly irreducible, meaning that there exists a constant $R > 0$ such that for any $z \in K \cap \Lambda$ with $|z| \geqslant R$, there exists a path with positive probability in $K \cap B(z,R)$ which starts in $z + K$ (i.e., at some point of the set $z + K$) and ends at z.

The latter condition is required in the proof of Theorem 1.3 and Theorem 1.4 when K is not $\mathcal{C}^2$. There are several simple situations where it is satisfied, in particular when $\mathbf{P}(X \in -K) > 0$. Actually, one can prove that such a condition is always satisfied when K is $\mathcal{C}^2$ or even $\mathcal{C}^1$, as the boundary of the cone gets flatter away from zero. Hence, for simplicity we choose to keep it regardless of the boundary regularity, even if, as said before, it becomes superfluous when K is $\mathcal{C}^2$.

When K is convex, on each point σ of $\partial\Sigma$ there exists a non-trivial closed ball B in $\mathbf{S}^{d-1}$ such that $B \cap \Sigma = \sigma$. Hence, by standard analytic results [GT01, Theorem 6.13], Σ is regular for the Dirichlet problem. In particular (see for example the introduction of [BS97]), there exists a unique function u solution to the Dirichlet problem (1.4).

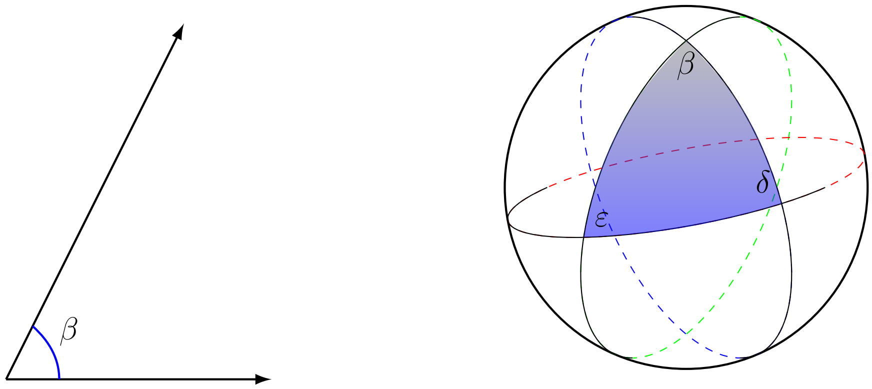

Figure 1.1. In dimension 2, Σ is an arc of circle and the cone K is a wedge of opening β. In dimension 3, any section $\Sigma \subset \mathbf{S}^2$ defines a cone. The picture on the right gives the example of a spherical triangle on the sphere $\mathbf{S}^2$, corresponding to the orthant $K = \mathbf{N}^3$ (after possible decorrelation of the coordinates, see (H2)*).*

Our next assumption (M1) involves the quantity

$$q := \sup_{\sigma \in \partial\Sigma} q_\sigma \geqslant 1,$$

that we now define. For each point $\sigma \in \partial\Sigma$, we define

$$K_\sigma := \left\{u \in \mathbf{R}^d : \exists\, t > 0, \sigma + tu \in K\right\}. \tag{1.5}$$

By convexity of K, the set K_σ is a convex cone, which represents the cone tangent to K at σ. Let q_σ denote the exponent of K_σ. Note that we always have $1 \leqslant q_\sigma \leqslant p$, since $K \subset K_\sigma$ and K_σ is included in a half-space. When K is $\mathcal{C}^2$ at σ, K_σ is precisely a half-space, which yields $q_\sigma = 1$.

We shall also assume a moment condition on the increments, which depends on the asymptotic shape of the cone K:

(M1) $\mathbf{E}[|X|^{r(p)}] < \infty$ for some $r(p) > p + q + d - 2 + (2-p)^+$ and $\mathbf{E}[|X|^{2+\delta}] < \infty$ for some $\delta > 0$. If the boundary of K is $\mathcal{C}^2$ (which implies $q = 1$), the strict inequality for $r(p)$ may be replaced by a weak inequality.

In the case where the cone is $\mathcal{C}^2$ or when considering asymptotic results inside the cone, the latter moment condition can be replaced by the following assumption of the local structure of the distribution of the increments:

(M2) $\mathbf{P}(X = x) \leqslant |x|^{-p-d+1} f(|x|)$ for some function f which is decreasing and such that $u^{(3-p)\vee 1} f(u) \to 0$ as $u \to \infty$. In the particular case $p = 2$, we require that $u \log u f(u) \to 0$.

In this paper, we do not require the existence of a bigger cone K' with $\partial K \setminus \{0\} \subset \operatorname{int}(K')$, such that the réduite u can be extended to a harmonic function on K'. This necessary condition in [DW15] is removed in [DW19] under the moment assumption (M1).

Main results

Our first main result is the asymptotics of the Green function (1.2) in the regime where the endpoint tends to infinity while staying far from the boundary.

Theorem 1.1. — *Set $r_1(p) = p+d-2+(2-p)^+$ and assume that either $\mathbf{E}|X|^{r_1(p)}$ is finite or* (M2) *holds. In the case $p = 2$, additionally assume that $\mathbf{E}[|X|^d \log|X|]$ is finite.*

(a) *If there exists $\alpha > 0$ such that $|y| \to \infty$ with $\operatorname{dist}(y, \partial K) \geqslant \alpha|y|$, then*

$$G_K(x, y) \sim cV(x)\frac{u(y)}{|y|^{2p+d-2}}. \tag{1.6}$$

(b) *If $\mathbf{E}|X|^r$ is finite for some $r > r_1(p)$, there exists $\rho > 0$ such that* (1.6) *holds uniformly for $|y| \to \infty$ with $\operatorname{dist}(y, \partial K) \geqslant |y|^{1-\rho}$.*

We will construct an example showing that the moment assumptions of Theorem 1.1 are optimal (see Section 5). We now turn to the Green function asymptotics along the boundary. In the case when the cone is a half-space, we obtain the following:

Theorem 1.2. — *Assume that $K = \{x : x_d > 0\}$ and that $\mathbf{E}|X|^{d+1} < \infty$. Assume also that $x = (0, \dots, 0, x_d)$ with $x_d = o(|y|)$. Then*

$$G_K(x, y) \sim c\frac{V(x)V'(y)}{|y|^d}.$$

Here, V' is the harmonic function for the random walk $\{-S'(n)\}$ killed at leaving K. Furthermore, uniformly in $x, y \in K$,

$$G_K(x, y) \leqslant C\frac{V(x)V'(y)}{|y|^d}.$$

Theorem 1.2 appears to be not only an extension of Uchiyama's results [Uch14], but will be one of the crucial tools to derive the boundary asymptotics of the Green function in the general convex case (Theorem 1.3 below).

When K is not a half-space but a general convex cone, we first introduce

$$K_\rho := \left\{y \in K : \operatorname{dist}(y, \partial K) \geqslant R|y|^{1-\rho}\right\} \tag{1.7}$$

as well as the stopping time

$$\zeta_y := \inf\left\{n \geqslant 1 : y + S'(n) \in K_\rho\right\}, \tag{1.8}$$

for $y \in K$. We denote by y_ρ the random element $y + S'(\zeta_y)$.

Theorem 1.3. — *Suppose that $|y|$ goes to infinity with $y/|y|$ converging to $\sigma \in \partial\Sigma$. Assume* (H1)–(H5) *and* $\mathbf{E}|X|^{r(p)} < \infty$ *for some* $r(p) > p+q_\sigma+d-2+(2-p)^+$*, then, for ρ small enough,*

$$G_K(x,y) \sim \frac{V(x)\mathbf{E}\left[u(y_\rho), \tau'_y > \zeta_y\right]}{|y|^{p+q+d-2}}.$$

If $q_\sigma = 1$, then the latter asymptotics can be improved as

$$G_K(x,y) \sim \frac{V(x)c_\sigma\left(\operatorname{dist}(y,\partial K)\right)}{|y|^{p+d-1}},$$

with c_σ a positive function which is asymptotically linear, and the moment assumption can be replaced by $\mathbf{E}|X|^{p+d-1+(2-p)^+} < \infty$ or by (M2).

Let us comment on three different aspects of Theorem 1.3. First, we will construct an example showing that our hypotheses are optimal (see Section 5). Moreover, in the above result, the convergence is uniform on all $\sigma \in \Sigma$. Finally, Theorem 1.3 easily implies the identification of the Martin boundary of S killed when exiting K, answering the uniqueness problem of the discrete harmonic functions formulated on page 560.

Theorem 1.4. — *Assume* (H1)–(H5) *and* (M1)*. The Martin kernel of S killed on the boundary of K is reduced to one point, which corresponds to the function V in* (1.3)*. In particular, there is up to a scaling constant a unique harmonic function positive in K and with Dirichlet boundary conditions in ∂K. If K is $\mathcal{C}^2$,* (M1) *can be changed into the local condition* (M2).

Towards a Ney and Spitzer theorem in cones

Ney and Spitzer consider in [NS66] random walks with non-zero drift in $\mathbf{Z}^d$ and prove that the Martin boundary is homeomorphic to the unit sphere $\mathbf{S}^{d-1}$. In [IRL10, IR09], Ignatiouk-Robert and Loree prove that for random walks in $\mathbf{N}^d$ with a drift whose entries are all non-zero, the Martin boundary is homeomorphic to $\mathbf{S}^{d-1} \cap \mathbf{R}^d_+$. However, the question of a general non-zero drift (i.e., with zero entries allowed) is left open in [IRL10, IR09]. Our results should allow to complete the picture; this will be the topic of future research.

Description of the methods used in our proofs

One of the standard approaches to the analysis of Green functions is based on local limit theorems for the process under consideration. For random walks confined to cones, one can apply local limit theorems from [DW15]. Since these results are applicable for $n > \varepsilon|y|^2$ only, one gets an asymptotically sharp lower estimate for G_K. One way to obtain an upper bound is to get a good control over $\mathbf{P}(x + S(n) = y, \tau_x > n)$ for $n \ll |y|^2$. Caravenna and Doney [CD19] have used this approach to obtain necessary and sufficient conditions for validity of the local renewal theorem for one-dimensional non-restricted random walks.

To control local large deviation probabilities in our model, we use recent results obtained in Raschel and Tarrago [RT20], based on heat kernel estimates. These results, which are improvements of the local limit theorems of [DW15], lead to Theorem 1.1 (b) and to the first claim in Theorem 1.3.

The analysis of local probabilities requires a slightly stronger moment assumptions than in Theorem 1.1 (a) and in the second half of Theorem 1.3 correspondingly. In order to derive these results, we use a different approach, where we control the whole sum $\sum_{n\leqslant \varepsilon|y|^2} \mathbf{P}(x + S(n) = y, \tau_x > n)$ instead of controlling every summand. This part is based on the functional limit theorem for walks in cones obtained in Duraj and Wachtel [DW20]. As we have mentioned before, this approach requires less moments, but one needs to impose stronger regularity conditions on the boundary of the cone.

Structure and sketch of the results

Our paper is organized as follows:

- Section 2: proof of Theorem 1.1 on the Green function asymptotics in the interior domain.
- Section 3: proof of Theorem 1.2 on the Green function asymptotics along the boundary in the case of the half-space; this result has its own interest and will also be crucially used in the next section, in the general convex case.
- Section 4: proof of Theorem 1.3 on the Green function asymptotics along the boundary in the general case; proof of Theorem 1.4 on the structure of the Martin boundary (uniqueness problem).
- Section 5: optimality of the moment assumptions in Theorems 1.1 and 1.3.
- Appendix A: proof of various lower bounds on the survival probability, which are used when showing Theorem 1.3.

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2. Asymptotics of the Green function far from the boundary

In this section, we prove Theorem 1.1. Let us start by introducing the key ideas.

Sketch of the proof

The proof runs as follows. Fix some $\varepsilon > 0$ and split $G_K(x,y)$ into two parts:

(2.1)
$$\begin{aligned} G_K(x,y) &= \sum_{n<\varepsilon|y|^2} \mathbf{P}\left(x+S(n)=y, \tau_x>n\right) + \sum_{n\geqslant\varepsilon|y|^2} \mathbf{P}\left(x+S(n)=y, \tau_x>n\right) \\ &=: S_1(x,y,\varepsilon)+S_2(x,y,\varepsilon). \end{aligned}$$

The main idea is that the first term will be negligible, meaning that

$$\lim_{\varepsilon\to 0}\limsup_{|y|\to\infty} \frac{|y|^{2p+d-2}}{u(y)} S_1(x,y,\varepsilon)=0, \tag{2.2}$$

while the second term $S_2(x,y,\varepsilon)$ will provide the main contribution in the Green function asymptotics. The asymptotic analysis (2.2) of $S_1(x,y,\varepsilon)$ is very different under the hypotheses (a) and (b) of Theorem 1.1; on the contrary, the study of $S_2(x,y,\varepsilon)$ is based on local limit theorems for $\{S(n)\}$ conditioned to stay in K.

Asymptotics of $S_2(x,y,\varepsilon)$

The study of the asymptotic behaviour of $S_2(x,y,\varepsilon)$ relies on local limit theorems for random walks in cones. Under the conditions of part (a) of the theorem it suffices to use [DW15, Theorem 5], which states that uniformly in $y\in K$,

$$n^{p/2+d/2}\mathbf{P}\left(x+S(n)=y,\tau_x>n\right)=\varkappa H_0 V(x) u\left(\frac{y}{\sqrt{n}}\right)e^{-|y|^2/2n}+o(1), \tag{2.3}$$

where

$$H_0 := \left(\int_K u(y)e^{-|y|^2/2}dy\right)^{-1}$$

and $\varkappa$ is an absolute constant such that $\mathbf{P}(\tau_x>n)\sim\varkappa V(x)n^{-p/2}$. Indeed, this local limit theorem implies that

$$\begin{aligned} S_2(x,y,\varepsilon) &= \varkappa H_0 V(x)\sum_{n\geqslant\varepsilon|y|^2}\frac{1}{n^{p/2+d/2}}u\left(\frac{y}{\sqrt{n}}\right)e^{-|y|^2/2n}+o\left(\sum_{n\geqslant\varepsilon|y|^2}\frac{1}{n^{p/2+d/2}}\right) \\ &= \varkappa H_0 V(x)u(y)\sum_{n\geqslant\varepsilon|y|^2}\frac{1}{n^{p+d/2}}e^{-|y|^2/2n}+o\left(|y|^{-p-d+2}\right) \\ &= \varkappa H_0 V(x)u(y)|y|^{-2p-d+2}\int_\varepsilon^\infty z^{-p-d/2}e^{-1/(2z)}dz+o\left(|y|^{-p-d+2}\right). \end{aligned}$$

Letting here $\varepsilon \to 0$ and recalling that $u(y) \geqslant c(\alpha)|y|^p$ for $\mathrm{dist}(y, \partial K) \geqslant \alpha|y|$, we obtain

$$\lim_{\varepsilon \to 0} \lim_{|y| \to \infty} \frac{|y|^{2p+d-2}}{u(y)} S_2(x, y, \varepsilon) = \varkappa H_0 V(x) \int_0^\infty z^{-p-d/2} e^{-1/(2z)} dz. \tag{2.4}$$

In the part (b) we are relaxing the restriction on y. More precisely we assume that $\mathrm{dist}(y, \partial K) \geqslant |y|^{1-\rho}$ with some sufficiently small $\rho > 0$. It turns out that [DW15, Theorem 5] is not sufficient anymore. An improved version of the local limit theorem has been obtained in [RT20]. By Proposition 4.3 in that paper,

$$\frac{n^{p/2+d/2}}{u\left(\frac{y}{\sqrt{n}}\right)} \mathbf{P}\left(x + S(n) = y, \tau_x > n\right) = \varkappa H_0 V(x) e^{-|y|^2/2n} + o(1)$$

uniformly in $y \in K^A_{n,\varepsilon}$, where

$$K^A_{n,\varepsilon} = \left\{ z \in K : |z| \leqslant A\sqrt{n},\, \mathrm{dist}(z, \partial K) \geqslant n^{1/2-\varepsilon} \right\} \tag{2.5}$$

and $A = \varepsilon^{-1}$.

As $|y| \to \infty$ with $\mathrm{dist}(y, \partial K) \geqslant |y|^{1-\rho}$ and ρ small enough, one has $y \in K^A_{n,\varepsilon}$ for all $\varepsilon|y|^2 \leqslant n \leqslant |y|^{2+\varepsilon'}$, for some $\varepsilon' > 0$. Hence,

$$\begin{aligned}
S_{21}(x, y, \varepsilon) &:= \sum_{\varepsilon|y|^2 \leqslant n \leqslant |y|^{2+\varepsilon'}} \mathbf{P}\left(x + S(n) = y, \tau_x > n\right) \\
&= \varkappa H_0 V(x) \sum_{\varepsilon|y|^2 \leqslant n \leqslant |y|^{2+\varepsilon'}} \frac{1}{n^{p/2+d/2}} u\left(\frac{y}{\sqrt{n}}\right) e^{-|y|^2/2n} \\
&\quad + o\left(u(y) \sum_{\varepsilon|y|^2 \leqslant n \leqslant |y|^{2+\varepsilon'}} n^{-p-d/2} \right) \\
&= \varkappa H_0 V(x) u(y) \sum_{n \geqslant \varepsilon|y|^2} \frac{1}{n^{p+d/2}} e^{-|y|^2/2n} + o\left(u(y)|y|^{-2p-d+2} \right) \\
&= \varkappa H_0 V(x) u(y) |y|^{-2p-d+2} \int_\varepsilon^\infty z^{-p-d/2} e^{-1/(2z)} dz + o\left(u(y)|y|^{-2p-d+2} \right).
\end{aligned}$$

Furthermore, using (2.3), we have

$$\begin{aligned}
S_{22}(x, y, \varepsilon) &:= \sum_{n > |y|^{2+\varepsilon'}} \mathbf{P}\left(x + S(n) = y, \tau_x > n\right) \\
&= \sum_{n > |y|^{2+\varepsilon'}} \left[\varkappa H_0 V(x) n^{-p-d/2} u(y) e^{-|y|^2/2n} + o\left(n^{-p/2-d/2}\right) \right] \\
&= O\left(u(y) \sum_{n > |y|^{2+\varepsilon'}} n^{-p-d/2} \right) + o\left(\sum_{n > |y|^{2+\varepsilon'}} n^{-p/2-d/2} \right) \\
&= o\left(u(y)|y|^{-2p-d+2} \right) + o\left(|y|^{-p-d+2-\varepsilon'(p/2+d/2-1)} \right).
\end{aligned}$$

As a result, we have

$$S_2(x,y,\varepsilon) = \varkappa H_0 V(x) u(y) |y|^{-2p-d+2} \int_\varepsilon^\infty z^{-p-d/2} e^{-1/(2z)} dz \\ + o\left(u(y)|y|^{-2p-d+2}\right) + o\left(|y|^{-p-d+2-\varepsilon'(p/2+d/2-1)}\right).$$

Letting here $\varepsilon \to 0$ and recalling that $u(y) \geqslant c|y|^{p-\rho}$ for $\mathrm{dist}(y,\partial K) \geqslant |y|^{1-\rho}$, see [DW15, Lemma 19] and [Var99], we obtain that for ρ small enough,

$$\lim_{\varepsilon\to 0} \lim_{\substack{|y|\to\infty \\ \mathrm{dist}(y,\partial K) > |y|^{1-\rho}}} \frac{|y|^{2p+d-2}}{u(y)} S_2(x,y,\varepsilon) = \varkappa H_0 V(x) \int_0^\infty z^{-p-d/2} e^{-1/(2z)} dz. \tag{2.6}$$

Asymptotics of $S_1(x,y,\varepsilon)$ in case (a)

Proof of Theorem 1.1 (a). — Let us prove the first part of Theorem 1.1. It remains to show that (2.2) holds. Fix additionally some small $\delta > 0$ and define

$$\Theta_y := \inf\{n \geqslant 1 : x + S(n) \in B_{\delta,y}\}, \tag{2.7}$$

where $B_{\delta,y}$ denotes the ball of radius $\delta|y|$ around the point y. Then we have

$$\begin{aligned} S_1(x,y,\varepsilon) &= \sum_{n<\varepsilon|y|^2} \mathbf{P}\left(x+S(n)=y, \tau_x > n \geqslant \Theta_y\right) \\ &= \sum_{n<\varepsilon|y|^2} \sum_{k=1}^{n} \sum_{z\in B_{\delta,y}} \mathbf{P}\left(x+S(k)=z, \tau_x > k = \Theta_y\right) \\ &\qquad \mathbf{P}\left(z+S(n-k)=y, \tau_z > n-k\right) \\ &\leqslant \sum_{k<\varepsilon|y|^2} \sum_{z\in B_{\delta,y}} \mathbf{P}\left(x+S(k)=z, \tau_x > k = \Theta_y\right) \sum_{j<\varepsilon|y|^2-k} \mathbf{P}\left(z+S(j)=y\right) \\ &\leqslant \mathbf{E}\left[G^{\left(\varepsilon|y|^2\right)}\left(y-x-S(\Theta_y)\right); \tau_x > \Theta_y, \Theta_y \leqslant \varepsilon|y|^2\right], \end{aligned} \tag{2.8}$$

where

$$G^{(t)}(z) := \sum_{n<t} \mathbf{P}(S(n)=z).$$

We first focus on the case $d \geqslant 3$. Then, according to [Uch98, Theorem 2], for all $z \in \mathbf{Z}^d$,

$$G(z) := G^{(\infty)}(z) \leqslant \frac{C}{1+|z|^{d-2}}, \tag{2.9}$$

provided that $\mathbf{E}|X|^{s_d} < \infty$, where $s_d = 2+\varepsilon$ for $d = 3,4$ and $s_d = d-2$ for $d \geqslant 5$. Since $r_1(p) = p+d-2+(2-p)^+ > s_d$, (2.9) yields

$$\begin{aligned} S_1(x,y,\varepsilon) &\leqslant C\mathbf{E}\left[\frac{1}{1+|y-x-S(\Theta_y)|^{d-2}}; \tau_x > \Theta_y, \Theta_y \leqslant \varepsilon|y|^2\right] \\ &\leqslant C\mathbf{P}\left(|y-x-S(\Theta_y)| \leqslant \delta^2|y|, \tau_x > \Theta_y, \Theta_y \leqslant \varepsilon|y|^2\right) \\ &\quad + \frac{C(\delta)}{|y|^{d-2}}\mathbf{P}\left(\tau_x > \Theta_y, \Theta_y \leqslant \varepsilon|y|^2\right). \end{aligned} \tag{2.10}$$

Noting now that $|y-x-S(\Theta_y)| \leqslant \delta^2|y|$ yields $|X(\Theta_y)| > \delta(1-\delta)|y|$ and using our moment assumption, we conclude that

$$\begin{aligned}(2.11)\quad \mathbf{P}\left(|y-x-S(\Theta_y)| \leqslant \delta^2|y|, \tau_x > \Theta_y, \Theta_y < \varepsilon|y|^2\right) &\leqslant \sum_{k<\varepsilon|y|^2} \mathbf{P}\left(|X(k)| > \delta(1-\delta)|y|, \tau_x > k = \Theta_y\right) \\ &\leqslant \mathbf{P}\left(|X| > \delta(1-\delta)|y|\right) \sum_{k<\varepsilon|y|^2} \mathbf{P}\left(\tau_x > k-1\right).\end{aligned}$$

Using [DW15, Theorem 1], one can easily obtain

$$\Sigma(y) := \sum_{k<\varepsilon|y|^2} \mathbf{P}\left(\tau_x > k-1\right) \leqslant C(x) \begin{cases} 1, & p > 2, \\ \log|y|, & p = 2, \\ |y|^{2-p}, & p < 2. \end{cases} \tag{2.12}$$

Combining (2.11) and (2.12), we conclude that

$$\mathbf{P}\left(|y-x-S(\Theta_y)| \leqslant \delta^2|y|, \tau_x > \Theta_y, \Theta_y < \varepsilon|y|^2\right) = o\left(|y|^{-d-p+2}\right) \tag{2.13}$$

under the moment conditions of part (a).

Recalling that V is harmonic for $S(n)$ killed at leaving K, we obtain

$$\begin{aligned}\mathbf{P}\left(\tau_x > \Theta_y, \Theta_y < \varepsilon|y|^2\right) &= \sum_{k<\varepsilon|y|^2} \sum_{z:|z-y|\leqslant\delta|y|} \mathbf{P}\left(\tau_x > k, \Theta_y = k, x+S(k) = z\right) \\ &= \sum_{k<\varepsilon|y|^2} \sum_{z:|z-y|\leqslant\delta|y|} \frac{V(x)}{V(z)} \mathbf{P}^{(V)}\left(\Theta_y = k, x+S(k) = z\right) \\ &\leqslant \frac{V(x)}{\min_{z\in K:|z-y|\leqslant\delta|y|} V(z)} \mathbf{P}^{(V)}\left(\Theta_y < \varepsilon|y|^2\right),\end{aligned}$$

where $\mathbf{P}^{(V)}$ is the Doob h-transform for $S(n)$ killed at leaving K. More precisely, it is defined through

$$\mathbf{P}^{(V)}(A) = \frac{1}{V(x)} \mathbf{E}\Big[V(x+S(n))\mathbf{1}_A; \tau_x > n\Big],$$

for every $n \geqslant 1$ and every $A \in \sigma(S(1), S(2), \ldots, S(n))$.

It follows from the assumption $\mathrm{dist}(y, \partial K) \geqslant \alpha|y|$ and [DW15, Lemma 13] that for $\delta > 0$ sufficiently small,

$$\min_{z\in K:|z-y|\leqslant\delta|y|} V(z) \geqslant C|y|^p.$$

As a result,

$$|y|^p \mathbf{P}\left(\tau_x > \Theta_y, \Theta_y < \varepsilon|y|^2\right) \leqslant C(x) \mathbf{P}^{(V)}\left(\max_{n<\varepsilon|y|^2} |x+S(n)| > (1-\delta)|y|\right).$$

Applying now the functional limit theorem for $S(n)$ under $\mathbf{P}^{(V)}$, see [DW20, Theorem 2 and Corollary 3], we conclude that

$$\lim_{\varepsilon\to 0} \limsup_{|y|\to\infty} |y|^p \mathbf{P}\left(\tau_x > \Theta_y,\ \Theta_y < \varepsilon|y|^2\right) = 0. \tag{2.14}$$

Note that the functional limit theorem from [DW20] only requires $p \vee (2+\varepsilon)$-moments. Combining (2.10)–(2.14), we infer that (2.2) is valid under the assumption $\mathbf{E}|X_1|^{r_1(p)} < \infty$ in all dimensions $d \geqslant 3$.

Assume now that (M2) holds. It is clear that this restriction implies $\mathbf{E}|X_1|^p < \infty$. Therefore, [DW15, Theorem 5] is still applicable and (2.4) remains valid for all random walks satisfying (M2). In order to show that (2.2) remains valid as well, we notice that

$$\begin{aligned} &S_1(x,y,\varepsilon) \\ &\quad \leqslant C\mathbf{E}\left[\frac{1}{1+|y-x-S(\Theta_y)|^{d-2}}; |y-x-S\left(\Theta_y\right)| \leqslant \delta^2|y|, \tau_x > \Theta_y, \Theta_y \leqslant \varepsilon|y|^2\right] \\ &\qquad\qquad + \frac{C(\delta)}{|y|^{d-2}}\mathbf{P}\left(\tau_x > \Theta_y, \Theta_y \leqslant \varepsilon|y|^2\right). \end{aligned}$$

In view of (2.14), we have to estimate the first term on the right-hand side only. For any z such that $|z-y| \leqslant \delta^2|y|$, we have

$$\begin{aligned} &\mathbf{P}\left(x+S\left(\Theta_y\right)=z, \tau_x > \Theta_y, \Theta_y \leqslant \varepsilon|y|^2\right) \\ &\qquad \leqslant \sum_{k=1}^{\varepsilon|y|^2} \sum_{z' \in K\setminus B_{\delta,y}} \mathbf{P}\Big(x+S(k-1)=z', \tau_x > k-1\Big)\mathbf{P}\left(X(k)=z-z'\right). \end{aligned}$$

Since $|z-z'| > \delta(1-\delta)|y|$, we infer from (M2) that

$$\begin{aligned} &\mathbf{P}\left(x+S(\Theta_y)=z, \tau_x > \Theta_y, \Theta_y \leqslant \varepsilon|y|^2\right) \\ &\qquad \leqslant C(\delta)|y|^{-p-d+1} f\left(\delta(1-\delta)|y|\right) \sum_{k=1}^{\varepsilon|y|^2} \mathbf{P}\left(\tau_x > k-1\right) \\ &\qquad = C(\delta)|y|^{-p-d+1} f\left(\delta(1-\delta)|y|\right)\Sigma(y). \end{aligned} \tag{2.15}$$

For every positive integer m, there are $O(m^{d-1})$ lattice points z such that $|z-y| \in (m, m+1]$. Then, using (2.15), we obtain

$$\begin{aligned} &\mathbf{E}\left[\frac{1}{1+|y-x-S\left(\Theta_y\right)|^{d-2}}; |y-x-S(\Theta_y)| \leqslant \delta^2|y|, \tau_x > \Theta_y, \Theta_y \leqslant \varepsilon|y|^2\right] \\ &\qquad \leqslant C(\delta)|y|^{-p-d+1} f\left(\delta(1-\delta)|y|\right)\Sigma(y) \sum_{m=1}^{\delta^2|y|} \frac{m^{d-1}}{1+m^{d-2}} \\ &\qquad \leqslant C(\delta)|y|^{-p-d+3} f\left(\delta(1-\delta)|y|\right)\Sigma(y). \end{aligned}$$

Taking into account (2.12), we conclude that

$$\begin{aligned} &\mathbf{E}\left[\frac{1}{1+|y-x-S\left(\Theta_y\right)|^{d-2}}; |y-x-S\left(\Theta_y\right)| \leqslant \delta^2|y|, \tau_x > \Theta_y, \Theta_y \leqslant \varepsilon|y|^2\right] \\ &\qquad = o\left(|y|^{-p-d+2}\right). \end{aligned}$$

This completes the proof of the theorem for $d \geqslant 3$.

We now focus on $d = 2$; in this case, we cannot use the full Green function. We will obtain bounds for $G^{(t)}(x)$ directly from the local limit theorem for unrestricted walks. More precisely, we shall use Propositions 9 and 10 from Chapter 2 in Spitzer's book [Spi76], which assert that as $n \to \infty$,

$$\mathbf{P}(S(n) = z) = \frac{1}{2\pi n} e^{-|z|^2/2n} + \frac{\rho(n,z)}{|z|^2 \vee n}, \tag{2.16}$$

where as $n \to \infty$,

$$\sup_{z \in \mathbf{Z}^2} \rho(n,z) \to 0.$$

This asymptotic representation implies that for all $t \geqslant 2$,

$$\sup_{z \in \mathbf{Z}^2} G^{(t)}(z) \leqslant C \log t. \tag{2.17}$$

Furthermore, for $|z| \to \infty$ and $t \leqslant a|z|^2$, one has

$$G^{(t)}(z) \leqslant \sum_{n=1}^{a|z|^2} \frac{1}{2\pi n} e^{-|z|^2/2n} + o(1) = \frac{1}{2\pi} \int_0^a \frac{1}{v} e^{-1/2v} dv + o(1).$$

As a result,

$$\sup_{z \in \mathbf{Z}^2} G^{\left(a|z|^2\right)}(z) \leqslant C(a) < \infty. \tag{2.18}$$

Using (2.17) and (2.18), we obtain

$$\begin{aligned} S_1(x,y,\varepsilon) \leqslant C \log |y| \mathbf{P}\left(|y - x - S(\Theta_y)| \leqslant \delta^2 |y|, \tau_x > \Theta_y, \Theta_y \leqslant \varepsilon |y|^2\right) \\ + C(\varepsilon) \mathbf{P}\left(\tau_x > \Theta_y, \Theta_y \leqslant \varepsilon |y|^2\right). \end{aligned}$$

According to (2.13),

$$\begin{aligned} \mathbf{P}\left(|y - x - S(\Theta_y)| \leqslant \delta^2 |y|, \tau_x > \Theta_y, \Theta_y \leqslant \varepsilon |y|^2\right) &= o\left(|y|^{-r_1(p)} \mathbf{E}\left[\tau_x; \tau_x < |y|^2\right]\right) \\ &= o\left(|y|^{-p} / \log |y|\right). \end{aligned}$$

Combining this with (2.14), we conclude that (2.2) holds for $d = 2$. The proof of Theorem 1.1 (a) is completed. ☐

Preliminary estimates for the proof of Theorem 1.1 (b)

In this part, we give some bounds on the local probability $\mathbf{P}(x + S(n) = y, \tau_x > n)$, when $|x - y|$ is between the order of fluctuations $n^{1/2}$ and $n^{1/2+\kappa}$, for some κ small enough. The main result will be given in Proposition 2.4; it needs three lemmas, stated as Lemmas 2.1, 2.2 and 2.3.

We will use the coupling of Götze and Zaitsev (see [GZ09, Theorem 4] and [DW15, Lemma 17]) for random walks having increments satisfying to (M1). Introduce a standard Brownian motion $\{B(t)\}_{t \geqslant 0}$ in $\mathbf{R}^d$ with unit covariance matrix and zero drift. Suppose that X has moments of order $r(p)$, with $r(p) > p + d - 2 + (2 - p)^+$

and $r(p) > 2 + \delta$. By [GZ09, Theorem 4], there exists a constant K such that for $\gamma \leqslant 1/2 - 1/r(p)$,

$$\mathbf{P}\left(\sup_{0 \leqslant t \leqslant n} |S(\lfloor t \rfloor) - B(t)| \geqslant n^{1/2-\gamma}\right) \leqslant K n^{-r}, \tag{2.19}$$

with

$$r = r(p)(1/2 - \gamma) - 1. \tag{2.20}$$

In the proof of Lemma 2.1 below, we will use several estimates from [RT20] on the transition probabilities of a Brownian motion in a cone. Those estimates come from general Gaussian estimates for the heat kernel in a Lipschitz domain, see [GSC11, Section 6] for general statements. In particular, we need to introduce the Brownian counterpart of the exit time τ_x in (1.1), i.e.,

$$\tau_x^{\mathrm{bm}} = \inf\left\{t > 0 : x + B(t) \notin K\right\}. \tag{2.21}$$

The first inequality from [GSC11, Theorem 5.11] gives an upper bound for the transition probabilities in K for the Brownian motion started at $y \in K$ killed outside K:

$$\mathbf{P}\left(y + B(1) \in dz, \tau_y^{\mathrm{bm}} > 1\right) \leqslant C\mathbf{P}\left(\tau_y^{\mathrm{bm}} > 1\right) \exp\left(-|z-y|^2/c\right) dz, \tag{2.22}$$

for some positive constants c and C. The survival time of the Brownian motion in K is well estimated by the réduite u, as the following inequality from [GSC11, Theorem 5.4] shows:

$$\mathbf{P}\left(\tau_y^{\mathrm{bm}} > 1\right) \leqslant C u(y). \tag{2.23}$$

Finally, define, using our notation (2.5),

$$K_{n,\varepsilon} := K_{n,\varepsilon}^{\infty} = \left\{z \in K : \mathrm{dist}\,(z, \partial K) \geqslant n^{1/2-\varepsilon}\right\}.$$

Lemma 2.1. — *There exist $\kappa, \varepsilon, c, C > 0$ such that for all n large enough and $A\sqrt{n} \leqslant t \leqslant n^{1/2+\kappa}$,*

$$\mathbf{P}\left(|S(n)| > t, \tau_y > n\right) \leqslant C\left(u\left(y/\sqrt{n}\right) \exp\left(-t^2/(cn)\right) + n^{-r}\right)$$

and for $y \in K_{n,\varepsilon}$ such that $|y| \leqslant n^{1/2+\kappa}$,

$$\mathbf{P}\left(\tau_y > n\right) \leqslant C u\left(y/\sqrt{n}\right).$$

Proof. — Choose $x_0 \in \mathbf{R}^d$ and $R > 0$ such that

$$|x_0| = 1, \quad x_0 + K \subset K \quad \text{and} \quad \mathrm{dist}\,(R x_0 + K, \partial K) > 1.$$

Let $y \in K_{n,\varepsilon}^A$ and set $y^+ := y + R x_0 n^{1/2-\gamma}$. Using the same construction as in the proof of [DW15, Lemma 20], we get

$$\begin{aligned}&\mathbf{P}\left(|S(n)| > t, \tau_y > n\right)\\ &\qquad \leqslant \int_{\left|z - y/\sqrt{n}\right| > t/\sqrt{n} - 2Rn^{-\gamma}} \mathbf{P}\left(y^+/\sqrt{n} + B(1) \in dz, \tau_{y^+/\sqrt{n}}^{\mathrm{bm}} > 1\right) + O\left(n^{-r}\right).\end{aligned}$$

Using (2.22) yields

$$\int_{\left|z-y/\sqrt{n}\right|>t/\sqrt{n}-2Rn^{-\gamma}} \mathbf{P}\left(y^+/\sqrt{n}+B(1)\in dz, \tau^{\mathrm{bm}}_{y^+/\sqrt{n}}>1\right) \leqslant C\mathbf{P}\left(\tau^{\mathrm{bm}}_{y^+/\sqrt{n}}>1\right)\int_{\left|z-y/\sqrt{n}\right|>t/\sqrt{n}-2Rn^{-\gamma}} C\exp\left(-\left|z-y^+/\sqrt{n}\right|^2/c\right)dz. \tag{2.24}$$

By the local Hölder continuity of the survival probability $\mathbf{P}(\tau^{\mathrm{bm}}_x>1)$ in x (see [RT20, Proposition A.1]), there exist $\alpha,\chi,C_\alpha>0$ such that

$$\mathbf{P}\left(\tau^{\mathrm{bm}}_{y^+/\sqrt{n}}>1\right)\leqslant \mathbf{P}\left(\tau^{\mathrm{bm}}_{y/\sqrt{n}}>1\right)+C_\alpha\left(|y|/\sqrt{n}\right)^\chi n^{-\alpha\gamma}.$$

Hence, using (2.23) yields

$$\mathbf{P}\left(\tau^{\mathrm{bm}}_{y^+/\sqrt{n}}>1\right)\leqslant Cu(y/\sqrt{n})+C_\alpha n^{\chi\kappa-\alpha\gamma}.$$

By [DW15, Lemma 19], one has

$$u(x)\geqslant cd(x,\partial K)^p, \tag{2.25}$$

so that $u(y/\sqrt{n})\geqslant \operatorname{dist}(y/\sqrt{n},K)^p\geqslant n^{-p\varepsilon}$. Choosing κ such that $\alpha\gamma-\chi\kappa>0$ and then ε such that $\varepsilon\leqslant(\alpha\gamma-\chi\kappa)/p$ yields that for some $C>0$ and $y\in K_{n,\varepsilon}$ with $|y|\leqslant n^{1/2+\kappa}$,

$$\mathbf{P}\left(\tau^{\mathrm{bm}}_{y^+/\sqrt{n}}>1\right)\leqslant Cu\left(y/\sqrt{n}\right).$$

Hence, integrating in (2.24) over the angular coordinates gives

$$\int_{\left|z-y/\sqrt{n}\right|>t/\sqrt{n}-2Rn^{-\gamma}} \mathbf{P}\left(y^+/\sqrt{n}+B(1)\in dz, \tau^{\mathrm{bm}}_{y^+/\sqrt{n}}>1\right) \leqslant Cu\left(y/\sqrt{n}\right)\int_{z>t/\sqrt{n}-4Rn^{-\gamma}}\exp\left(-|z|^2/c\right)dz$$

for some $C>0$. The latter inequality for $t=0$ gives the second inequality of Lemma 2.1. For the first one, notice that there exists $C>0$ such that $\int_x^\infty\exp(-z^2)dz \leqslant C\exp(-x^2)$. Choosing $\kappa<\gamma$ yields $\exp((t/\sqrt{n}-4Rn^{-\gamma})^2/c)\sim\exp((t/\sqrt{n})^2/c)$ for $t\leqslant n^{1/2+\kappa}$, and finally we obtain that for some constant $C>0$,

$$\int_{\left|z-y/\sqrt{n}\right|>t/\sqrt{n}-2n^{-\gamma}} \mathbf{P}\left(y^+/\sqrt{n}+B(1)\in dz, \tau^{\mathrm{bm}}_{y^+/\sqrt{n}}>1\right) \leqslant Cu\left(y/\sqrt{n}\right)\exp\left(\left(t/\sqrt{n}\right)^2/c\right). \qquad \square$$

We can extend the latter result by relaxing the condition $y\in K_{n,\varepsilon}$.

Lemma 2.2. — *Let $x\in K$. There exists $C>0$ such that for all $t\leqslant n^{1/2+\kappa}$ (κ being as in Lemma 2.1),*

$$\mathbf{P}\left(|S(n)|\geqslant t,\tau_x\geqslant n\right)\leqslant C\left(V(x)n^{-p/2}\exp\left(-t^2/(cn)\right)+n^{-r}\right).$$

Proof. — Introduce the stopping time

$$t_{x,\varepsilon}(n):=\inf\left\{m\geqslant 1: x+S(m)\in K_{n,\varepsilon}\right\} \tag{2.26}$$

and $x_\varepsilon(n) := x + S(t_{x,\varepsilon}(n))$. Then, applying the proof steps of [DW15, Section 4] to Lemma 2.1, we get

$$\begin{aligned}&\mathbf{P}\left(|S(n)| \geqslant t, \tau_x \geqslant n\right)\\&\leqslant C\left(n^{-p/2}\exp\left(-t^2/(cn)\right) \times \mathbf{E}\left[u\left(x_\varepsilon(n)\right); \tau_x > t_{x,\varepsilon}(n), t_{x,\varepsilon}(n) \leqslant n^{1-\varepsilon}\right] + n^{-r}\right)\\&\qquad + n^{-p/2}O\left(\mathbf{E}\left[|x_\varepsilon(n)|^p; |x_\varepsilon(n)| > \theta_n\sqrt{n}, \tau_x > t_{x,\varepsilon}(n), t_{x,\varepsilon}(n) \leqslant n^{1-\varepsilon}\right]\right)\\&\qquad + O\left(\exp\left(-Cn^{\varepsilon'}\right)\right),\end{aligned}$$

where $\theta_n = n^{-\varepsilon/8}$ and ε' is small enough. Using Lemma A.1 with $\alpha = p$ and $q = r(p)$ now gives

$$\begin{aligned}n^{-p/2}\mathbf{E}&\left[|x_\varepsilon(n)|^p; |x_\varepsilon(n)| > \theta_n\sqrt{n}, \tau_x > t_{x,\varepsilon}(n), t_{x,\varepsilon}(n) \leqslant n^{1-\varepsilon}\right]\\&= o\left(n^{-\left(p+d-2+(2-p)^+\right)/2}\right),\end{aligned}$$

since $p + d - 2 + (2-p)^+ < r(p)$. Since $(p + d - 2 + (2-p)^+)/2 \geqslant r$, see (2.20), Lemma 2.1 yields

$$n^{-p/2}\mathbf{E}\left[|x_\varepsilon(n)|^p; |x_\varepsilon(n)| > \theta_n\sqrt{n}, \tau_x > t_{x,\varepsilon}(n), t_{x,\varepsilon}(n) \leqslant n^{1-\varepsilon}\right] = o\left(n^{-r}\right)$$

for $t \leqslant n^{1/2+\kappa}$. Since, by [DW15, Lemma 21],

$$\lim_{n\to\infty}\mathbf{E}\left[u\left(x_\varepsilon(n)\right); \tau_x > t_{x,\varepsilon}(n), t_{x,\varepsilon}(n) \leqslant n^{1-\varepsilon}\right] = V(x),$$

the result is deduced. □

For the next lemma, we need some bounds from [DW15, Lemmas 27 and 29]. There exist positive constants a and C such that for all $u \geqslant 0$,

$$\limsup_{n\to\infty} n^{d/2} \sup_{|z-x|\geqslant u\sqrt{n}} \mathbf{P}\left(x + S(n) = z\right) \leqslant C\exp\left(-au^2\right). \tag{2.27}$$

In particular, there exists $C(x) > 0$ such that

$$\sup_{y\in K} \mathbf{P}\left(x + S(n) = y, \tau_x \geqslant n\right) \leqslant C(x)n^{-p/2-d/2}. \tag{2.28}$$

Lemma 2.3. — *There exist C and n_0 such that for $n \geqslant n_0$, all $y \in K_{n,\varepsilon}$ with $|y| \leqslant n^{1/2+\kappa}$ and all $z \in K$,*

$$\mathbf{P}\left(y + S(n) = z, \tau_y > n\right) \leqslant Cn^{-d/2}u\left(y/\sqrt{n}\right).$$

Proof. — Let $m := \lfloor n/2\rfloor$. Then

$$\begin{aligned}&\mathbf{P}\left(y + S(n) = z, \tau_y > n\right)\\&\quad= \sum_{z'\in K}\mathbf{P}\left(y + S(m) = z', \tau_y > m\right)\mathbf{P}\left(z' + S(n-m) = z, \tau_{z'} > n - m\right)\\&\quad\leqslant C\mathbf{P}\left(\tau_y > m\right)m^{-d/2},\end{aligned}$$

where we have used (2.27) with $u = 0$ to bound $\mathbf{P}(z' + S(m) = z, \tau_{z'} > n - m)$. Thus, by Lemma 2.1, there exists n_0 such that for $n \geqslant n_0$ and $y \in K_{n,\varepsilon}$ with $|y| \leqslant n^{1/2+\kappa}$,

$$\mathbf{P}\left(y + S(n) = z, \tau_y > n\right) \leqslant Cn^{-d/2}u\left(y/\sqrt{n}\right).$$

□

Putting the previous results together yields the following estimate on the local probability at middle range.

Proposition 2.4. — *Let $x \in K$. There exists $C > 0$ such that for all $y \in K_{n,\varepsilon}$ such that $|y-x| \leqslant n^{1/2+\kappa}$,*

$$\mathbf{P}\left(x+S(n)=y, \tau_x > n\right) \leqslant CV(x)n^{-p/2-d/2}\left(u(y)n^{-p/2}\exp\left(-|x-y|^2/(cn)\right)+n^{-r}\right).$$

Proof. — Let $m := \lfloor n/2 \rfloor$. Then we have

$$\begin{aligned}
&\mathbf{P}\left(x+S(n)=y, \tau_x > n\right)\\
&= \sum_{z\in K\,:\,|z-x|\geqslant|y-x|/2} \mathbf{P}\left(x+S(m)=z, \tau_x>m\right)\mathbf{P}\left(y+S'(n-m)=y, \tau'_y > n-m\right)\\
&+ \sum_{z\in K\,:\,|z-x|<|y-x|/2} \mathbf{P}\left(x+S(m)=z, \tau_x>m\right)\mathbf{P}\left(y+S'(n-m)=y, \tau'_y > n-m\right)\\
&=: M_1+M_2.
\end{aligned}$$

By Lemma 2.2 and Lemma 2.3, the first sum is bounded from above by

$$\begin{aligned}
M_1 &\leqslant Cu\left(y/\sqrt{n}\right)n^{-d/2}\mathbf{P}\left(|S(n)|>|x-y|/2, \tau_x>n\right)\\
&\leqslant CV(x)u\left(y/\sqrt{n}\right)n^{-d/2}\left(n^{-p/2}\exp\left(-|y-x|^2/(cn)\right)+n^{-r}\right),
\end{aligned}$$

where we have used in the last inequality the hypothesis $|y-x|/2 \leqslant n^{1/2+\kappa}$ in order to apply Lemma 2.2. Similarly, by (2.28) and Lemma 2.1, the second sum is bounded by

$$\begin{aligned}
M_2 &\leqslant CV(x)n^{-d/2-p/2}\mathbf{P}\left(|S'(m)|>|x-y|/2, \tau'_y>n\right)\\
&\leqslant CV(x)n^{-d/2-p/2}\left(u\left(y/\sqrt{n}\right)\exp\left(-|y-x|^2/(cn)\right)+n^{-r}\right)\\
&\leqslant CV(x)u(y)n^{-d/2-p/2}\left(n^{-p/2}\exp\left(-|y-x|^2/(cn)\right)+n^{-r}\right),
\end{aligned}$$

where we have used in the last inequality the hypothesis that $|y-x|/2 \leqslant n^{1/2+\kappa}$ in order to apply Lemma 2.1, as well as the fact that $u(y) \geqslant 1$ for $y \in K_{n,\varepsilon}$ and n large enough. The result is then deduced by summing the bounds of M_1 and M_2. □

Asymptotics of $S_1(x,y,\varepsilon)$ in case (b)

Proof of Theorem 1.1(b). — We now prove Theorem 1.1 under the hypothesis (b). Without loss of generality, we assume that $d \geqslant 2$. We have to show that (2.2) holds for y satisfying $\operatorname{dist}(y,\partial K) \geqslant |y|^{1-\rho}$. Our strategy is to decompose $S_1(x,y,\varepsilon)$ as a sum of three terms:

$$\begin{aligned}
(2.29)\quad & S_1\left(x,y,\varepsilon\right)\\
&= \Sigma_1+\Sigma_2+\Sigma_3 := \left(\sum_{n=0}^{N_1}+\sum_{n=N_1+1}^{N_2}+\sum_{n=N_2}^{\varepsilon|x-y|^2}\right)\mathbf{P}\left(x+S(n)=y, \tau_x>n\right),
\end{aligned}$$

with N_1 of the form $|y-x|^{2-\nu}$ and N_2 to be defined later. We begin by giving an estimate of the truncated Green function Σ_1.

Proposition 2.5. — *Let $\nu > 0$ and suppose that $\mathbf{E}|X|^{r+(2-p)^+} < \infty$. Then, for all $a < r$,*

$$\sum_{n=0}^{|y-x|^{2-\nu}} \mathbf{P}\left(x+S(n)=y, \tau_x > n\right) = o\left(|x-y|^a\right).$$

Proof. — Following the proof of [DW15, Lemma 24], we introduce the stopping time

$$\mu := \inf\left\{i \geqslant 1 : |X(i)| \geqslant |y-x|^{1-\nu/\alpha}\right\},$$

where α is large enough and will be chosen later. Let $n \leqslant |y-x|^{2-\nu}$. Then

$$\begin{aligned}\mathbf{P}&\left(x+S(n)=y, \tau_x > n\right) \\ &= \mathbf{P}\left(x+S(n)=y, \tau_x > n, \mu > n\right) + \mathbf{P}\left(x+S(n)=y, \tau_x > n, \mu \leqslant n\right).\end{aligned}$$

On the one hand, using Fuk–Nagaev inequalities [FN71] as in [DW15, Corollary 23] yields

$$\begin{aligned}\mathbf{P}&\left(x+S(n)=y, \tau_x > n, \mu > n\right) \\ &\leqslant \mathbf{P}\left(|S(n)| \geqslant |x-y|/2, \sup_{k\leqslant n}|X(k)| \leqslant |y-x|^{1-\nu/\alpha}\right) \\ &\leqslant \left(\frac{n\sqrt{d}e}{|x-y|^{2-\nu/\alpha}/2}\right)^{|x-y|^{\nu/\alpha}/\left(2\sqrt{d}\right)} \\ &\leqslant \left(\frac{|x-y|^{2-\nu}\sqrt{d}e}{|x-y|^{2-\nu/\alpha}/2}\right)^{2|x-y|^{\nu/\alpha}/\left(2\sqrt{d}\right)} \\ &\leqslant \exp\left(-C|x-y|^{\nu/\alpha}\right)\end{aligned}$$

for y large enough. On the other hand, recall that since X admits moments of order $r(p) := r+(2-p)^+$,

$$\begin{aligned}\mathbf{P}&\left(x+S(n)=y, \tau_x > n, \mu \leqslant n\right) \\ &\leqslant \sum_{k=1}^{n} \mathbf{P}\left(\tau_x > k-1, |X(k)| \geqslant |y-x|^{1-\nu/\alpha}, y+S'(n-k) = x+S(k)\right) \\ &\leqslant CV(x)\frac{\mathbf{E}\left[|X|^{r(p)}\right]}{|y-x|^{(1-\nu/\alpha)r(p)}}\sum_{k=1}^{n} k^{-p/2}(n+1-k)^{-d/2},\end{aligned}$$

where we have used the Markov property of the random walk, applied (2.27) with $u=0$ to $S'(n-k)$ and then (2.28) in the last inequality. Hence, we get

$$\sum_{n=0}^{|y-x|^{2-\nu}} \mathbf{P}\left(x+S(n)=y,\tau_x>n\right) \leqslant |y-x|^{2-\nu}\exp\left(-C|x-y|^{\nu/\alpha}\right) + CV(x)\frac{\mathbf{E}\left[|X|^{r(p)}\right]}{|y-x|^{(1-\nu/\alpha)r(p)}}\sum_{n=1}^{|y-x|^{2-\nu}}\sum_{k=1}^{n}k^{-p/2}(n+1-k)^{-d/2} \leqslant C'|y-x|^{-(1-\nu/\alpha)r(p)}\sum_{k=1}^{|y-x|^{2-\nu}}k^{-p/2}\sum_{k=1}^{|y-x|^{2-\nu}}k^{-d/2}.$$

Since $d\geqslant 2$, we have the following elementary estimate, for some constant $C>0$,

$$\sum_{k=1}^{|y-x|^{2-\nu}}k^{-p/2}\sum_{k=1}^{|y-x|^{2-\nu}}k^{-d/2}\sim C\log|y-x|^{\mathbf{1}_{d=2}+\mathbf{1}_{p=2}}\left(|y-x|^{2-\nu}\right)^{(1-p/2)\wedge 0}.$$

Hence,

$$\sum_{n=0}^{|y-x|^{2-\nu}} \mathbf{P}\left(x+S(n)=y,\tau_x>n\right) \leqslant C|y-x|^{-(1-\nu/\alpha)r(p)+(2-\nu)((1-p/2)\wedge 0)}\log|y-x|^{\mathbf{1}_{d=2}+\mathbf{1}_{p=2}}.$$

Finally, since $r(p)=r+(2-p)^+$, for $a<r$ and α large enough, we conclude the proof of Proposition 2.5. □

We now conclude the proof of Theorem 1.1 (b).

Lemma 2.6. — *Suppose that* $\mathbf{E}[|X|^{r(p)}]<\infty$ *with* $r(p)>p+d-2+(2-p)^+$. *Then,*

$$\lim_{\varepsilon\to 0}\limsup_{\substack{|y|\to\infty\\ \operatorname{dist}(y,\partial K)>|y|^{1-\rho}}}\frac{|y|^{2p+d-2}}{u(y)}S_1(x,y,\varepsilon)=0.$$

Proof. — Our starting point is the three-term decomposition (2.29). Since $\operatorname{dist}(y,\partial K)>|y|^{1-\rho}$, we have $u(y)\geqslant |y|^{p-p\rho}$ by (2.25). Hence, it suffices to prove that

$$\limsup_{\substack{|y|\to\infty\\ \operatorname{dist}(y,\partial K)>|y|^{1-\rho}}}|y|^{p+d-2+p\rho}\left(\Sigma_1+\Sigma_2\right)=0 \tag{2.30}$$

and

$$\lim_{\varepsilon\to 0}\limsup_{\substack{|y|\to\infty\\ \operatorname{dist}(y,\partial K)>|y|^{1-\rho}}}\frac{|y|^{2p+d-2}}{u(y)}\Sigma_3=0. \tag{2.31}$$

In order to prove (2.30), we start by the following estimate, obtained in Proposition 2.5, for ρ small enough:

$$\Sigma_1=o\left(|x-y|^{-p/2-d/2-p\rho}\right).$$

We now study Σ_2. Let $\nu > 0$ be such that $(2-\nu)(1/2+\kappa) > 1$, with κ as in Lemma 2.1. Suppose that $\delta < \nu$. With c as in Lemma 2.1, introduce

$$N_2 := \inf\left\{n \geqslant 1 : \exp\left(-\frac{|y-x|^2}{cn}\right) \geqslant n^{-r+p\rho}\right\}.$$

Recall that $r = r(p)(1/2-\gamma)-1$, see (2.20), and that $r > p/2$ for $d \geqslant 2$ and γ small enough, so that N_2 exists as soon as ρ is small enough. Furthermore, for $d \geqslant 2$ and y large enough, $N_2 \geqslant N_1$, since for K large enough,

$$\exp\left(-\frac{|y-x|^2}{c\frac{|y-x|^2}{K\log|y-x|}}\right) = |y-x|^{-K/c} \leqslant \left(\frac{|y-x|^2}{K\log|x-y|}\right)^{-r+p/2}.$$

By our choice of ν and $\delta < \nu$, $|y-x| \leqslant n^{1/2+\kappa}$ for $n \geqslant |y-x|^{2-\delta}$ and y large enough. Applying Proposition 2.4 to Σ_2 then yields

$$\begin{aligned}\Sigma_2 &\leqslant CV(x)\varepsilon|x-y|^2\left(|y-x|^{2-\nu}\right)^{-r-p/2-d/2}\\ &\leqslant C\varepsilon V(x)u(y)|y-x|^{-(2r+p+d-2)+f(\nu)} \leqslant C\frac{V(x)}{A^2}|y-x|^{-(2r+p+d-2)+f(\nu)},\end{aligned}$$

where $f : \mathbf{R} \to \mathbf{R}$ is linear. Since $r > p\rho$, choosing ν small enough yields

$$\Sigma_2 \leqslant C\varepsilon V(x)|y-x|^{-(r+p+d-2+u)},$$

with $u = 2r - p\rho > 0$, for y large enough. Hence (2.30) is proved.

We turn to the term Σ_3. First, by the choice of N_2 and Proposition 2.4,

$$\Sigma_3 \leqslant CV(x)u(y)\sum_{n=N_2+1}^{\varepsilon\lfloor|y-x|^2\rfloor} n^{-p-d/2}\exp\left(-\frac{|y-x|^2}{cn}\right).$$

Set $g_{k,B}(t) = t^{-k}\exp(-B/t)$, with $B, k > 0$. Then

$$g'_{k,B}(t) = \left(Bt^{-k-2} - kt^{-k-1}\right)\exp(-B/t),$$

and thus $g_{k,B}$ is increasing on $[0, B/k]$. Applying the latter property to $k = p + d/2$ and $B = |y-x|^2/c$ yields that if $\varepsilon^{-1} > c(p+d/2)$ (which we assume from now on), then

$$\Sigma_3 \leqslant C\varepsilon|y|^{-2p-d+2}u(y)\exp\left(-\varepsilon^{-1}/c\right).$$

This implies (2.31), thereby completing the proof of Lemma 2.6. □

3. Boundary asymptotics of the Green function: the half-space case

In this section, we shall consider the particular cone

$$K = \left\{x \in \mathbf{R}^d : x_d > 0\right\}.$$

Since the rotations of the space do not affect our moment assumptions, the results of this section remain valid for any half-space in $\mathbf{R}^d$. For this very particular cone, we have

- $u(x) = x_d$;
- $\tau_x = \inf\{n \geqslant 1 : x_d + S_d(n) \leqslant 0\}$;
- $V(x)$ depends on x_d only and is proportional to the renewal function of ladder heights of the random walk $\{S_d(n)\}$, which increases with x_d and is non-zero at 0. This property implies that $V(x)$ is separated from zero.

In other words, the exit problem from K is actually a one-dimensional problem. This allows us to use existing results for one-dimensional walks.

The proof of Theorem 1.2 is based on the following simple generalization of known results for cones.

Lemma 3.1. — *Assume that $\mathbf{E}|X|^{2+\delta} < \infty$. Then, uniformly in $x \in K$ with $x_d = o(\sqrt{n})$,*

(a) $\mathbf{P}(\tau_x > n) \sim \varkappa V(x) n^{-1/2}$;

(b) $\left(\frac{x+S([nt])}{\sqrt{n}}\right)_{t \in [0,1]}$ *conditioned on $\{\tau_x > n\}$ converges weakly to the Brownian meander in K;*

(c) $\sup_{y \in K} \left| n^{1/2+d/2} \mathbf{P}\left(x + S(n) = y, \tau_x > n\right) - cV(x) \frac{y_d}{\sqrt{n}} e^{-|y-x|^2/2n} \right| \to 0.$

Furthermore, uniformly in $x \in K$,

$$\mathbf{P}(\tau_x > n) \leqslant CV(x) n^{-1/2}.$$

Proof. — The first statement is the well-known result for one-dimensional random walks, see [Don12, Corollary 3]. The second and third statements for fixed starting points x have been proved in [DW20] and in [DW15], respectively. To consider the case of growing x_d, one has to make only one change: Lemma 24 from [DW15] should be replaced by the estimate

$$\lim_{n \to \infty} \frac{1}{V(x)} \mathbf{E}\left[|x + S(\nu_n)| \,; \tau_x > \nu_n, |x + S(\nu_n)| > \theta_n \sqrt{n}, \nu_n \leqslant n^{1-\varepsilon} \right] = 0$$

uniformly in $x_d \leqslant \theta_n \sqrt{n}/2$. If $x_d \geqslant n^{1/2-\varepsilon}$ then $\nu_n = 0$ and the expectation equals zero. If $x_d \leqslant n^{1/2-\varepsilon}$ then one repeats the proof of [DW15, Lemma 24] with p replaced by 1 and uses the part (a) of the lemma to obtain an estimate for the sum $\sum_{j \leqslant n^{1-\varepsilon}} \mathbf{P}(\tau_x > j-1)$ uniform in x_d. (In [DW15], the Markov inequality has been used, since one does not have the statement (a) in general cones.) The last statement was proven in [DW16, Lemma 3]. □

Lemma 3.2. — *Uniformly in y with $y_d = o(\sqrt{n})$,*

$$\mathbf{P}\left(x + S(n) = y, \tau_x > n\right) \sim c \frac{V(x)V'(y)}{n^{1+d/2}} e^{-|y|^2/2n}.$$

Furthermore, uniformly in $n \geqslant 1$ and $x, y \in K$,

$$\mathbf{P}\left(x + S(n) = y, \tau_x > n\right) \leqslant C \frac{V(x)V'(y)}{n^{1+d/2}}.$$

Proof. — Set $m := \lfloor \frac{n}{2} \rfloor$ and write

$$\begin{aligned} \text{(3.1)}\quad & \mathbf{P}\left(x+S(n)=y,\tau_x>n\right) \\ &= \sum_{z\in K} \mathbf{P}\Big(x+S(n-m)=z,\tau_x>n-m\Big)\mathbf{P}\left(z+S(m)=y,\tau_z>m\right) \\ &= \sum_{z\in K} \mathbf{P}\Big(x+S(n-m)=z,\tau_x>n-m\Big)\mathbf{P}\left(y+S'(m)=z,\tau'_y>m\right), \end{aligned}$$

where we recall that $S'=-S$ is the reverse random walk and

$$\tau'_y := \inf\Big\{n\geqslant 1 : y+S'(n)\notin K\Big\}.$$

Applying Lemma 3.1 (c) to the random walk $\{S'(n)\}$, we obtain

$$\begin{aligned} &\mathbf{P}\left(x+S(n)=y,\tau_x>n\right) \\ &\qquad = \frac{cV'(y)}{m^{1+d/2}}\sum_{z\in K} z_d e^{-|z-y|^2/2m}\mathbf{P}\Big(x+S(n-m)=z,\tau_x>n-m\Big) \\ &\qquad + o\Big(V'(y)m^{-1/2-d/2}\mathbf{P}\left(\tau_x>n-m\right)\Big). \end{aligned}$$

Using now Lemma 3.1 (a), we get

$$\begin{aligned} &\mathbf{P}\left(x+S(n)=y,\tau_x>n\right) \\ &\qquad = \frac{cV'(y)V(x)}{m^{1/2+d/2}(n-m)^{1/2}}\mathbf{E}_x\left[\frac{S_d(n-m)}{\sqrt{m}}e^{-|S(n-m)-y|^2/2m}\middle|\tau_x>n-m\right] \\ &\qquad + o\left(\frac{V(x)V'(y)}{m^{-1/2-d/2}(n-m)^{1/2}}\right). \end{aligned}$$

It follows from Lemma 3.1 (b) that

$$\mathbf{E}_x\left[\frac{S_d(n-m)}{\sqrt{m}}e^{-|S(n-m)-y|^2/2m}\middle|\tau_x>n-m\right] \sim \mathbf{E}\left[M_{K,d}(1)e^{-\left|M_K-y/\sqrt{m}\right|^2/2}\right],$$

where $M_K(t)=(M_{K,1}(t),M_{K,2}(t),\ldots,M_{K,d}(t))$ is the meander in K.

Since $K=\mathbf{R}^{d-1}\times\mathbf{R}_+$, all coordinates of M_K are independent. Furthermore, $M_{K,1}(t),\ldots,M_{K,d-1}(t)$ are Brownian motions and $M_{K,d}(t)$ is the one-dimensional Brownian meander. Combining these observations with $y_d=o(\sqrt{n})$, we conclude that

$$\begin{aligned} &\mathbf{E}\left[M_{K,d}(1)e^{-\left|M_K(1)-y/\sqrt{m}\right|^2/2}\right] \\ &\qquad \sim \mathbf{E}\left[M_{K,d}(1)e^{-M_{K,d}(1)^2/2}\right]\prod_{i=1}^{d-1}\mathbf{E}\left[e^{-\left(M_{K,i}(1)-y_i/\sqrt{m}\right)^2/2}\right] \\ &\qquad = C\prod_{i=1}^{d-1}e^{-y_i^2/4m} \sim Ce^{-|y|^2/2n}. \end{aligned}$$

This completes the proof of the first statement.

To prove the uniform bound, we first notice that (3.1) implies that

$$\mathbf{P}\left(x+S(n)=y, \tau_x>n\right) \leqslant \sup_{z \in K} \mathbf{P}\Big(x+S(n-m)=z, \tau_x>n-m\Big) \mathbf{P}\left(\tau_y^{\prime}>m\right).$$

Applying the uniform upper bound from Lemma 3.1 and the local limit theorem (2.3), we get

$$\mathbf{P}\left(x+S(n)=y, \tau_x>n\right) \leqslant C \frac{V(x)}{(n-m)^{1/2+d/2}} \frac{V^{\prime}(y)}{m^{1/2}}.$$

Recalling that $m=\lfloor\frac{n}{2}\rfloor$ completes the proof of the Lemma 3.2. □

Proof of Theorem 1.2. — If y is such that $y_d \geqslant \alpha|y|$ for some $\alpha>0$, then it suffices to repeat the proof of Theorem 1.1. We thus consider the boundary case $y_d=o(|y|)$. Using Lemma 3.2, one easily obtains

$$\begin{aligned} \lim_{\varepsilon \to 0} \lim_{|y| \to \infty} \frac{|y|^d}{V(x) V^{\prime}(y)} S_2(x, y, \varepsilon) &= c \lim_{\varepsilon \to 0} \lim_{|y| \to \infty} \sum_{n \geqslant \varepsilon|y|^2}|y|^d n^{-1-\frac{d}{2}} e^{-\frac{|y|^2}{2 n}} \\ &= c \int_0^{\infty} v^{-1-d/2} e^{-\frac{1}{2 v}} d v, \end{aligned}$$

and the last integral is finite. It follows that the asymptotic relation will be proven if we show that

$$\lim_{\varepsilon \to 0} \lim_{|y| \to \infty} \frac{|y|^d}{V(x) V^{\prime}(y)} S_1(x, y, \varepsilon)=0. \tag{3.2}$$

Using an appropriate rotation, we can reduce everything to the case $y_k=o(|y|)$ for any $k=2, \ldots, d-1$ and $y_1 \sim|y|$. This also implies $y_d=o(|y|)$.

We first split the probability $\mathbf{P}(x+S(n)=y, \tau_x>n)$ into two parts:

$$\begin{aligned} \mathbf{P}&\left(x+S(n)=y, \tau_x>n, \max_{k \leqslant n}\left|X_1(k)\right| \leqslant \gamma y_1\right) \\ &\qquad +\mathbf{P}\left(x+S(n)=y, \tau_x>n, \max_{k \leqslant n}\left|X_1(k)\right|>\gamma y_1\right), \end{aligned}$$

where $\gamma \in(0,1)$. Introduce the stopping time

$$\sigma_\gamma:=\inf \Big\{k \geqslant 1: |X_1(k)|>\gamma y_1\Big\}.$$

Then, by the Markov property,

$$\begin{aligned} \mathbf{P}&\left(x+S(n)=y, \tau_x>n, \max_{k \leqslant n}\left|X_1(k)\right|>\gamma y_1\right) \\ &= \sum_{k=1}^n \mathbf{P}\Big(x+S(n)=y, \tau_x>n, \sigma_\gamma=k\Big) \\ &\leqslant \sum_{k=1}^n \mathbf{P}\left(\tau_x>k-1\right) \mathbf{P}\left(\left|X_1\right|>\gamma y_1\right) \max_z \mathbf{P}(S(n-k)=z). \end{aligned}$$

Using now the bounds $\mathbf{P}(\tau_x > k) \leqslant CV(x)k^{-1/2}$ and $\max_z \mathbf{P}(S(k) = z) \leqslant Ck^{-d/2}$, we obtain

$$\begin{aligned}\mathbf{P}&\left(x + S(n) = y, \tau_x > n, \max_{k\leqslant n} |X_1(k)| > \gamma y_1\right)\\ &\leqslant CV(x)\mathbf{P}\left(|X_1| > \gamma y_1\right) \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \frac{1}{(n-k+1)^{d/2}}\\ &\leqslant CV(x)\mathbf{P}\left(|X_1| > \gamma y_1\right) \frac{(\log n)^{\mathbf{1}_{d=2}}}{\sqrt{n}}.\end{aligned}$$

Here, in the last step we have split the sum

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \frac{1}{(n-k+1)^{d/2}} \quad \text{into} \quad \sum_{k=1}^{\frac{n}{2}} \quad \text{and} \quad \sum_{k=\frac{n}{2}}^{n}$$

and used elementary inequalities. This implies that

$$\begin{aligned}&\sum_{n=1}^{\varepsilon|y|^2} \mathbf{P}\left(x + S(n) = y, \tau_x > n, \max_{k\leqslant n} |X_1(k)| > \gamma y_1\right)\\ &\qquad\leqslant C\sqrt{\varepsilon}V(x)\mathbf{P}\left(|X_1| > \gamma y_1\right)|y|\left(\log|y|\right)^{\mathbf{1}_{d=2}}.\end{aligned} \tag{3.3}$$

As a result, for all random walks satisfying

$$\mathbf{E}\left[|X|^{d+1}\left(\log|X|\right)^{\mathbf{1}_{d=2}}\right] < \infty,$$

we have

$$\sum_{n=1}^{\varepsilon|y|^2} \mathbf{P}\left(x + S(n) = y, \tau_x > n, \max_{k\leqslant n} |X_1(k)| > \gamma y_1\right) = o\left(\frac{V(x)}{|y|^d}\right). \tag{3.4}$$

In order to estimate $\mathbf{P}(x + S(n) = y, \tau_x > n, \max_{k\leqslant n} |X_1(k)| \leqslant \gamma y_1)$, we shall perform the following change of measure:

$$\overline{\mathbf{P}}\left(X(k) \in dz\right) = \frac{e^{hz_1}}{\varphi(h)}\mathbf{P}\left(X(k) \in dz, |X_1(k)| \leqslant \gamma y_1\right),$$

where

$$\varphi(h) = \mathbf{E}\left[e^{hX_1}; |X_1| \leqslant \gamma y_1\right].$$

Therefore,

$$\begin{aligned}\mathbf{P}&\left(x + S(n) = y, \tau_x > n, \max_{k\leqslant n} |X_1(k)| \leqslant \gamma y_1\right)\\ &\qquad= e^{-hy_1}\varphi^n(h)\overline{\mathbf{P}}\left(x + S(n) = y, \tau_x > n\right).\end{aligned} \tag{3.5}$$

According to [FN71, Equation (21)],

$$\begin{aligned}&e^{-hy_1}\varphi^n(h)\\ &\leqslant \exp\left\{-hy_1 + hn\mathbf{E}\left[X_1; |X_1| \leqslant \gamma y_1\right] + \frac{e^{h\gamma y_1} - 1 - h\gamma y_1}{\gamma^2 y_1^2} n\mathbf{E}\left[X_1^2; |X_1| \leqslant \gamma y_1\right]\right\}.\end{aligned}$$

Choosing

$$h=\frac{1}{\gamma y_1}\log\left(1+\frac{\gamma y_1^2}{n\mathbf{E}\left[X_1^2;|X_1|\leqslant\gamma y_1\right]}\right) \tag{3.6}$$

and noting that

$$\left|\mathbf{E}\left[X_1;|X_1|\leqslant\gamma y_1\right]\right|=\left|\mathbf{E}\left[X_1;|X_1|>\gamma y_1\right]\right|\leqslant\frac{1}{\gamma y_1}\mathbf{E}\left[X_1^2\right]=\frac{1}{\gamma y_1},$$

we conclude that uniformly for $n\leqslant\gamma|y|^2$, it holds

$$e^{-hy_1}\varphi^n(h)\leqslant\left(\frac{en}{\gamma y_1^2}\right)^{1/\gamma}.$$

Plugging this into (3.5), we obtain that uniformly for $n\leqslant\gamma|y|^2$,

$$\begin{aligned}\mathbf{P}\left(x+S(n)=y,\tau_x>n,\max_{k\leqslant n}|X_1(k)|\leqslant\gamma y_1\right)\\ \leqslant C(\gamma)\left(\frac{n}{|y|^2}\right)^{1/\gamma}\overline{\mathbf{P}}\left(x+S(n)=y,\tau_x>n\right).\end{aligned} \tag{3.7}$$

According to [Ess68, Theorem 6.2], there exists an absolute constant C such that

$$\sup_z\overline{\mathbf{P}}(S(n)=z)\leqslant\frac{C}{n^{d/2}}\chi^{-d/2},$$

where

$$\chi:=\sup_{u\geqslant 1}\frac{1}{u^2}\inf_{|t|=1}\overline{\mathbf{E}}\Big[\left(t,X(1)-X(2)\right);|X(1)-X(2)|\leqslant u\Big].$$

Since h defined in (3.6) converges to zero as $|y|\to\infty$ uniformly in $n\leqslant\gamma|y|^2$,

$$\begin{aligned}&\overline{\mathbf{E}}\Big[\left(t,X(1)-X(2)\right);|X(1)-X(2)|\leqslant u\Big]\\ &\qquad\to\mathbf{E}\Big[\left(t,X(1)-X(2)\right);|X(1)-X(2)|\leqslant u\Big]\end{aligned}$$

for every fixed u. Since $S(n)$ is truly d-dimensional under the original measure,

$$\inf_{|t|=1}\mathbf{E}\Big[\left(t,X(1)-X(2)\right);|X(1)-X(2)|\leqslant u\Big]>0$$

for all large values u. As a result, there exists $\chi_0>0$ such that $\chi\geqslant\chi_0$ for all $|y|$ large enough and all $n\leqslant\gamma|y|^2$. Consequently,

$$\sup_z\overline{\mathbf{P}}(S(n)=z)\leqslant\frac{C\chi_0^{-d/2}}{n^{d/2}}. \tag{3.8}$$

Combining this bound with (3.7), we obtain for all $r\in(0,1)$ and $\gamma<2/d$,

$$\begin{aligned}\sum_{n=1}^{|y|^{2-r}}\mathbf{P}\left(x+S(n)=y,\tau_x>n,\max_{k\leqslant n}|X_1(k)|\leqslant\gamma y_1\right)\\ \leqslant C(\gamma)\chi_0^{-d/2}y|^{-2/\gamma}\sum_{n=1}^{|y|^{2-r}}n^{1/\gamma-d/2}\leqslant C(\gamma)\chi_0^{-d/2}|y|^{-2/\gamma}|y|^{(2-r)(1/\gamma-d/2+1)},\end{aligned} \tag{3.9}$$

for all $n \leqslant \gamma|y|^2$. If we choose γ so small that $r(1/\gamma - d/2+1) > 2$, then

$$\sum_{n=1}^{|y|^{2-r}} \mathbf{P}\left(x+S(n)=y, \tau_x>n, \max_{k\leqslant n}|X_1(k)|\leqslant \gamma y_1\right) = o\left(\frac{1}{|y|^d}\right). \tag{3.10}$$

In the case $n \geqslant |y|^{2-r}$, we cannot ignore the condition $\tau_x > n$. By the Markov property at times $n/3$ and $2n/3$ and by (3.8),

$$\begin{aligned}\overline{\mathbf{P}}(x+S(n)=y,\tau_x>n) &\leqslant \sum_{z,z'}\overline{\mathbf{P}}(x+S(n/3)=z,\tau_x>n/3)\,\overline{\mathbf{P}}(z+S(n/3)=z')\\ &\qquad \overline{\mathbf{P}}(z'+S(n/3)=y,\tau_{z'}>n/3)\\ &= \sum_{z,z'}\overline{\mathbf{P}}(x+S(n/3)=z,\tau_x>n/3)\\ &\qquad \overline{\mathbf{P}}(z+S(n/3)=z')\,\overline{\mathbf{P}}(y+S'(n/3)=z',\tau_y>n/3)\\ &\leqslant \frac{C}{n^{d/2}}\overline{\mathbf{P}}(\tau_x>n/3)\,\overline{\mathbf{P}}\left(\tau'_y>n/3\right).\end{aligned}$$

Therefore, it remains to show that, uniformly in $n \in [|y|^{2-r}, |y|^2]$,

$$\overline{\mathbf{P}}(\tau_x > n/3) \leqslant C\frac{1+x_d}{\sqrt{n}}. \tag{3.11}$$

Indeed, from this estimate and from the corresponding estimate for the reverse walk, we get

$$\overline{\mathbf{P}}(x+S(n)=y,\tau_x>n) \leqslant C\frac{(x_d+1)(y_d+1)}{n^{d/2+1}}.$$

With the help of (3.7), this implies that

$$\begin{aligned}\sum_{n=|y|^{2-r}}^{\varepsilon|y|^2} \mathbf{P}\left(x+S(n)=y,\tau_x>n,\max_{k\leqslant n}|X_1(k)|\leqslant\gamma y_1\right)&\\ \leqslant C\varepsilon^{1/\gamma-d/2}(x_d+1)(y_d+1)|y|^{-d}&.\end{aligned} \tag{3.12}$$

Combining this with (3.4) and (3.10), we would obtain (3.2).

To derive (3.11), we first estimate some moments of the random walk $S_d(n)$ under $\overline{\mathbf{P}}$. By definition of this probability measure,

$$\overline{\mathbf{E}}[X_d] = \frac{1}{\varphi(h)}\mathbf{E}\left[X_d e^{hX_1}; |X_1| \leqslant \gamma y_1\right].$$

For the expectation on the right-hand side, we have the representation

$$\begin{aligned}\mathbf{E}\left[X_d e^{hX_1};|X_1|\leqslant\gamma y_1\right] &= \mathbf{E}\Big[X_d;|X_1|\leqslant\gamma y_1\Big] + h\mathbf{E}\Big[X_dX_1;|X_1|\leqslant\gamma y_1\Big]\\ &\quad + \mathbf{E}\left[X_d\left(e^{hX_1}-1-hX_1\right);|X_1|\leqslant\gamma y_1\right]\\ &= -\mathbf{E}\Big[X_d;|X_1|>\gamma y_1\Big] - h\mathbf{E}\Big[X_dX_1;|X_1|>\gamma y_1\Big]\\ &\quad + \mathbf{E}\left[X_d\left(e^{hX_1}-1-hX_1\right);|X_1|\leqslant\gamma y_1\right].\end{aligned}$$

In the last step, we have used the equalities $\mathbf{E}[X_d] = \mathbf{E}[X_dX_1] = 0$. If

$$\mathbf{E}|X|^{3+\delta} < \infty, \tag{3.13}$$

then by the Markov inequality,

$$\mathbf{E}\Big[X_d; |X_1| > \gamma y_1\Big] + h\mathbf{E}\Big[X_d X_1; |X_1| > \gamma y_1\Big] = o\left(y_1^{-2}\right) = o\left(n^{-1}\right).$$

Therefore,

$$\mathbf{E}\left[X_d e^{hX_1}; |X_1| \leqslant \gamma y_1\right] = o\left(n^{-1}\right) + \mathbf{E}\left[X_d\left(e^{hX_1} - 1 - hX_1\right); |X_1| \leqslant \gamma y_1\right].$$

It is obvious that $|e^x - 1 - x| \leqslant \frac{x^2}{2} e^{|x|}$. Therefore,

$$\begin{aligned}
&\left|\mathbf{E}\left[X_d\left(e^{hX_1} - 1 - hX_1\right); |X_1| \leqslant \gamma y_1\right]\right| \\
&\qquad \leqslant \frac{h^2}{2}\mathbf{E}\left[|X_d| X_1^2 e^{h|X_1|}; |X_1| \leqslant \gamma y_1\right] \\
&\qquad \leqslant \frac{e}{2} h^2 \mathbf{E}|X_d| X_1^2 + h^2 e^{h\gamma y_1}\mathbf{E}\left[|X_d| X_1^2; |X_1| > \frac{1}{h}\right] \\
&\qquad \leqslant \frac{e}{2} h^2 \mathbf{E}|X_d| X_1^2 + h^{2+\delta} e^{h\gamma y_1}\mathbf{E}|X_d||X_1|^{2+\delta} \\
&\qquad \leqslant \frac{e}{2} h^2 \mathbf{E}|X|^3 + h^{2+\delta} e^{h\gamma y_1}\mathbf{E}|X|^{3+\delta}.
\end{aligned}$$

In the last step, we have used Hölder's inequality. It is immediate from the definition of h that $h^2 \leqslant cn^{-1}$. Further, if $n \geqslant |y|^{2-r}$ with some $r < \frac{\delta}{2}$, then $h^{2+\delta} e^{h\gamma y_1} = o(n^{-1})$. From these estimates and from (3.13), we obtain that uniformly in $n \in [|y|^{2-r}, |y|^2]$,

$$\left|\mathbf{E}\left[X_d e^{hX_1}; |X_1| \leqslant \gamma y_1\right]\right| \leqslant \frac{c}{n}. \tag{3.14}$$

By the same arguments,

$$\begin{aligned}
\varphi(h) &= \mathbf{E}\left[e^{hX_1}; |X_1| \leqslant \gamma y_1\right] \\
&= \mathbf{P}\left(|X_1| \leqslant \gamma y_1\right) + h\mathbf{E}\Big[X_1; |X_1| \leqslant \gamma y_1\Big] + \mathbf{E}\Big[e^{hX_1} - 1 - hX_1; |X_1| \leqslant \gamma y_1\Big] \\
&= 1 - \mathbf{P}\left(|X_1| > \gamma y_1\right) - h\mathbf{E}\Big[X_1; |X_1| > \gamma y_1\Big] + \mathbf{E}\Big[e^{hX_1} - 1 - hX_1; |X_1| \leqslant \gamma y_1\Big] \\
&= 1 + o\left(n^{-1}\right).
\end{aligned} \tag{3.15}$$

Combining this with (3.14), we finally obtain

$$\left|\overline{\mathbf{E}} X_d\right| \leqslant \frac{c_1}{n}. \tag{3.16}$$

We now turn to the second and third moments of X_d under $\overline{\mathbf{P}}$. Using (3.15) and the moment assumption, we have

$$\begin{aligned}
\overline{\mathbf{E}} X_d^2 &= \frac{1}{\varphi(h)}\mathbf{E}\left[X_d^2 e^{hX_1}; |X_1| \leqslant \gamma y_1\right] = (1 + o(1))\,\mathbf{E}\left[X_d^2 e^{hX_1}; |X_1| \leqslant \gamma y_1\right] \\
&= \mathbf{E}\left[X_d^2; |X_1| \leqslant \gamma y_1\right] + o(1) + O\left(\mathbf{E}\left[X_d^2\left(e^{hX_1} - 1\right); |X_1| \leqslant \gamma y_1\right]\right) \\
&= 1 + o(1) + O\left(h e^{h\gamma y_1}\right).
\end{aligned}$$

Noting that $he^{h\gamma y_1} = o(1)$ for all $n \geqslant |y|^{2-r}$, we get

$$\overline{\mathbf{E}} X_d^2 = 1 + o(1). \tag{3.17}$$

Similarly,

$$\begin{aligned}\overline{\mathbf{E}}|X_d|^3 &= (1+o(1))\mathbf{E}\left[|X_d|^3 e^{hX_1}; |X_1| \leqslant \gamma y_1\right]\\ &\leqslant c\left(\mathbf{E}\left[|X_d|^3; |X_1| \leqslant 1/h\right] + e^{h\,\gamma y_1}\mathbf{E}\left[|X_d|^3; |X_1| > 1/h\right]\right)\\ &\leqslant c\left(\mathbf{E}|X_d|^3 + h^\delta e^{h\gamma y_1}\mathbf{E}|X_d|^{3+\gamma}\right).\end{aligned}$$

Using once again the fact that $h^\delta e^{h\gamma y_1} = o(1)$ for $n \geqslant |y|^{2-r}$, we arrive at

$$\overline{\mathbf{E}}|X_d|^3 \leqslant c_3. \tag{3.18}$$

Now we can derive (3.11). First, it follows from (3.16) that

$$\overline{\mathbf{P}}\left(\tau_x > n/3\right) \leqslant \overline{\mathbf{P}}\left(\tau^0_{x+c_1} > n/3\right),$$

where

$$\tau^0_y := \inf\left\{k \geqslant 1 : y + S^0_d(k) \leqslant 0\right\} \quad \text{and} \quad S^0_d(k) := S_d(k) - k\overline{\mathbf{E}}X_d.$$

Applying [DSW18, Lemma 25] to the random walk S^0_d, we have

$$\overline{\mathbf{P}}\left(\tau^0_y > k\right) \leqslant \frac{\overline{\mathbf{E}}\left[y + S^0_d(k); \tau^0_y > k\right]}{\overline{\mathbf{E}}\left[(y + S^0_d(k))^+\right]}.$$

Relations (3.17) and (3.18) allow the application of the central limit theorem to the walk $S^0_d(k)$, which gives $\overline{\mathbf{E}}[(y + S^0_d(k))^+] \geqslant c\sqrt{k}$. Consequently,

$$\overline{\mathbf{P}}\left(\tau^0_y > k\right) \leqslant \frac{C}{\sqrt{k}}\overline{\mathbf{E}}\left[y + S^0_d(k); \tau^0_y > k\right].$$

Further, by the optional stopping theorem,

$$\overline{\mathbf{E}}\left[y + S^0_d(k); \tau^0_y > k\right] = y - \overline{\mathbf{E}}\left[y + S^0_d\left(\tau^0_y\right); \tau^0_y \leqslant k\right] \leqslant y - \overline{\mathbf{E}}\left[y + S^0_d\left(\tau^0_y\right)\right].$$

We now use inequality [Mog73, Equation (7)], which states that there exists an absolute constant A such that

$$-\overline{\mathbf{E}}\left[y + S^0_d\left(\tau^0_y\right)\right] \leqslant A\frac{\overline{\mathbf{E}}|X_d|^3}{\overline{\mathbf{E}}X_d^2}.$$

Combining this with (3.17) and (3.18), we finally get

$$\overline{\mathbf{P}}\left(\tau^0_y > k\right) \leqslant \frac{C(y+1)}{\sqrt{k}},$$

which implies (3.11).

In order to prove a uniform upper estimate, we notice that (3.3), (3.12) with $\varepsilon = 1$ and (3.9) imply that

$$\sum_{n=1}^{|y|^2}\mathbf{P}\left(x + S(n) = y, \tau_x > n\right) \leqslant C\frac{(x_d+1)(y_d+1)}{|y|^d}.$$

Recalling that V and V' are separated from zero and asymptotically linear, we obtain

$$\sum_{n=1}^{|y|^2}\mathbf{P}\left(x + S(n) = y, \tau_x > n\right) \leqslant C\frac{V(x)V'(y)}{|y|^d}.$$

It remains to notice that the uniform upper bound from Lemma 3.2 implies that

$$\sum_{n>|y|^2} \mathbf{P}\left(x+S(n)=y, \tau_x>n\right) \leqslant CV(x)V'(y) \sum_{n>|y|^2} n^{-1-d/2} \leqslant C\frac{V(x)V'(y)}{|y|^d}. \qquad \square$$

4. Boundary asymptotics of the Green function: the general case

The proof of Theorem 1.3 consists in splitting the Green function $G_K(x,y)$ in (1.2) as a sum of two terms, the first (resp. second) one being given by the contribution in the large deviation (resp. asymptotic) regime.

The main difficulty is to prove that the first term is actually dominated by the second one; in order to achieve this, we use a coupling of the random walk with a Brownian motion, with stronger bounds than the ones initially used in [DW15]. The drawback is the need of stronger moment assumptions on the increments, which is the main reason why the assumption (M1) is used instead of the classical moment condition of [DW15], namely $\mathbf{E}|X|^{r(p)}<\infty$ with $r(p)=p$ if $p>2$ and $r(p)=2+\delta$ for some $\delta>0$ if $p\leqslant 2$.

The arguments to show Theorem 1.3 are very different in the $\mathcal{C}^2$ regular case, and two proofs are provided in this section.

Exact asymptotics with $\mathcal{C}^2$-regularity

In this part, we will assume that the cone is $\mathcal{C}^2$. Before starting the proof of Theorem 1.3, we need to introduce some notation. Let $|y|\to\infty$ in such a way that $\operatorname{dist}(y,\partial K)=o(|y|)$. Let $y_\perp\in\partial K$ be defined by the relation

$$\operatorname{dist}(y,\partial K)=|y-y_\perp|.$$

Set $\sigma(y):=y_\perp/|y|\in\partial\Sigma$ and assume that $\sigma(y)$ converges as $|y|\to\infty$ to some $\overline{\sigma}\in\partial\Sigma$. Let H_y denote a tangent hyperplane at point $y_\perp$. Let P_n be the distribution of the linear interpolation of $t\mapsto (y+S(nt))/\sqrt{n}$ conditioned to stay in the half-space K_y containing the cone K and having boundary H_y. Then $P_n\to P$ weakly on $\mathcal{C}([0,1])$. Denote

$$A_n:=\Big\{f\in\mathcal{C}([0,1]): f(k/n)\in K \text{ for all } 1\leqslant k\leqslant n\Big\}.$$

Then

$$\liminf A_n \supseteq \Big\{f\in\mathcal{C}([0,1]): f(t)\in K \text{ for all } t\in(0,1]\Big\}$$

and

$$\limsup \overline{A}_n \subseteq \Big\{f\in\mathcal{C}([0,1]): f(t)\in \overline{K} \text{ for all } t\in(0,1]\Big\},$$

where $\overline{A}$ denotes the closure of A.

Denote for every fixed n by $[0,1]\ni t\mapsto S(nt)$ the linear interpolation of $\{S(k)\}_{k\leqslant n}$. The conditions to apply [Dur78, Theorem 2.3] are met. This leads to an invariance principle: $[0,1]\ni r\mapsto \frac{y+S(nr)}{\sqrt{n}}$ converges weakly as $\frac{n}{|y|^2}\to t$ to the Brownian meander

$(B_r)_{r\leqslant 1}$ inside the cone K started at $\frac{\sigma}{\sqrt{t}}$. In particular, with $T_y := \inf\{n \geqslant 1 : y + S(n) \notin K_y\}$,

$$\mathbf{P}\left(\frac{y+S(n)}{\sqrt{n}} \in B \middle| T_y > n\right) \sim Q_{\sigma,t}(B) = \int_B q_{\sigma,t}(z)dz, \quad \frac{n}{|y|^2} \to t, \tag{4.1}$$

where $q_{\sigma,t}(z)$ is the density of the Brownian meander in K, started at $\frac{\sigma}{\sqrt{t}}$ and evaluated at time 1. [Dur78, Theorem 2.3] leads to

$$\mathbf{P}\left(\tau_y > n | T_y > n\right) \to c_{\sigma,t} := \int_K q_{\sigma,t}(z)dz. \tag{4.2}$$

The relations (4.1) and (4.2) imply that

$$V(y) \geqslant c|y|^{p-1}\left(1 + \operatorname{dist}(y, \partial K)\right). \tag{4.3}$$

Indeed, by the harmonicity of V, one has for all $n \geqslant 1$,

$$V(y) = \mathbf{E}\Big[V\left(y + S(n)\right); \tau_y > n\Big].$$

Fix now some $\varepsilon > 0$ and note that choosing $n = \lfloor |y|^2 \rfloor$, it follows that $V(z) \sim u(z)$ uniformly as $z \to \infty$ as long as the distance of z to ∂K is at least $\varepsilon|z|$, see [DW15, Lemma 13]. We obtain, as $|y| \to \infty$ and $\varepsilon \to 0$,

$$V(y) \geqslant \mathbf{P}\left(T_y > \lfloor |y| \rfloor^2\right) c_{\sigma,1}|y|^p \int_K u(z) q_{\sigma,1}(z)dz.$$

Due to results for one-dimensional random walks, we arrive at

$$\mathbf{P}\left(T_y > \lfloor |y| \rfloor^2\right) \geqslant c\frac{1 + \operatorname{dist}\left(y, \partial K\right)}{|y|},$$

which establishes (4.3).

Before proving Theorem 1.3, we record an auxiliary estimate needed in its proof.

Lemma 4.1. — *Define*

$$\phi_\sigma(t) := c_{\sigma,t} \int_K u(z) e^{-\frac{|z|^2}{2}} q_{\sigma,t}(z)dz.$$

Then there exists some $c > 0$ such that as $t \to 0$, $\phi_\sigma(t) = o(e^{-c/t})$.

Proof. — First, due to the invariance principle for the half-space, it holds

$$c_{\sigma,t} = \mathbf{P}_\sigma\left(\tau^{\mathrm{me}} > t\right) = \mathbf{P}_{\sigma/\sqrt{t}}\left(\tau^{\mathrm{me}} > 1\right),$$

where $\tau^{\mathrm{me}} := \inf\{t > 0 : M^\sigma(t) \notin K_y\}$. Here $M^\sigma(t)$ is a Brownian meander in K_y, whereas we will denote the Brownian meander in K by $M^\sigma_K(t)$. Since $|\sigma| = 1$ and K is contained in K_y, it is clear that $c_{\sigma,t} \to 1$ as $t \to 0$.

Then we have

$$\phi_\sigma(t) \leqslant C\mathbf{E}_{\sigma/\sqrt{t}}\left[u\left(M^\sigma_K(1)\right) e^{-\frac{\left|M^\sigma_K(1)\right|^2}{2}}\right] \leqslant C\mathbf{E}_{\sigma/\sqrt{t}}\left[u\left(M^\sigma(1)\right) e^{-\frac{|M^\sigma(1)|^2}{2}}\right].$$

The second inequality can be easily justified using the invariance principles for meanders in K and K_y as well as the fact that $c_{\sigma,t}$ is bounded away from zero, since $c_{\sigma,t} \to 1$. It follows that

$$\phi_\sigma(t) \leqslant C\mathbf{E}_{\sigma/\sqrt{t}}\left[e^{-\frac{|M^\sigma(1)|^2}{4}}\right].$$

Due to rotational invariance of Brownian motion, the expectation above doesn't depend on σ, so that we can choose $\sigma = (1, 0, \dots, 0)$ and $K_y = \mathbf{R}^{d-1} \times \mathbf{R}_+$. The first $d-1$ coordinates become independent Brownian motions, whereas the last one is a 1-dimensional Brownian meander (see [DIM77] for its density). This finishes the proof. □

Proof of Theorem 1.3 when K is $\mathcal{C}^2$. — To estimate the contribution coming from large values of n, one does not need the limit theorems from the previous paragraph: quite rough estimates turn out to be sufficient.

Set $m := \lfloor n/2 \rfloor$. Then, applying the Markov property at time m and inverting the time in the second part of the path, we obtain

$$\begin{aligned}
&\mathbf{P}\left(x + S(n) = y, \tau_x > n\right) \\
&\qquad = \sum_{z \in K} \mathbf{P}\left(x + S(m) = z, \tau_x > m\right) \mathbf{P}\left(y + S'(n-m) = z, \tau'_y > n - m\right) \\
&\qquad \leqslant \max_{z \in K} \mathbf{P}\left(x + S(m) = z, \tau_x > m\right) \mathbf{P}\left(\tau'_y > n - m\right).
\end{aligned}$$

By [DW15, Theorem 5],

$$\max_{z \in K} \mathbf{P}\left(x + S(m) = z, \tau_x > m\right) \leqslant C\frac{V(x)}{m^{p/2+d/2}}.$$

Furthermore, due to results for the one-dimensional walks (see for example [DW16, Lemma 3]),

$$\mathbf{P}\left(\tau'_y > n - m\right) \leqslant \mathbf{P}\left(T'_y > n - m\right) \leqslant C\frac{1 + \operatorname{dist}(y, \partial K)}{\sqrt{n-m}}. \tag{4.4}$$

Combining these estimates, we obtain

$$\mathbf{P}\left(x + S(n) = y\right) \leqslant CV(x)\left(1 + \operatorname{dist}(y, \partial K)\right) n^{-(p+d+1)/2}.$$

Consequently, for $A \geqslant 2$ and $|y| \geqslant 1$,

$$\begin{aligned}
\sum_{n \geqslant A|y|^2} \mathbf{P}\left(x + S(n) = y\right) &\leqslant CV(x)\left(1 + \operatorname{dist}(y, \partial K)\right) \sum_{n \geqslant A|y|^2} n^{-(p+d+1)/2} \\
&\leqslant CV(x)A^{-(p+d-1)/2}\frac{1 + \operatorname{dist}(y, \partial K)}{|y|^{p+d-1}}.
\end{aligned} \tag{4.5}$$

We now turn to the middle part, namely, $n \in (\varepsilon|y|^2, A|y|^2)$. Using again the Markov property at time $m = \lfloor n/2 \rfloor$ and applying [DW15, Theorem 5], we obtain

$$\begin{aligned}
&\mathbf{P}\left(x+S(n)=y, \tau_x>n\right) \\
&= \sum_{z\in K} \mathbf{P}\left(x+S(m)=z, \tau_x>m\right) \mathbf{P}\left(y+S'(n-m)=z, \tau'_y>n-m\right) \\
&= \frac{\varkappa H_0 V(x)}{m^{p/2+d/2}} \sum_{z\in K} \left(u\left(\frac{z}{\sqrt{m}}\right) e^{-\frac{|z|^2}{2m}} + o(1)\right) \mathbf{P}\left(y+S'(n-m)=z, \tau'_y>n-m\right) \\
&= \frac{\varkappa H_0 V(x)}{m^{p/2+d/2}} \mathbf{E}\left[u\left(\frac{S'(n-m)}{\sqrt{m}}\right) e^{-\frac{|S'(n-m)|^2}{2m}}; \tau'_y>n-m\right] + o\left(\frac{\mathbf{P}\left(\tau'_y>n-m\right)}{m^{p/2+d/2}}\right).
\end{aligned}$$

Taking into account (4.4), we have

$$\begin{aligned}
&\mathbf{P}\left(x+S(n)=y, \tau_x>n\right) \\
&= \frac{\varkappa H_0 V(x)}{m^{p/2+d/2}} \mathbf{E}\left[u\left(\frac{S'(n-m)}{\sqrt{m}}\right) e^{-\frac{|S'(n-m)|^2}{2m}}; \tau'_y>n-m\right] + o\left(\frac{1+\operatorname{dist}(y,\partial K)}{n^{(p+d+1)/2}}\right).
\end{aligned}$$

Next, it follows from (4.1) and (4.2) that if $\frac{n}{|y^2|} \sim t$, then

$$\mathbf{E}\left[u\left(\frac{S'(n-m)}{\sqrt{m}}\right) e^{-\frac{|S'(n-m)|^2}{2m}}; \tau'_y>n-m\right] \sim \mathbf{P}\left(T'_y>n-m\right)\phi_\sigma(t/2).$$

Since T'_y is an exit time from a half-space,

$$\mathbf{P}\left(T'_y>k\right) \sim v'(y)k^{-1/2},$$

where $v'(y)$ is the positive harmonic function for S' killed at leaving the half-space K_σ. As a result,

$$\mathbf{P}\left(x+S(n)=y, \tau_x>n\right) = C_0 \frac{V(x)v'(y)}{n^{(p+d+1)/2}} \phi_\sigma\left(\frac{n}{|y|^2}\right) + o\left(\frac{1+\operatorname{dist}(y,\partial K)}{n^{(p+d+1)/2}}\right),$$

where

$$C_0 := \varkappa H_0 2^{(p+d+1)/2}.$$

This representation implies that

$$\begin{aligned}
&\sum_{n=\varepsilon|y|^2}^{A|y|^2} \mathbf{P}(x+S(n)=y, \tau_x>n) \\
&\qquad = C_0 V(x)v'(y) \sum_{n=\varepsilon|y|^2}^{A|y|^2} n^{-(p+d+1)/2} \phi_\sigma\left(\frac{n}{2|y|^2}\right) + o\left(\frac{1+\operatorname{dist}(y,\partial K)}{n^{(p+d-1)/2}}\right) \\
&\qquad = C_0 \frac{V(x)v'(y)}{|y|^{p+d-1}} \int_\varepsilon^A \phi_\sigma(t/2)t^{-(p+d+1)/2}dt + o\left(\frac{1+\operatorname{dist}(y,\partial K)}{|y|^{p+d-1}}\right).
\end{aligned}$$

Combining this with (4.5) and letting $A \to \infty$, one can easily obtain

$$\lim_{|y|\to\infty} \frac{|y|^{p+d-1}}{V(x)v'(y)} S_2(x,y,\varepsilon) = C_0 \int_\varepsilon^\infty \phi_\sigma(t/2)t^{-(p+d+1)/2}dt.$$

From Lemma 4.1, it follows that

$$\lim_{\varepsilon\to 0}\lim_{|y|\to\infty}\frac{|y|^{p+d-1}}{V(x)v'(y)}S_2(x,y,\varepsilon)=C_0\int_0^\infty \phi_\sigma(t/2)t^{-(p+d+1)/2}dt. \tag{4.6}$$

It remains to estimate $S_1(x,y,\varepsilon)$. We shall use the same strategy as in the proof of Theorem 1.1, but instead of the Green function for the whole space, we shall use the Green function for the half-space K_y. More precisely, recalling the definition of Θ_y from (2.7),

$$\begin{aligned}
&S_1(x,y,\varepsilon)=\sum_{n<\varepsilon|y^2|}\mathbf{P}\left(x+S(n)=y,\tau_x>n\geqslant\Theta_y\right)\\
&=\sum_{n<\varepsilon|y^2|}\sum_{k=1}^{n}\sum_{z\in B_{\delta,y}}\mathbf{P}\left(x+S(n)=z,\tau_x>k=\Theta_y\right)\mathbf{P}\left(z+S(n-k)=y,\tau_z>n-k\right)\\
&=\sum_{k<\varepsilon|y|^2}\sum_{z\in B_{\delta,y}}\mathbf{P}\left(x+S(n)=z,\tau_x>k=\Theta_y\right)\sum_{j<\varepsilon|y|^2-k}\mathbf{P}\left(z+S(j)=y,\tau_z>j\right)\\
&\leqslant\sum_{k<\varepsilon|y|^2}\sum_{z\in B_{\delta,y}}\mathbf{P}\left(x+S(n)=z,\tau_x>k=\Theta_y\right)\sum_{j<\varepsilon|y|^2}\mathbf{P}\left(y+S'(j)=z,T'_y>j\right)\\
&=\mathbf{E}\left[G_{\varepsilon,y}\left(x+S\left(\Theta_y\right)\right);\tau_x>\Theta_y,\Theta_y\leqslant\varepsilon|y|^2\right],
\end{aligned}$$

where

$$G_{\varepsilon,y}(z)=\sum_{j<\varepsilon|y|^2}\mathbf{P}\left(y+S'(j)=z,T'_y>j\right).$$

Applying the inequality of Theorem 1.2 yields that

$$G_{\varepsilon,y}(z)\leqslant C\frac{V(z)v'(y)}{|z|^d}\quad\text{and}\quad G_{\varepsilon,y}(z)\leqslant C\frac{V(z)v'(y)}{|y|^d}.$$

Hence, combining both inequalities, bounding $V(z)$ by $C(1+\mathrm{dist}(z,H_y))$ and applying (2.18) to the random walk $S'(n)$ for $d=2$, we obtain

$$G_{\varepsilon,y}(z)\leqslant C\frac{v'(y)\left(1+\mathrm{dist}(z,H_y)\right)}{1+|z-y|^d}\wedge 1.$$

Therefore,

$$\begin{aligned}
S_1(x,y,\varepsilon)\leqslant{}& C\mathbf{P}\left(|y-x-S(\Theta_y)|\leqslant\delta^2|y|,\tau_x>\Theta_y,\Theta_y\leqslant\varepsilon|y|^2\right)\\
&+C(\delta)\frac{v'(y)}{|y|^d}\mathbf{E}\left[\left(1+\mathrm{dist}\left(x+S\left(\Theta_y\right),H_y\right)\right);\tau_x>\Theta_y,\Theta_y\leqslant\varepsilon|y|^2\right].
\end{aligned}\tag{4.7}$$

The first term has been estimated in (2.13) for random walks having finite moments of order $r_2(p):=p+d-1+(2-p)^+$:

$$\mathbf{P}\left(|y-x-S\left(\Theta_y\right)|\leqslant\delta^2|y|,\tau_x>\Theta_y,\Theta_y\leqslant\varepsilon|y|^2\right)=o\left(|y|^{-p-d+1}\right). \tag{4.8}$$

In order to estimate the second term in (4.7), we shall perform again the change of measure with the harmonic function V:

$$\mathbf{E}\left[\left(1+\operatorname{dist}\left(x+S\left(\Theta_y\right),H_y\right)\right);\tau_x>\Theta_y,\Theta_y\leqslant\varepsilon|y|^2\right]$$
$$=V(x)\mathbf{E}^{(V)}\left[\frac{1+\operatorname{dist}\left(x+S\left(\Theta_y\right),H_y\right)}{V\left(x+S\left(\Theta_y\right)\right)};\Theta_y\leqslant\varepsilon|y|^2\right].$$

Applying now (4.3), we obtain

$$\mathbf{E}\left[\left(1+\operatorname{dist}\left(x+S\left(\Theta_y\right),H_y\right)\right);\tau_x>\Theta_y,\Theta_y\leqslant\varepsilon|y|^2\right]$$
$$\leqslant CV(x)|y|^{-p+1}\mathbf{P}^{(V)}\left(\Theta_y\leqslant\varepsilon|y|^2\right).$$

From this estimate and (2.14), we conclude that

$$\lim_{\varepsilon\to0}\lim_{|y|\to\infty}|y|^{p-1}\mathbf{E}\left[\left(1+\operatorname{dist}\left(x+S\left(\Theta_y\right),H_y\right)\right);\tau_x>\Theta_y,\Theta_y\leqslant\varepsilon|y|^2\right]=0.$$

Combining this estimate with (4.7) and (4.8) as well as [DW15, Lemma 13], we get

$$\lim_{\varepsilon\to0}\lim_{|y|\to\infty}\frac{|y|^{p+d-1}}{v'(y)}S_1(x,y,\varepsilon)=0. \tag{4.9}$$

Since $v'(y)$ is the harmonic function on a half-space, it is bounded from below by a positive number (see the discussion at the beginning of Section 3), and (4.9) and (4.6) yield the desired result for the case $\mathbf{E}[|X|^{r_2(p)}]<\infty$ due to classical results for one-dimensional random walks.

Assume now that (M2) holds. It is easy to see from the above proof that

$$\lim_{\varepsilon\to0}\lim_{|y|\to\infty}\frac{|y|^{p+d-1}}{V(x)v'(y)}S_2(x,y,\varepsilon)=C_0\int_0^\infty\phi_\sigma(t)t^{-(p+d+1)/2}dt. \tag{4.10}$$

Therefore we focus on the asymptotics of $S_1(x,y,\varepsilon)$ in the following. With similar steps as above it holds

$$S_1(x,y,\varepsilon)\leqslant C(\delta)v'(y)\mathbf{E}\Bigg[\frac{1+\operatorname{dist}\left(x+S\left(\Theta_y\right),H_y\right)}{1+\left|x+S\left(\Theta_y\right)-y\right|^d};\left|y-x-S\left(\Theta_y\right)\right|\leqslant\delta^2|y|,$$
$$\tau_x>\Theta_y,\Theta_y\leqslant\varepsilon|y|^2\Bigg]+C(\delta)\frac{v'(y)}{|y|^d}\mathbf{E}\left[\left(1+\operatorname{dist}\left(x+S\left(\Theta_y\right),H_y\right)\right);\tau_x>\Theta_y,\Theta_y\leqslant\varepsilon|y|^2\right].$$

The second summand can be treated just as above with the help of (4.3), so that we need to show

$$\mathbf{E}\left[\frac{1+\operatorname{dist}\left(x+S\left(\Theta_y\right),H_y\right)}{1+\left|=x+S\left(\Theta_y\right)-y\right|^d};\left|y-x-S\left(\Theta_y\right)\right|\leqslant\delta^2|y|,\tau_x>\Theta_y,\Theta_y\leqslant\varepsilon|y|^2\right]$$
$$=O\left(|y|^{-p-d+1}\right).$$

It holds

$$1+\operatorname{dist}\left(x+S\left(\Theta_y\right),H_y\right)\leqslant1+\left|S\left(\Theta_y\right)-y\right|+\left|y-y_\perp\right|=o(|y|)+\left|S\left(\Theta_y\right)-y\right|.$$

To complete the proof, we now show for $r = d-1$ and $r = d$,

$$\begin{aligned}
&S_{2,r}(x,y,\varepsilon)\\
&\quad:=\mathbf{E}\left[\frac{1}{1+|x+S\left(\Theta_y\right)-y|^r};|y-x-S\left(\Theta_y\right)|\leqslant\delta^2|y|,\tau_x>\Theta_y,\Theta_y\leqslant\varepsilon|y|^2\right]\\
&\quad=o\left(|y|^{-p-d+1}\right).
\end{aligned}$$

With a similar calculation as in the proof of Theorem 1.1 (using (2.15)), we obtain

$$\begin{aligned}
&\mathbf{E}\left[\frac{1}{1+|y-x-S\left(\Theta_y\right)|^{d-1}};|y-x-S\left(\Theta_y\right)|\leqslant\delta^2|y|,\tau_x>\Theta_y,\Theta_y\leqslant\varepsilon|y|^2\right]\\
&\qquad\leqslant C(\delta)|y|^{-p-d+1}f\left(\delta\left(1-\delta\right)|y|\right)\mathbf{E}\left[\tau_x;\tau_x<|y|^2\right]\sum_{m=1}^{\delta^2|y|}\frac{m^{d-1}}{m^{d-1}}\\
&\qquad\leqslant C(\delta)|y|^{-p-d+2}f\left(\delta(1-\delta)|y|\right)|y|^{(2-p)^+}.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\mathbf{E}\left[\frac{1}{1+|y-x-S\left(\Theta_y\right)|^{d}};|y-x-S\left(\Theta_y\right)|\leqslant\delta^2|y|,\tau_x>\Theta_y,\Theta_y\leqslant\varepsilon|y|^2\right]\\
&\qquad\leqslant C(\delta)|y|^{-p-d+1}f\left(\delta(1-\delta)|y|\right)\mathbf{E}\left[\tau_x;\tau_x<|y|^2\right]\sum_{m=1}^{\delta^2|y|}\frac{m^{d-1}}{m^{d}}\\
&\qquad\leqslant C(\delta)\log(|y|)|y|^{-p-d+2}f\left(\delta(1-\delta)|y|\right)|y|^{(2-p)^+}.
\end{aligned}$$

This finishes the proof of Theorem 1.3 when K is $\mathcal{C}^2$. □

Exact asymptotics in the general case

We now turn to the general convex case, without assuming that the boundary is $\mathcal{C}^2$. Recall from (1.7) the definition of

$$K_\rho := \left\{y\in K : \mathrm{dist}(y,\partial K)\geqslant|y|^{1-\rho}\right\},$$

where ρ is given in Theorem 1.1. Further, for $y\in K$, $\zeta_y := \inf\{n\geqslant 0 : y+S'(n)\in K_\rho\}$ was introduced in (1.8).

Proof of Theorem 1.3 in the general case. — Set $\varepsilon = \rho/2$ and split the Green function as

$$\begin{aligned}
&G_K(x,y) = \\
&= \sum_{1 \leqslant n < |y-x|^{2-2\varepsilon}} \mathbf{P}\left(y+S'(n)=x, \tau'_y > n\right) \\
&\quad + \sum_{n \geqslant |y-x|^{2-2\varepsilon}} \mathbf{P}\left(y+S'(n)=x, \tau'_y > n, \zeta_y > |y-x|^{2-3\varepsilon}\right) \\
&\quad + \sum_{n \geqslant |y-x|^{2-2\varepsilon}} \mathbf{P}\left(y+S'(n)=x, \tau'_y > n, \zeta_y \leqslant |y-x|^{2-3\varepsilon}, \left|S'_{\zeta_y}\right| > |y-x|^{1-\varepsilon/\alpha}\right) \\
&\quad + \sum_{n \geqslant |y-x|^{2-2\varepsilon}} \mathbf{P}\left(y+S'(n)=x, \tau'_y > n, \zeta_y \leqslant |y-x|^{2-3\varepsilon}, \left|S'_{\zeta_y}\right| \leqslant |y-x|^{1-\varepsilon/\alpha}\right) \\
&:= Z_1 + Z_2 + Z_3 + Z_4,
\end{aligned}$$

with α to be fixed later. We shall study successively the terms Z_1, Z_2, Z_3 and Z_4.

Study of Z_1 and Z_2.

It follows from Proposition 2.4 that $Z_1 \leqslant C|y-x|^{-a}$, for some parameter $a > p+q+d-2+(2-p)^+$. In order to analyse Z_2, we need the preliminary estimates (4.11) and (4.12) below. To that purpose, remark that $\zeta_y = t'_{y,\rho/2}(|y|^{2-\rho})$, see (2.26). Hence, recalling that $\varepsilon = \rho/2$ and noting that $t'_{y,\varepsilon}(n)$ is increasing in n, we get with [DW15, Lemma 14]

$$\mathbf{P}\left(\zeta_y > n^{1-\varepsilon}, \tau'_y > n\right) \leqslant \mathbf{P}\left(t'_{y,\varepsilon}(n) > n^{1-\varepsilon}, \tau'_y > n\right) \leqslant C\exp\left(n^{-\varepsilon}\right) \tag{4.11}$$

for $n \geqslant |y|^{2-2\varepsilon}$. Applying Lemma A.1 to the stopping time ζ_y and using the moment condition $\mathbf{E}|X|^{r(p)} < \infty$, we obtain that there exist $C > 0$ and $\alpha > 0$ such that

$$\mathbf{P}\left(\left|S'_{\zeta_y}\right| \geqslant |y-x|^{1-\varepsilon/\alpha}, \zeta_y \leqslant |y-x|^{2-\varepsilon}, \tau'_y > |y-x|^{2-\varepsilon}\right) \leqslant C|y-x|^{-s}, \tag{4.12}$$

with $s > (2-2\varepsilon)(p+q+d-4+(2-p)^+)/2$.

Let us now write

$$\begin{aligned}
Z_2 = &\sum_{n \geqslant |y-x|^{2-2\varepsilon}} \mathbf{P}\left(x+S(n)=y, \tau_x > n, \zeta_y \geqslant n^{1-\varepsilon}\right) \\
&\qquad + \sum_{n \geqslant |y-x|^{2-2\varepsilon}} \mathbf{P}\left(x+S(n)=y, \tau_x > n, |y-x|^{2-3\varepsilon} \leqslant \zeta_y \leqslant n^{1-\varepsilon}\right).
\end{aligned}$$

By (4.11), the first term is bounded by $\sum_{n\geqslant|y-x|^{2-2\varepsilon}} C\exp(-n^{\varepsilon}) \leqslant C\exp(-|y-x|^{\varepsilon'})$ for some $C>0$ and some $0<\varepsilon'<\varepsilon$. Moreover, by (2.28) and (4.11),

$$\sum_{n\geqslant|y-x|^{2-2\varepsilon}} \mathbf{P}(x+S(n)=y,\tau_x>n,|y-x|^{2-3\varepsilon}\leqslant\zeta_y\leqslant n^{1-\varepsilon}) =$$
$$\sum_{n\geqslant|y-x|^{2-2\varepsilon}} \mathbf{E}\Big[x+S\left(n-\zeta_y\right)=y+S'\left(\zeta_y\right);$$
$$\tau_x>n-\zeta_y,\tau'_y>\zeta_y,|y-x|^{2-3\varepsilon}\leqslant\zeta_y\leqslant n^{1-\varepsilon}\Big]$$
$$\leqslant \sum_{n\geqslant|y-x|^{2-2\varepsilon}} C\left(n-n^{1-\varepsilon}\right)^{-d/2-p/2}\mathbf{P}\left(\zeta_y>|y-x|^{2-3\varepsilon},\tau'_y\geqslant|y-x|^{2-3\varepsilon}\right)$$
$$\leqslant C\exp\left(-|y-x|^{\varepsilon(2-3\varepsilon)}\right),$$

so that, for ε small enough,

$$Z_2\leqslant C\exp\left(-|y-x|^{\varepsilon'}\right),$$

for some constant $C>0$ and $0<\varepsilon'<\varepsilon$.

Study of Z_3

By (2.28), we have for $n\geqslant|x-y|^{2-2\varepsilon}$ and y large enough

$$\mathbf{P}\left(y+S'(n)=x,\tau'_y>n,\zeta_y\leqslant|y-x|^{2-3\varepsilon},\left|S'_{\zeta_y}\right|\geqslant|y-x|^{1-\varepsilon/\alpha}\right)$$
$$\leqslant \mathbf{E}\left[(n-\zeta_y)^{-p/2-d/2};\tau'_y>\zeta_y,\zeta_y\leqslant|y-x|^{2-3\varepsilon},\left|S'_{\zeta_y}\right|\geqslant|y-x|^{1-\varepsilon/\alpha}\right]$$
$$\leqslant Cn^{-p/2-d/2}\mathbf{P}\left(\tau'_y>\zeta_y,\zeta_y\leqslant|y-x|^{2-3\varepsilon},\left|S'_{\zeta_y}\right|\geqslant|y-x|^{1-\varepsilon/\alpha}\right)$$
$$\leqslant Cn^{-p/2-d/2}|y-x|^{s},$$

where we applied (4.12) on the last equality. Hence,

$$Z_3\leqslant C\sum_{n=|x-y|^{2-2\varepsilon}}^{\infty} n^{-p/2-d/2}|y-x|^{s}\leqslant|y-x|^{-(2-2\varepsilon)(p/2+d/2-1)-s}.$$

By the definition of s given in (4.12),

$$(2-2\varepsilon)(p/2+d/2-1)+s>(p+d-2)+(p+q+d-4+2(1-p/2)^{+})+g(\varepsilon)$$
$$=p+q+d-2+\left(p+d-4+2(1-p/2)^{+}\right)+g(\varepsilon),$$

with g linear. Since $p+d-4+2(1-p/2)^{+}>0$ for all $p\geqslant1$ and $d>2$ (the case $d=1$ and $d=2$ are given by the C^2-boundary case),

$$Z_3=o\left(|x-y|^{-b}\right),$$

with $b>p+q+d-2$ for ε small enough.

Study of Z_4

Since $|x-y|^{2-2-\varepsilon}=o(|x-y|^2$, we have for any fixed $c>0$ and y large enough

$$Z_4 \geqslant \mathbf{E}\left[S_2(x,y+S'(\zeta_y,c)\,;\tau'_y>\zeta_y,\zeta_y\leqslant|y-x|^{2-3\varepsilon},|S(\zeta_y)|\leqslant|y-x|^{1-\varepsilon/\alpha}\right],$$

where $S_2(x,y,c)$ is defined in (2.1). Then, (2.6) yields for y large enough

$$\begin{aligned}Z_4 \geqslant C\mathbf{E}\Big[u\left(y+S'(\zeta_y)\right)|y+S'(\zeta_y)|^{-2p-d+2}\,;\tau'_y>\zeta_y,\\ \zeta_y\leqslant|y-x|^{2-3\varepsilon},|S(\zeta_y)|\leqslant|y-x|^{1-\varepsilon/\alpha}\Big]\\ \geqslant C|y|^{-2p-d+2}|y|^{p-q''+q'+O(\varepsilon)}\mathbf{P}\left(\tau'_y>\zeta_y,\zeta_y\leqslant|y-x|^{2-3\varepsilon},|S(\zeta_y)|\leqslant|y-x|^{1-\varepsilon/\alpha}\right),\end{aligned}$$

for some $q''>q'>q$ small enough, where we have used the fact that $y+S'(\zeta_y)\in K_\rho$, $|S(\zeta_y)|\leqslant|y-x|^{1-\varepsilon/\alpha}$ and Lemma A.5 to give a lower bound on $u(y+S'(\zeta_y))$. Hence, using again (4.11) and (4.12) gives

$$Z_4 \geqslant |y|^{-p-q''+q'-d+2+O(\varepsilon)}\left(\mathbf{P}\left(\tau'_y>|y-x|^{2-3\varepsilon}\right)-C\exp\left(-|y|^{\varepsilon}\right)-K|y-x|^{-s}\right),$$

for some $s>0$. With y going to infinity with $y/|y|$ going to $\sigma\in\partial\Sigma$, Lemma A.4 gives then that $\mathbf{P}(\tau'_y>|y-x|^{2-3\varepsilon})\geqslant C|y-x|^{-q'/2(2-3\varepsilon)}$ and $s>q'$ for ε and $q'>q$ small enough, which yields

$$Z_4\geqslant c|y|^{-p-q''-d+2+O(\varepsilon)}.$$

Hence, for ε and $q''>q$ small enough,

$$Z_1+Z_2+Z_3=o(Z_4).$$

Moreover, by Theorem 1.1,

$$\begin{aligned}\mathbf{E}\left[G_K\left(x,y+S'(\zeta_y)\right);\tau'_y>\zeta_y,\zeta_y\leqslant|y-x|^{2-3\varepsilon},|S(\zeta_y)|\leqslant|y-x|^{1-\varepsilon/\alpha}\right]\\ \sim V(x)|y|^{-2p-q+2}\mathbf{E}\Big[u\left(y+S'(\zeta_y)\right),\tau'_y>\zeta_y,\zeta_y\\ \leqslant|y-x|^{2-3\varepsilon},|S(\zeta_y)|\leqslant|y-x|^{1-\varepsilon/\alpha}\Big],\end{aligned}$$

as y goes to infinity. Since we also have

$$\begin{aligned}\mathbf{E}\left[u\left(y+S'(\zeta_y)\right);\tau'_y>\zeta_y,\left(\zeta_y\geqslant|y-x|^{2-3\varepsilon}\right)\cup\left(|S\left(\zeta_y\right)|\geqslant|y-x|^{1-\varepsilon/\alpha}\right)\right]\\ =o\left(\mathbf{E}\left[u\left(y+S'(\zeta_y)\right);\tau'_y>\zeta_y,\zeta_y\leqslant|y-x|^{2-3\varepsilon},|S(\zeta_y)|\leqslant|y-x|^{1-\varepsilon/\alpha}\right]\right)\end{aligned}$$

for the same reasons as before, the result is deduced. □

The uniqueness of the harmonic function is then a straightforward deduction of the latter theorem together with Martin boundary theory.

Corollary 4.2. — *The Martin boundary of S killed on the boundary of K is reduced to a singleton, and there exists a unique harmonic function (up to multiplication by a constant).*

Proof. — Let $x_0, x \in K$ and let (y_n) be a sequence in K going to infinity. Then, by Theorems 1.1 and 1.3, as $n \to \infty$,

$$\frac{G_K(x, y_n)}{G_K(x_0, y_n)} \to \frac{V(x)}{V(x_0)}.$$

The Martin boundary is thus reduced to a singleton. □

5. Optimality of the moment conditions

In this section, we prove that the assumptions of Theorems 1.1 and 1.3 are optimal.

Uchiyama [Uch98] has shown, see Theorem 2 there, that if $d \geqslant 5$ and $\mathbf{E}|X|^{d-2} < \infty$, then

$$G(z) \sim \frac{c}{|z|^{d-2}}$$

as $|z| \to \infty$. The same asymptotics is valid when $d = 4$ or $d = 3$, provided that respectively $\mathbf{E}|X|^2 \log |X| < \infty$ or $\mathbf{E}|X|^2 < \infty$.

Uchiyama mentions also that this moment condition is optimal: for any $\varepsilon > 0$, there exists a random walk satisfying $\mathbf{E}|X|^{d-2-\varepsilon} < \infty$ and

$$\limsup_{|z| \to \infty} |z|^{d-2} G(z) = \infty.$$

Uchiyama considers dimensions 4 and 5 only, but it is quite simple to show that this statement holds in every dimension $d \geqslant 5$. We now give an example in our setting of a random walk which shows the optimality of Uchiyama's condition and of the moment condition in Theorem 1.1. Our example is just a multidimensional variation of the classical Williamson example, see [Wil68].

Let d be greater than 4 and consider X with the following distribution. For every $n \geqslant 1$ and for every basis vector e_k put

$$\mathbf{P}\left(X = \pm 2^n e_k\right) = \frac{q_n}{2d},$$

where the sequence q_n is such that

$$\sum_{n=1}^{\infty} q_n = 1 \quad \text{and} \quad q_n \sim \frac{c \log n}{2^{n(d-2)}}.$$

Clearly,

$$\mathbf{E}|X|^{d-2} = \infty \quad \text{and} \quad \mathbf{E}\frac{|X|^{d-2}}{\log^{1+\varepsilon} |X|} < \infty.$$

Using now the obvious inequality $G(x) \geqslant \mathbf{P}(X = x)$, we conclude that for every $j = 1, \ldots, d$,

$$\lim_{n \to \infty} 2^{(d-2)n} G\left(\pm 2^n e_j\right) = \infty.$$

If we have a cone K such that $p \geqslant 2$ and $e_j \in \Sigma$ for some j, then, choosing $q_n \sim \frac{c \log n}{2^{n(p+d-2)}}$, we also have

$$\lim_{n \to \infty} 2^{(p+d-2)n} G_K\left(e_j, (1 + 2^n) e_j\right) = \infty.$$

Therefore, the finiteness of $\mathbf{E}|X|^{r_1(p)}$ cannot be replaced by a weaker moment assumption.

But Uchiyama shows that the moment assumption $\mathbf{E}|X|^{d-2}$ is not necessary, as it can be replaced by $\mathbf{P}(X = x) = o(|x|^{-d-2})$, which implies the existence of the second moment only. In Theorem 1.1 we have a similar situation: the moment condition $\mathbf{E}|X|^{r_1(p)} < \infty$ is not necessary and can be replaced by the assumption (M2), which yields the finiteness of $\mathbf{E}|X|^{p\vee 2}$ only. It has been shown in [DW15] that if $p > 2$, the condition $\mathbf{E}|X|^p < \infty$ is an optimal moment condition for the existence of the harmonic function $V(x)$.

Clearly, one can adapt the random walk from the example above to show that the moment assumption in the second statement of Theorem 1.3 is minimal as well. Indeed, it suffices to take $q_n \sim \frac{c\log n}{2^{n(p+d-1)}}$ and to assume that one of the vectors $\pm e_j$ belongs to the boundary of the cone K.

In order to show that the moment conditions in the first claim of Theorem 1.3 are nearly minimal we consider the cone $K = \mathbf{R}^d_+$, $d \geqslant 3$. Clearly, $p = d$ for this cone. Set $\sigma = (1, 0, 0, \dots, 0)$. Then one has $K_\sigma = \mathbf{R} \times \mathbf{R}^{d-1}_+$ and $q_\sigma = d - 1$ because K_σ here is an intersection of $d - 1$ half-spaces. We assume again that

$$\mathbf{P}\left(X = \pm 2^n e_k\right) = \frac{q_n}{2d}.$$

This time we choose $q_n \sim c\frac{\log n}{2^{3(d-1)n}}$. Denoting by $\mathbf{1}$ the vector $e_1 + \dots + e_d$, we obtain that as $n \to \infty$,

$$G_K\left(\mathbf{1}, \mathbf{1} + 2^n e_1\right) \gg 2^{-3(d-1)n}.$$

Moreover, it is rather simple to see that $\mathbf{E}[u(y_\rho), \tau'_y > \zeta_y]$ converges to a positive constant for $\mathbf{1} + 2^n e_1$. As a result, the first statement may fail for a random walk with $\mathbf{E}|X|^{3d-3} = \infty$. Remark that the first statement requires not only finiteness of moment of order $p + q_\sigma + d - 2 + (2-p)^+$, but also finiteness of some moment strictly greater than $p + q_\sigma + d - 2 + (2-p)^+$. We conjecture that this condition is actually sharp when $d \geqslant 3$.

Appendix A. Boundary asymptotics of the survival probability

The goal of this section is to collect lower bounds on the survival probability at time $n \geqslant 1$ of the random walk starting at x when $n = o(|x|^2)$ and $x \to \infty$ while $\frac{x}{|x|} \to \sigma \in \partial K$. Those bounds are used in the proof of our main results. The strategy of the proof is to compare the tangent cone at σ with some smaller cones included in K. Let us give a first recall a useful result from [RT20, Lemma B.1].

Lemma A.1. — *Let $0 \leqslant r \leqslant p$ and $A > 0$, and suppose that the increment X admits moments of order $\kappa > r$. Set*

$$S(x,n)^+ = \sup_{1 \leqslant \ell \leqslant n^{1-\varepsilon}} |S(\ell)| \mathbf{1}_{\tau_x > \ell}.$$

Then, for each $s < (\kappa - r)/2$ *and* $\beta \in ((p/2-1) \wedge 0, p/2)$, *there exists* $C > 0$ *such that*

$$\mathbf{E}\left[\left(S(x,n)^+\right)^r ; S(x,n)^+ \geqslant n^{1/2-\varepsilon/8}\right] \leqslant C n^{-s} n^{1-(p/2-\beta)}(1+|x|)^{p-2\beta}$$

for all $x \in K$. *In particular, uniformly on* $x \in K$, $|x| \leqslant A\sqrt{n}$,

$$\mathbf{E}\left[\left(S(x,n)^+\right)^r ; S(x,n)^+ \geqslant n^{1/2-\varepsilon/8}\right] \leqslant C n^{-s+1}.$$

Recall from (H5) that a random walk S is strongly irreducible in a cone K if there exists a constant $R > 0$ such that for any $z \in K \cap \Lambda$ with $|z| \geqslant R$, there exists a path with positive probability in $K \cap B(z, R)$ which starts in $z + K$ and ends at z. If K is a cone with exponent q such that S is strongly irreducible in K, then there exists $c > 0$ such that for all $z \in K$ and all $n \geqslant 1$,

$$\mathbf{P}\left(\tau_z > n\right) \geqslant c n^{-q/2}. \tag{A.1}$$

See [RT20, Lemma 4.5] for a proof of this fact.

We now prove that a tangent cone can be well approximated by a smaller cone included in the original cone. We recall that K_σ denotes the tangent cone to K at σ, see (1.5), and for $\alpha > 0$ we set

$$K_{\sigma,\alpha} = \left\{x \in K_\sigma : \alpha|x - \sigma| \leqslant \operatorname{dist}\left(x, \partial K_\sigma\right)\right\}.$$

Notice that for α small enough, $K_{\sigma,\alpha}$ is a non-empty cone. For $\varepsilon > 0$, let

$$V_\varepsilon(\sigma) = B(\sigma, \varepsilon) \cap K \quad \text{and} \quad \partial V_\varepsilon(\sigma) = B(\sigma, \varepsilon) \cap \partial K.$$

Hereafter, $(z - \sigma) + K_{\sigma,\alpha}$ denotes the translated version of $K_{\sigma,\alpha}$ with origin at z.

Lemma A.2. — *For all* $\alpha > 0$ *sufficiently small, there exist* $\varepsilon, \alpha' > 0$ *such that for all* $y \in \partial V_\varepsilon(\sigma)$ *and all* $z \in (y - \sigma) + K_{\sigma,\alpha} \cap B(\sigma, \varepsilon)$, *one has* $z \in K$ *and*

$$\operatorname{dist}(z, \partial K) \geqslant \alpha'|z - y|.$$

The proof of the above lemma uses a few basic facts from convex analysis and subgradients, see [HUL01, Section D] for more information on this topic. Recall that for a convex function $\phi : C \to \mathbf{R}$ defined on an open convex set $C \subset \mathbf{R}^{d-1}$, we define the subgradient $\partial\phi(x)$ of ϕ at $x \in C$ by

$$\partial\phi(x) = \left\{v \in \mathbf{R}^{d-1} : \forall\, u \in C,\, \phi(u) - \phi(x) \geqslant \langle v, u - x\rangle\right\}.$$

The subgradient is upper-semicontinuous in the following sense: if $x_n \to x$ and $v_n \to v$ with $v_n \in \partial\phi(x_n)$ for any n, then $v \in \partial\phi(x)$.

For $s \in \mathbf{R}^{d-1}$, the convex function ϕ admits a directional derivative $\phi_s(x)$ at any point $x \in C$, and we have

$$\phi_s(x) = \max_{v \in \partial\phi(x)} \langle v, s\rangle.$$

Note that the upper-semicontinuity of the subgradient implies a uniform upper-semicontinuity of the directional derivatives.

Lemma A.3. — *Let* $x \in C$ *and* $\varepsilon > 0$. *There exists a neighborhood* V *of* x *such that*

$$\phi_s(u) \leqslant \phi_s(x) + \varepsilon$$

for all $u \in V$ *and* $s \in \mathbf{S}^{d-2}$.

Proof. — Let us prove the statement by contradiction. Assume the existence of a sequence (x_n, s_n) in $C \times \mathbf{S}^{d-2}$ such that $x_n \to x$ and $\phi_{s_n}(x_n) > \phi_{s_n}(x) + \varepsilon$. Up to taking a subsequence, we can assume that $s_n \to s \in \mathbf{S}^{d-2}$. For each n, let v_n be the maximizer of $\langle v, s_n\rangle$ for $v \in \partial\phi_{x_n}$. Since $\partial\phi$ is uniformly bounded on a neighborhood of x, we can assume by compactness that v_n converges to a vector v_0. By upper-semicontinuity of $\partial\phi$, one has $v_0 \in \partial\phi(x)$. Then we have

$$\phi_{s_n}(x_n) = \langle v_n, s_n\rangle \to \langle v_0, s\rangle \leqslant \max_{v \in \partial\phi(x)} \langle v, s\rangle \leqslant \phi_s(x), \tag{A.2}$$

and on the other hand

$$\langle v_n, s_n\rangle = \phi_{s_n}(x_n) \geqslant \phi_{s_n}(x) + \varepsilon.$$

Since $s \mapsto \phi_s(x)$ is continuous and $s_n \to s$, $\phi_{s_n}(x_n) \geqslant \phi_s(x) + \varepsilon/2$ for n large enough, and by (A.2) we get a contradiction. ☐

Proof of Lemma A.2. — Up to an isometry of $\mathbf{R}^d$, we can assume $\sigma = 0$ and that $(0, \dots, 0, 1)$ is a vector pointing inside K. Let V be a neighborhood of 0 in $H_d := \{x \in \mathbf{R}^d : x_d = 0\}$ such that there exists a convex function $\phi : V \to \mathbf{R}$ with Lipschitz constant M whose graph is locally the boundary of K around σ. We further assume that there exists $\varepsilon > 0$ such that

$$\Big\{(y, t) \in V \times \mathbf{R} : \phi(y) < t < \phi(y) + \varepsilon\Big\} \subset K.$$

Such ε always exists if we assume V small enough.

Note that the tangent cone of K at σ is exactly the set

$$K_\sigma = \Big\{(y, x_d) \in \mathbf{R}^{d-1} \times \mathbf{R} : x_d \geqslant \phi_y(0)\Big\}.$$

Let α be small enough so that $K_{\sigma,\alpha}$ is non-empty. For $\beta > 0$, set

$$\widetilde{K}_\beta := \Big\{(y, x_d) \in \mathbf{R}^{d-1} \times \mathbf{R} : x_d \geqslant \phi_y(0) + \beta|y|\Big\}.$$

Then, $(\widetilde{K}_\beta)_{\beta>0}$ is a decreasing sequence of cones and $\bigcup_{\beta>0} \widetilde{K}_\beta = K_\sigma$, hence there exists $\alpha' > 0$ such that $K_{\sigma,\alpha} \subset \widetilde{K}_{\alpha'}$. By Lemma A.3, let $\varepsilon' < \varepsilon$ be such that $B_{\mathbf{R}^{d-1}}(0, \varepsilon') \subset V$ is a neighborhood of 0, with the property that for each $y \in B_{\mathbf{R}^{d-1}}(0, \varepsilon')$ and $s \in \mathbf{S}^{d-2}$, we have

$$\phi_s(y) \leqslant \phi_s(0) + \alpha'/2. \tag{A.3}$$

Let $z \in y + \widetilde{K}_{\alpha'} \cap B(y, \varepsilon'/2)$ with $y = (y_1, \phi(y_1)) \in \partial K$ and $y_1 \in B_{\mathbf{R}^{d-1}}(0, \varepsilon'/2)$. Writing $z = (z_1, z_2) \in \mathbf{R}^{d-1} \times \mathbf{R}$, we have on the first hand

$$z_2 - \phi(y_1) \geqslant \phi_{z_1-y_1}(0) + \alpha'|z_1 - y_1|.$$

On the other hand, integrating (A.3) on the segment $[y_1, z_1] \subset B_{\mathbf{R}^{d-1}}(0, \varepsilon')$ yields

$$\phi(z_1) - \phi(y_1) = \int_0^1 \phi_{z_1-y_1}(y_1 + t(z_1 - y_1))\, dt \leqslant \phi_{z_1-y_1}(0) + \alpha'/2|z_1 - y_1|.$$

Hence, $z_2 \geqslant \phi(z_1) + \alpha'/2|z_1 - y_1|$, which yields

$$\phi(z_1) + \alpha'/2|z_1 - y_1| < z_2 < \phi(z_1) + \varepsilon/2 \tag{A.4}$$

by the choice of ε'. Since $z_1 \in V$, $(z_1, u) \in K$ for all $u \in (\phi(z_1), \phi(z_1) + \varepsilon)$, which implies that $(z_1, z_2) \in K$. Therefore, for $y \in \partial V_{\varepsilon'/2}(\sigma)$ we have $(y - \sigma) + \widetilde{K}_{\alpha'} \cap$

$B(\sigma, \varepsilon'/2) \subset K$. Since $K_{\sigma,\alpha} \subset \widetilde{K}_{\alpha'}$, we also have $(y-\sigma) + K_{\sigma,\alpha} \cap B(\sigma, \varepsilon'/2) \subset K$ for all $y \in \partial V_{\varepsilon'/2}(\sigma)$.

Since ϕ is Lipschitz with Lipschitz constant $M > 0$ on $B_{\mathbf{R}^{d-1}}(0, \varepsilon')$, standard geometric arguments yield that for $c = \sin(\arctan(1/M))$,

$$d(z, \partial K) \geqslant c\,(z_2 - \phi(z_1))\,,$$

when $z = (z_1, z_2) \in K$ with $z_1 \in B_{\mathbf{R}^{d-1}}(0, \varepsilon'/2)$ and $z_2 \leqslant \varepsilon'/2$. Thus, (A.4) yields that

$$d(z, \partial K) \geqslant \frac{c\alpha'}{2}|z_1 - y_1|. \tag{A.5}$$

Since the Lipschitz property also yields $|z_1 - y_1| \geqslant |\phi(z_1) - \phi(y_1)|/M$, we deduce that

$$d(z, \partial K) \geqslant \frac{c\alpha'}{2M}|\phi(z_1) - \phi(y_1)|.$$

Hence, for $c' = \min\{c, \frac{c\alpha'}{2M}\}$,

$$d(z, \partial K) \geqslant \frac{c'}{2}\,(z_2 - \phi(z_1) + \phi(z_1) - \phi(y_1)) \geqslant \frac{c'}{2}|z_2 - \phi(y_1)|. \tag{A.6}$$

Let t be such that $|y - z| \leqslant t \max\{|y_1 - z_1|, |z_2 - y_2|\}$. Then, since $y_2 = \phi(y_1)$,

$$d(z, \partial K) \geqslant \frac{c'}{2t}|y - z|.$$

This concludes the proof of the second statement. ☐

Proposition A.4. — *Suppose* (H5) *that S is strongly irreducible in K. Let $\sigma \in \partial K$ and q_σ the exponent associated to the corresponding tangent cone K_σ. Then, for all $q' > q_\sigma$ and $\varepsilon > 0$ small enough, there exists $c > 0$ such that for all x large enough with $\frac{x}{|x|} \to \sigma$ and for all $n \leqslant |x|^{2-\varepsilon}$,*

$$\mathbf{P}(\tau_x > n) > cn^{-q'/2}.$$

Proof. — Let $q' > q_\sigma$ be small enough, and let $\alpha > 0$ be such that $q_{K_{\sigma,\alpha}} = q'$, where $q_{K_{\sigma,\alpha}}$ denotes the exponent of the cone $K_{\sigma,\alpha}$. Such α exists, since $K_{\sigma,\alpha} \cap \mathbf{S}^{d-1}$ converges in Hausdorff distance to K_σ as α goes to zero. Similarly to the proof of Lemma A.2, assume without loss of generality that $K \subset \mathbf{R}^{d-1} \times \mathbf{R}^+$, $\sigma = (1, 0, \ldots, 0)$ and $v = (0, \ldots, 0, 1)$ is a vector pointing towards the interior of K. For $x \in K$, let $x_{/\!/}$ be the projection of x on ∂K along $(0, \ldots, 0, 1)$. As x goes to infinity while $x/|x| \to \sigma$, $x_{/\!/}/|x|$ converges to σ and $|x/|x| - x_{/\!/}| \to 0$.

By Lemma A.2, there exist $\eta, \alpha' > 0$ such that for all $z \in \partial V_\eta(\sigma)$ and all $u \in (z - \sigma) + K_{\sigma,\alpha} \cap B(z, \eta)$, $\mathrm{dist}(u, \partial K) \geqslant \alpha'|u - z|$. For α small enough and $|x|$ large enough, $x/|x| \in (x_{/\!/}/|x| - \sigma) + K_{\sigma,\alpha}$, with $x_{/\!/}/|x| \in \partial V_\eta(\sigma)$, which yields then that

$$\mathrm{dist}(x, \partial K) \geqslant \alpha'|x - x_{/\!/}|. \tag{A.7}$$

For α small enough so that $v + \sigma$ points towards the interior of $K_{\sigma,\alpha}$, let $t > 0$ be such that the harmonic function $V_{K_{\sigma,\alpha}}(\sigma + tv)$ is positive. The existence of t is guaranteed by [DW15, Theorem 1], which gives also $c > 0$ such that $\mathbf{P}(\tau_{tv, K_{\sigma,\alpha}-\sigma} > n) \geqslant cn^{-q'/2}$ for all $n \geqslant 1$. Hence, for x such that $|x - x_{/\!/}| > t$, $x - tv \in x_{/\!/} - \sigma + K_{\sigma,\alpha}$ and

$$\mathbf{P}(\tau_{x, x_{/\!/}-\sigma+K_{\sigma,\alpha}} > n) \geqslant \mathbf{P}(\tau_{x, x-tv+K_{\sigma,\alpha}} > n) \geqslant cn^{-q'/2}. \tag{A.8}$$

Let us assume from now on that $|x - x_{/\!/}| \geqslant t$. Suppose first that $n \geqslant |x - x_{/\!/}|^{2-\varepsilon}$. Thanks to the moments assumption (M1), we can apply the first part of Lemma A.1 to the random walk in $K' := x_{/\!/} - \sigma + K_{\sigma,\alpha}$ with $r = 0$, $\kappa > q' + 2$ small enough and ε' small enough to get

$$\mathbf{P}; \left(\sup_{1 \leqslant l \leqslant n} |S(l)| \geqslant n^{1/2+\varepsilon'}, \tau_{x, K'} \geqslant n \right) \leqslant Cn^{-s},$$

with $s > q'/2$, for $n \geqslant |x - x_{/\!/}|^{2-\varepsilon}$. Hence, the latter inequality together with (A.8) yields

$$\mathbf{P} \left(\sup_{1 \leqslant l \leqslant n} |S(l)| \leqslant n^{1/2+\varepsilon'}, \tau_{x, K'} \geqslant n \right) \geqslant cn^{-q'/2},$$

for $n \geqslant |x - x_{/\!/}|^{2-\varepsilon}$ and some $c > 0$. Since $n \leqslant |x|^{2-\varepsilon}$, choosing ε' small enough implies that

$$\mathbf{P} \left(\sup_{1 \leqslant l \leqslant n} |S(l)| \leqslant |x|^{1-\varepsilon''}, \tau_{x, K'} \geqslant n \right) \geqslant cn^{-q'}.$$

Since, by Lemma A.2, $(x_{/\!/} - \sigma) + K_{\sigma,\alpha} \cap B(x_{/\!/}, \eta|x|) \subset K$, the latter inequality implies that

$$\mathbf{P}\left(\tau_{x,K} \geqslant n\right) \geqslant \mathbf{P} \left(\sup_{1 \leqslant l \leqslant n} |S(l)| \leqslant |x|^{1-\varepsilon''}, \tau_{x, K'} \geqslant n \right) \geqslant c'n^{-q'}$$

for all $n \geqslant |x - x_{/\!/}|^{2-\varepsilon}$ and x large enough.

Suppose now that $n \leqslant |x - x_{/\!/}|^{2-\varepsilon}$. Applying Doob and Rosenthal inequalities together with (A.7) gives

$$\begin{aligned}
\mathbf{P}\left(\tau_x \leqslant n\right) &\leqslant \mathbf{P}\left(\sup_{1 \leqslant k \leqslant n} |S_k| \geqslant \operatorname{dist}(x, \partial K) \right) \\
&\leqslant \mathbf{P}\left(\sup_{1 \leqslant k \leqslant n} |S_k| \geqslant \alpha' \left| x - x_{/\!/} \right| \right) \\
&\leqslant \frac{2n\mathbf{E}\left[|X|^2\right]}{\alpha'^2 \left| x - x_{/\!/} \right|^2} \\
&\leqslant Cn^{-\varepsilon/(2-\varepsilon)}.
\end{aligned}$$

Hence, there exist $c, N > 0$ such that $\mathbf{P}(\tau_x > n) \geqslant c$ for $n > N$ with $n \leqslant |x - x_{/\!/}|^{2-\varepsilon}$.

Suppose finally that $|x - x_{/\!/}| \leqslant t$. By the proof of [DW15, Lemma 14] and the strong irreducibility of S in K, there exist $c, \rho, n_0 > 0$ such that for x large enough, we have

$$\mathbf{P}\left(\left| x + S(n_0) - (x + S(n_0))_{/\!/} \right| \geqslant t, |S(n_0)| \leqslant n_0 R \right) \geqslant \rho.$$

Hence, for $n \geqslant n_0$, by the Markov property and the first part of the proof,

$$\begin{aligned}\mathbf{P}\left(\tau_x > n\right) &\geqslant \mathbf{E}\left[\tau_{x+S(n_0)} \geqslant n - n_0; \left|x + S\left(n_0\right) - \left(x + S\left(n_0\right)\right)_{/\!/}\right| \geqslant t, \left|S\left(n_0\right)\right| \leqslant n_0 R\right] \\ &\geqslant c\rho\left(n - n_0\right)^{-q'}.\end{aligned}$$

This gives the result for n large enough. □

We also give an asymptotic lower bound of the réduite u along the boundary, which is sharper than (2.25).

Lemma A.5. — *Let $\sigma \in \partial\Sigma$ and $q'' > q' > q_\sigma$. Then there exists $c > 0$ such that uniformly on x going to infinity while $x/|x| \to \sigma$ and $\operatorname{dist}(x, \partial K) = o(|x|)$,*

$$u(x) \geqslant c|x|^{p-q''} \operatorname{dist}(x, \partial K)^{q'}.$$

Proof. — We use the same notations as in the previous proof and take $\alpha > 0$ such that $q_{K_{\sigma,\alpha}} = q'$. Let x going to infinity with $x/|x| \to \sigma$. By Lemma A.2, there exists $\varepsilon > 0$ such that for $t > 0$ and x large enough,

$$\mathbf{P}\left(\tau_x^{\mathrm{bm}} \geqslant t\right) \geqslant \mathbf{P}\left(\tau > t, \sup_{0 \leqslant u \leqslant t} |B(u)| \leqslant \varepsilon\left|x - x_{/\!/}\right|\right),$$

where we have set $\tau := \tau^{\mathrm{bm}}_{x, x_{/\!/} - \sigma + K_{\sigma,\alpha}}$.

Let us show that $\mathbf{P}(\tau > t, \sup_{0 \leqslant u \leqslant t} |B(u)| > \varepsilon|x - x_{/\!/}|)$ is negligible in comparison with $\mathbf{P}(\tau > t)$, by adapting the reflection principle to a Brownian motion in a cone. By conditioning on the last time θ when B reaches the sphere of radius $\varepsilon|x - x_{/\!/}|$, we get

$$\begin{aligned}&\mathbf{P}\left(\tau > t, \sup_{0 \leqslant u \leqslant t} |B(u)| > \varepsilon\left|x - x_{/\!/}\right|, |B(t)| \leqslant \varepsilon\left|x - x_{/\!/}\right|\right) \\ &\qquad\qquad \leqslant \mathbf{P}\Big(\theta < t < \tau, \Big\langle B(t) - B(\theta), B(\theta)\Big\rangle < 0\Big).\end{aligned}$$

We denote by B^θ the process $B(\theta + u) - B(\theta)$, which is a Brownian meander independent of $(B(u))_{0 \leqslant u \leqslant \theta}$, see for example [RY05]. Denote by K^+ (resp. K^-) the intersection of K with the set

$$\left\{v \in \mathbf{R}^d : \langle v, B(\theta)\rangle > \langle B(\theta), B(\theta)\rangle\right\} \quad \left(\text{resp. } \left\{v \in \mathbf{R}^d : \langle v, B(\theta)\rangle < \langle B(\theta), B(\theta)\rangle\right\}\right),$$

and let s denote the symmetry with respect to the hyperplane $B(\theta) + B(\theta)^\perp$. Since we have $s(K^-) \subset K^+$,

$$\Big(B(\theta) + B^\theta(u)\Big)_{0 \leqslant u \leqslant t-\theta} \subset K^- \quad \text{implies that} \quad s\left(\Big(B(\theta) + B^\theta(u)\Big)_{0 \leqslant u \leqslant t-\theta}\right) \subset K^+.$$

Moreover, s turns a negative meander into a positive one, and is thus measure preserving. Therefore,

$$\begin{aligned}&\mathbf{P}\left(s\left(\left\{\Big(B(\theta) + B^\theta(u)\Big)_{0 \leqslant u \leqslant t-\theta} \subset K^-\right\}\right)\right) \\ &\qquad\qquad = \mathbf{P}\left(\left\{\Big(B(\theta) + B^\theta(u)\Big)_{0 \leqslant u \leqslant t-\theta} \subset K^-\right\}\right).\end{aligned}$$

This implies that

$$\mathbf{P}\Big(\theta < t < \tau, \Big\langle B(t) - B(\theta), B(\theta)\Big\rangle < 0\Big) \leqslant \mathbf{P}\Big(\theta < t < \tau, \Big\langle B(t) - B(\theta), B(\theta)\Big\rangle > 0\Big).$$

Since $\langle B(t) - B(\theta), B(\theta)\rangle > 0$ implies that $|B(t)| > |B(\theta)|$, we get finally

$$\mathbf{P}\left(\tau > t, \sup_{0\leqslant u\leqslant t} |B(u)| > \varepsilon \left|x - x_{/\!/}\right|, |B(t)| \leqslant \varepsilon \left|x - x_{/\!/}\right|\right) \\ \leqslant \mathbf{P}\left(\tau > t, \sup_{0\leqslant u\leqslant t} |B(u)| > \varepsilon|x - x_{/\!/}|, |B(t)| > \varepsilon|x - x_{/\!/}|\right).$$

Since

$$\mathbf{P}\left(\tau > t, \sup_{0\leqslant u\leqslant t} |B(u)| > \varepsilon \left|x - x_{/\!/}\right|, |B(t)| > \varepsilon \left|x - x_{/\!/}\right|\right) \\ = \mathbf{P}\left(\tau > t, |B(t)| > \varepsilon \left|x - x_{/\!/}\right|\right),$$

we have

$$\begin{aligned} &\mathbf{P}\left(\tau > t, \sup_{0\leqslant u\leqslant t} |B(u)| > \varepsilon \left|x - x_{/\!/}\right|\right) \\ &\qquad = \mathbf{P}\left(\tau > t, \sup_{0\leqslant u\leqslant t} |B(u)| > \varepsilon \left|x - x_{/\!/}\right|, |B(t)| \leqslant \varepsilon \left|x - x_{/\!/}\right|\right) \\ &\qquad + \mathbf{P}\left(\tau > t, \sup_{0\leqslant u\leqslant t} |B(u)| > \varepsilon \left|x - x_{/\!/}\right|, |B(t)| > \varepsilon \left|x - x_{/\!/}\right|\right) \\ &\qquad \leqslant 2\mathbf{P}\left(\tau > t, \sup_{0\leqslant u\leqslant t} |B(u)| > \varepsilon \left|x - x_{/\!/}\right|, |B(t)| > \varepsilon \left|x - x_{/\!/}\right|\right) \\ &\qquad \leqslant 2\mathbf{P}\left(\tau > t, |B(t)| > \varepsilon \left|x - x_{/\!/}\right|\right). \end{aligned}$$

Therefore, using [DW15, Lemma 18] yields for $t = o(|x - x_{/\!/}|^2)$,

$$\mathbf{P}\left(\tau > t, \sup_{0\leqslant u\leqslant t} |B(u)| > \varepsilon \left|x - x_{/\!/}\right|\right) = o\left(\mathbf{P}(\tau > t)\right),$$

and finally, introducing the réduite $u_{K_{\sigma,\alpha}}$ of the cone $K_{\sigma,\alpha}$,

$$\mathbf{P}\left(\tau > t, \sup_{0\leqslant u\leqslant t} |B(u)| \leqslant \varepsilon \left|x - x_{/\!/}\right|\right) \sim \frac{u_{K_{\sigma,\alpha}}\left(x - x_{/\!/}\right)}{t^{q'/2}},$$

uniformly for t and x such that $x - x_{/\!/} = o(\sqrt{t})$. Since by Lemma A.2,

$$\mathbf{P}\left(\tau > t, \sup_{0\leqslant u\leqslant t} |B(u)| \leqslant \varepsilon \left|x - x_{/\!/}\right|\right) \leqslant \mathbf{P}\left(\tau_x^{\mathrm{bm}} > t\right),$$

and $u_{K_{\sigma,\alpha}}(x - x_{/\!/}) \geqslant c\,\mathrm{dist}(x - x_{/\!/}, \partial K_{\sigma,\alpha})^{q'}$ by (2.25), we have

$$\mathbf{P}\left(\tau_x^{\mathrm{bm}} > t\right) \geqslant \frac{c\,\mathrm{dist}\left(x - x_{/\!/}, \partial K_{\sigma,\alpha}\right)^{q'}}{t^{q'/2}}.$$

Usual Gaussian estimates in K (see for example [RT20, Appendix A]) yields therefore

$$\frac{u(x)}{t^{p/2}} \geqslant c\mathbf{P}\left(\tau_x^{\mathrm{bm}} > t\right) \geqslant \frac{c\,\mathrm{dist}\left(x - x_{/\!/}, \partial K_{\sigma,\alpha}\right)^{q'}}{t^{q'/2}}$$

for x going to infinity with $x/|x| \to \sigma$, $\mathrm{dist}(x - x_{/\!/}, \partial K_{\sigma,\alpha}) = o(|x|)$ and $x - x_{/\!/} = o(\sqrt{t})$. Hence, evaluating the above inequality at $t = |x|^{2(p-q'')/(p-q')}$ for $q'' > q'$ small enough gives for any $q' > q$ the existence of $c > 0$ such that

$$u(x) \geqslant c|x|^{p-q''}\,\mathrm{dist}\left(x - x_{/\!/}, \partial K_{\sigma,\alpha}\right)^{q'}. \qquad \square$$

BIBLIOGRAPHY

[AD01] Larbi Alili and Ronald A. Doney, *Martin boundaries associated with a killed random walk*, Ann. Inst. Henri Poincaré, Probab. Stat. **37** (2001), no. 3, 313–338. ↑560

[AGKV05] Valeriĭ I. Afanasyev, Jochen Geiger, Götz-Dietrich Kersting, and Vladimir A. Vatutin, *Criticality for branching processes in random environment*, Ann. Probab. **33** (2005), no. 2, 645–673. ↑560

[BBMKM16] Alin Bostan, Mireille Bousquet-Mélou, M. Kauers, and Stephen Melczer, *On* 3-*dimensional lattice walks confined to the positive octant*, Ann. Comb. **20** (2016), no. 4, 661–704. ↑560

[BBMM21] Alin Bostan, Mireille Bousquet-Mélou, and Stephen Melczer, *Counting walks with large steps in an orthant*, J. Eur. Math. Soc. (JEMS) **23** (2021), no. 7, 2221–2297. ↑563

[BBO05] Philippe Biane, Philippe Bougerol, and Neil O'Connell, *Littelmann paths and Brownian paths*, Duke Math. J. **130** (2005), no. 1, 127–167. ↑561

[Bia91] Philippe Biane, *Quantum random walk on the dual of SU*(n), Probab. Theory Relat. Fields **89** (1991), no. 1, 117–129. ↑561

[Bia92] ———, *Équation de Choquet–Deny sur le dual d'un groupe compacts*, Probab. Theory Relat. Fields **94** (1992), no. 1, 39–51. ↑561

[BMM10] Mireille Bousquet-Mélou and Marni Mishna, *Walks with small steps in the quarter plane*, Algorithmic probability and combinatorics. Papers from the AMS special sessions, Chicago, IL, USA, October 5–6, 2007 and Vancouver, BC, Canada, October 4–5, 2008, Contemporary Mathematics, vol. 520, American Mathematical Society, 2010, pp. 1–39. ↑560

[BMS15] Aymen Bouaziz, Sami Mustapha, and Mohamed Sifi, *Discrete harmonic functions on an orthant in* $\mathbb{Z}^d$, Electron. Commun. Probab. **20** (2015), article no. 52 (13 pages). ↑561

[BS97] Rodrigo Bañuelos and Robert G. Smits, *Brownian motion in cones*, Probab. Theory Relat. Fields **108** (1997), no. 3, 299–319. ↑562, 563

[BT12] Sylvain Billiard and Viet Chi Tran, *A general stochastic model for sporophytic self-incompatibility*, J. Math. Biol. **64** (2012), no. 1-2, 163–210. ↑560

[CB83] Jacob W. Cohen and Onno J. Boxma, *Boundary value problems in queueing system analysis*, North-Holland Mathematics Studies, vol. 79, North-Holland, 1983. ↑560

[CD19] Francesco Caravenna and Ronald A. Doney, *Local large deviations and the strong renewal theorem*, Electron. J. Probab. **24** (2019), article no. 72 (48 pages). ↑566

[CdL13] Rama Cont and Adrien de Larrard, *Price dynamics in a Markovian limit order market*, SIAM J. Financial Math. **4** (2013), 1–25. ↑560

[DHRS18] Thomas Dreyfus, Charlotte Hardouin, Julien Roques, and Michael F. Singer, *On the nature of the generating series of walks in the quarter plane*, Invent. Math. **213** (2018), no. 1, 139–203. ↑560

[DIM77] Richard T. Durrett, Donald L. Iglehart, and Douglas R. Miller, *Weak convergence to Brownian meander and Brownian excursion*, Ann. Probab. **5** (1977), 117–129. ↑590

[Don98] Ronald A. Doney, *The Martin boundary and ratio limit theorems for killed random walks*, J. Lond. Math. Soc. **58** (1998), no. 3, 761–768. ↑560

[Don12] ———, *Local behaviour of first passage probabilities*, Probab. Theory Relat. Fields **152** (2012), no. 3-4, 559–588. ↑580

[DSW18] Denis Denisov, Alexander Sakhanenko, and Vitali Wachtel, *First-passage times for random walks with nonidentically distributed increments*, Ann. Probab. **46** (2018), no. 6, 3313–3350. ↑587

[Dur78] Richard T. Durrett, *Conditioned limit theorems for some null recurrent Markov processes*, Ann. Probab. **6** (1978), 798–828. ↑588, 589

[DW10] Denis Denisov and Vitali Wachtel, *Conditional limit theorems for ordered random walks*, Electron. J. Probab. **15** (2010), article no. 11. ↑560

[DW15] ———, *Random walks in cones*, Ann. Probab. **43** (2015), no. 3, 992–1044. ↑562, 564, 566, 567, 568, 569, 570, 571, 572, 573, 574, 575, 577, 580, 588, 589, 590, 591, 593, 595, 599, 603, 605

[DW16] ———, *An exact asymptotics for the moment of crossing a curved boundary by an asymptotically stable random walk*, Theory Probab. Appl. **60** (2016), no. 3, 481–500. ↑580, 590

[DW19] ———, *Alternative constructions of a harmonic function for a random walk in a cone*, Electron. J. Probab. **24** (2019), article no. 92 (26 pages). ↑562, 564

[DW20] Jetlir Duraj and Vitali Wachtel, *Invariance principles for random walks in cones*, Stochastic Processes Appl. **130** (2020), no. 7, 3920–3942. ↑566, 570, 571, 580

[EK08] Peter Eichelsbacher and W. König, *Ordered random walks*, Electron. J. Probab. **13** (2008), 1307–1336. ↑560

[Ess68] Carl-Gustav Esseen, *On the concentration function of a sum of independent random variables*, Z. Wahrscheinlichkeitstheor. Verw. Geb. **9** (1968), 290–308. ↑584

[FIM17] Guy Fayolle, Roudolf Iasnogorodski, and Vadim Malyshev, *Random walks in the quarter plane. Algebraic methods, boundary value problems, applications to queueing systems and analytic combinatorics*, Probability Theory and Stochastic Modelling, vol. 40, Springer, 2017, second edition, previously published with the subtitle Algebraic methods, boundary value problems and applications. ↑560

[FN71] Dao Ha Fuk and Sergey V. Nagaev, *Probabilistic inequalities for sums of independent random variables*, Theory Probab. Appl. **16** (1971), no. 4, 643–660. ↑577, 583

[GSC11] Pavel Gyrya and Laurent Saloff-Coste, *Neumann and Dirichlet heat kernels in inner uniform domains*, Astérisque, vol. 336, Société Mathématique de France, 2011. ↑562, 573

[GT01] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, reprint of the 1998 ed., Classics in Mathematics, Springer, 2001. ↑563

[GZ09] Friedrich Götze and Andrei Yu Zaĭtsev, *The accuracy of approximation in the multidimensional invariance principle for sums of independent identically distributed random vectors with finite moments*, Zap. Nauchn. Semin. (POMI) **368** (2009), no. 15, 110–121, 283–284. ↑572, 573

[HUL01] Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal, *Fundamentals of Convex Analysis*, Grundlehren. Text Editions, vol. 305-306, Springer, 2001. ↑600

[IR08] Irina Ignatiouk-Robert, *Martin boundary of a killed random walk on a half-space*, J. Theor. Probab. **21** (2008), no. 1, 35–68. ↑560

[IR09] ———, *Martin boundary of a killed random walk on* $\mathbb{Z}_+^d$, https://arxiv.org/abs/0909.3921, 2009. ↑560, 565

[IR21] ———, *Harmonic functions of random walks in a semigroup via ladder heights*, J. Theor. Probab. **34** (2021), no. 1, 34–80. ↑561

[IRL10] Irina Ignatiouk-Robert and Christophe Loree, *Martin boundary of a killed random walk on a quadrant*, Ann. Probab. **38** (2010), no. 3, 1106–1142. ↑560, 565

[LR16] Cédric Lecouvey and Kilian Raschel, *t-Martin boundary of killed random walks in the quadrant*, Séminaire de Probabilités XLVIII, Lecture Notes in Mathematics, vol. 2168, Springer, 2016, pp. 305–323. ↑561

[Mog73] Anatolii A. Mogul'skii, *Absolute estimates for moments of certain boundary functionals*, Theory Probab. Appl. **18** (1973), 340–347. ↑587

[MS19] Sami Mustapha and Mohamed Sifi, *Discrete harmonic functions in Lipschitz domains*, Electron. Commun. Probab. **24** (2019), article no. 58 (15 pages). ↑561

[NS66] Peter E. Ney and Frank L. Spitzer, *The Martin boundary for random walk*, Trans. Am. Math. Soc. **121** (1966), 116–132. ↑560, 565

[PW92] Massimo A. Picardello and Wolfgang Woess, *Martin boundaries of Cartesian products of Markov chains*, Nagoya Math. J. **128** (1992), 153–169. ↑561

[Ras11] Kilian Raschel, *Green functions for killed random walks in the Weyl chamber of Sp*(4), Ann. Inst. Henri Poincaré, Probab. Stat. **47** (2011), no. 4, 1001–1019. ↑560, 561

[Ras14] ———, *Random walks in the quarter plane, discrete harmonic functions and conformal mappings. With an appendix by S. Franceschi*, Stochastic Processes Appl. **124** (2014), no. 10, 3147–3178. ↑561

[RT20] Kilian Raschel and Pierre Tarrago, *Boundary behavior of random walks in cones*, Markov Process. Relat. Fields **26** (2020), no. 4, 711–756. ↑561, 566, 568, 573, 574, 599, 600, 606

[RY05] Daniel Revuz and Marc Yor, *Continuous martingales and Brownian motion*, 3rd. corrected printing ed., Grundlehren der Mathematischen Wissenschaften, vol. 293, Springer, 2005. ↑604

[Saw97] Stanley A. Sawyer, *Martin boundaries and random walks*, Harmonic functions on trees and buildings. Workshop on harmonic functions on graphs, New York, NY, October 30–November 3, 1995, Contemporary Mathematics, vol. 206, American Mathematical Society, 1997, pp. 17–44. ↑560

[Spi76] Frank L. Spitzer, *Principles of Random Walk*, 2nd ed., Graduate Texts in Mathematics, vol. 34, Springer, 1976. ↑572

[Ste90] John R. Stembridge, *Nonintersecting paths, Pfaffians, and plane partitions*, Adv. Math. **83** (1990), no. 1, 96–131. ↑560

[Uch98] Kôhei Uchiyama, *Green's functions for random walks on* $\mathbb{Z}^N$, Proc. Lond. Math. Soc. **77** (1998), no. 1, 215–240. ↑569, 598

[Uch14] ———, *Green's functions of random walks on the upper half plane*, Tôhoku Math. J. **66** (2014), no. 2, 289–307. ↑565

[Var99] Nicolas Th. Varopoulos, *Potential theory in conical domain*, Math. Proc. Camb. Philos. Soc. **125** (1999), no. 2, 335–384. ↑561, 569

[Var09] ———, *The discrete and classical Dirichlet problem*, Milan J. Math. **77** (2009), 397–436. ↑561

[Wil68] John A. Williamson, *Random walks and Riesz kernels*, Pac. J. Math. **25** (1968), 393–415. ↑598

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